Critical Fermi surface states in 2+1 dimensions.

Part II

Max Metlitski

KITP, September 16, 2011

Critical Fermi surface states in 2+1 d. Part II.

- "Theory" of critical Fermi surface states in 2+1d

  - gapless boson interacting with the Fermi surface
    - gauge field (spinon Fermi-surface, half-filled Landau level)
    - order parameter (nematic transition)
  - two patch theory
    - scaling forms
    - failure of large N expansion
    - a better controlled model
      - ε – expansion (aka Nayak-Wilczek expansion)
      - the MIT double scaling limit

- Pairing instabilities of critical Fermi surface states

Dynamical scaling

$$L = \sum_s f_s^\dagger (\partial_\tau + v_F(-is\partial_x - \frac{\partial_y^2}{2K}))f_s + v_F a \sum_s s f_s^\dagger f_s + \frac{1}{2e^2}(\partial_y a)^2$$

- How to scale time?

$$k_y \to s k_y,\ k_x \to s^2 k_x,\ \tau \to s^z \tau$$

- Choose z to leave the gauge-fermion coupling invariant (marginal)

$$z = 3$$

- Fermion kinetic term is irrelevant under such scaling

- Define the theory via $\eta \to 0^+$ limit $f_s^\dagger \partial_\tau f_s \to \eta f_s^\dagger \partial_\tau f_s$

Problem

$$L = \sum_s f_s^\dagger (\eta \partial_\tau + (-is\partial_x - \partial_y^2)) f_s + a \sum_s s f_s^\dagger f_s + \frac{1}{2e^2} (\partial_y a)^2$$

- No expansion parameter

$$[e^2] = \frac{q_y^3}{\omega}$$ - dimensionfull

- Theory is strongly coupled

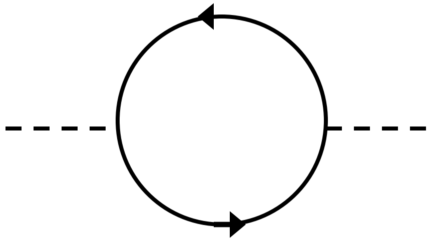
- Usual approach: large- N expansion

$$L = \sum_s f_s^\dagger (\eta \partial_\tau + (-is\partial_x - \partial_y^2)) f_s + a \sum_s s f_s^\dagger f_s + \frac{N}{2e^2} (\partial_y a)^2$$

$$S_{\mathrm{eff}}[a] \sim N\Gamma[a]$$ - use saddle point approximation

- Actually, fails for this problem *S. S. Lee (2009)*

Sanity check: one loop results

$\Pi^0(\omega, \vec{q}) =$  $= c_b N \frac{|\omega|}{|q_y|}$

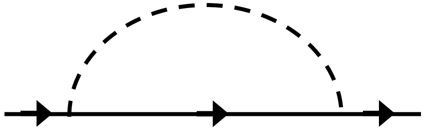
$$D^{-1}(\omega, q_y) = N\left(c_b \frac{|\omega|}{|q_y|} + \frac{q_y^2}{e^2}\right)$$ - Landau damping $z = 3$

- Landau damping comes from the two-patch regime

Sanity check: one loop results

- Fermion self-energy at criticality

$$\Sigma(\omega, \vec{k}) =$$



$$= -i \frac{c_f}{N} sgn(\omega) |\omega|^{2/3}$$

$$G_s^{-1}(\omega, \vec{k}) = -i \frac{c_f}{N} sgn(\omega) |\omega|^{2/3} + s k_x + k_y^2$$

- Respects the scaling

$$k_y \to s k_y, \; k_x \to s^2 k_x, \; \tau \to s^z \tau \qquad z = 3$$

Scaling properties

$$L = \sum_s f_s^\dagger(\eta\partial_\tau + (-is\partial_x - \partial_y^2))f_s + a\sum_s s f_s^\dagger f_s + \frac{1}{2e^2}(\partial_y a)^2$$

- Shift symmetry + Ward-Identities constrain the RG properties severely
- Only two anomalous dimensions

η_f - fermion anomalous dimension

z - dynamical critical exponent

$$f = Z_f^{1/2} f_r, \quad e^2 = Z_e e_r^2$$

$$\eta_f = -\Lambda\frac{\partial}{\partial\Lambda}\log Z_f \qquad\qquad z - 3 = -\Lambda\frac{\partial}{d\Lambda}\log Z_e$$

Scaling forms: gauge field

- $$D^{-1}(\omega, \vec{q}) \sim |\vec{q}|^{z-1} f\left(\frac{|\omega|}{|\vec{q}|^z}\right)$$

- Simple Landau-damped frequency dependence consistent with scaling form

$$D^{-1}(\omega, \vec{q}) - D^{-1}(\omega = 0, \vec{q}) \sim \frac{|\omega|}{|\vec{q}|}$$

- Static behaviour

$$D^{-1}(\omega = 0, \vec{q}) \sim |\vec{q}|^{z-1}$$

Scaling forms: fermions

- $$G^{-1}(\omega, \vec{k}) \sim k^{1-\eta_f/2} g\left(\frac{|\omega|}{|k|^{z/2}}\right), \quad k = k_x + \frac{k_y^2}{2K}$$

- "Fermionic dynamical exponent" is half the "bosonic dynamical exponent"

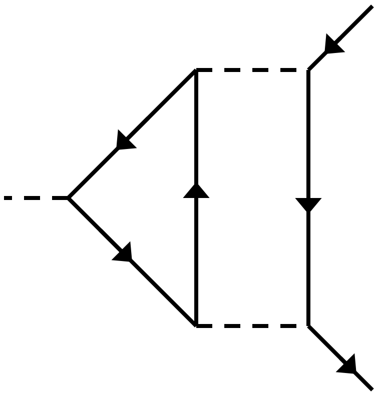
- Static behaviour: $G^{-1}(0, \vec{k}) \sim k^{1-\eta_f/2}$

- Dynamic behaviour: $G^{-1}(\omega, 0) \sim \omega^{(2-\eta_f)/z}$

M.M and S. Sachdev (2010)

Failure of large-N expansion

- Can we systematically compute anomalous dimensions in large-N limit?
- Large-N expansion fails at higher loops:



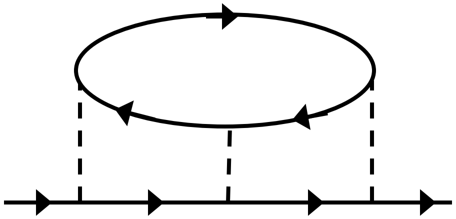
$$\sim \frac{1}{\eta \omega^{1/3}}$$

-infra-red divergence

$$G_0(\omega, \vec{k}) = \frac{1}{-i\eta\omega + k_x + k_y^2}$$

S. S. Lee (2009)

Failure of large-N expansion



$$\sim \frac{1}{\eta\omega^{1/3}} \times \omega^{2/3}$$

-infra-red divergence

$$G_0(\omega, \vec{k}) = \frac{1}{-i\eta\omega + k_x + k_y^2}$$

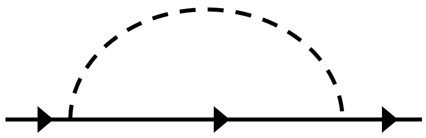
S. S. Lee (2009)

Failure of large-N expansion

- Wrong dynamical scaling of bare fermion Green's function

$$G_0(\omega, \vec{k}) = \frac{1}{-i\eta\omega + k_x + k_y^2}$$

- Solution: dress by one-loop self-energy

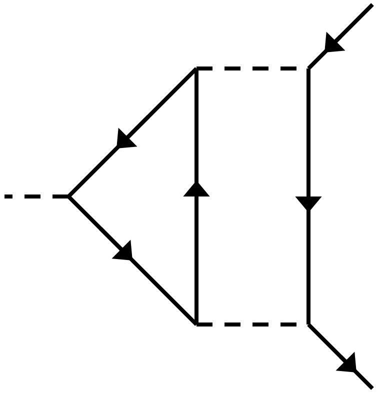


$$G_1(\omega, \vec{k}) = \frac{1}{-i\frac{c_f}{N} sgn(\omega)|\omega|^{2/3} + k_x + k_y^2}$$

- Traded small parameter $\eta \to \frac{1}{N}$

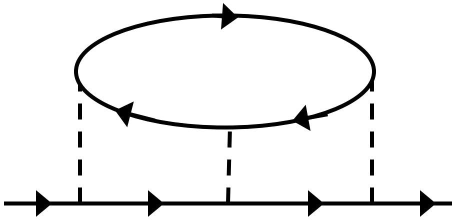
S. S. Lee (2009)

Violation of large-N counting



$$\sim \frac{1}{N\eta} \rightarrow O(1)$$

same as leading order!



$$\sim \frac{1}{N^2\eta} \rightarrow O\left(\frac{1}{N}\right)$$

S. S. Lee (2009)

Violation of large-N counting

$$G^{-1}(\omega, 0) = -i\eta\omega - i\frac{c_f}{N}|\omega|^{2/3} sgn(\omega)$$

- Crossover scale: $\Lambda \sim \left(\frac{c_f}{\eta N}\right)^3 \stackrel{N\to\infty}{\to} 0$

- Limits $N \to \infty$ and $\omega \to 0$ do not commute.

