A supersymmetric model for lattice fermions

Liza Huijse – Harvard University
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KITP

K. Schoutens, UvA,
P. Fendley (Uva), J. Halverson (UVa), N. Moran (ENS),
J. Vala (Maynooth), D. Mehta (Syracuse),
B. Bauer (ETH/Station Q), M. Troyer (ETH), E. Berg (Harvard)

Introduction

Strongly interacting electron systems

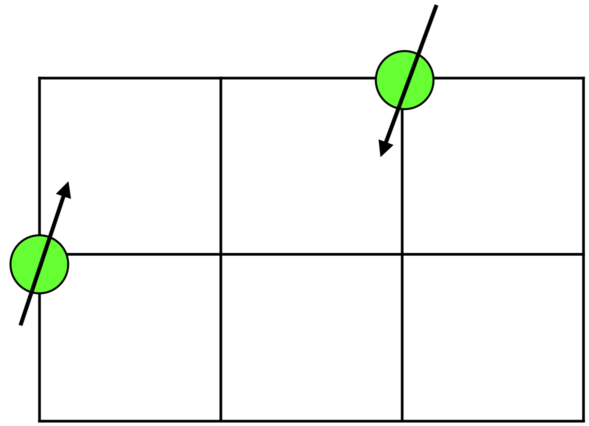
Key examples:

- High T_c superconductors
- Heavy fermion compounds

Challenge conventional theoretical techniques

Introduction

Lattice models



configurations:

electrons located on the sites of an ionic lattice in a solid

Hamiltonian:

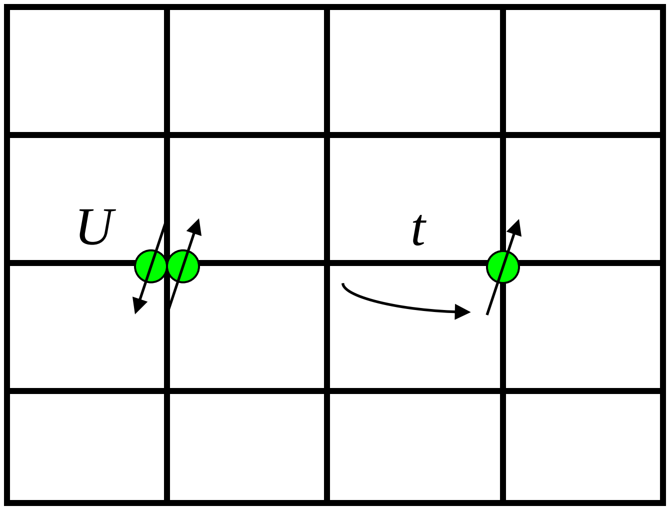
typically a sum of kinetic (hopping) terms and short range repulsive interactions

Introduction

Hubbard model (1963)

Coulomb repulsion → onsite repulsion *U*

$$H = -t \sum_{\langle ij \rangle} c^{\dagger}_{i\sigma} c_{j\sigma} + U \sum_{i} n_{i\uparrow} n_{i\downarrow}$$



$$n_{i\sigma} = c^{+}_{i\sigma} c_{i\sigma}$$

$$\left\{ c^{+}_{i\sigma}, c_{j\sigma} \right\} = \delta_{ij}$$

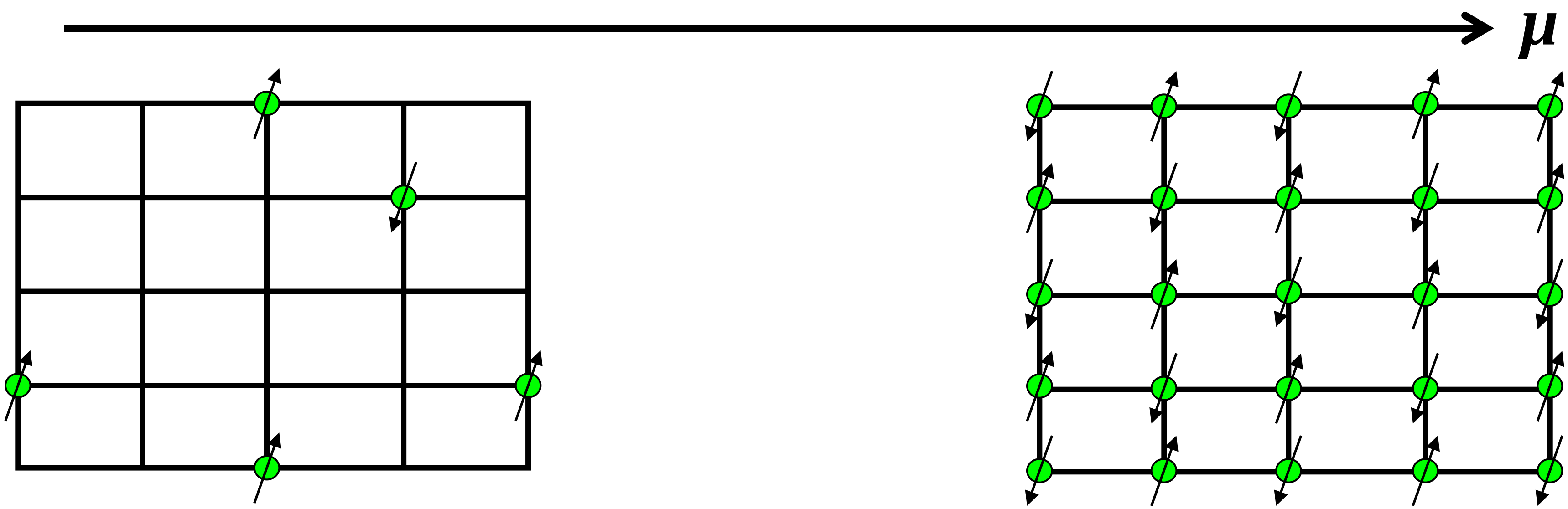
$$\sigma = \uparrow, \downarrow$$

Introduction

Hubbard model

- Kinetics dominated n<<1, U<<t → Fermi liquid
- Interaction dominated $U>>t$

  → Mott insulator at half filling



Introduction

Strongly interacting electron systems

Challenge: Intermediate densities

Conventional techniques fail

- Mean field results are unreliable
- Bethe Ansatz does not work in $D > 1$
- Quantum Monte Carlo suffers from sign problem
- …

→ Too difficult

Our work

A model for strongly interacting fermions

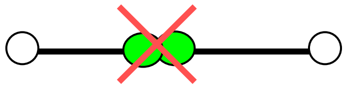
1. Simplifications/adjustments
2. Fine tuning

→ Exact result for strongly interacting fermions in $D > 1$
(not accessible via conventional techniques)

