Fermions, fermions, fermions!

Jan Zaanen
A black hole full of answers

Jan Zaanen

A facet of string theory, the currently favoured route to a ‘theory of everything’, might help to explain some properties of exotic matter phases — such as some peculiarities of high-temperature superconductors.

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Plan

Koenraad: AdS/CFT and the emergent Fermi liquid


2. Geometrizing the fermion signs: the Ceperley path integral and the conformal Feynmannian backflow state.
The quantum in the kitchen: Landau’s miracle

Electrons are waves

Pauli exclusion principle: every state occupied by one electron

Unreasonable: electrons strongly interact!!

Landau’s Fermi-liquid: the highly collective low energy quantum excitations are like electrons that do not interact.
‘Shankar/Polchinski’ functional renormalization group

**UV:** weakly interacting Fermi gas

Integrate momentum shells: functions of running coupling constants

**All interactions (except marginal Hartree) irrelevant** => Scaling limit might be perfectly ideal Fermi-gas
The end of weak coupling

Strong interactings:

Fermi gas as UV starting point does not make sense!

=> ‘emergent’ Fermi liquid fixed point remarkably resilient (e.g. 3He)

=> Non Fermi-liquid/non ‘Hartree-Fock’ (BCS etc) states of fermion matter?
Fermion sign problem

Imaginary time path-integral formulation

\[ Z = \text{Tr} \exp(-\beta \hat{H}) = \int dR \rho(R, R; \beta) \]

\[ R = (r_1, \ldots, r_N) \in \mathbb{R}^{N_d} \]

\[ \rho_{B/F}(R, R; \beta) = \frac{1}{N!} \sum_p (-1)^p \rho_D(R, PR; \beta) \]

\[ = \frac{1}{N!} \sum_p (-1)^p \int_{R \to PR} D\mathbf{R}(\tau) \exp \left\{ -\frac{1}{h} \int_0^{h/T} d\tau \left( \frac{m}{2} \dot{\mathbf{R}}^2(\tau) + V(\mathbf{R}(\tau)) \right) \right\} \]

Boltzmannons or Bosons:
- integrand non-negative
- probability of equivalent classical system: (crosslinked) ringpolymers

Fermions:
- negative Boltzmann weights
- non probablistic: NP-hard problem (Troyer, Wiese)!!!
Phase diagram high Tc superconductors


‘Stripy stuff’, spontaneous currents, phase fluctuations..

\[ \text{BCS} = \sum_k \varepsilon_k c_k^\dagger c_k \]

The return of normalcy

\[ \hat{n}_{\text{WCd}} = \Pi \left( n_\uparrow^\varepsilon_\uparrow \varepsilon_\downarrow c_{\uparrow \downarrow}^\dagger c_{\downarrow \uparrow} \right) |\varepsilon_\uparrow, \varepsilon_\downarrow, c_{\uparrow \downarrow}, c_{\downarrow \uparrow}, \rangle \]
Fermionic quantum phase transitions in the heavy fermion metals


JZ, Science 319, 1205 (2008)

\[ m^* = \frac{1}{E_F} \]

\[ E_F \rightarrow 0 \Rightarrow m^* \rightarrow \infty \]
A UV that is strongly interacting critical …

UV governed by conformal invariance, no Fermi-surface, no Fermi energy

Emergent heavy Fermi-liquid in the IR that can disappear at a QPT

This is effortlessly encoded in Koenraad’s AdS/CFT!
Emergent non Fermi-liquids with Fermi-surfaces

Hong Liu’s emergent 2D CFT “AdS/ARPES”:

- Truly critical: renormalized Fermi energy is zero.
- But the Fermi surface is remembered as singularity structure in momentum space.

How can a state be conformal while it remembers the reciprocal length Fermi momentum? Bosons cannot!!

Critical Fermi-surface (Senthil): phenomenological.
Gauge theories (Sung-Sik, Subir): Shankar/Polchinski attitude.
Geometrizing Fermi-Dirac statistics

Ceperley’s path integral: encoding Fermi-Dirac statistics in geometry: the nodal hypersurface.

- Fermi-liquid: Fermi-energy is encoded in the local geometry but the Fermi-surface is encoded globally

  F. Krueger et al., arXiv:0802.2455

- Feynmannian backflow ansatz: nodal geometry turns fractal, the state is conformal but room for globally encoded Fermi-surface information.

