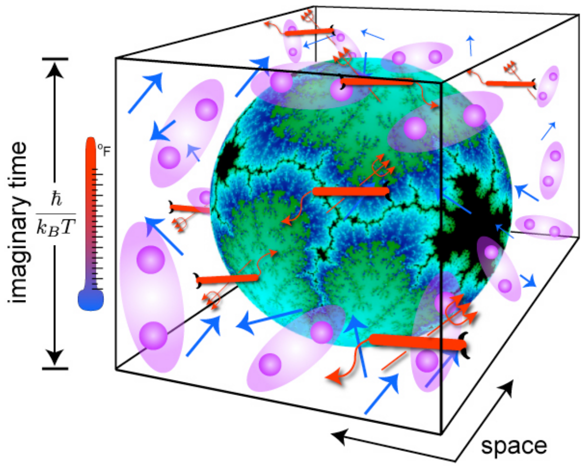
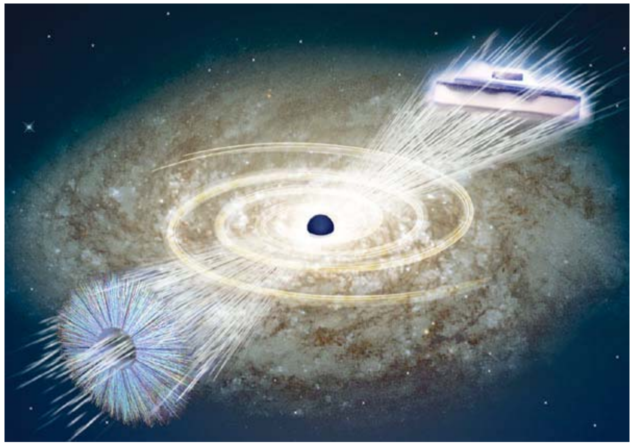
Fermions, fermions, fermions!

Jan Zaanen





Nederlandse Organisatie voor Wetenschappelijk Onderzoek

Universiteit Leiden
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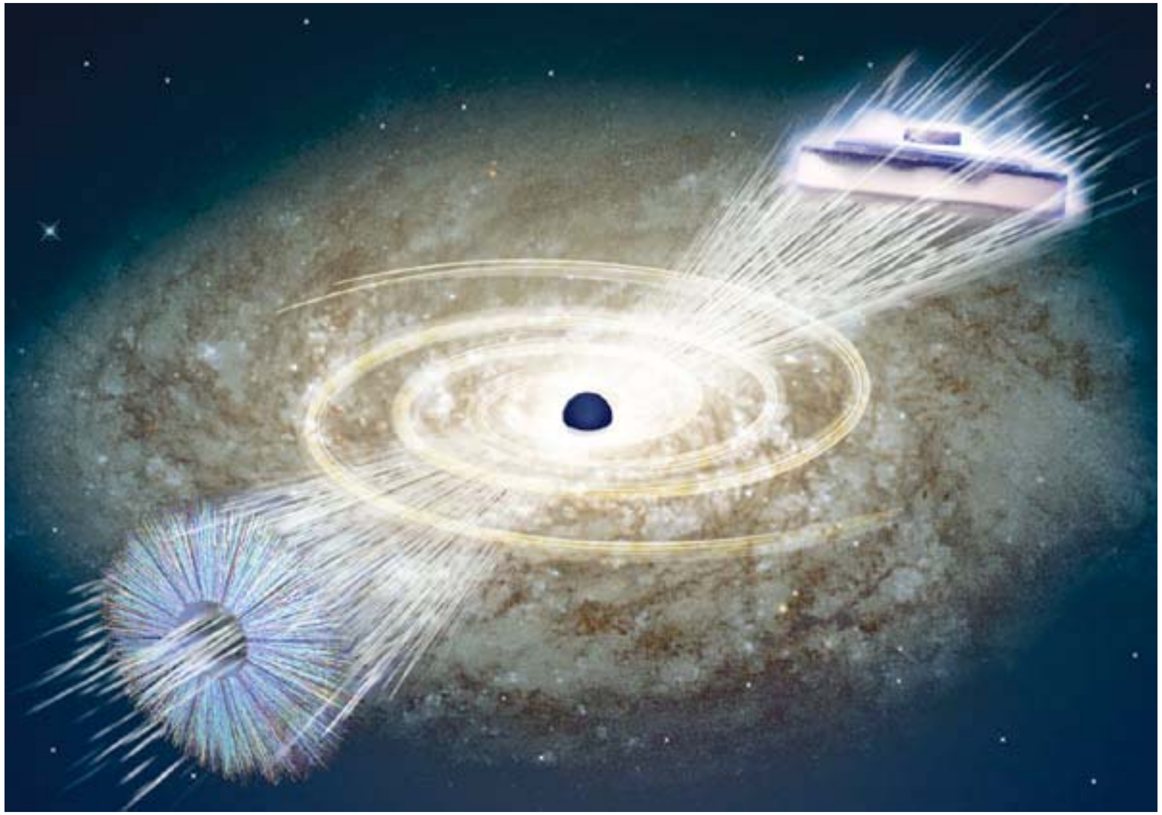
THEORETICAL PHYSICS

A black hole full of answers

Jan Zaanen

A facet of string theory, the currently favoured route to a 'theory of everything', might help to explain some properties of exotic matter phases — such as some peculiarities of high-temperature superconductors.

NATURE|Vol 448|30 August 2007



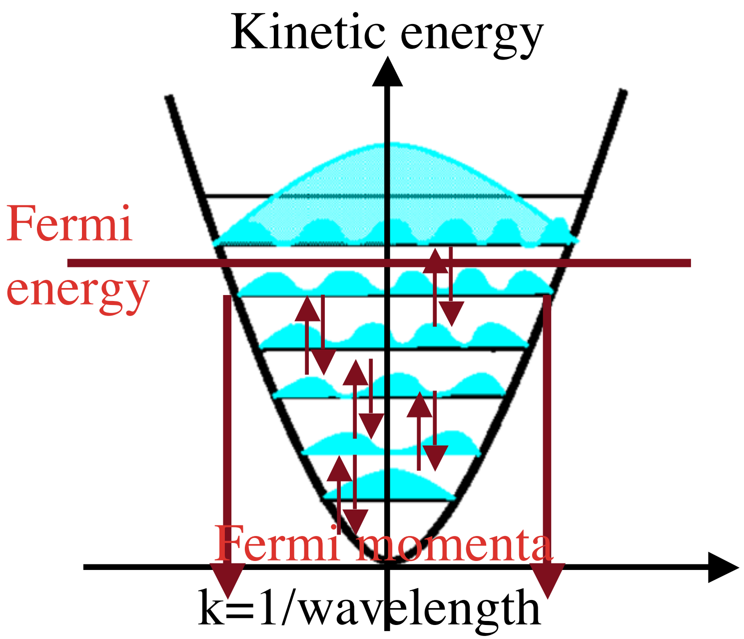
Plan

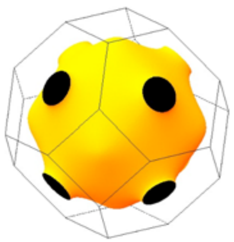
Koenraad: AdS/CFT and the emergent Fermi liquid