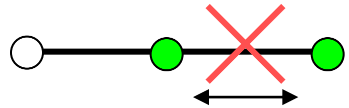
The model

Hardcore spinless fermions

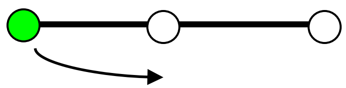
- spinless fermions

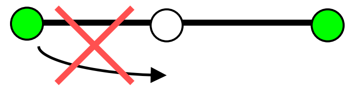


- hardcore $V_1 \to \infty$



- hopping t

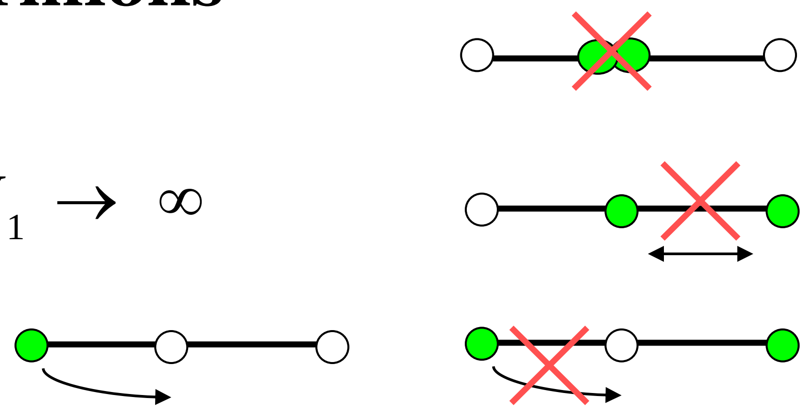


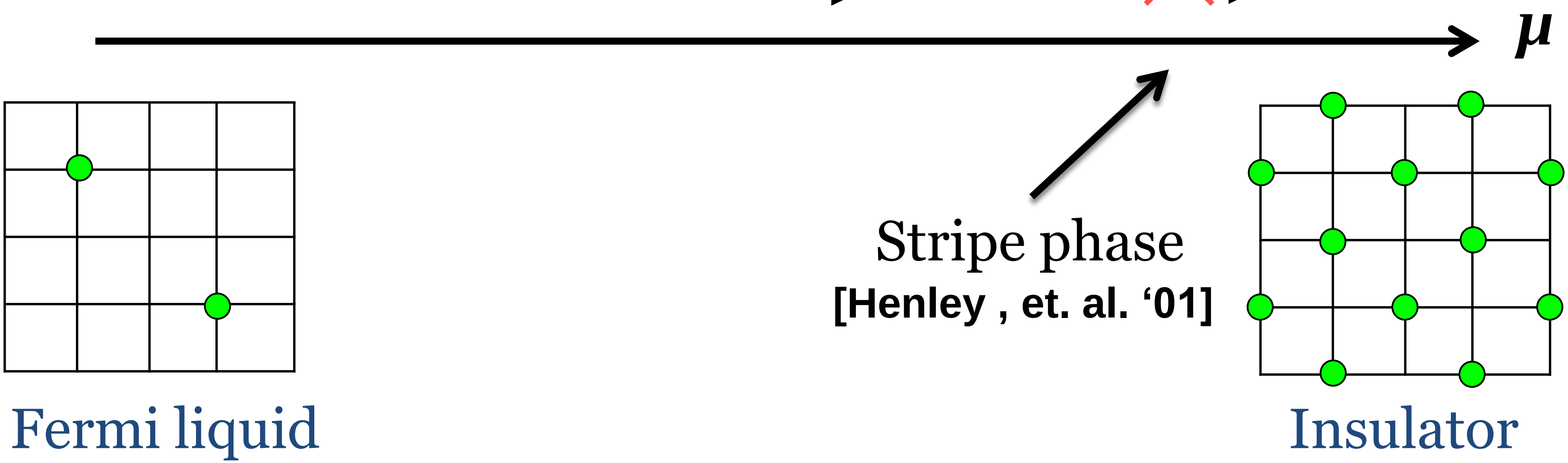


The model

Hardcore spinless fermions

- spinless fermions
- hardcore $V_1 \to \infty$
- hopping t

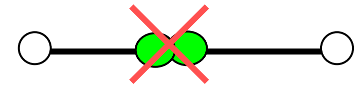




The model

Hardcore spinless fermions

- spinless fermions



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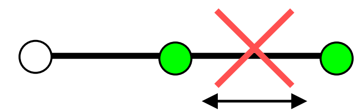
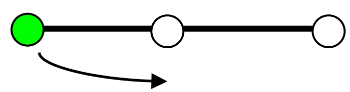
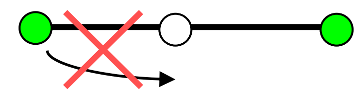
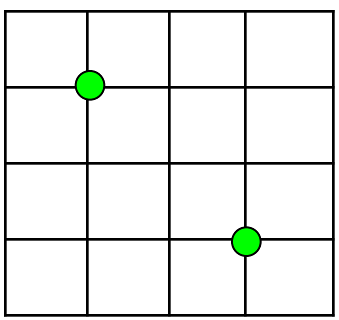


- hopping t





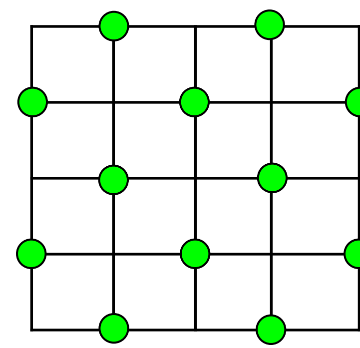
μ



Fermi liquid

Supersymmetry

[Fendley, et. al. ‘03]



Insulator

Plan of the talk

Benefit of supersymmetry is twofold:

- Powerful tools
- Subtle interplay between kinetic and potential terms leading to quantum criticality and superfrustration

- The model: definition & supersymmetry basics

- Superfrustration: extensive gs entropy
- Quantum criticality (1+1D)
- Quantum ground states as tilings

- Conclusions

Supersymmetry

Algebraic structure

supercharges Q^{+}, $Q^{-}=(Q^{+})^{\dagger}$ and fermion number N_f :

$$(Q^{+})^{2}=0,\quad (Q^{-})^{2}=0,\quad [N_f, Q^{\pm}]=\pm Q^{\pm}$$

Hamiltonian defined as

$$H=\left\{Q^{+}, Q^{-}\right\}$$

satisfies

$$[H, Q^{+}]=[H, Q^{-}]=0,\quad [H, N_f]=0$$

Supersymmetry

Spectrum:

- $E \geq 0$ for all states
- $E > 0$ pair into doublets (superpartners)

$$(|\psi\rangle, Q^{+}|\psi\rangle), \qquad Q^{-}|\psi\rangle = 0$$

- $E = 0$ states are singlets $Q^{+}|\psi\rangle = Q^{-}|\psi\rangle = 0$

High energy physics:

symmetry between bosonic and fermionic particles

Here:

- particles are spinless fermions (f)
- symmetry between "bosonic" (f even) and "fermionic" ($f \pm 1$ odd) states

The model

The supersymmetric lattice model

Supercharges for hardcore spinless fermions:

$$Q^+ = \sum_i c_i^\dagger \prod_{j \text{ next to } i} (1 - n_j), \quad Q^- = (Q^+)^\dagger, \quad n_j = c_j^\dagger c_j$$

Hamiltonian for 1D chain $H = \{Q^+, Q^-\}$

$$H = \sum_i [(1 - n_{i-1}) c_i^\dagger c_{i+1} (1 - n_{i+2}) + \text{h.c.}] + \sum_i n_{i-1} n_{i+1} - 2N_f + L$$

Hamiltonian for general lattice

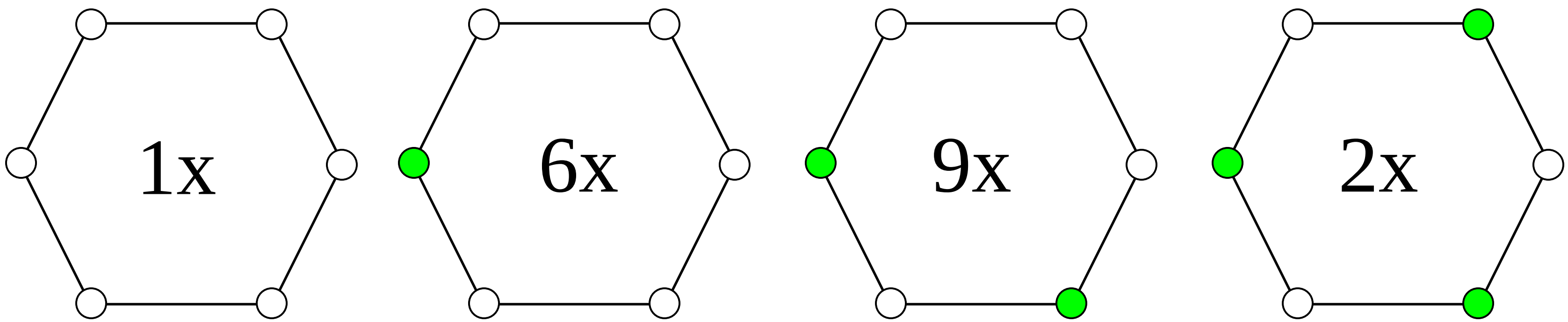
$$H = \sum_{<ij>} P_{<i>} c_i^\dagger c_j P_{<j>} + \sum_i P_{<i>} \qquad P_{<i>} = \prod_{j \text{ next to } i} (1 - n_j)$$

[Fendley - Schoutens - de Boer 2003]

Supersymmetry: example

6-site chain

Possible configurations for hardcore spinless fermions

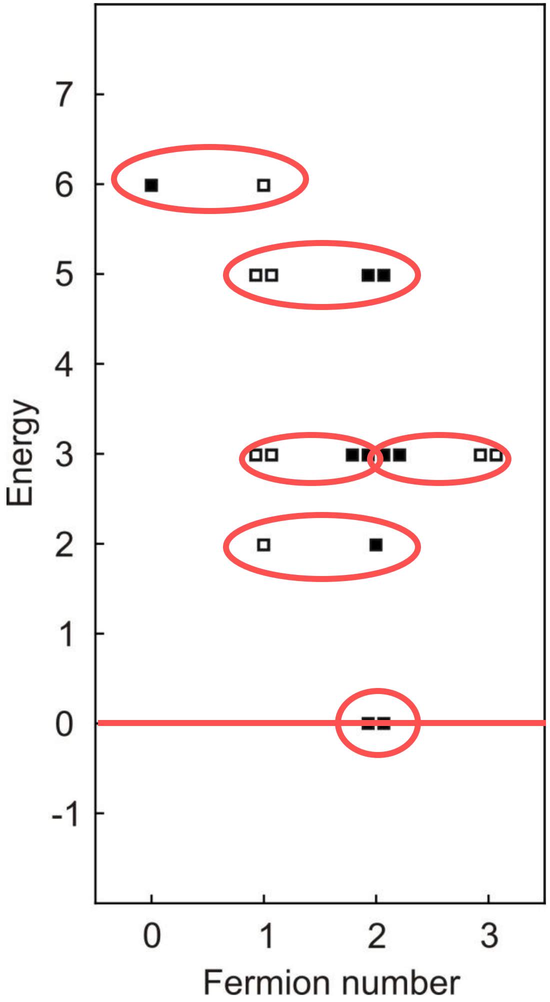


$|0\rangle$ $\quad c_i^+|0\rangle$ $\quad c_i^+c_{i+2}^+|0\rangle$ $\quad c_i^+c_{i+2}^+c_{i+4}^+|0\rangle$

Supersymmetry: example

Manifestly supersymmetric spectrum

- Energy is positive definite
- E>0 states form pairs between "fermionic" and "bosonic" states
- E=0 states are singlets



Witten index: superfrustration

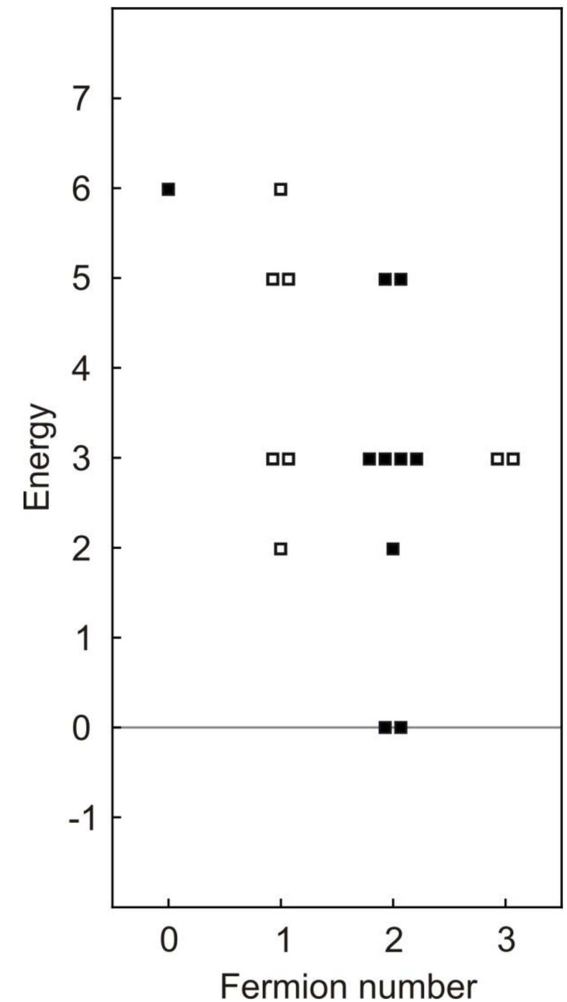
Powerful tool: Witten index

$$W = \mathrm{Tr}\,(-1)^{N_f}$$

"bosonic" states contribute +1, "fermionic" states contribute -1, so all superpartners cancel