  F. Krueger, JZ, arXiv:0804.2161
The nodal hypersurface

Antisymmetry of the wave function

\[ \Psi(r_1, \ldots, r_i, \ldots, r_j, \ldots, r_N) = -\Psi(r_1, \ldots, r_j, \ldots, r_i, \ldots, r_N) \]

Free Fermions

\[ \Psi_0(R) \sim \text{Det} \left( e^{ik_i r_j} \right)_{ij} \]

Pauli hypersurface

\[ P = \bigcup_{i \neq j} P_{ij} \]

\[ P_{ij} = \{ R \in \mathbb{R}^{Nd} | r_i = r_j \} \]

\[ \text{dim} P = N d - d \]

Nodal hypersurface

\[ \Omega = \{ R \in \mathbb{R}^{Nd} | \Psi(R) = 0 \} \]

\[ \text{dim} \Omega = N d - 1 \]

Test particle
Constrained path integrals

Formally we can solve the sign problem!!

\[
\rho_F(R, R'; \beta) = \frac{1}{N!} \sum_{\mathcal{P}, \text{even}} \int_{\gamma: R \to R'} DR(\tau) \exp \left\{ -\frac{1}{\hbar} \int_0^{\hbar/T} d\tau \left( \frac{m}{2} \dot{R}^2(\tau) + V(R(\tau)) \right) \right\}
\]

\[
\Gamma(R, R') = \{ \gamma : R \to R' | \rho_F(R, R(\tau); \tau) \neq 0 \}
\]

Self-consistency problem:
Path restrictions depend on \( \rho_F \) !

Ceperley path integral: Fermi gas in momentum space

Single particle propagator:

\[ g(k, k', \tau) = 2\pi \delta(k - k') e^{-\frac{|k|^2\tau}{2\hbar m}} \]

\textbf{single particle momentum conserved}

N particle density matrix:

\[ \rho_F(K, K', \tau) = \frac{1}{N!} \sum_P (-1)^P \prod_{i=1}^N g(k_i, k'_i, \tau) \]

\[ = \frac{1}{N!} e^{-\sum_{i=1}^N \frac{|k_i|^2\tau}{2\hbar m}} \sum_P (-1)^P \prod_{i=1}^N 2\pi \delta(k_i - k'_i) \]

\[ = \prod_{k_1 \neq k_2 \neq \ldots \neq k_N} 2\pi \delta(k_i - k'_i) e^{-\frac{|k_i|^2\tau}{2\hbar m}} \]

\textbf{‘harmonic potential’}

\[ \tau \]

\[ 1 \text{-cycles} \]
Fermi gas = cold atom Mott insulator in harmonic trap!

\[ \rho_F (K, K''; \tau) = \prod_{k_1 \neq k_2 \neq \ldots \neq k_N} 2\pi \delta(k_i - k_i') e^{-\frac{|k_i|^2 \tau}{2\hbar m}} \]

**Reading the worldline picture**

**Fermi-energy: confinement energy imposed by local geometry**

\[ l^2(\tau) = \langle (\mathbf{r}_i(\tau) - \mathbf{r}_i(0))^2 \rangle = 2dD\tau = 2d\frac{\hbar}{2m}\tau \]

\[ l^2(\tau_c) \sim r_s^2 \rightarrow \tau_c \sim \frac{1}{2d} \frac{2m}{\hbar} n^{-2/d} \]

\[ \hbar\omega_c = \frac{\hbar}{\tau_c} \sim d\frac{\hbar^2}{2m} n^{2/d} \sim E_F \]

**Fermi surface encoded globally:** \( \rho_F = Det\left(e^{ik_i r_j}\right) = 0 \)

Change in coordinate of one particle changes the nodes everywhere

**Finite T:** \( \rho_F = (4\pi\lambda\beta)^{-dN/2} Det\left[\exp\left(-\frac{(r_i - r_{j0})^2}{4\lambda\tau}\right)\right] \)

\[ \lambda = \frac{\hbar^2}{2M} \]

**Non-locality length:** \( \lambda_{nl} = v_F \tau_{inel} = v_F \left(\frac{E_F}{k_B T}\right) \left(\frac{\hbar}{k_B T}\right) \)

Average node to node spacing

\[ \sim r_s = \left(\frac{V}{N}\right)^{1/d} = n^{-1/d} \]
Key to fermionic quantum criticality

At the QCP scale invariance, no $E_F$ → Nodal surface has to become fractal !!!
Fractal Cauliflower (romanesco)
Quantum Matter
Geometrizing Fermi-Dirac statistics

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Vacuum structure

Long time, zero temperature:

\[ \rho_F(R,R(\tau);\tau \to \infty) = \Psi^*(R)\Psi(R(\infty)) \]

IR fermionic information encoded in the ground state wavefunction.