1. Why condensed matter needs AdS/CFT: the Fermion signs.

2. Geometrizing the fermion signs: the Ceperley path integral and the conformal Feynmannian backflow state.

The quantum in the kitchen: Landau's miracle





Fermi surface of copper

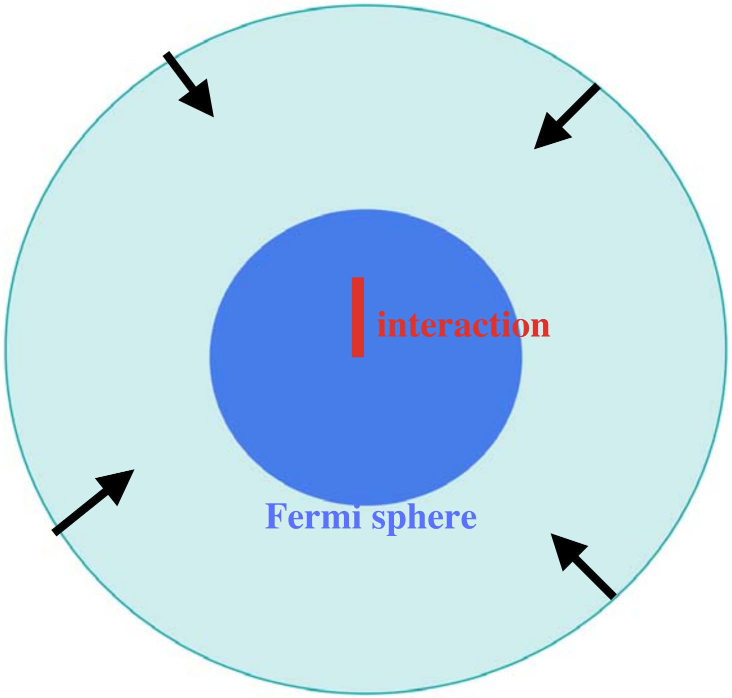
Electrons are waves

Pauli exclusion principle: every state occupied by one electron

Unreasonable: electrons strongly interact !!

Landau's Fermi-liquid: the highly collective low energy quantum excitations are like electrons that do not interact.

‘Shankar/Polchinski’ functional renormalization group

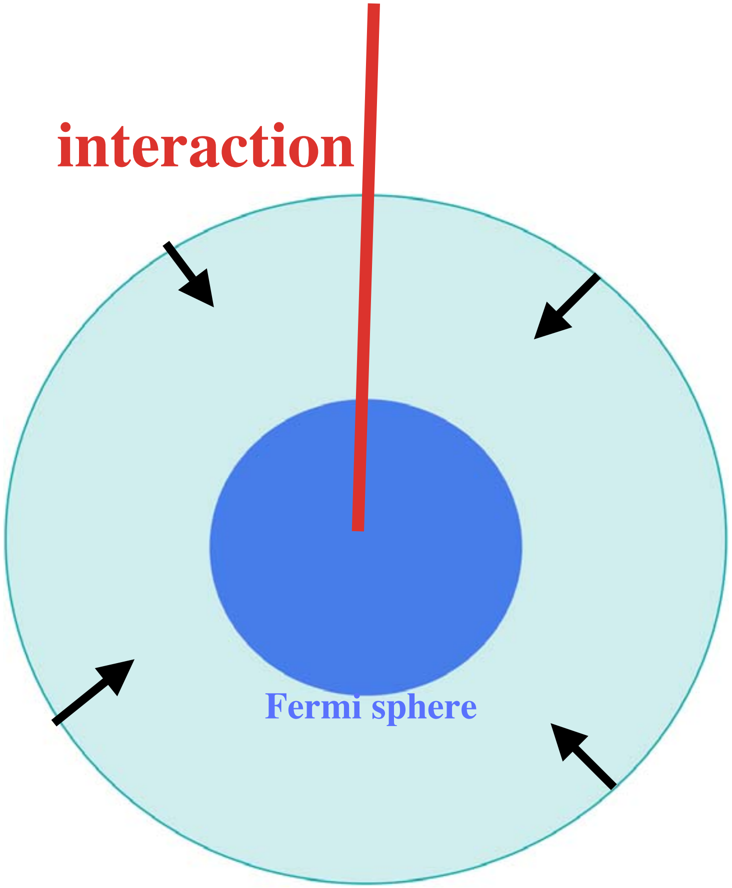


UV: weakly interacting Fermi gas

Integrate momentum shells: functions of running coupling constants

All interactions (except marginal Hartree) irrelevant => **Scaling limit might be perfectly ideal Fermi-gas**

The end of weak coupling



Strong interactings:

Fermi gas as UV starting point does not make sense!

=> 'emergent' Fermi liquid fixed point remarkably resilient (e.g. 3He)

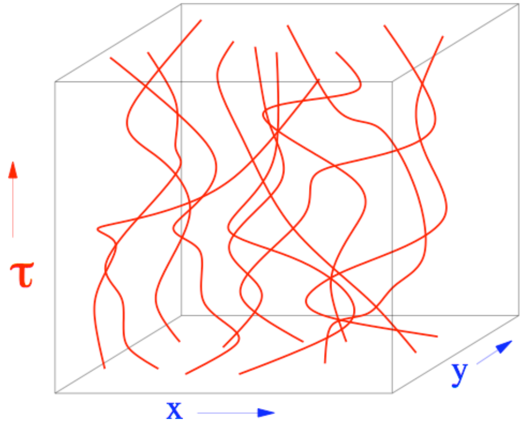
=> Non Fermi-liquid/non 'Hartree-Fock' (BCS etc) states of fermion matter?

Fermion sign problem

Imaginary time path-integral formulation

$$\begin{aligned}\mathcal{Z} &= \operatorname{Tr}\exp(-\beta\hat{\mathcal{H}}) \\ &= \int \mathrm{d}\mathbf{R}\,\rho(\mathbf{R},\mathbf{R};\beta)\end{aligned}$$

$$\mathbf{R} = (\mathbf{r}_1,\ldots,\mathbf{r}_N) \in \mathbb{R}^{Nd}$$



$$\begin{aligned}\rho_{B/F}(\mathbf{R},\mathbf{R};\beta) &= \frac{1}{N!}\sum_{\mathcal{P}}(\pm 1)^{\mathcal{P}}\rho_D(\mathbf{R},\mathcal{P}\mathbf{R};\beta) \\ &= \frac{1}{N!}\sum_{\mathcal{P}}(\pm 1)^{\mathcal{P}}\int_{\mathbf{R}\to\mathcal{P}\mathbf{R}}\mathcal{D}\mathbf{R}(\tau)\exp\left\{-\frac{1}{\hbar}\int_0^{\hbar/T}\mathrm{d}\tau\left(\frac{m}{2}\dot{\mathbf{R}}^2(\tau)+V(\mathbf{R}(\tau))\right)\right\}\end{aligned}$$