$$\Rightarrow W = \#GS_B - \#GS_F$$

$|W|$ is lower bound to number of ground states



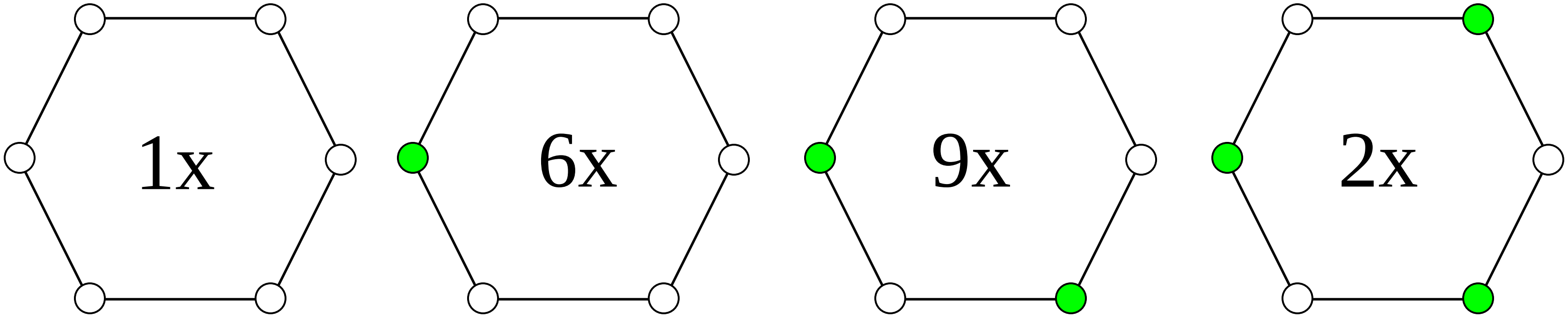
[Witten, '82]

Witten index: example

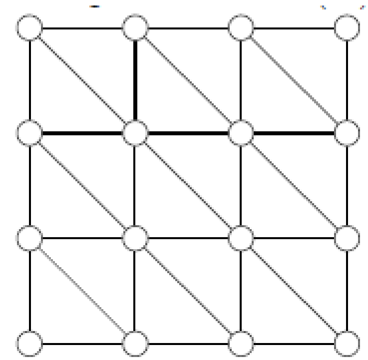
Purely combinatorial problem

$$W = \mathrm{Tr}\,(-1)^{N_f}$$

$$W = 1 - 6 + 9 - 2 = 2$$



Witten index



Triangular lattice

$N \times M$ sites with periodic BC

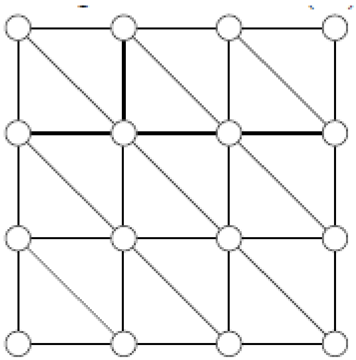
	1	2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1	1	1
2	1	-3	-5	1	11	9	-13	-31	-5	57
3	1	-5	-2	7	1	-14	1	31	-2	-65
4	1	1	7	-23	11	25	-69	193	-29	-279
5	1	11	1	11	36	-49	211	-349	811	-1064
6	1	9	-14	25	[illegible]	[illegible]	[illegible]	-415	1462	-4911
7	1	-13	1	-69	[illegible]	[illegible]	[illegible]	3403	-7055	5237
8	1	-31	31	193	[illegible]	[illegible]	[illegible]	881	-28517	50849
9	1	-5	-2	-29	881	1462	-7055	-28517	31399	313315
10	1	57	-65	-279	-1064	-4911	5237	50849	313315	950592
11	1	67	1	859	1651	12607	32418	159083	499060	2011307
12	1	-47	130	-1295	-589	-26006	-152697	-535895	-2573258	-3973827
13	1	-181	1	-77	-1949	67523	330331	-595373	-10989458	-49705161
14	1	-87	-257	3641	12611	-139935	-235717	5651377	4765189	-232675057
15	1	275	-2	-8053	-32664	272486	-1184714	-1867189	134858383	-702709340

$$|W| \sim 1.14^{NM}$$

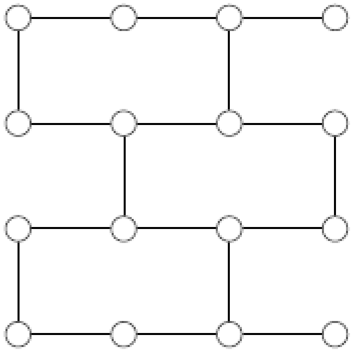
[van Eerten, '05]

Superfrustration - examples

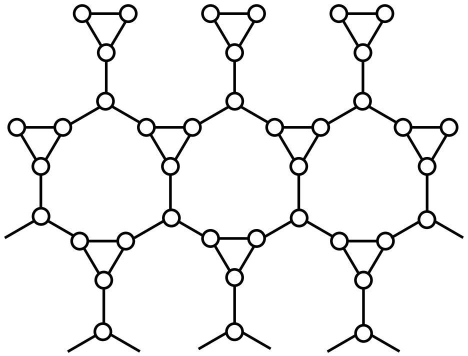
Tri



Hex



Martini



$|W| \sim 1.14^L$

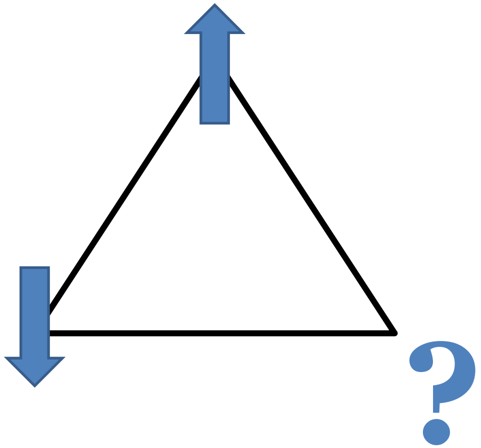
$|W| \sim 1.2^L$

$S_{gs} \sim 0.16$ L

L is number of sites

[van Eerten, '05; Fendley, Schoutens, '05]

Superfrustration



Frustration
Competing terms in hamiltonian
→ multiple ground states

Supersymmetry
Subtle competition between kinetic and potential terms
→ for 2D lattices exponential ground state degeneracy

Violation of 3rd law of thermodynamics
Exponential number of ground states
→ finite zero temperature entropy

Superfrustration

´3-rule´

• repulsive interactions favor 3-site interparticle distance
• chemical potential favors higher densities

Combined with kinetic terms
→ quantum charge frustration at *intermediate densities*

[Fendley, Schoutens, ‘05]

Quantum criticality

1D chain

Periodic chain of length L:

2 gs   for L mod 3 = 0;
1 gs   otherwise

- Fermion number in ground state: $f = [L/3]$
- Bethe Ansatz solution
- Continuum limit: N=(2,2) SCFT with central charge c=1 → quantum critical, emergent spacetime supersymmetry

[Fendley-Schoutens-deBoer '03, Fendley-Nienhuis-Schoutens '03, Beccaria-DeAngelis '05, Fendley-Hagendorf '10 & '11, LH '11]

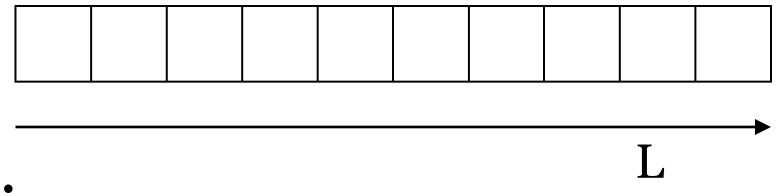
Relations/extensions

- XXZ spin chain – exact mapping
- SUSY Matrix models (via mapping to XXZ spin chain) of Veneziano-Wosiek
- Generalize hard-core constraint
  - Allow k particles to be nearest neighbors, but not k+1: M_k susy model ↔ k-th SCFT minimal model

[Fendley , Nienhuis , Schoutens '03]

Square ladder

Periodic ladder of length L:



3 gs for L mod 4 = 0;
1 gs otherwise

- Fermion number in ground state: $f = [L/2]$
- Continuum theory: $\mathcal{N}$=(2,2) SCFT with c=3/2
  - Witten index: k+1
  - Spectral flow
  - Entanglement entropy consistent with c=3/2
  - Operator content (scaling dimensions)

[with B. Bauer, M. Troyer, K. Schoutens (in progress)]

Entanglement entropy

Von Neumann entropy:

$$S_A = -\mathrm{Tr}\rho_A \ln \rho_A$$

With reduced density matrix:

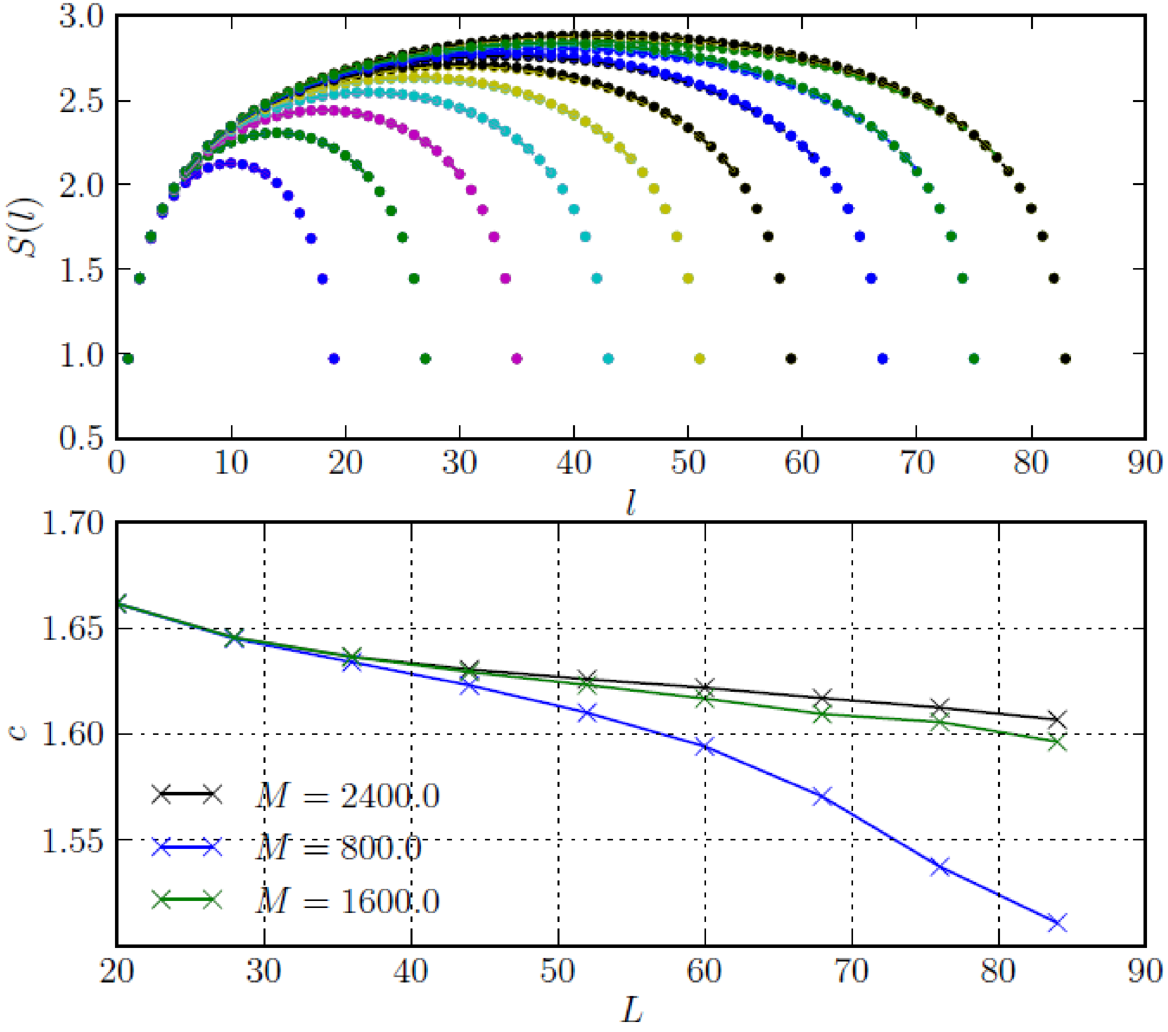
$$\rho_A = \mathrm{Tr}_B \rho$$

For a critical system with finite length L:

$$S(l_A) = \frac{c}{3} \ln(\frac{L}{\pi} \sin(\frac{l_A \pi}{L})) + b$$

[Calabrese, Cardy '04]

Entanglement entropy



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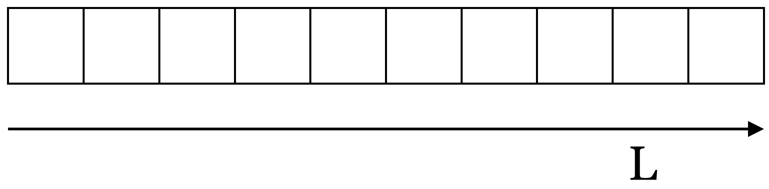
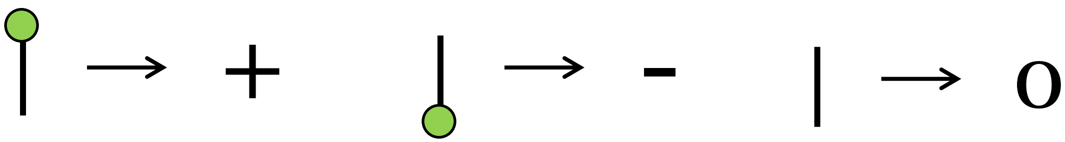
Square ladder





Maps particle configurations to: …+00-0-+0+…

c=3/2:

c=1 bosonic mode ~ charge, c=1/2 Ising mode ~ +/-

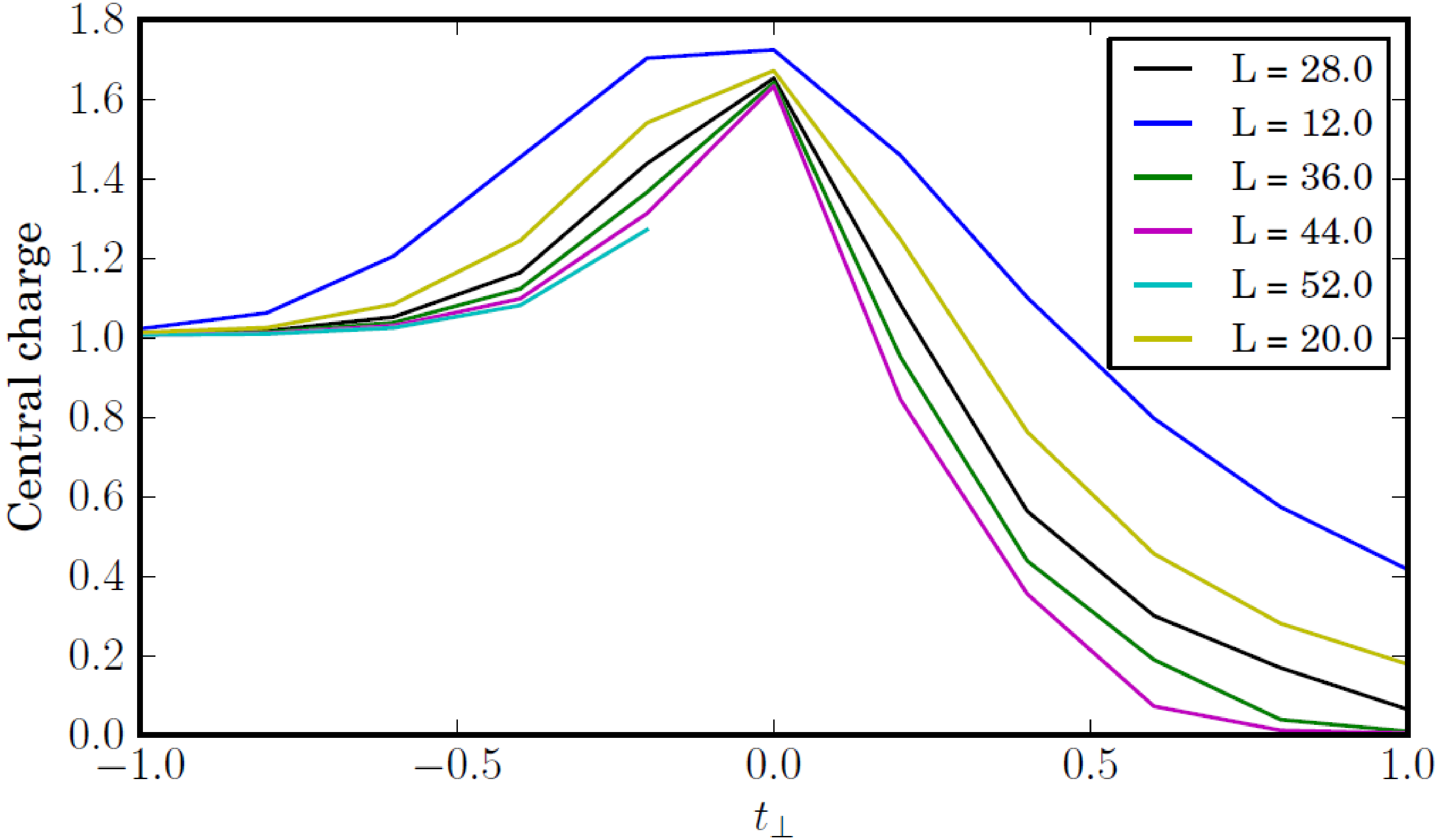
[with B. Bauer, M. Troyer, K. Schoutens (in progress)]

Perturbing away from c=3/2

- Change rung hopping: favors/disfavors singlets on rungs -> gaps Ising mode
- Stagger chemical potential on even/odd rungs: breaks translational invariance -> gaps bosonic charge mode