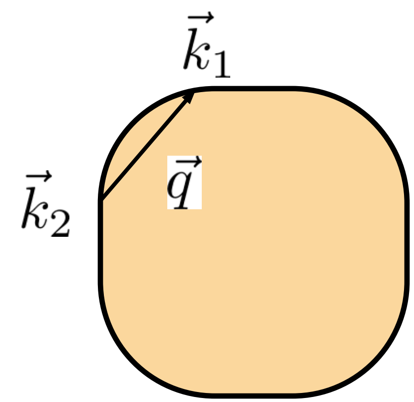
S. S. Lee (2009)

Genus expansion

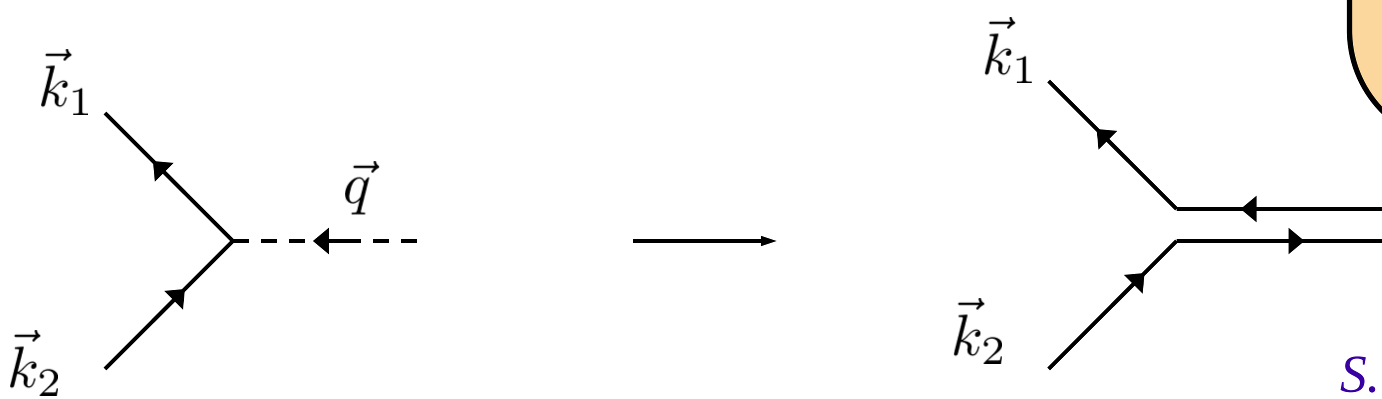
- A systematic way to count the power of N
- Where do extra powers of N come from?

$$G_1(\omega,\vec{k}) = \frac{1}{-i\frac{c_f}{N}sgn(\omega)|\omega|^{2/3}+k_x+k_y^2}$$

- Need to find the phase space for all fermions to be on the Fermi-surface
- Double-line representation



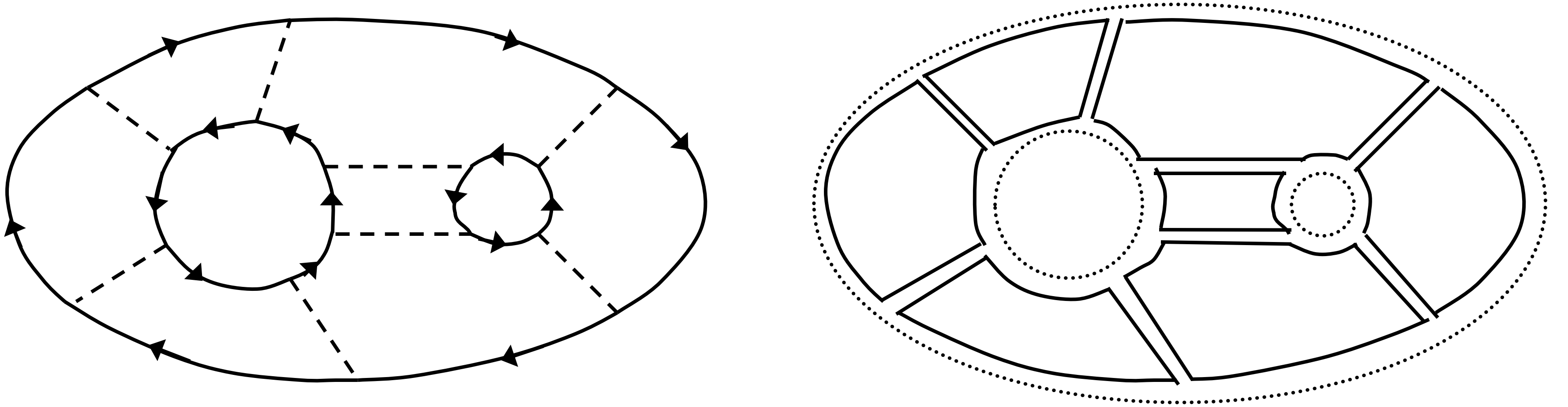
$$\vec{q} = \vec{k}_1 - \vec{k}_2$$



S. S. Lee (2009)

Genus expansion

- Go to double-line representation and classify diagrams by their topology

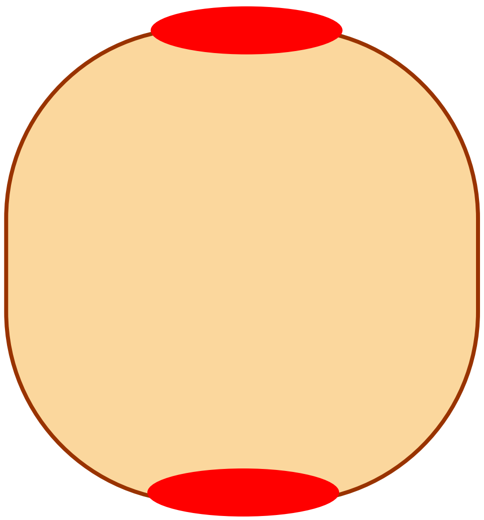


- Degree of a diagram in N is related to the genus of the surface on which it can be drawn
- At $N = \infty$ have to sum an infinite set of planar diagrams

S. S. Lee (2009)

One patch vs two patches

- For a theory with one patch all planar diagrams are finite due to kinematics



$$z = 3, \quad \eta_f = 0, \qquad N = \infty$$

S. S. Lee (2009)

- For a theory with two patches we find:

  i) divergences appear in planar graphs

  ii) large-N genus counting is violated

To three loop order:

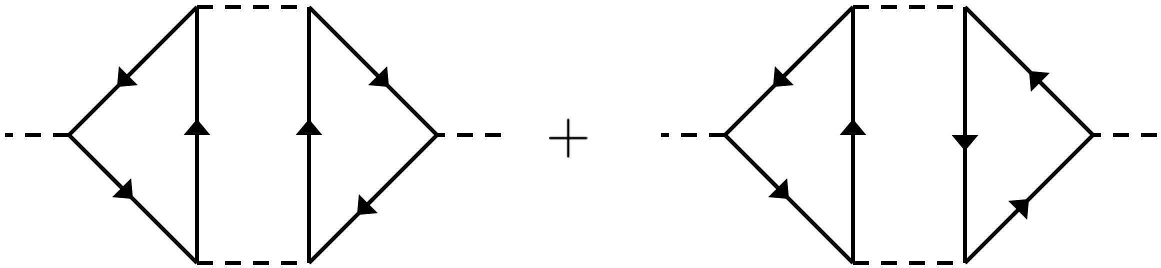
$$z = 3, \quad \eta_f \neq 0$$

η_f is not suppressed for $N \to \infty$

M. M. and S. Sachdev (2010)

Dynamical exponent at three loops

$$\delta^3\Pi(\omega = 0, q_y) =$$



Genus counting: planar $\sim O(N)$ non-planar $\sim O(1)$

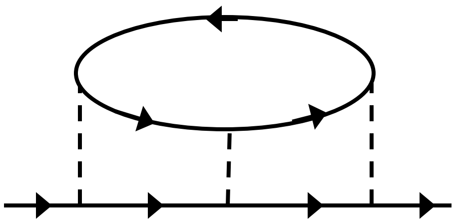
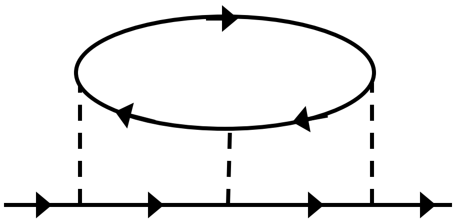
Actually: $\sim N q_y^{3/2}$ $\sim -N q_y^{3/2}$

Cancellation: $\sim N^{3/2} q_y^2$

- Divergences coming from outside of two-patch regime cancel!
- The contribution of the two-patch regime is finite. $\boxed{z = 3}$
- Violation of genus expansion.