Boltzmannons or Bosons:

- integrand non-negative
- probability of equivalent classical system: (crosslinked) ringpolymers

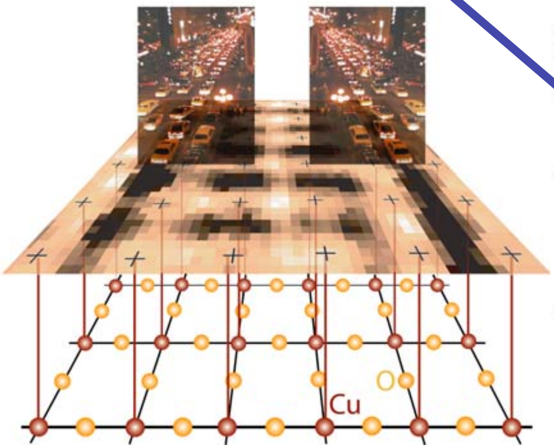
Fermions:

- negative Boltzmann weights
- non probablistic: NP-hard problem (Troyer, Wiese)!!!

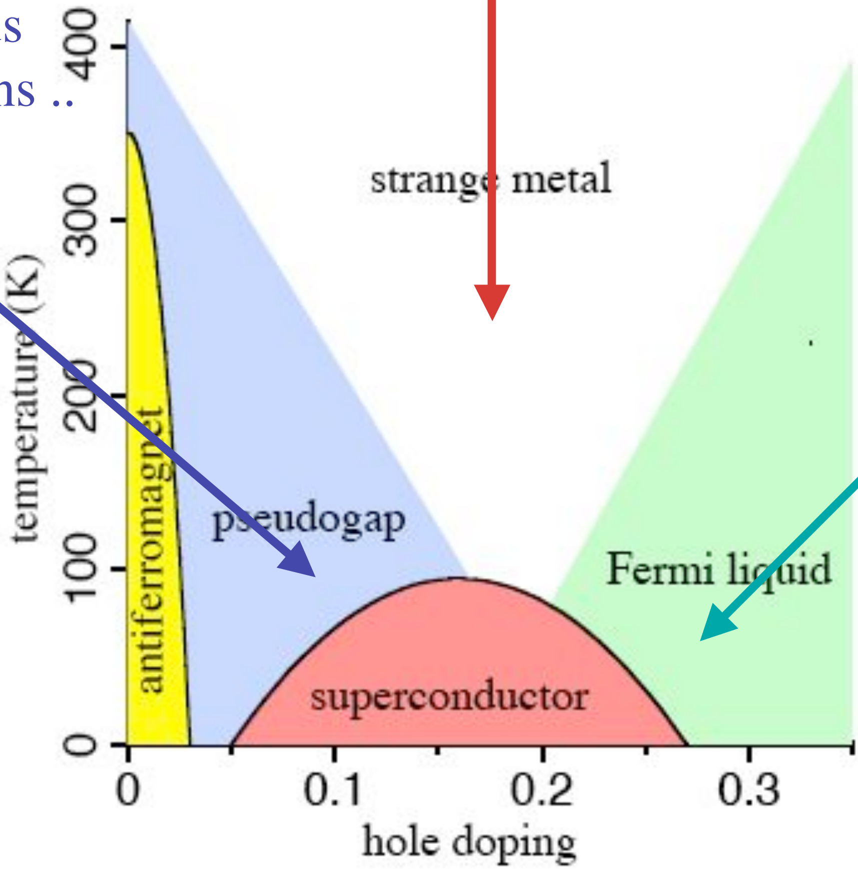
Phase diagram high Tc superconductors

Mystery quantum critical metal (Van der Marel et al. Nature 425, 271,2003; JZ, Nature 430, 512 ,2004)

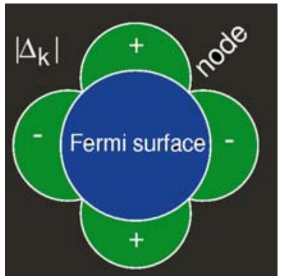
'Stripy stuff', spontaneous currents, phase fluctuations ..



JZ, Science **315**, 1372 (2007)