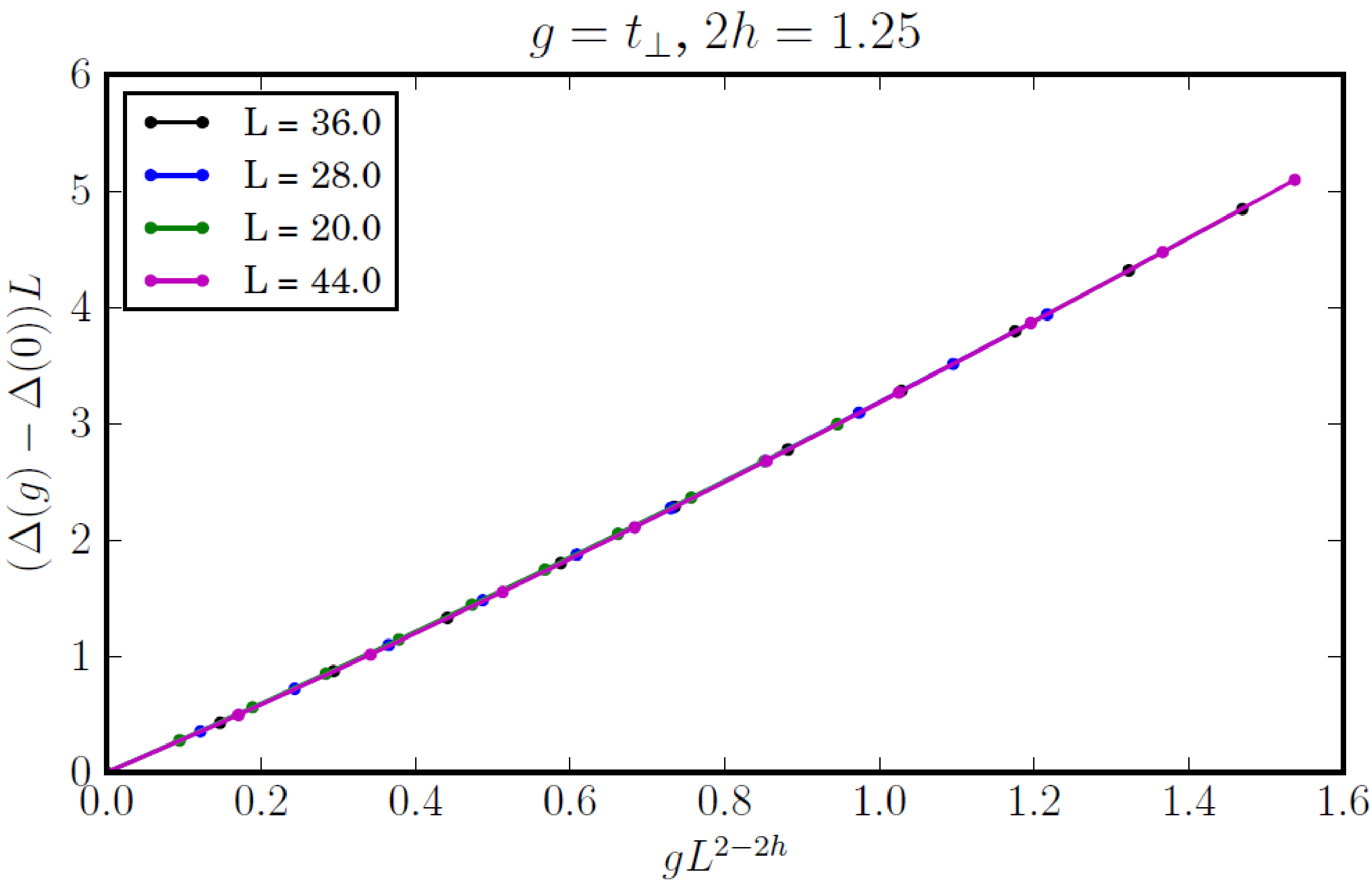
[with B. Bauer, M. Troyer, K. Schoutens (in progress)]

Central charge vs rung hopping perturbation



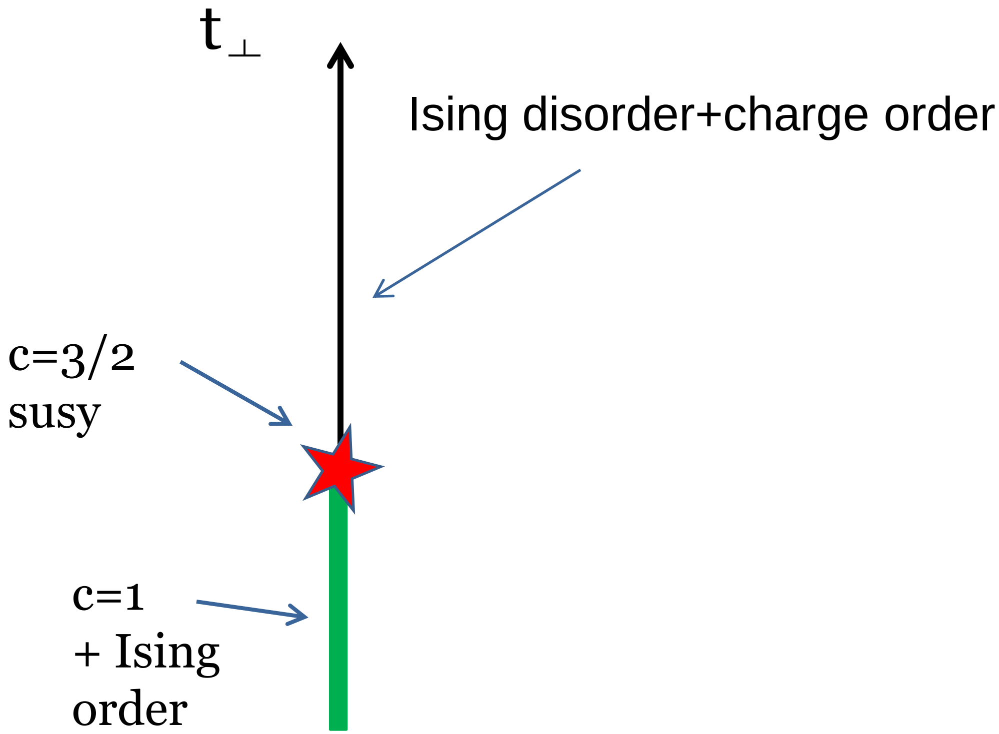
[with B. Bauer, M. Troyer, K. Schoutens (in progress)]

Gap scaling for rung hopping



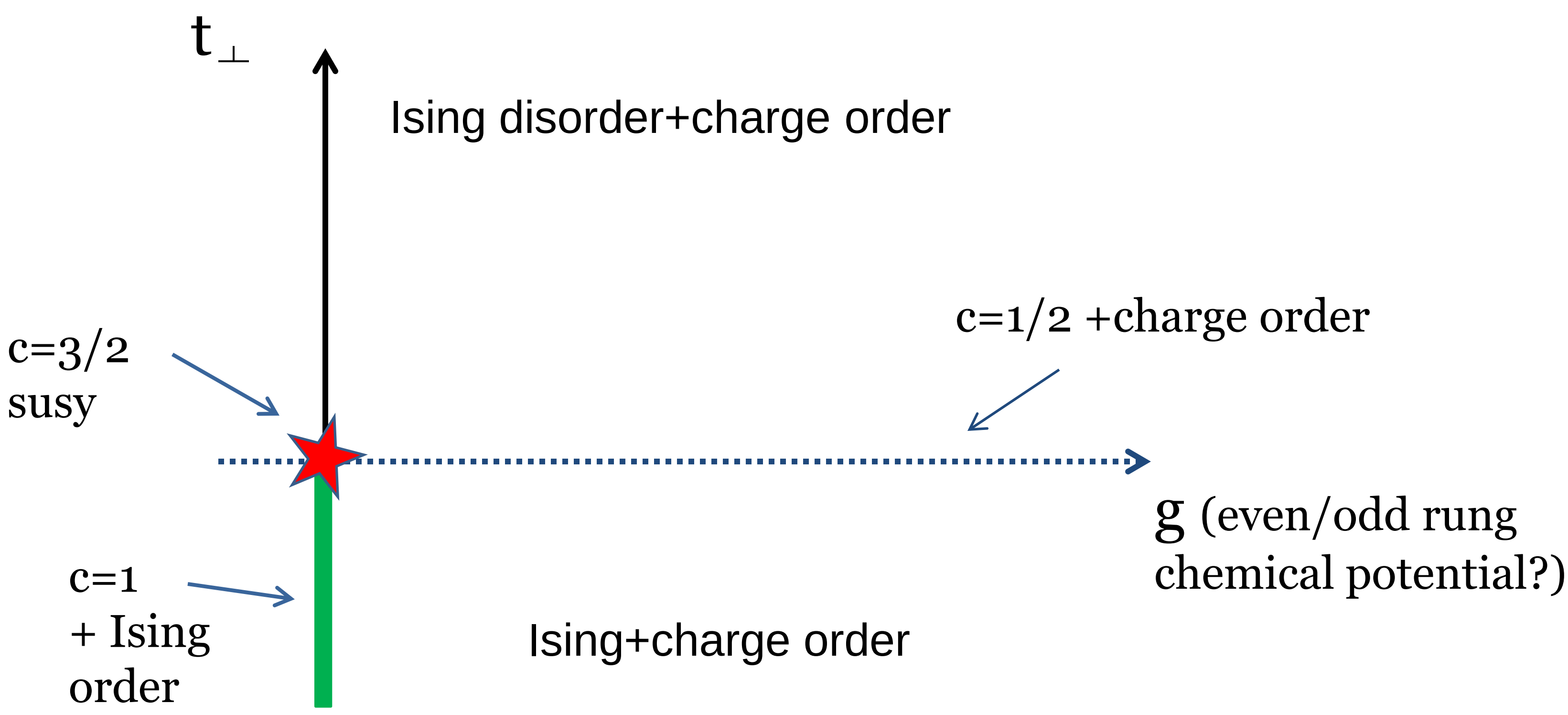
[with B. Bauer, M. Troyer, K. Schoutens (in progress)]

Phase diagram (tentative)



[with B. Bauer, M. Troyer, K. Schoutens (in progress)]

Phase diagram (very tentative)



[with B. Bauer, M. Troyer, K. Schoutens (in progress)]

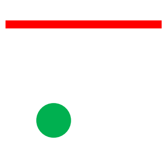
Susy staggering

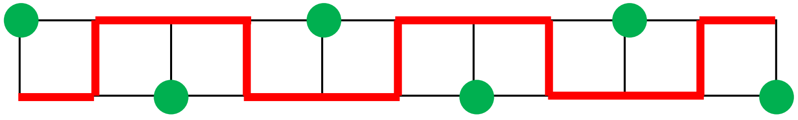
Susy preserving perturbation:

$$Q^+ = \sum \lambda_i^* c_i^\dagger P_{<i>}$$

$$Q^- = \sum \lambda_i c_i P_{<i>}$$

$$\lambda_i = \begin{cases} 1 & \text{for } i \in S_1 \\ y & \text{for } i \in S_2 \end{cases}$$