J. Rech, C. Pepin and A. Chubukov (2006)

Fermion anomalous dimension at three loops

$\delta^3\Sigma(\omega=0,\vec{k}) =$  $+$ 

Genus counting: planar non-planar

$$= (k_x + k_y^2)\log\left(\frac{\Lambda_y}{|k_x + k_y^2|^{1/2}}\right) \times \left[O(1) \quad + \quad O\left(\frac{1}{N^2}\log^3 N\right)\right]$$

$$\begin{aligned} \eta_f &= -0.06824, \quad N = 2 \\ \eta_f &= -0.10619, \quad N = \infty \end{aligned}$$

Remarks

Three loops: $z = 3$

$$\begin{aligned}\eta_f &= -0.06824, \quad N = 2 \\ \eta_f &= -0.10619, \quad N = \infty\end{aligned}$$

- Fermion anomalous dimension is not suppressed for large-N
- Anomalous dimension numerically small
- Is $z = 3$ to all orders?
- Further diagrams with singular contributions from outside two patch region? Do these always cancel?
- Does a sensible large N limit exist?

Extension: nematic transition

- $\phi -$ nematic order parameter

$$
\begin{aligned}
L_\psi &= f_{+\sigma}^\dagger \left(\partial_\tau - i v_F \partial_x - \frac{1}{2K} \partial_y^2 \right) f_{+\sigma} \\
&+ f_{-\sigma}^\dagger \left(\partial_\tau + i v_F \partial_x - \frac{1}{2K} \partial_y^2 \right) f_{-\sigma}
\end{aligned}
$$

$$
L_\phi = \frac{N}{2e^2} (\partial_y \phi)^2 + \frac{Nr}{2} \phi^2
$$

To three loops:

$\eta_f \to -\eta_f$

$$
L_{int} = \phi (f_{+\sigma}^\dagger f_{+\sigma} + f_{-\sigma}^\dagger f_{-\sigma})
$$

How to control the expansion?

$$S_a = \frac{1}{2e^2}\int d^2x d\tau (\nabla a)^2 \rightarrow \frac{1}{2e^2}\int \frac{d\omega d^2\vec{q}}{(2\pi)^3}|\vec{q}|^{1+\epsilon}|a(\vec{q},\omega)|^2$$

- Long-range interaction in the half-filled Landau level

$$S_{int} = \frac{1}{2}\int d^2x d^2x' d\tau (f^\dagger f)(\vec{x},\tau)\frac{1}{|\vec{x}-\vec{x}'|^{1+\epsilon}}(f^\dagger f)(\vec{x}',\tau)$$

$$\nabla \times \vec{a} = 2(2\pi) f^\dagger f$$

- $\epsilon = 0$ - Coulomb interaction

  $\epsilon = 1$ - contact interaction

B. I. Halperin, P. A. Lee, N. Read; C. Nayak and F. Wilczek (1994)

How to control the expansion?

$$L = \sum_s f_s^\dagger (\partial_\tau + v_F(-is\partial_x - \frac{\partial_y^2}{2K}))f_s + v_F a \sum_s s f_s^\dagger f_s + \frac{N}{2e^2} q_y^{1+\epsilon} a^2$$

- Perform more conventional scaling dictated by the fermion kinetic term:

$$\omega \to s^2\omega, \; k_x \to s^2 k_x, \; k_y \to s k_y$$

- Fermion-gauge field interactions are at tree level:

$\epsilon < 0$ - irrelevant

$\epsilon > 0$ - relevant

$\epsilon = 0$ - marginal

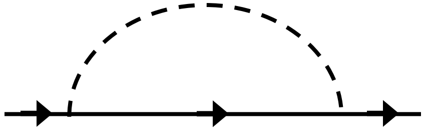
- Theory described by a single dimensionless coupling constant:

$$\alpha = \frac{e^2 v_F \Lambda_y^{-\epsilon}}{(2\pi)^2} \qquad\qquad \frac{d\alpha}{dl} = \frac{\epsilon}{2}\alpha$$

ε-expansion

- Quantum corrections to scaling

$$\Sigma(\omega, \vec{k}) =$$



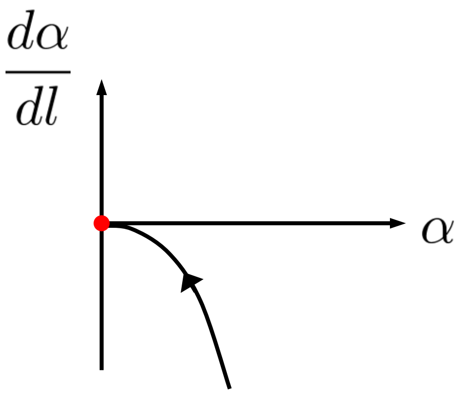
$$= -i\frac{\alpha}{N}\omega \log\frac{\Lambda_\omega}{|\omega|}$$

$$\frac{d\alpha}{dl} = \frac{\epsilon}{2}\alpha - \frac{\alpha^2}{N}$$

C. Nayak and F. Wilczek (1994)

ε-expansion

- $\epsilon = 0$ $$\frac{d\alpha}{dl} = -\frac{\alpha^2}{N}$$



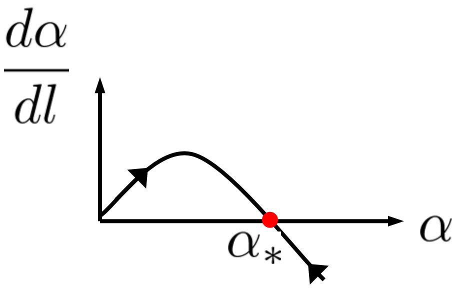
- interactions marginally irrelevant.
"Marginal Fermi-liquid."

$$\Sigma(\omega) = -i\frac{\alpha}{N}\omega \log \frac{\Lambda_\omega}{|\omega|}$$

C. Nayak and F. Wilczek (1994)

ε-expansion

- $\epsilon > 0$ $\qquad \dfrac{d\alpha}{dl} = \dfrac{\epsilon}{2}\alpha - \dfrac{\alpha^2}{N}$



- new fixed-point: $\alpha_* = \dfrac{N\epsilon}{2}$

$$\Sigma(\omega) \sim -i|\omega|^{1-\epsilon/2}\mathrm{sgn}(\omega)$$

C. Nayak and F. Wilczek (1994)

Beyond one-loop

- Two expansions have been suggested: $\alpha = \frac{e^2 v_F \Lambda_y^{-\epsilon}}{(2\pi)^2}$ $\alpha_* \approx \frac{N\epsilon}{2}$

  - $\epsilon \to 0, \; N$ - fixed, "perturbative" expansion

    C. Nayak and F. Wilczek (1994)

    potential difficulty: $D(\omega, q_y) = \frac{e^2}{N} \frac{1}{\frac{e^2 k_F}{2\pi} \frac{|\omega|}{|q_y|} + |q_y|^{1+\epsilon}}$

    systematic counting of powers of α needs to be done!

  - $\epsilon \to 0, \; N \to \infty, \; N\epsilon$ - fixed, $1/N$ - expansion.

    D. Mross, J. McGreevy, H. Liu and T. Senthil (2010)

    cannot be applied to the $\epsilon = 0$ case.

Response functions

- Same as before,

$$D^{-1}(\omega,\vec{q}) \sim |\vec{q}|^{z-1} f\left(\frac{|\omega|}{|\vec{q}|^{z}}\right)$$

$$G^{-1}(\omega,\vec{k}) \sim k^{1-\eta_f/2} g\left(\frac{|\omega|}{|k|^{z/2}}\right), \quad k = |\vec{k}| - k_F$$

- Except non-locality of the action constrains

$$z = 2 + \epsilon$$

- $\eta_f \sim \dfrac{1}{N^2} X(\epsilon N) \neq 0$ at three loop order.