Need a wave function ansatz!
Hydrodynamic backflow

Classical fluid: incompressible flow

Feynman-Cohen: mass enhancement in $^4$He

Wave function ansatz for "foreign" atom moving through He superfluid with velocity small compared to sound velocity:

$$g(r) \sim \frac{kr}{r^3} \rightarrow \Psi = \phi \exp[ik \left( r_A + \sum_{i \neq A} \frac{r_i - r_A}{r_i^3} \right)]$$

Backflow wavefunctions in Fermi systems

$$\psi_{bf}(R) \sim \text{Det} \left( e^{ik_i \tilde{r}_{ij}} \right)_{ij}$$

$$\tilde{r}_j = r_j + \sum_{l(\neq j)} \eta(r_{jl})(r_j - r_l)$$

Widely used for node fixing in QMC

→ Significant improvement of variational GS energies
Frank’s fractal nodes ...

Feynman’s fermionic backflow wavefunction:

\[ \psi_{bf}(R) \sim \text{Det} \left( e^{ik_i \tilde{r}_j} \right)_{ij} \]

\[ \tilde{r}_j = r_j + \sum_{l(\neq j)} \eta(r_{jl})(r_j - r_l) \]

\[ \eta(r) = \frac{a^3}{r^3 + r_0^3} \]

Frank Krüger
Extracting the fractal dimension

The Hausdorff dimension. The Hausdorff dimension of a metric space $X$, $\dim_H(X)$, is the infimum of the numbers $\alpha$ with the following property: For any $\epsilon > 0$ there is a $\delta > 0$ and a cover $\mathcal{U}$ of $X$ such that the sets $B \in \mathcal{U}$ all have diameter $|B|$ smaller than $\delta$ and

$$\sum_{B \in \mathcal{U}} (|B|)^\alpha < \epsilon.$$ 

The correlation integral:

$$C'(r) = \lim_{n \to \infty} \frac{1}{n^2} \sum_{i,j=1}^{n} \Theta(r - |r_i - r_j|)$$

$$= \int_0^r d^D r' c(r')$$

For fractals:

$$C'(r) \sim r^\nu, \quad \nu \leq \dim_H$$

Inequality very tight, relative error below 1%

Grassberger & Procaccia, PRL (1983)
Fractal dimension of the nodal surface

Calculate the correlation integral \( C(r) \sim r^\nu \) on random d=2 dimensional cuts

\[
Nd - 1 < D_H = N \nu_d < Nd
\]

\[
d - \frac{1}{N} < \nu_d < d
\]

\[
N = 13 : \quad \nu = 1.976 \pm 0.012 \\
\quad \rightarrow D_H = 25 + (0.69 \pm 0.16)
\]

\[
N = 29 : \quad \nu = 1.982 \pm 0.008 \\
\quad \rightarrow D_H = 57 + (0.48 \pm 0.23)
\]

Backflow turns nodal surface into a fractal !!!
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Turning on the backflow

Nodal surface has to become fractal !!!

Try backflow wave functions

\[ \psi_{bf}(R) \sim \text{Det} \left( e^{ik_i \tilde{r}_j} \right)_{ij} \]

\[ \tilde{r}_j = r_j + \sum_{l(\neq j)} \eta(r_{ji})(r_j - r_l) \]

\[ \eta(r) = \frac{a^3}{r^3 + r_0^3} \]

Collective (hydrodynamic) regime:

\[ a \gg r_s \]
MC calculation of $n(k)$

$$\frac{m}{m^*} \propto \left(1 - \frac{a}{a_c}\right)^3$$

Divergence of effective mass as $a \rightarrow a_c$
The fixed point Hamiltonian

\[ \psi_{bf}(\mathbf{R}) \sim \text{Det} \left( e^{i\mathbf{k}_l \mathbf{r}_j} \right)_{ij} \]

\[ \mathbf{\tilde{r}}_j = \mathbf{r}_j + \sum_{l(\neq j)} \eta(r_{jl})(\mathbf{r}_j - \mathbf{r}_l) \]

\[ \Rightarrow |k_1,\ldots,k_N\rangle_{bf} = \int_{q_1,\ldots,q_N} \Gamma_{q_1,\ldots,q_N} |k_1 + q_1,\ldots,k_N + q_N\rangle_{\text{bare}} \]

turns singular at the QPT.