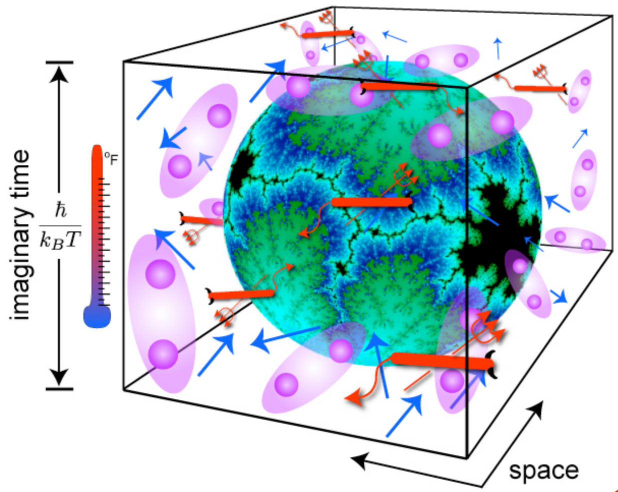
The return of normalcy

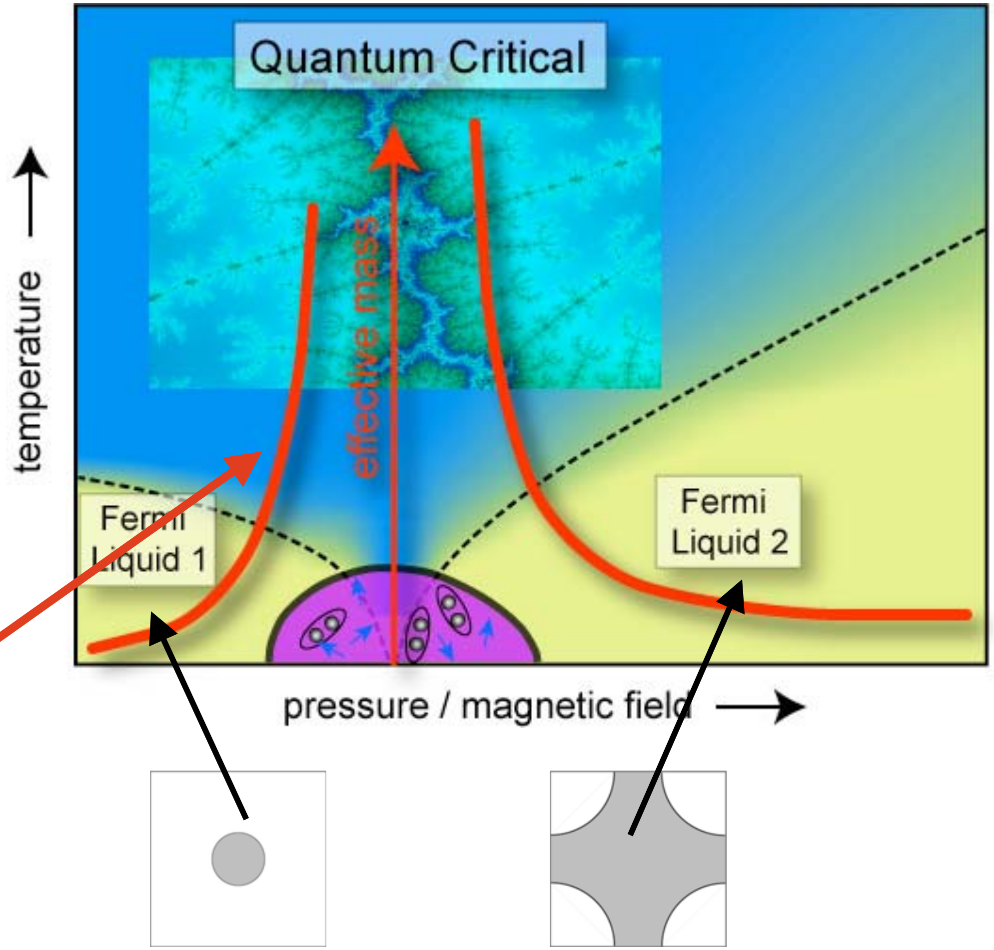


$$\Psi_{BCS} = \Pi_k \left(u_k + v_k c^{\dagger}_{k\uparrow} c^{\dagger}_{-k\downarrow} \right) |vac\rangle$$

Fermionic quantum phase transitions in the heavy fermion metals

JZ, Science 319, 1205 (2008)





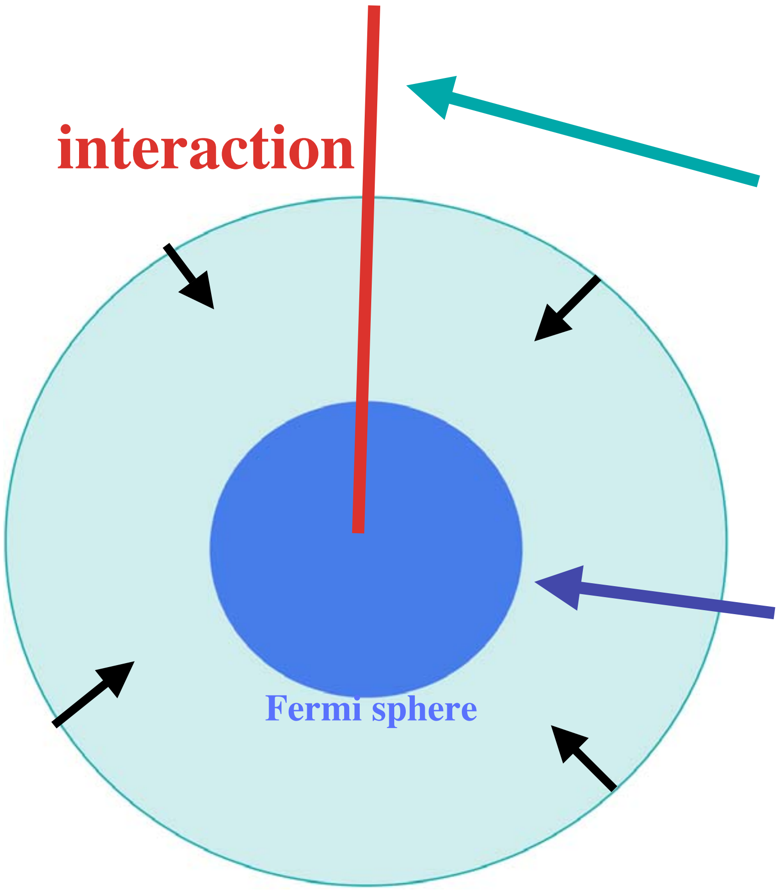
Paschen et al., Nature (2004)

QP effective mass

$$m^* = \frac{1}{E_F}$$

$$E_F \to 0 \Rightarrow m^* \to \infty$$

A UV that is strongly interacting critical …



UV governed by conformal invariance, no Fermi-surface, no Fermi energy

Emergent heavy Fermi-liquid in the IR that can disappear at a QPT

This is effortlessly encoded in Koenraad's AdS/CFT!

Emergent non Fermi-liquids with Fermi-surfaces

Hong Liu's emergent 2D CFT "AdS/ARPES":

- Truly critical: renormalized Fermi energy is zero.

- But the Fermi surface is remembered as singularity structure in momentum space.

How can a state be conformal while it remembers the reciprocal length Fermi momentum? Bosons cannot!!

Critical Fermi-surface (Senthil); phenomenological.

Gauge theories (Sung-Sik,Subir): Shankar/Polchinski attitude.

Geometrizing Fermi-Dirac statistics

Ceperley's path integral: encoding Fermi-Dirac statistics in geometry: the nodal hypersurface.

- Fermi-liquid: Fermi-energy is encoded in the local geometry but the Fermi-surface is encoded globally

F. Krueger et al., arXiv:0802.2455

- Feynmannian backflow ansatz: nodal geometry turns fractal, the state is conformal but room for globally encoded Fermi-surface information.

F. Krueger, JZ, arXiv:0804.2161

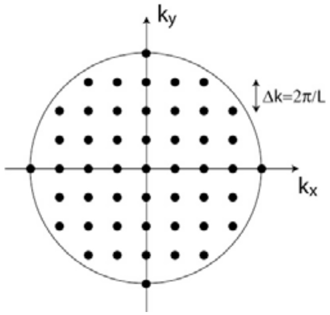
The nodal hypersurface

Antisymmetry of the wave function

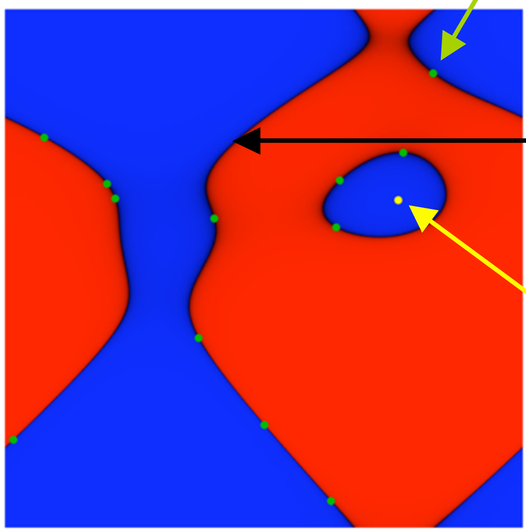
$$\Psi(\mathbf{r}_1,\ldots,\mathbf{r}_i,\ldots,\mathbf{r}_j,\ldots,\mathbf{r}_N) = -\Psi(\mathbf{r}_1,\ldots,\mathbf{r}_j,\ldots,\mathbf{r}_i,\ldots,\mathbf{r}_N)$$