D. Mross, J. McGreevy, H. Liu and T. Senthil (2010)

Critical Fermi surface states in 2+1 d. Part I.

- "Theory" of critical Fermi surface states in 2+1d

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    - gauge field (spinon Fermi-surface, half-filled Landau level)
    - order parameter (nematic transition)
  - two patch theory
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- Pairing instabilities of critical Fermi surface states

Pairing of critical Fermi surfaces

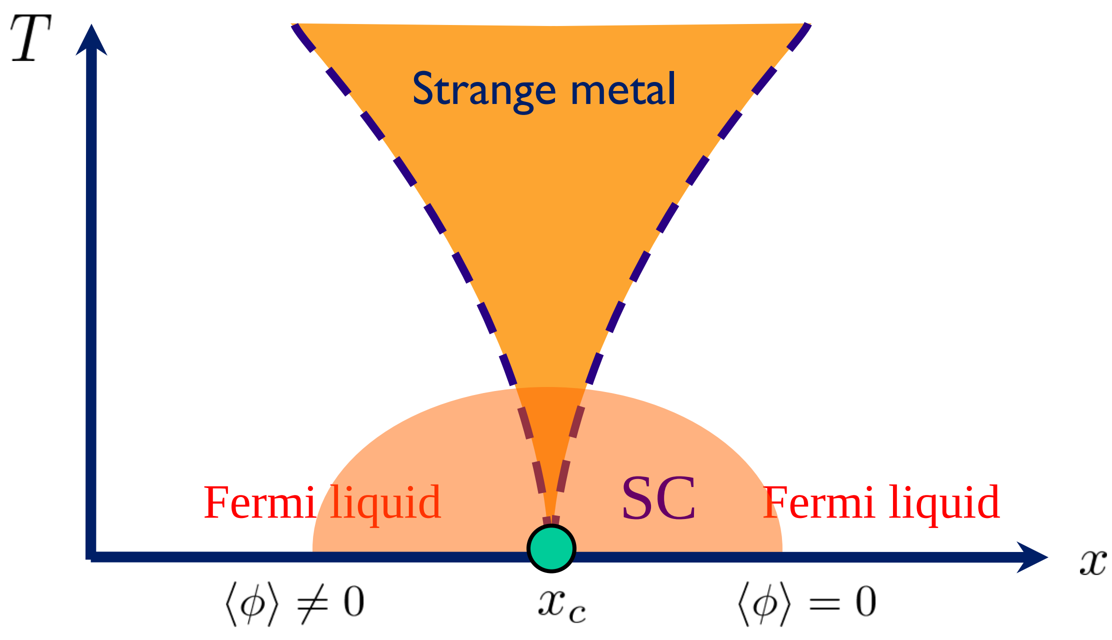
- A regular Fermi-liquid is unstable to arbitrarily weak attraction in the BCS channel.
- How about a critical Fermi surface?

Pairing instability of the nematic transition

- Nematic fluctuations lead to attraction in the BCS channel
- Fundamental problem: as one approaches the critical point

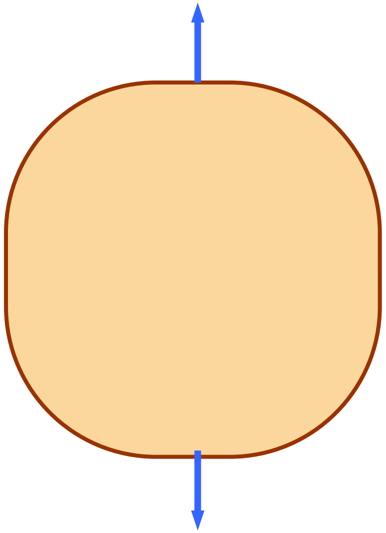
  the pairing glue becomes strong, but

  the quasiparticles are destroyed
- Who wins?



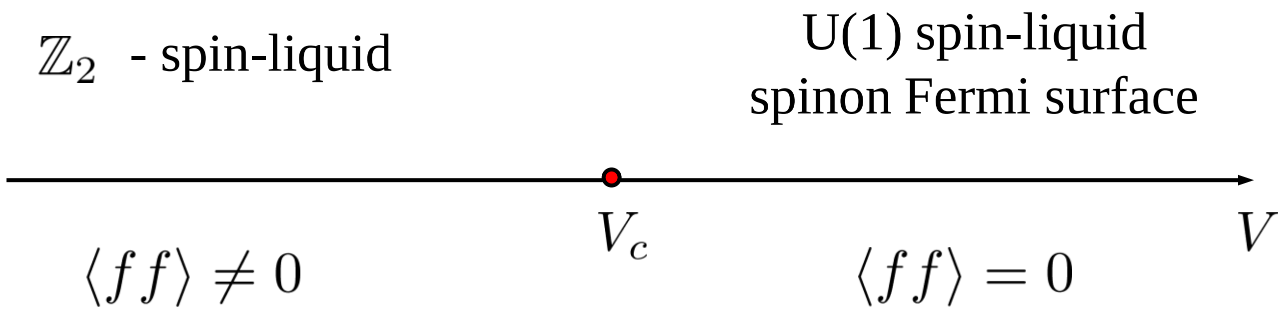
Pairing instabilities for spin/charge liquids

- Magnetic fluctuations mediate a long-range repulsion



- Can a short range attractive interaction V compete with this?
- What is the critical interaction strength?

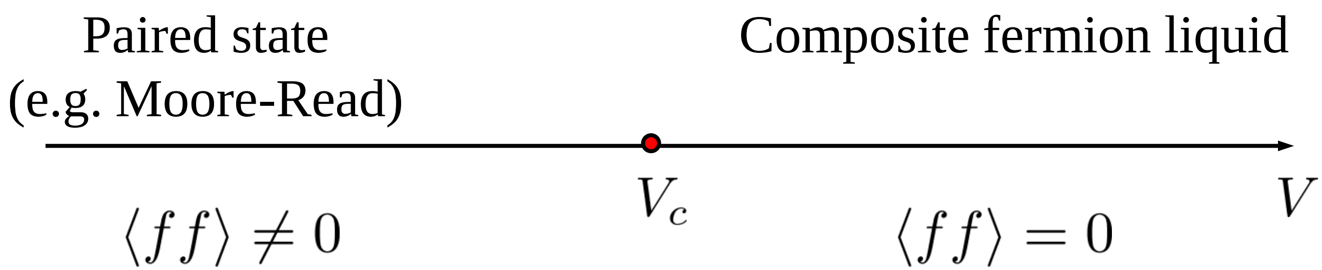
Pairing in a spin-liquid



Excitations:

Excitations:	gapped spinons gapped Abrikosov vortices (visons)	gapless spinons f_α gapless gauge (magnetic field) fluctuations

Pairing in a half-filled Landau level



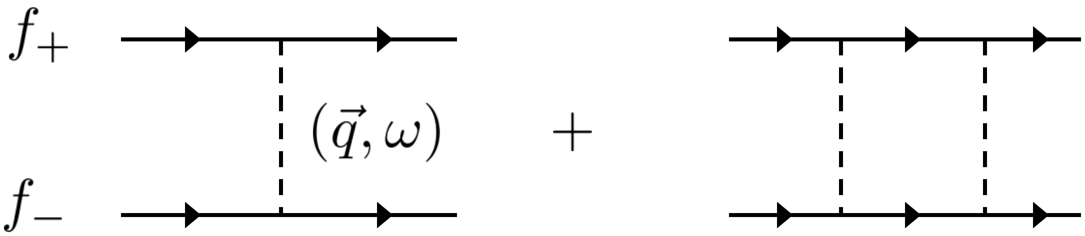
Excitations:	gapped fermions gapped vortices (charge $e/4$)	gapless composite fermions f_α gapless gauge (density) fluctuations

- Exact diagonalisation data may be interpreted as a transition/crossover

H.E.Rezayi and F.D.M. Haldane (2000), G. Moller, A. Wojs and N.R.Moller (2011)