It is the ground state of a Fermi-gas of backflow particles:

\[ H = \sum_k \varepsilon_k \hat{c}_k \hat{c}^+_k \]

Expressed in bare particles:

\[ H \propto \sum_k \varepsilon_k c_k^+ c_k + \sum_{N=2}^{\infty} \left( \frac{a}{r_s} \right)^N \sum_{\{kq\}} f(\{k,q\})(c^+_k c_k)^N \]

- At the critical point \( a \rightarrow r_s \) the fixed point Hamiltonian reveals a divergence in \( N \) where \( N \) refers to \( N \)-body interaction!

- No symmetry change, criticality is entirely of ‘statistical’ nature (information in nodal surface)!
Where is the Fermi-surface?

Fractality originating in the non-locality of the nodes:

\[ \rho_F = \text{Det} \left( \exp \left[ i k_i (r_j + \sum_j \eta(r_{ij})(r_i - r_j)) \right] \right) = 0 \]

=> Dynamics becomes conformal (vanishing of renormalized Fermi energy).

But Fermi-surface information is also globally wired into the nodes through backflow particle gas: \( |k| < k_F \).

Conjecture: there have to be singularities at the ‘remnant’ Fermi surface that are not conflicting with scale invariant dynamics because of the local-global dichotomy.
In conclusion …

Fermions at finite density: the fermion signs are wrecking established mathematical machinery, but it leaves room for BIG surprises.

AdS/CFT has started to show it muscles: the emergent Fermi-liquid (Koenraad), the singular-, marginal - … Fermi liquids (Hong).

Does a critical Fermion liquid need a Fermi-surface? The scale invariant backflow state suggests that the Fermi-surface is wired into the global structure of the nodal surface that is locally fractal.
Outlook

AdS/CFT is a very rich mathematical machine: rich enough to literally describe the fermion side of condensed matter?

- Magnetic fields: Landau quantization of fermions? Fractional quantum Hall??

- Bosonic (chiral) phase transitions: what happens with the coexisting Fermions?

- Hartnoll’s ‘hairy black hole’: is the superconductivity BCS like?

- Total currents versus fermionic currents: linear resistivity??

- Fermionic pair susceptibilities: BCS mechanism for fermionic quantum critical states??

- Fermions in the non-relativistic AdS/CFT ??
In conclusion …

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Quantum critical transport in heavy fermion systems

Blue = Fermi liquid:
\[ b \propto \Upsilon \]

Yellow = quantum critical regime:
\[ b \propto \Upsilon \]

Critical Cuprates are Planckian Dissipators

van der Marel, JZ, … Nature 2004:

Optical conductivity QC cuprates

Frequency less than temperature:

\[ \sigma_1(\omega, T) = \frac{1}{4\pi} \frac{\omega^2 \tau_r}{1 + \omega^2 \tau^2_r}, \quad \tau_r = A \frac{\hbar}{k_B T} \]

\[ \Rightarrow \left[ \frac{\hbar}{k_B T \sigma_1} \right] = \text{const.} \left(1 + A^2 \left[ \frac{\hbar \omega}{k_B T} \right]^2 \right) \]

A = 0.7: the normal state of optimally doped cuprates is a Planckian dissipator!
Why the Wilsonian renormalization group fails …

Phillips Chamon

(a) **Charge conservation** (‘hydrodynamics’) imposes engineering scaling dimensions on current

(b) Scale invariance: assume one diverging length scale.

\[
\sigma(\omega, T) = \frac{e^2}{\hbar} \left( \frac{k_B T}{\hbar c} \right)^{\frac{d-2}{2}} \Sigma \left( \frac{\hbar \omega}{k_B T} \right)
\Rightarrow \sigma_{DC}(T) = \frac{1}{\hbar} \Sigma(0) \left( \frac{k_B T}{\hbar c} \right)^{\frac{d-2}{2}}
\]

\[
\sigma_{DC} \propto \frac{1}{T} ??
\]

d = 2 or 3 implies that z < 0 !?
Empty
Empty