Free Fermions

$$\Psi_0(\mathbf{R}) \sim \mathrm{Det}\left(e^{i\mathbf{k}_i\mathbf{r}_j}\right)_{ij}$$



d=2



Pauli hypersurface

$$P = \bigcup_{i\neq j} P_{ij}$$

$$P_{ij} = \{\mathbf{R}\in\mathbb{R}^{Nd} | \mathbf{r}_i = \mathbf{r}_j\}$$

$$\dim P = Nd - d$$

Nodal hypersurface

$$\Omega = \{\mathbf{R}\in\mathbb{R}^{Nd} | \Psi(\mathbf{R}) = 0\}$$

$$\dim \Omega = Nd - 1$$

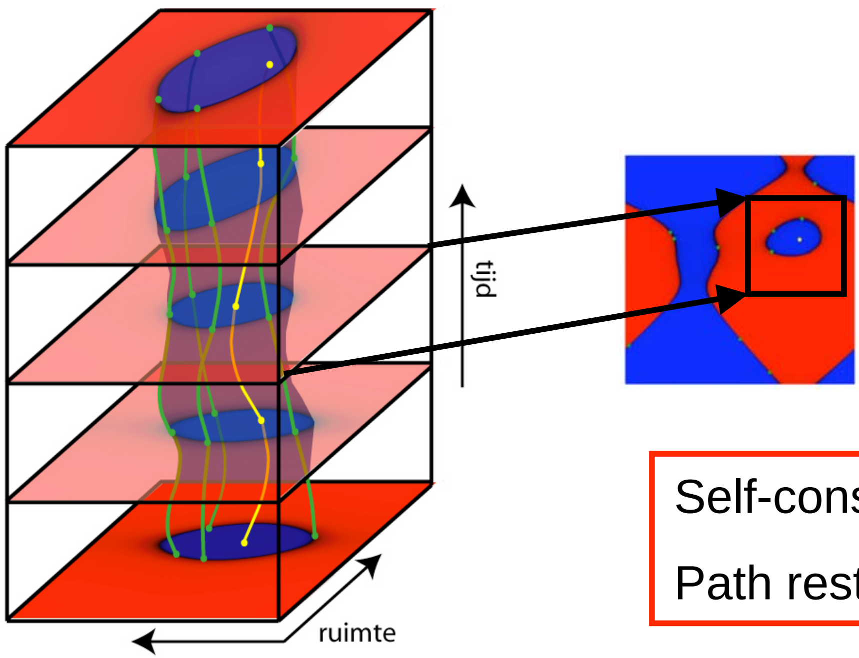
Test particle

Constrained path integrals

Formally we can solve the sign problem!!

$$\rho_F(\mathbf{R},\mathbf{R};\beta) = \frac{1}{N!}\sum_{\mathcal{P},\mathrm{even}} \int_{\gamma:\mathbf{R}\to\mathcal{P}\mathbf{R}}^{\gamma\in\Gamma(\mathbf{R},\mathcal{P}\mathbf{R})} \mathcal{D}\mathbf{R}(\tau)\exp\left\{-\frac{1}{\hbar}\int_0^{\hbar/T} \mathrm{d}\tau\left(\frac{m}{2}\dot{\mathbf{R}}^2(\tau)+V(\mathbf{R}(\tau))\right)\right\}$$

$$\Gamma(\mathbf{R},\mathbf{R}') = \{\gamma:\mathbf{R}\to\mathbf{R}' | \rho_F(\mathbf{R},\mathbf{R}(\tau);\tau)\neq 0\}$$



Ceperley, J. Stat. Phys. (1991)

Self-consistency problem:

Path restrictions depend on ρ_F !

Ceperley path integral: Fermi gas in momentum space

Sergei Mukhin

Single particle propagator:

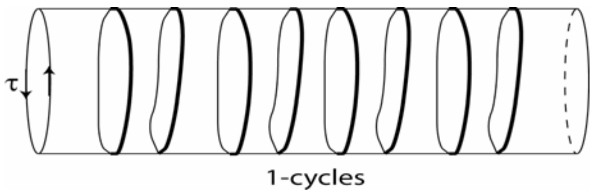
$$g(\boldsymbol{k}, \boldsymbol{k}', \tau) = 2\pi\delta(\boldsymbol{k} - \boldsymbol{k}')e^{-\frac{|\boldsymbol{k}|^2\tau}{2\hbar m}}$$

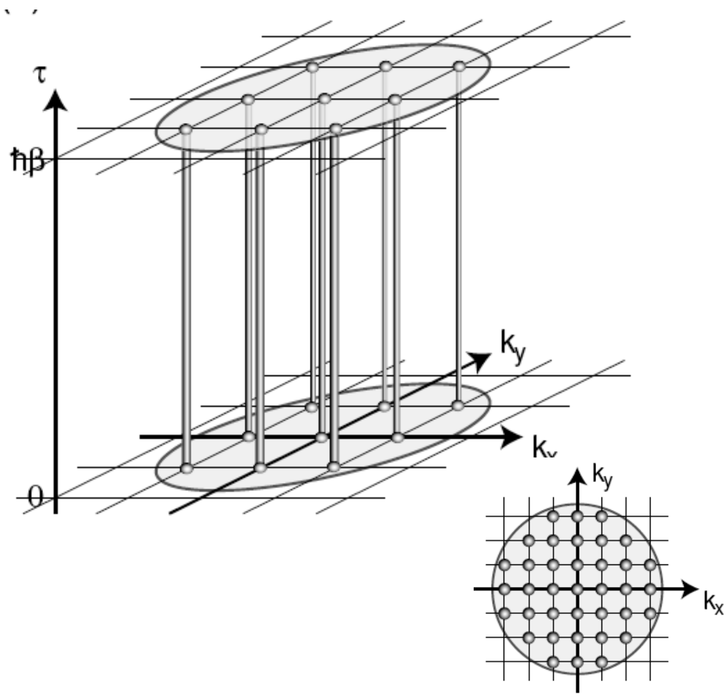
single particle momentum conserved

N particle density matrix:

$$\begin{aligned}\rho_F(\boldsymbol{K}, \boldsymbol{K}', \tau) &= \frac{1}{N!}\sum_{\mathcal{P}}(-1)^{\mathcal{P}}\prod_{i=1}^{N} g(\boldsymbol{k}_i, \boldsymbol{k}'_{p(i)}, \tau) \\ &= \frac{1}{N!}e^{-\sum_{i=1}^{N}\frac{|\boldsymbol{k}_i|^2\tau}{2\hbar m}}\sum_{\mathcal{P}}(-1)^{\mathcal{P}}\prod_{i=1}^{N}2\pi\delta(\boldsymbol{k}_i - \boldsymbol{k}'_{p(i)}) \\ &= \prod_{\boldsymbol{k}_1\neq\boldsymbol{k}_2\neq\ldots\neq\boldsymbol{k}_N} 2\pi\delta(\boldsymbol{k}_i - \boldsymbol{k}'_i)e^{-\frac{|\boldsymbol{k}_i|^2\tau}{2\hbar m}}\end{aligned}$$