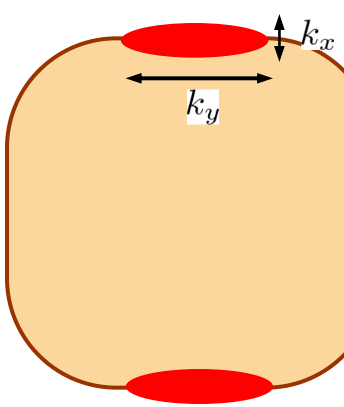
Pairing singularities

- Scattering amplitude in the BCS channel at $\epsilon = 0$

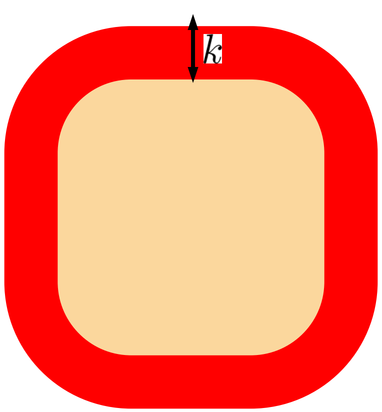


- In the regime $\omega \ll q_y^2$, the one loop diagram is enhanced by $-\frac{\alpha}{N} \log^2 \frac{q_y^2}{\omega}$
- $\delta L = g f^\dagger f^\dagger f f$

irrelevant in two-patch theory



marginal in Fermi-liquid theory

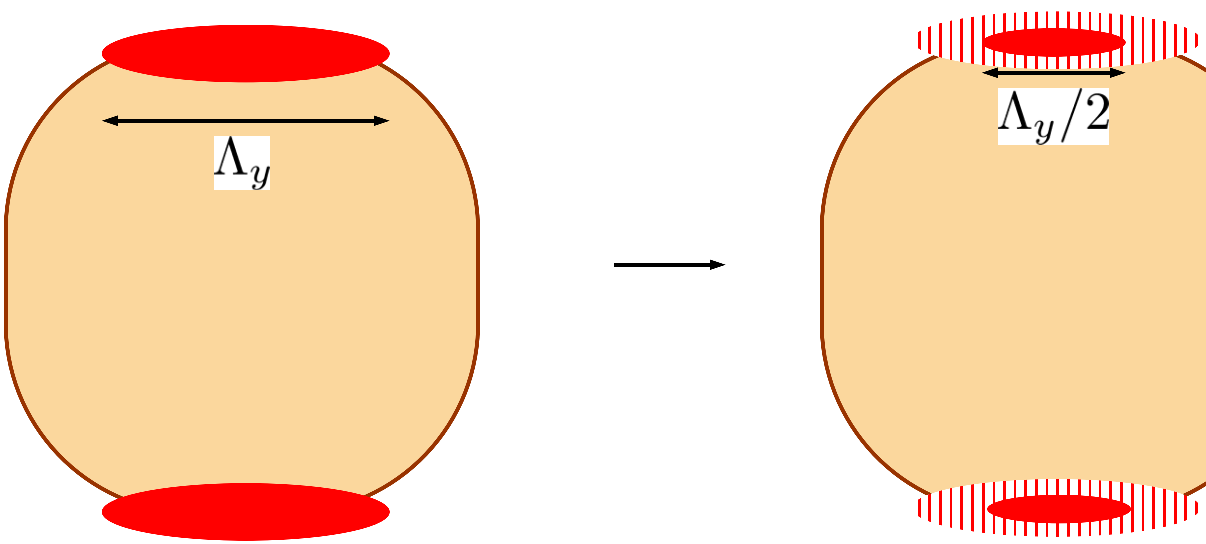


Conceptual difficulties with two-patch RG

- $\delta L = g f^\dagger f^\dagger f f$

$$\sim g\left(1 + \frac{\alpha}{N} \log^2 \frac{\Lambda}{\omega}\right)$$

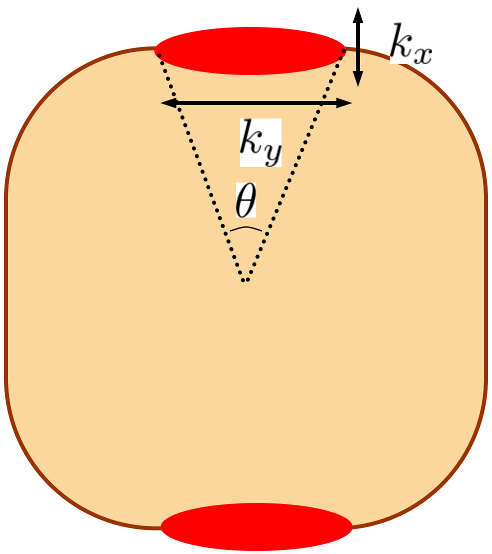
- Low-energy states on the Fermi-surface cannot be integrated out



Conceptual difficulties with two-patch RG

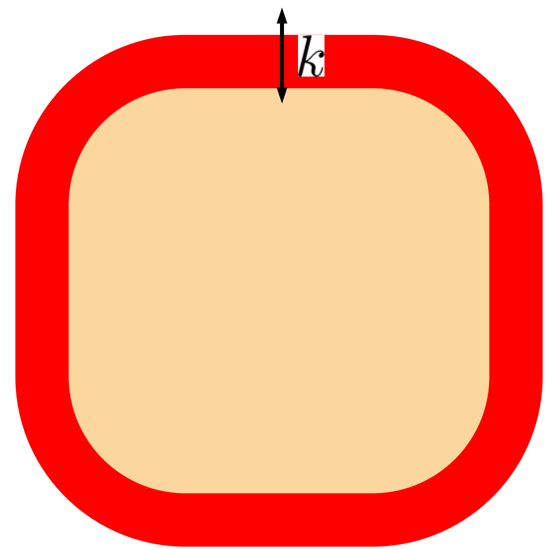
- Treatment of the pairing instability requires a marriage of two RG's:

$$k_y \to s k_y, \quad k_x \to s^2 k_x$$



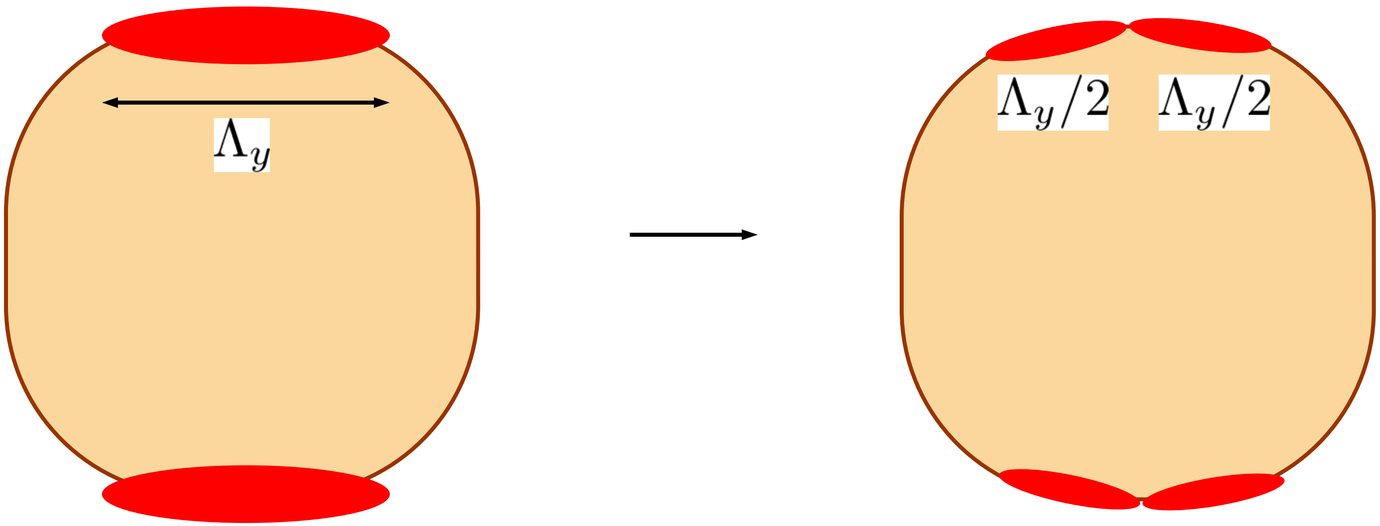
$$\theta \to s\theta$$

$$k \to sk$$



θ does not flow

Son's RG procedure



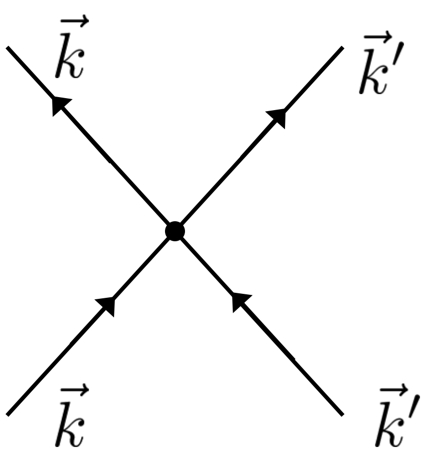
- Keep interpatch couplings!