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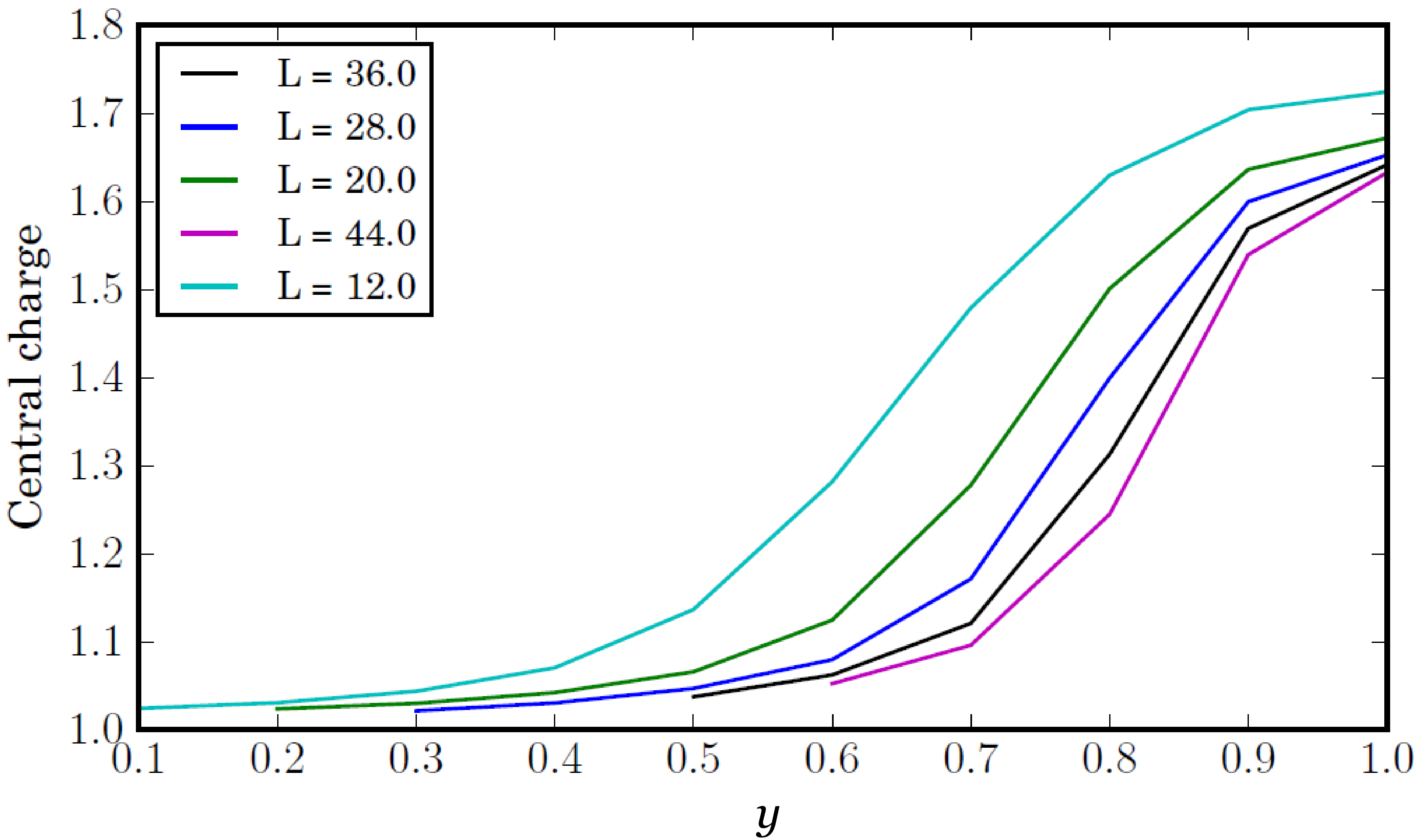
Central charge vs susy staggering



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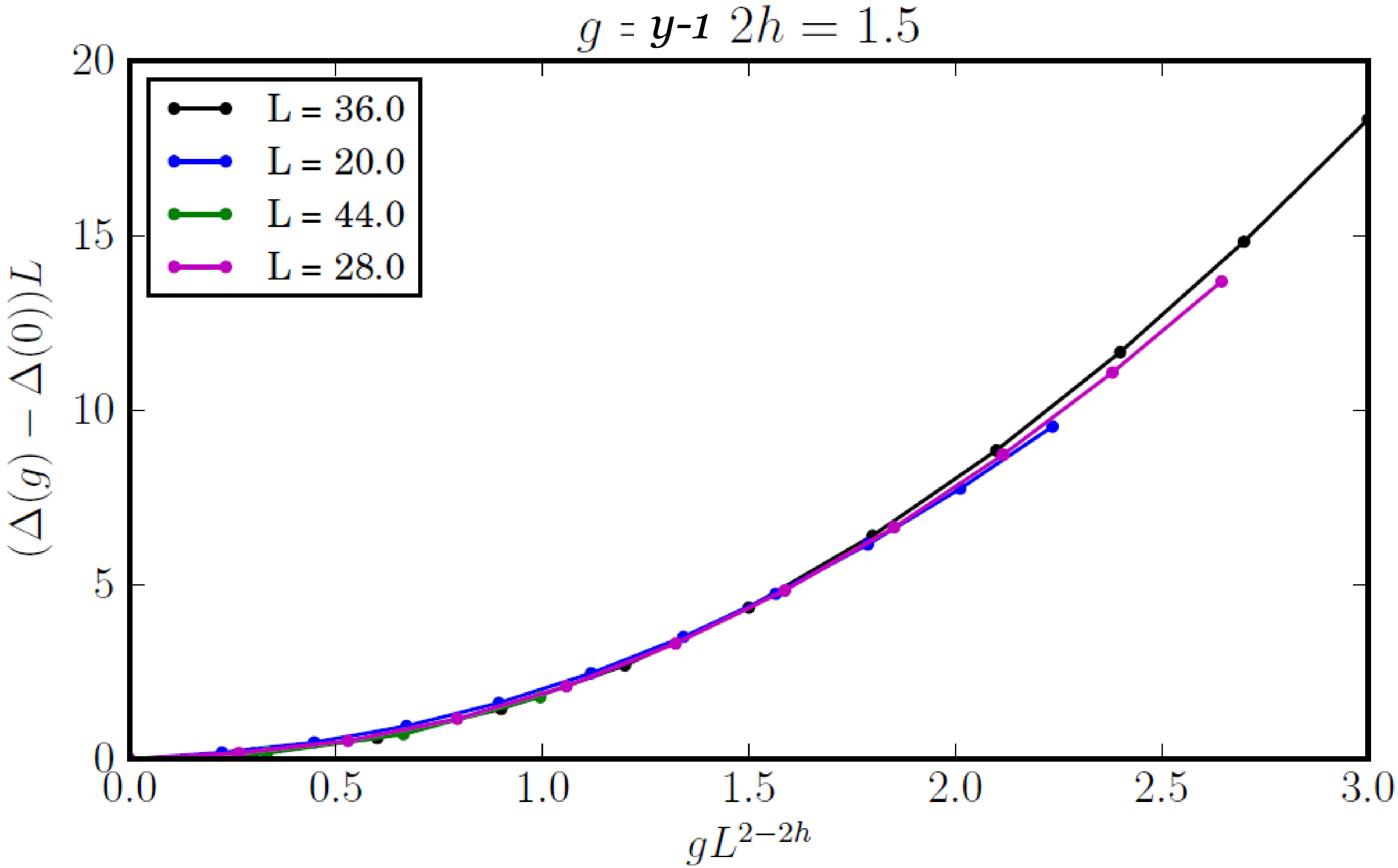
Gap scaling for susy staggering



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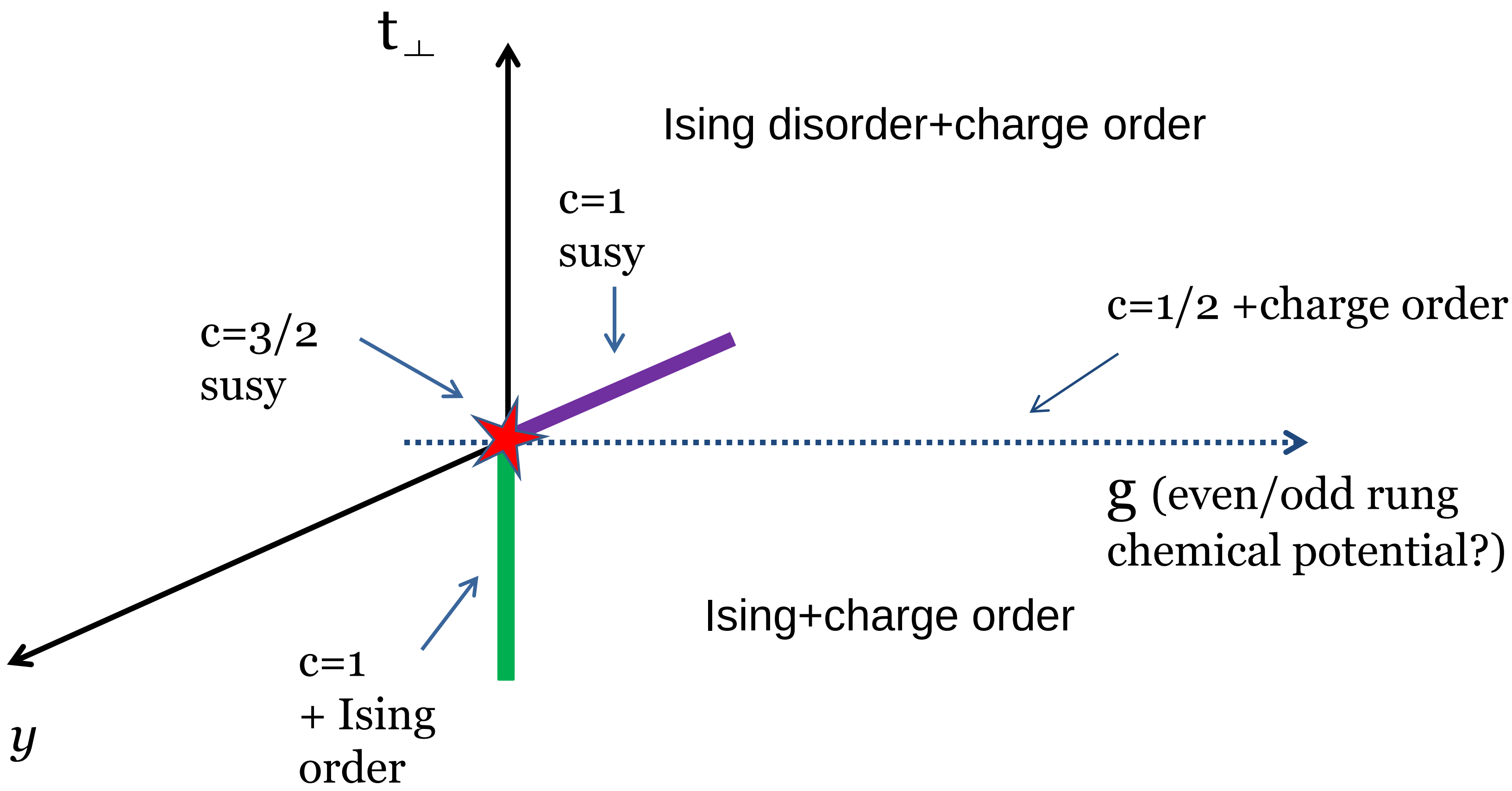
Phase diagram (very tentative)