'harmonic potential'

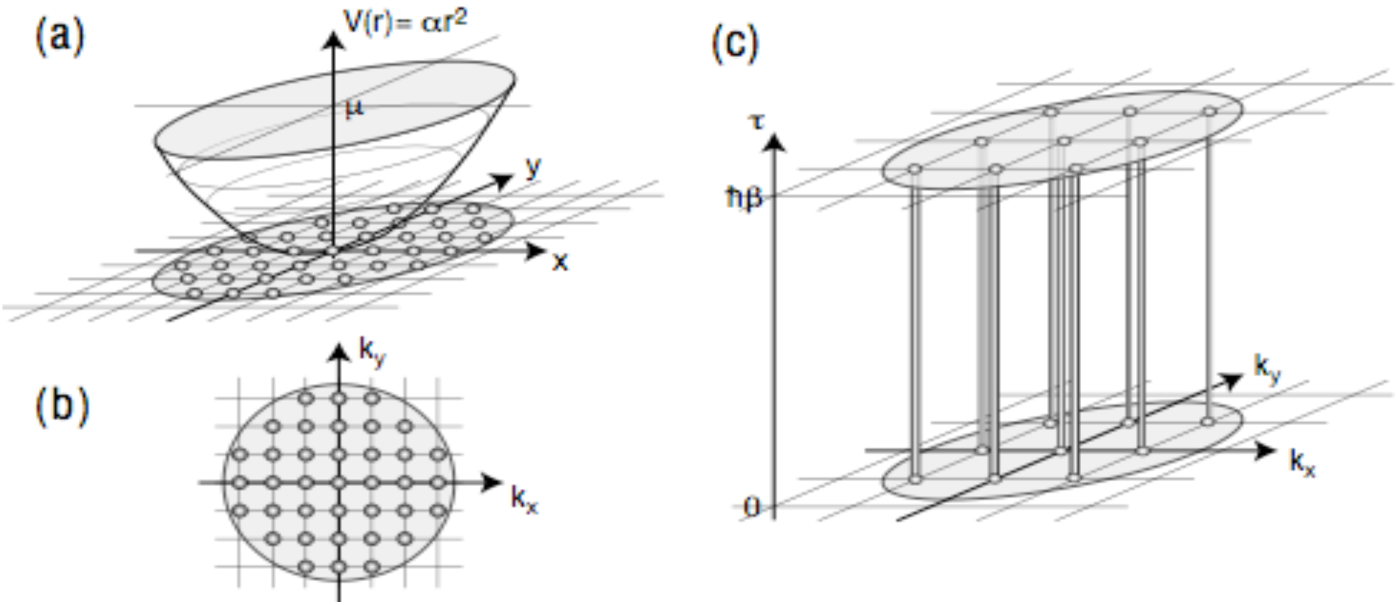




Fermi gas = cold atom Mott insulator in harmonic trap!

$$\rho_F(K,K";\tau) = \prod_{k_1 \neq k_2 \neq \cdots \neq k_N} 2\pi\delta\left(k_i - k_i^{"}\right) e^{-\frac{|k_i|^2 \tau}{2\hbar m}}$$

Mukhin, JZ, ..., Iranian J. Phys. (2008)



Reading the worldline picture

Fermi-energy: confinement energy imposed by local geometry

$$l^2(\tau) = \langle (\mathbf{r}_i(\tau) - \mathbf{r}_i(0))^2 \rangle = 2d\mathcal{D}\tau = 2d\frac{\hbar}{2m}\tau$$

$$l^2(\tau_c) \simeq r_s^2 \rightarrow \tau_c \simeq \frac{1}{2d}\frac{2m}{\hbar}n^{-2/d}$$

$$\hbar\omega_c = \frac{\hbar}{\tau_c} \simeq d\frac{\hbar^2}{2m}n^{2/d} \simeq E_F$$

Fermi surface encoded globally: $\rho_F = Det\left(e^{ik_i r_j}\right) = 0$

Change in coordinate of one particle changes the nodes everywhere

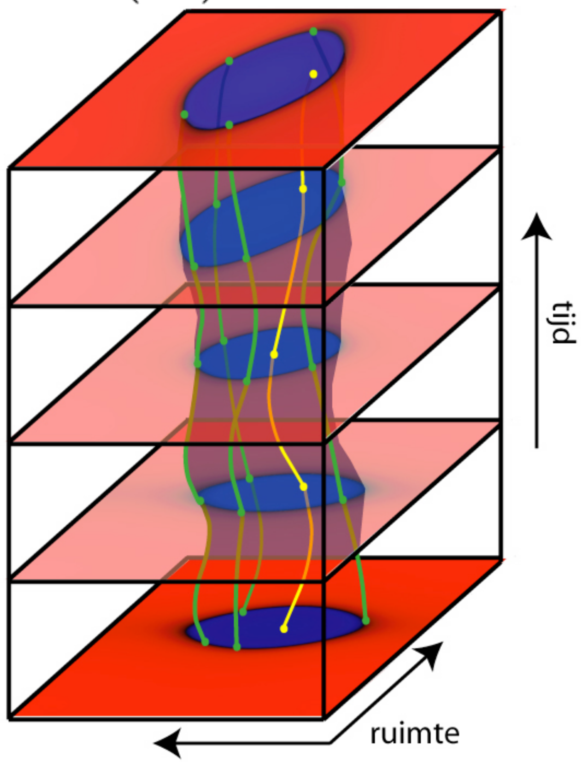
Finite T: $\rho_F = (4\pi\lambda\beta)^{-dN/2} Det\left[\exp\left(-\frac{(r_i - r_{j0})^2}{4\lambda\tau}\right)\right]$

$\lambda = \hbar^2 / (2M)$

Non-locality length: $\lambda_{nl} = v_F \tau_{inel} = v_F \left(\frac{E_F}{k_B T}\right)\left(\frac{\hbar}{k_B T}\right)$

Average node to node spacing

$$\sim r_s = \left(\frac{V}{N}\right)^{1/d} = n^{-1/d}$$



Key to fermionic quantum criticality

At the QCP scale invariance, no E_F → Nodal surface has to become fractal !!!

Fractal Cauliflower (romanesco)


Science
Quantum
Matter
AAAS

Geometrizing Fermi-Dirac statistics

Ceperley's path integral: encoding Fermi-Dirac statistics in geometry: the nodal hypersurface.