D. T. Son, Phys. Rev. D 59, 094019 (1999).

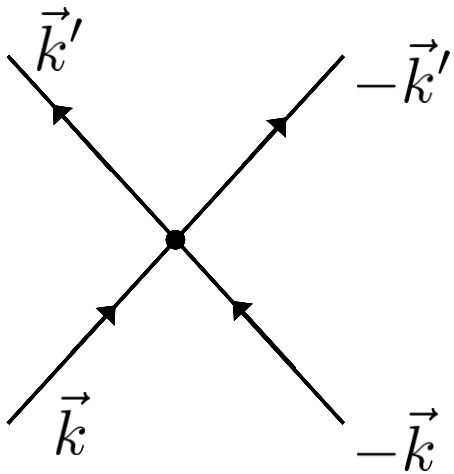
Perturbations

$$S_4 = -\frac{1}{4}\int \prod_{i=1}^{4} \frac{d^3k_i}{(2\pi)^3} U_{\alpha\beta;\gamma\delta}(\hat{k}_1, \hat{k}_2; \hat{k}_3, \hat{k}_4)\psi_\alpha^\dagger(k_1)\psi_\beta^\dagger(k_2)\psi_\gamma(k_3)\psi_\delta(k_4) \times(2\pi)^3\delta^3(k_1+k_2-k_3-k_4)$$

- Only two types of momentum conserving processes keep fermions on the FS

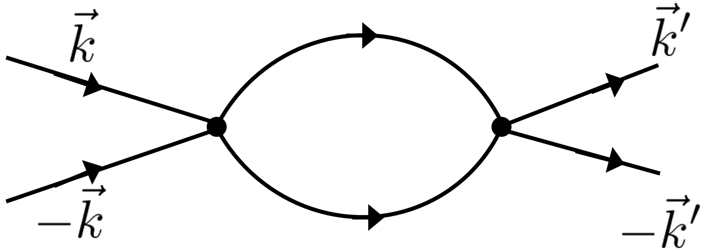


Forward-scattering $F^{s,a}(\vec{k}', \vec{k})$



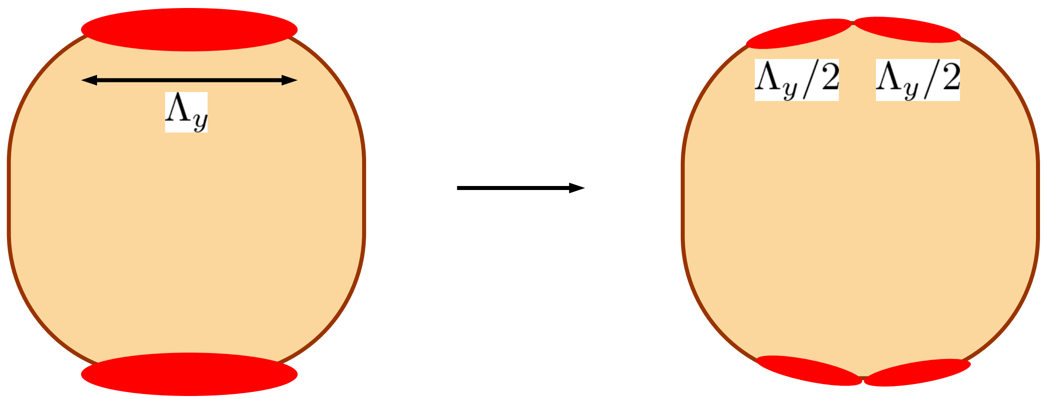
BCS scattering $V^{s,a}(\vec{k}', \vec{k})$

Fermi-liquid RG

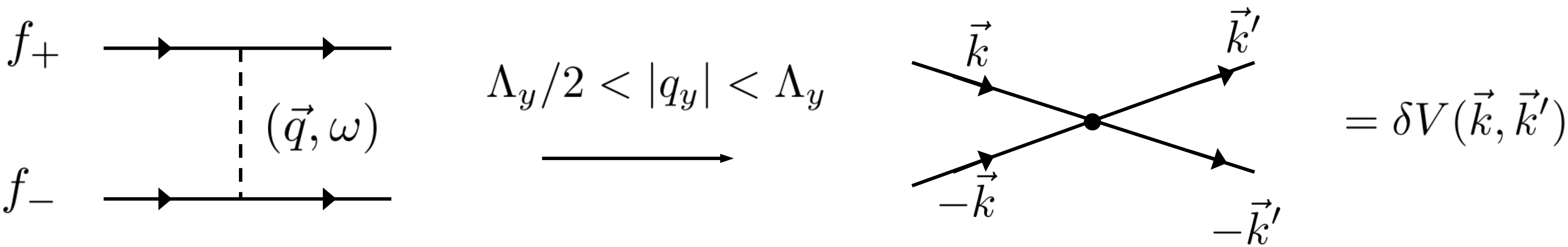


$$\frac{dV_m^{s,a}}{d\ell} = -(V_m^{s,a})^2$$

Son's RG



- Generation of inter-patch couplings:



- Generates an RG flow:

$$\frac{dV_m^{s,a}}{d\ell} = \mp \frac{1}{N} \alpha$$

Combined RG

$$\frac{dV_m^{s,a}}{d\ell} = \mp \frac{1}{N}\alpha - (V_m^{s,a})^2$$

$$\frac{d\alpha}{d\ell} = \frac{\epsilon}{2}\alpha - \frac{1}{N}\alpha^2$$

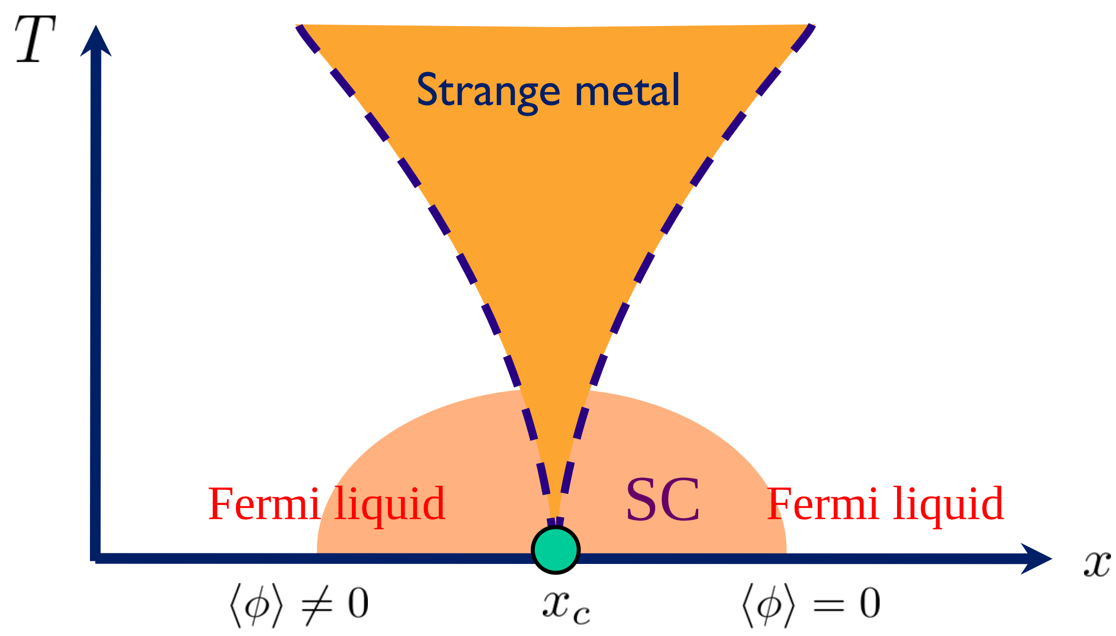
- (from intra-patch theory)

Pairing: Ising-nematic transition

$$\frac{dV_m^{s,a}}{d\ell} = -\frac{1}{N}\alpha - (V_m^{s,a})^2 \qquad \frac{d\alpha}{d\ell} = \frac{\epsilon}{2}\alpha - \frac{1}{N}\alpha^2$$

- Always flows to $V = -\infty$ (transition unstable to pairing)
- Pairing preempts the Non-Fermi-liquid physics