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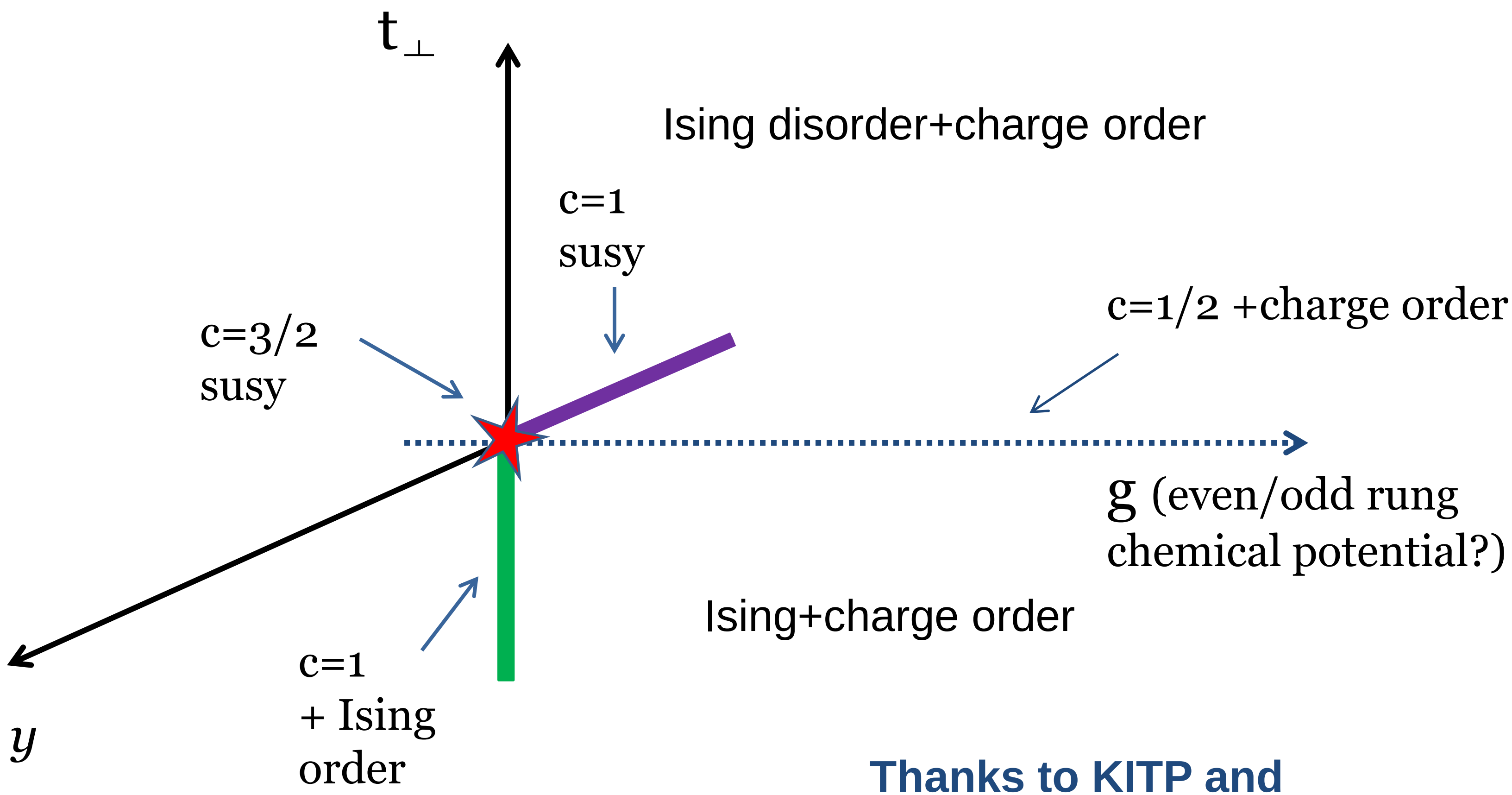


Thanks to KITP and
E. Berg, S. Trebst, J. McGreevy

[with B. Bauer, M. Troyer, K. Schoutens (in progress)]

Cohomology: quantum ground states as tilings

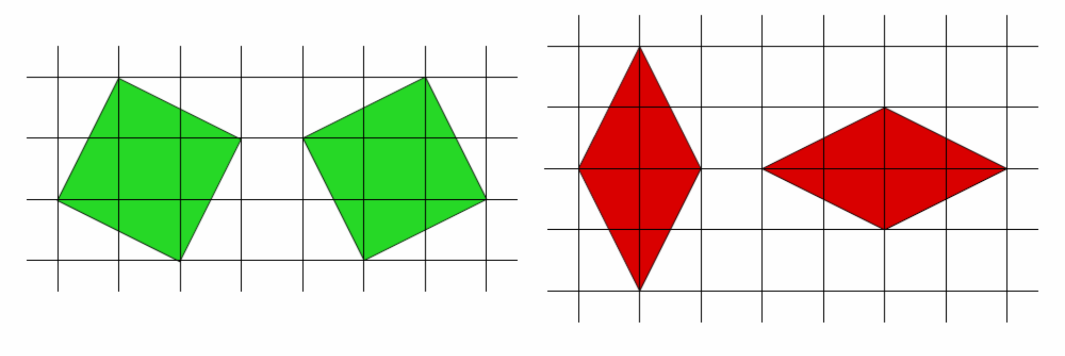
Powerful tool: Cohomology

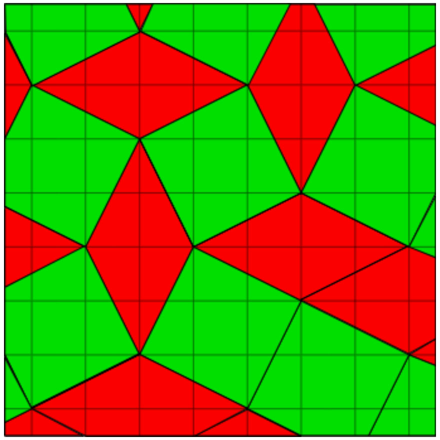
- Cohomology of Q

  GS are in 1-1 correspondence with cohomology elements
- More difficult to compute than Witten index
- But gives more information:
  - gives total number of gs
  - gives fermion number of gs
  - often gives relation between gs and geometric object

Square lattice (periodic BC)

- Cohomology of Q gives direct relation between ground states and rhombus tilings
- # of tiles = # of fermions





[LH - Schoutens '10]

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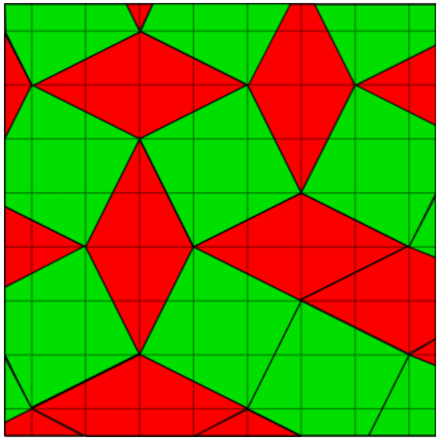


  - GS at intermediate filling

$$\frac{N_f}{L} \in \left[\frac{1}{5}, \frac{1}{4}\right] \cap \mathbb{Q}$$

  - Sub-extensive GS entropy

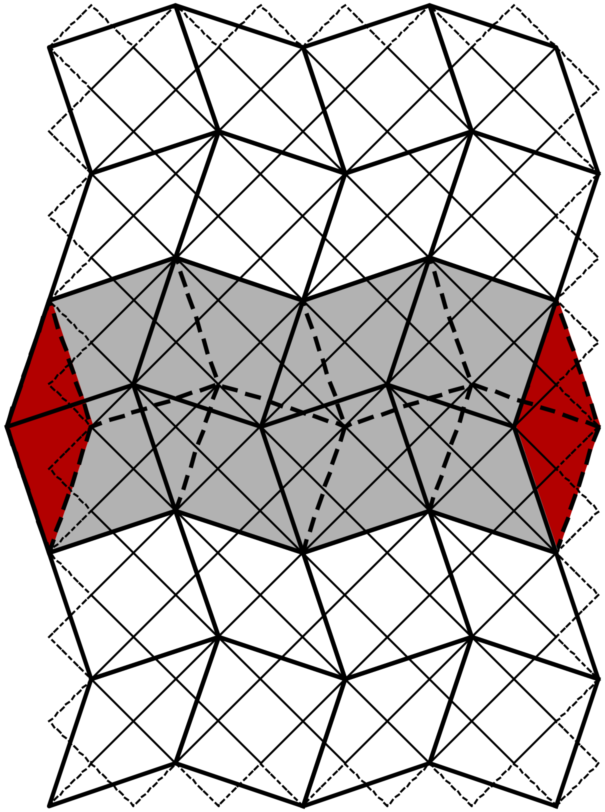
$$S_{gs} \sim 0.7\sqrt{L}$$

[LH - Schoutens '10]

Square lattice: edge states

- for 'diagonal' open boundary conditions there is a unique gs; expect that 'vanished' torus gs's form band of edge modes