- Fermi-liquid: Fermi-energy is encoded in the local geometry but the Fermi-surface is encoded globally

  F. Krueger et al., arXiv:0802.2455

- **Feynmannian backflow ansatz: nodal geometry turns fractal, the state is conformal but room for globally encoded Fermi-surface information.**

  F. Krueger, JZ, arXiv:0804.2161

Vacuum structure

Long time, zero temperature:

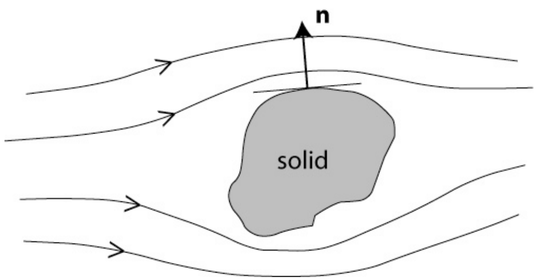
$$\rho_F\left(R,R(\tau);\tau\rightarrow\infty\right)=\Psi^*\left(R\right)\Psi\left(R(\infty)\right)$$

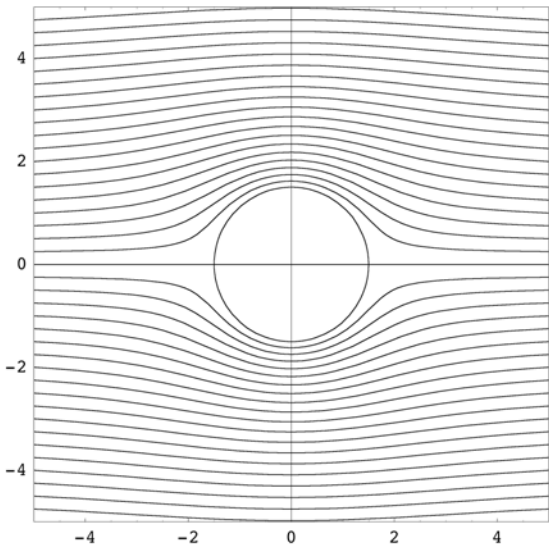
IR fermionic information encoded in the ground state wavefunction.

Need a wave function ansatz!

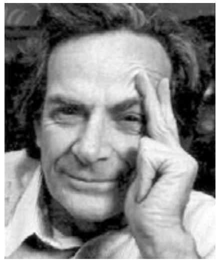
Hydrodynamic backflow

Classical fluid:
incompressible flow





Feynman-Cohen: mass enhancement in ^{4}He



Wave function ansatz for „foreign“ atom moving through He superfluid with velocity small compared to sound velocity:

$$g(\mathbf{r}) \sim \frac{\mathbf{k}\mathbf{r}}{r^3} \quad \rightarrow \quad \Psi = \phi \exp[i\mathbf{k}\left(\mathbf{r}_A + \sum_{i\neq A} \frac{\mathbf{r}_i - \mathbf{r}_A}{r_{iA}^3}\right)]$$

Backflow wavefunctions in Fermi systems

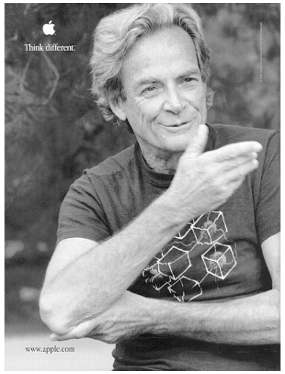
$$\begin{aligned} \psi_{bf}(\mathbf{R}) &\sim \mathrm{Det}\left(e^{i\mathbf{k}_i\tilde{\mathbf{r}}_j}\right)_{ij} \\ \tilde{\mathbf{r}}_j &= \mathbf{r}_j + \sum_{l(\neq j)} \eta(r_{jl})(\mathbf{r}_j - \mathbf{r}_l) \end{aligned}$$

Widely used for node fixing in QMC

→ Significant improvement of
variational GS energies

Frank's fractal nodes …

Feynman's fermionic backflow wavefunction:

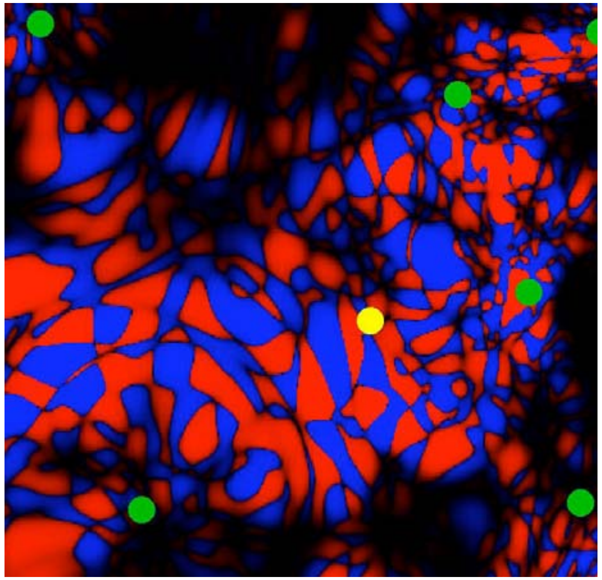


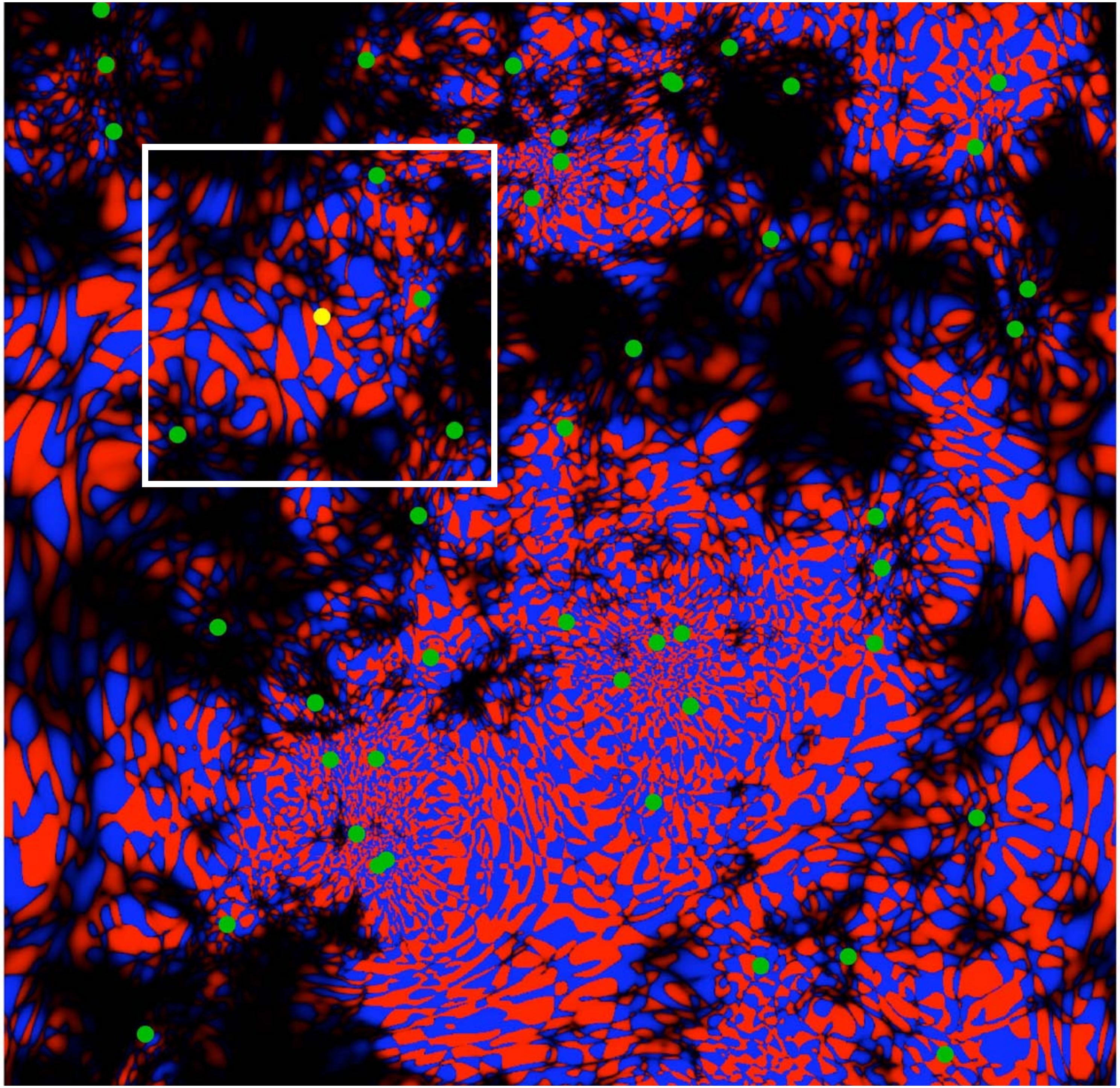
$$\psi_{bf}(\mathbf{R}) \sim \mathrm{Det}\left(e^{i\mathbf{k}_i\tilde{\mathbf{r}}_j}\right)_{ij}$$