$\Delta_{\text{pair}} \gg E_{\text{NFL}}$ whenever expansion controlled



Pairing: Ising-nematic transition

$$\frac{dV_m^{s,a}}{d\ell} = -\frac{1}{N}\alpha - (V_m^{s,a})^2 \qquad \frac{d\alpha}{d\ell} = \frac{\epsilon}{2}\alpha - \frac{1}{N}\alpha^2$$

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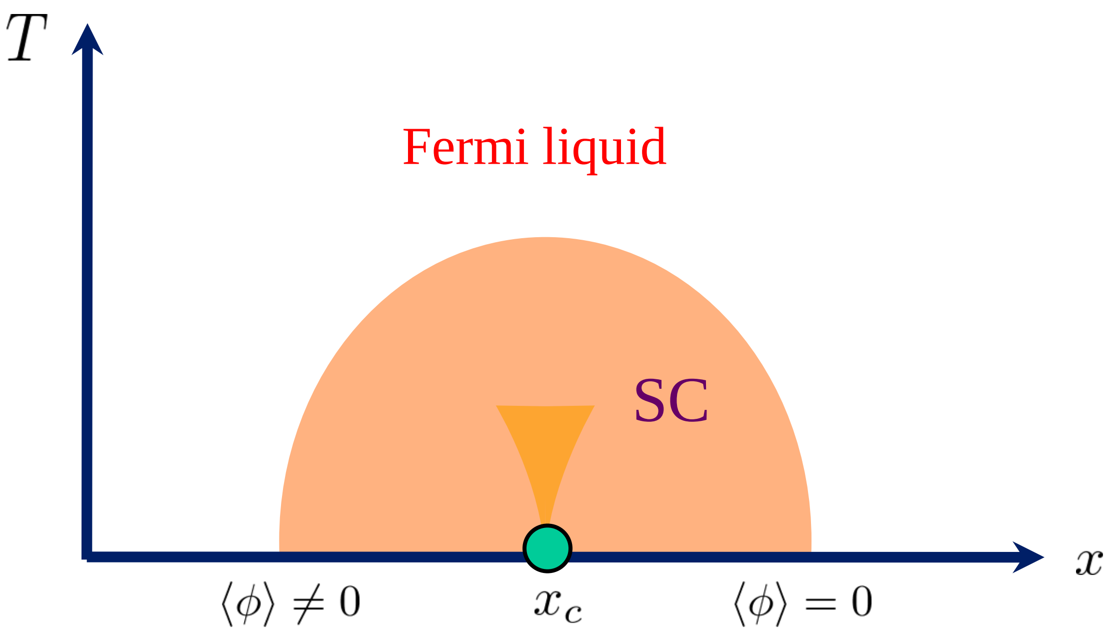
$\Delta_{\text{pair}} \gg E_{\text{NFL}}$ whenever expansion controlled



Pairing: Ising-nematic transition

$$\frac{dV_m^{s,a}}{d\ell} = -\frac{1}{N}\alpha - (V_m^{s,a})^2 \qquad\qquad \frac{d\alpha}{d\ell} = \frac{\epsilon}{2}\alpha - \frac{1}{N}\alpha^2$$

- $\epsilon = 0$

$$\Delta_{pair} \sim \Lambda \exp(-C\sqrt{N/\alpha}) \qquad\qquad E_{NFL} \sim \Lambda \exp(-N/\alpha)$$

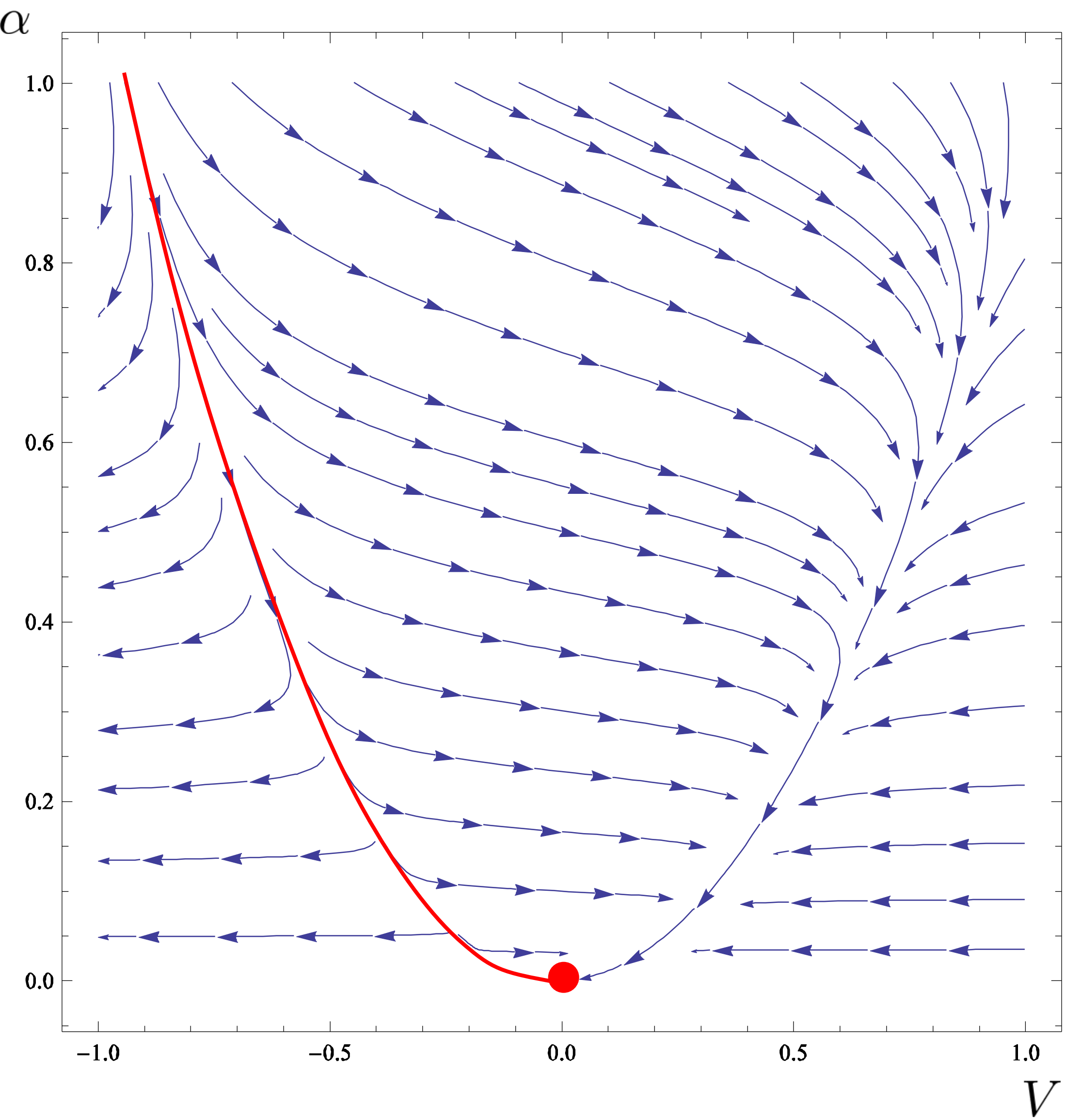
- similar to dense QCD in 3+1d (D. T. Son, (1999)).

- $\epsilon > 0, \quad \alpha \to 0$

$$\Delta_{pair} \sim \left(\frac{1}{\epsilon}\right)^{2/\epsilon} E_{NFL}$$

E. A. Yuzbashyan, unpublished (A. Chubukov, private communication)

Pairing: gauge field, ε =0



$$\frac{dV_m^{s,a}}{d\ell} = \frac{1}{N}\alpha - (V_m^{s,a})^2$$

$$\frac{d\alpha}{d\ell} = \frac{\epsilon}{2}\alpha - \frac{1}{N}\alpha^2$$

- $\epsilon = 0$
- Single fixed point (CFL)

$$V = 0, \ \alpha = 0$$

- Pairing gap onsets as

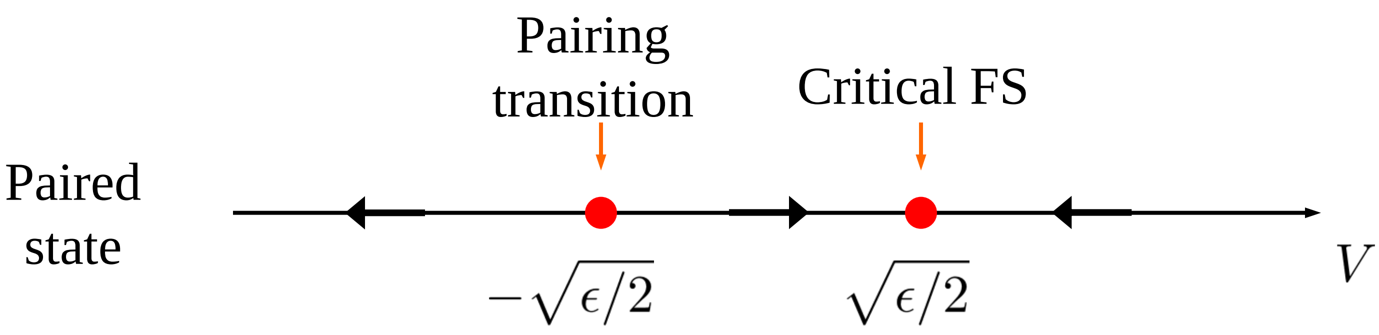
$$\Delta = \exp\left(-\frac{1}{16}\log^2(V_c - V)\right)$$