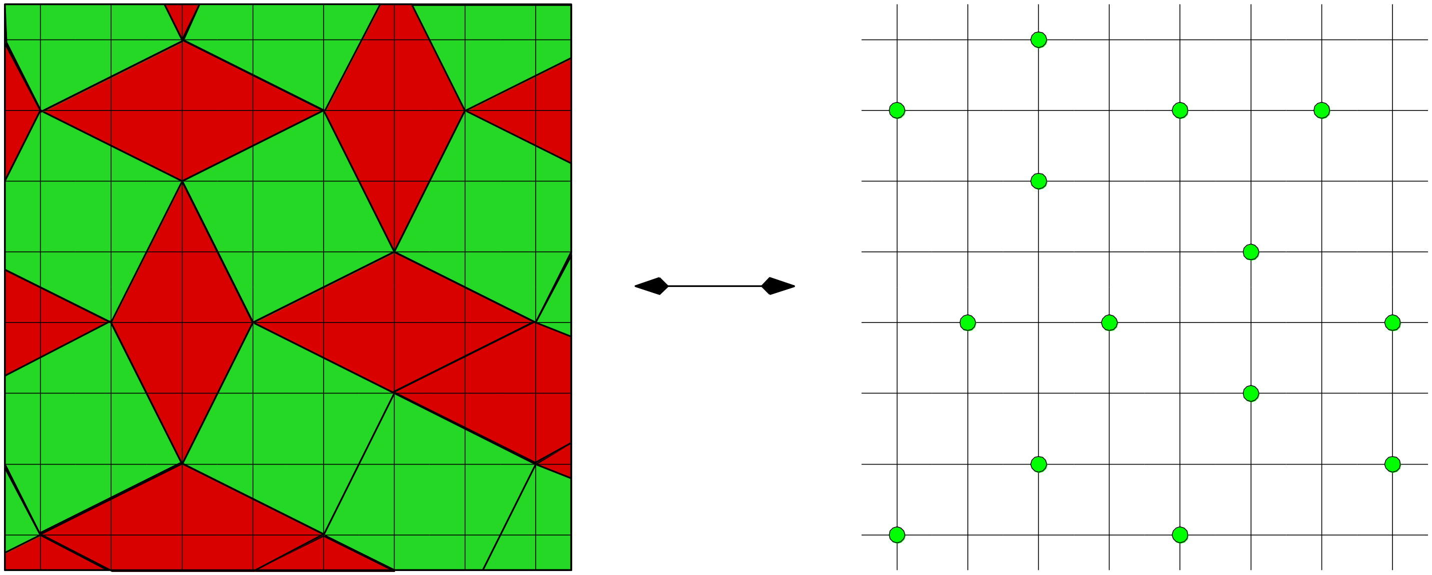
- explicit evidence for critical modes from ED studies of various ladder geometries



[LH - Halverson - Fendley - Schoutens '08]

Square lattice: ground states

- # gs grows exponentially with the linear size of the system
- zero energy ground states found at intermediate filling
- compelling evidence for critical edge modes
- what is the nature of these states?



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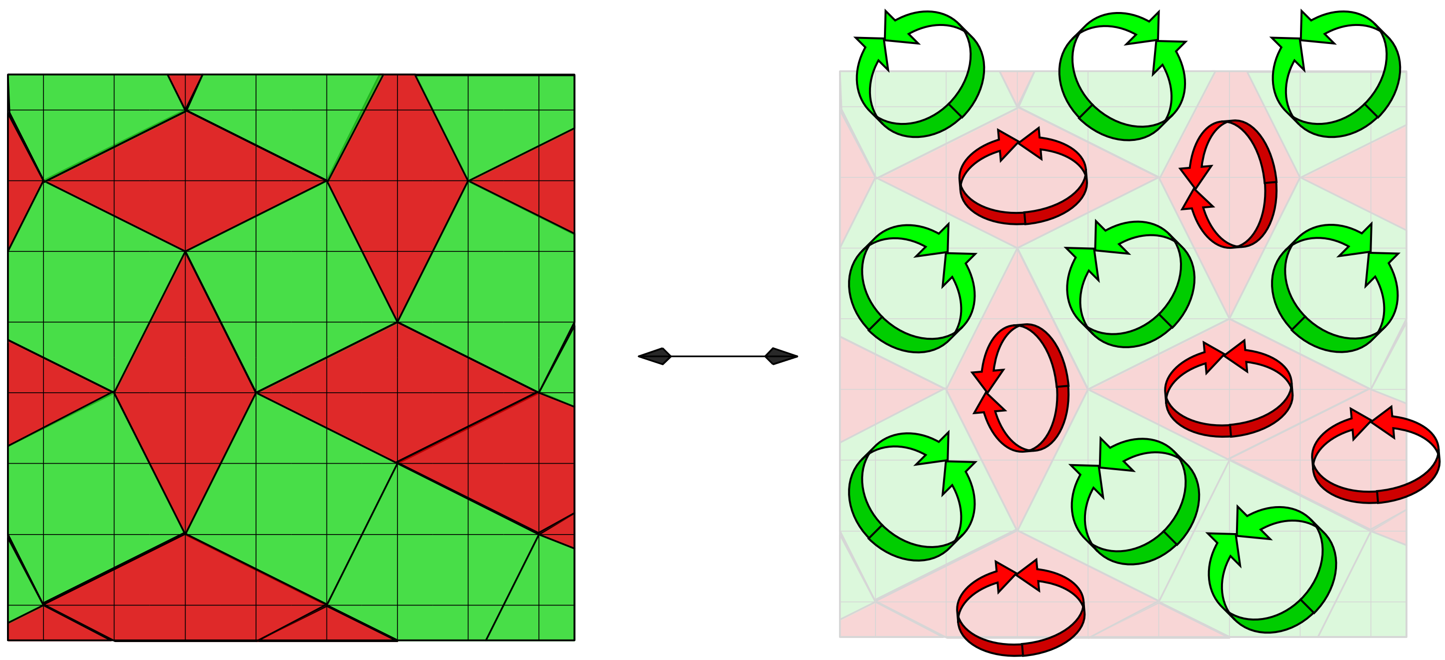
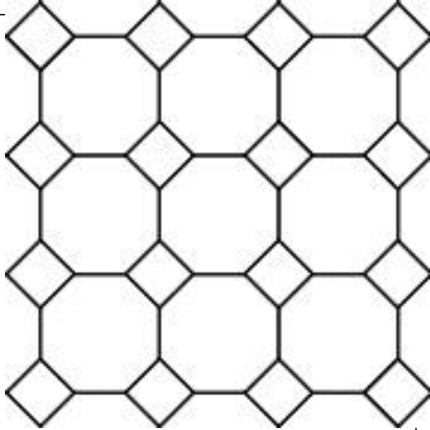


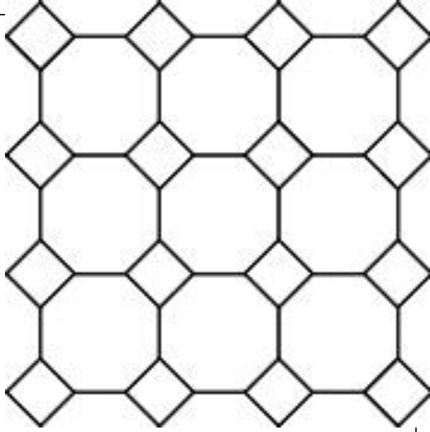
Octagon-square lattice



- $N \times M$ plaquettes with open bc : unique gs with one fermion per plaquette
- $N \times M$ plaquettes with closed bc: 2^M+2^N-1 gs

[Fendley, Schoutens, '05]

Octagon-square lattice

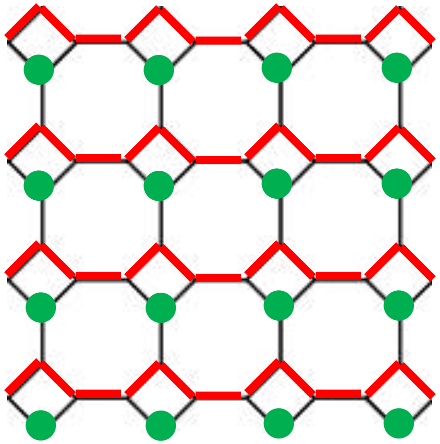


- $N \times M$ plaquettes with open bc : unique gs with one fermion per plaquette
- $N \times M$ plaquettes with closed bc: 2^M+2^N-1 gs
- Distinct topological sectors in extreme susy staggering limit (y<<1) (preliminary)

$$Q^+ = \sum \lambda_i^* c_i^\dagger P_{<i>}$$

$$Q^- = \sum \lambda_i c_i P_{<i>}$$

$$\lambda_i = \begin{cases} 1 & \text{for } i \in S_1 \\ y & \text{for } i \in S_2 \end{cases}$$

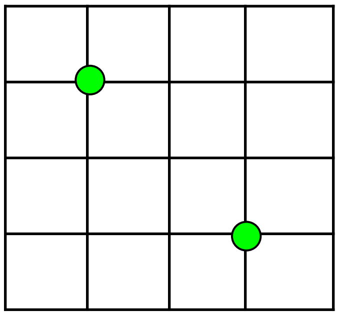


[Fendley, Schoutens, '05, With E. Berg (in progress)]

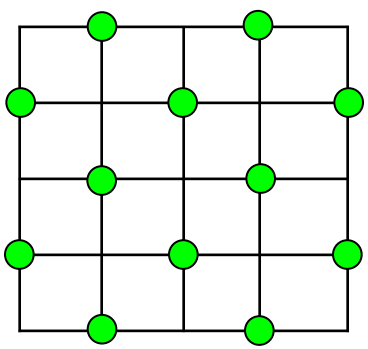
Conclusions

Exact results for strongly interacting fermions

μ



Supersymmetry



Techniques
Witten index
Cohomology
Spectral flow
Staggering
…

Features
Superfrustration
Quantum criticality
Topological phases
Competing orders
…

Connection to AdS/CFT ??

- Extensive gs entropy is typical feature
- In 1+1D emergent spacetime supersymmetry
- Low energy effective theories in 1+1D
  N=2 superconformal minimal models
- …

Thank you