M.M., D. Mross, S. Sachdev, T. Senthil, forthcoming

Pairing: gauge field, ε >0

$$\frac{dV_m^{s,a}}{d\ell} = \frac{1}{N}\alpha - (V_m^{s,a})^2$$

$$\frac{d\alpha}{d\ell} = \frac{\epsilon}{2}\alpha - \frac{1}{N}\alpha^2 \qquad \alpha \to \frac{N\epsilon}{2}$$

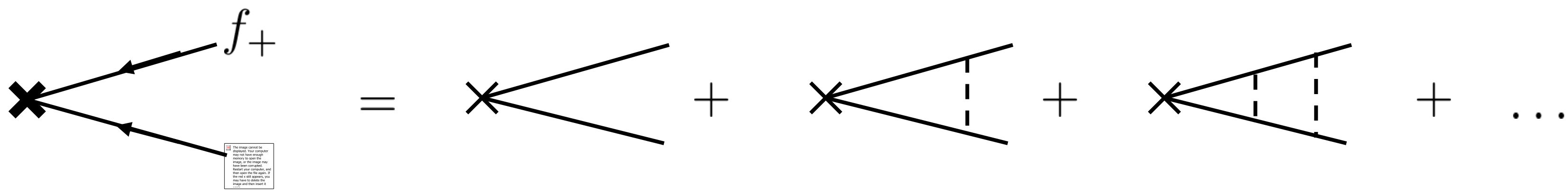


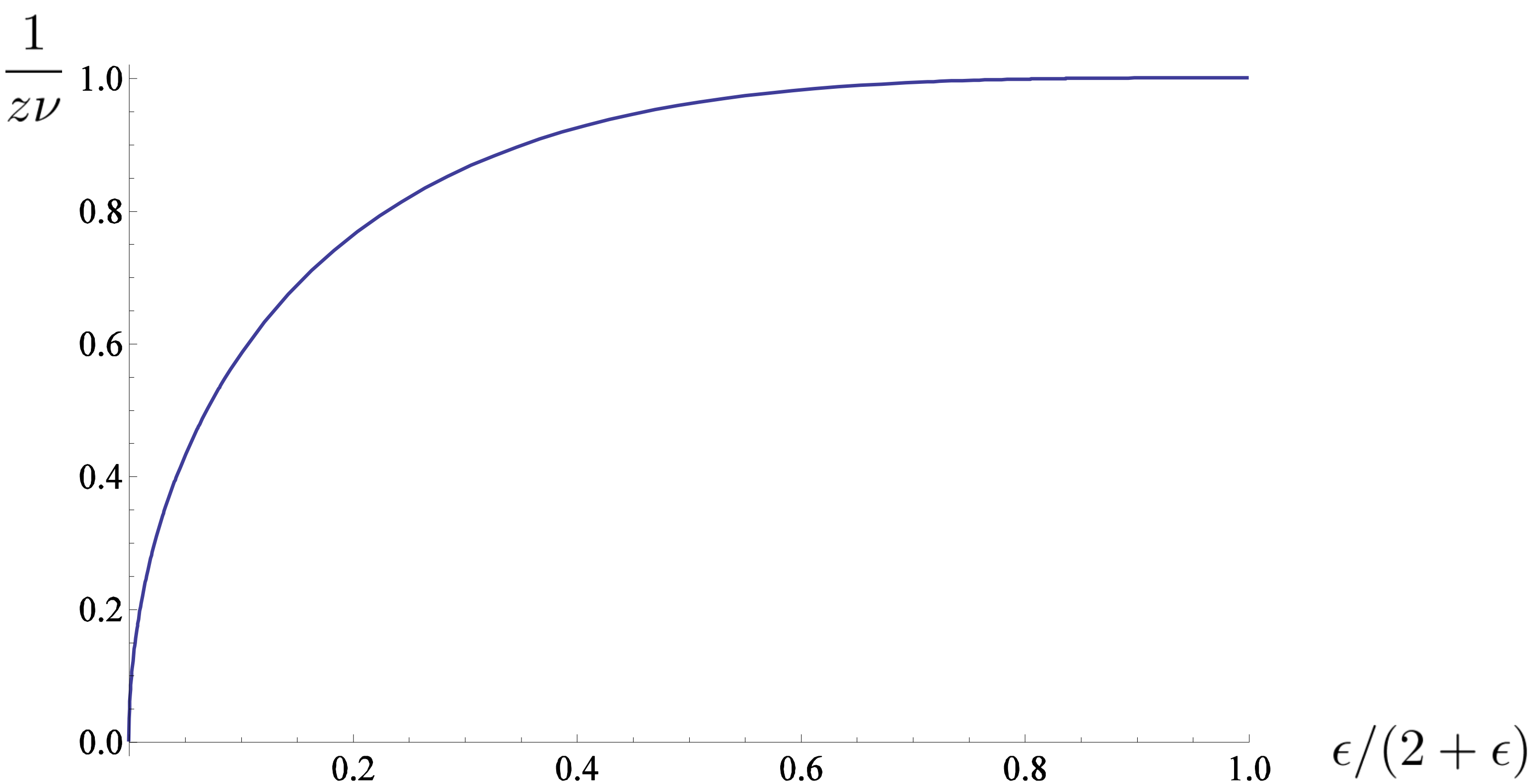
- Gap onsets as $\Delta = (V_c - V)^{z\nu}, \quad (z\nu)^{-1} = \sqrt{2\epsilon}$

M.M., D. Mross, S. Sachdev, T. Senthil, forthcoming

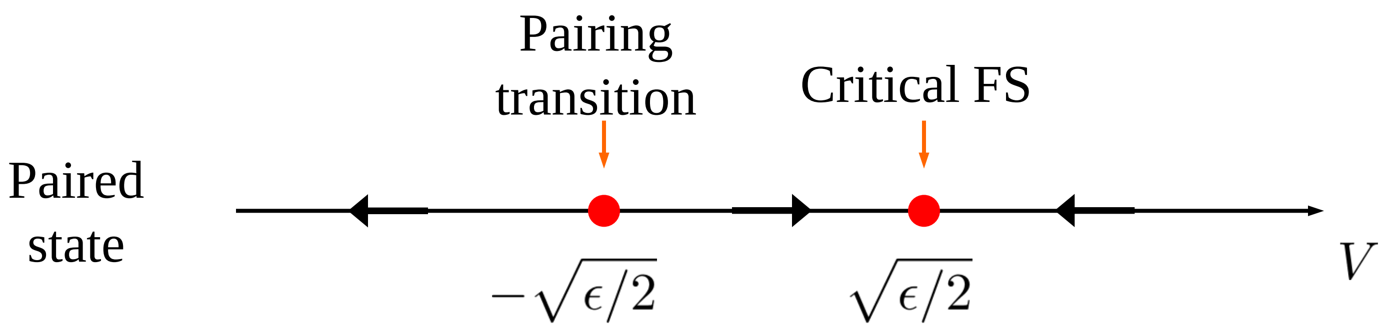
Eliashberg approximation

- All results for $\epsilon \ll 1$ can be reproduced by summing rainbow graphs in the Eliashberg approximation.





Open questions



- Can we make Son's RG more systematic?

  - explore anologies with problems in particle physics

$$\omega \to 0, \vec{q} \to 0, \quad |\omega| \to |\vec{q}|$$

- Are fermion and gauge field propagators different at the two fixed points?
- Properties of the paired state:
  - is the "superconductor" type I or type II
  - how does the vortex mass vanish at the pairing transition?

Conclusion

- Progress (and new challenges) in understanding critical fermi surface states.
- First theory of pairing transition out of a critical fermi-surface state.
- Lots of open questions

Thank you!