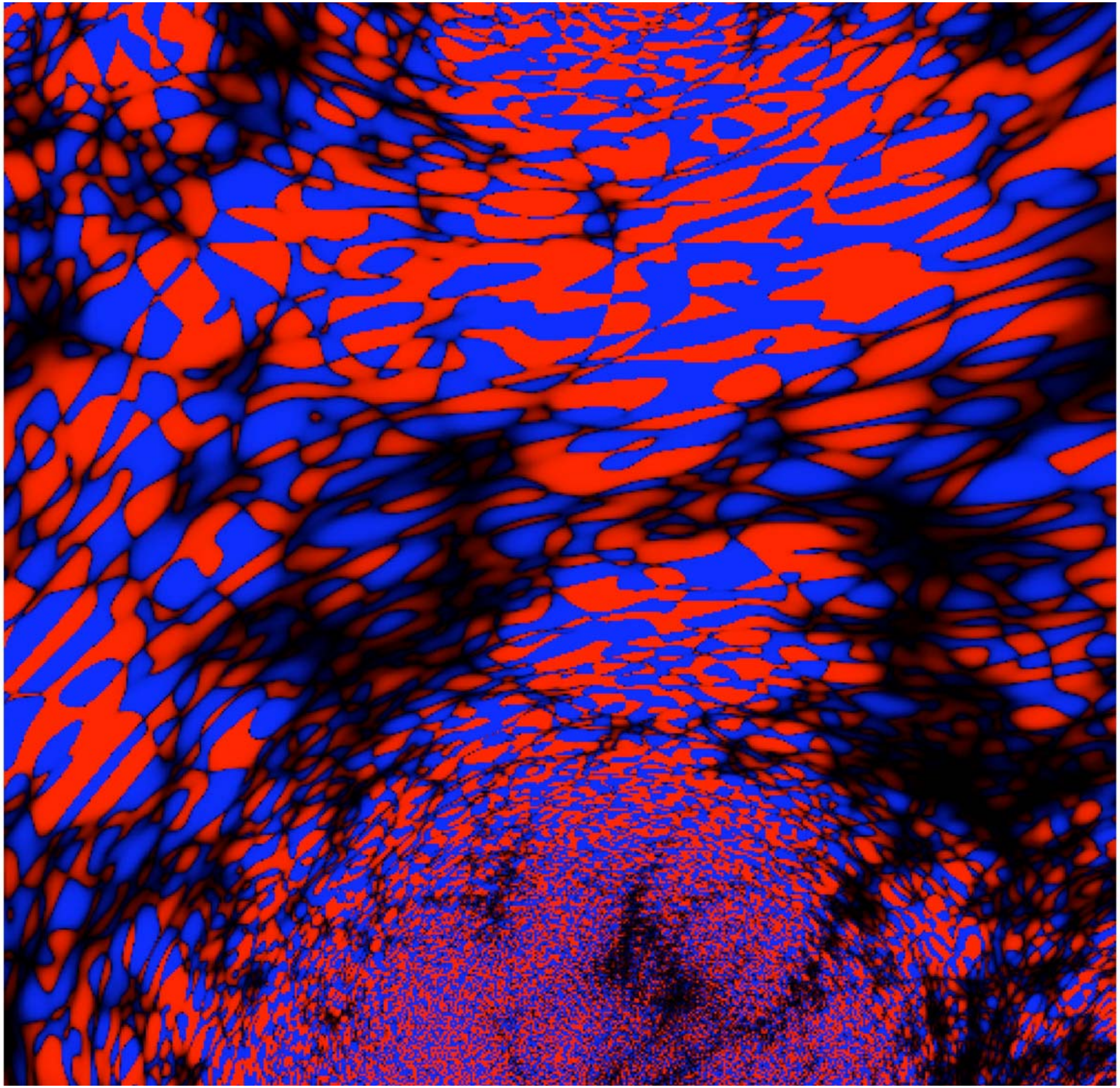
$$\tilde{\mathbf{r}}_j = \mathbf{r}_j + \sum_{l(\neq j)} \eta(r_{jl})(\mathbf{r}_j - \mathbf{r}_l)$$

$$\eta(r) = \frac{a^3}{r^3 + r_0^3}$$

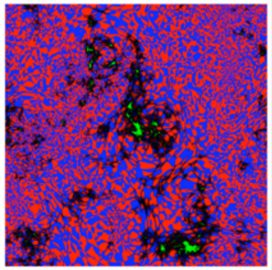
Frank Krüger







Extracting the fractal dimension



The Hausdorff dimension. The Hausdorff dimension of a metric space X, $\dim_H(X)$, is the infimimum of the numbers α with the following property: For any $\epsilon > 0$ there is a $\delta > 0$ and a cover $\mathfrak{U}$ of X such that the sets $B \in \mathfrak{U}$ all have diameter $|B|$ smaller than δ and

$$\sum_{B \in \mathfrak{U}} (|B|)^\alpha < \epsilon.$$

The correlation integral:

$$C(r) = \lim_{n\to\infty} \frac{1}{n^2} \sum_{i,j=1}^{n} \Theta(r - |\mathbf{r}_i - \mathbf{r}_j|)$$

$$= \int_0^r \mathrm{d}^D r' c(\mathbf{r}')$$

For fractals:

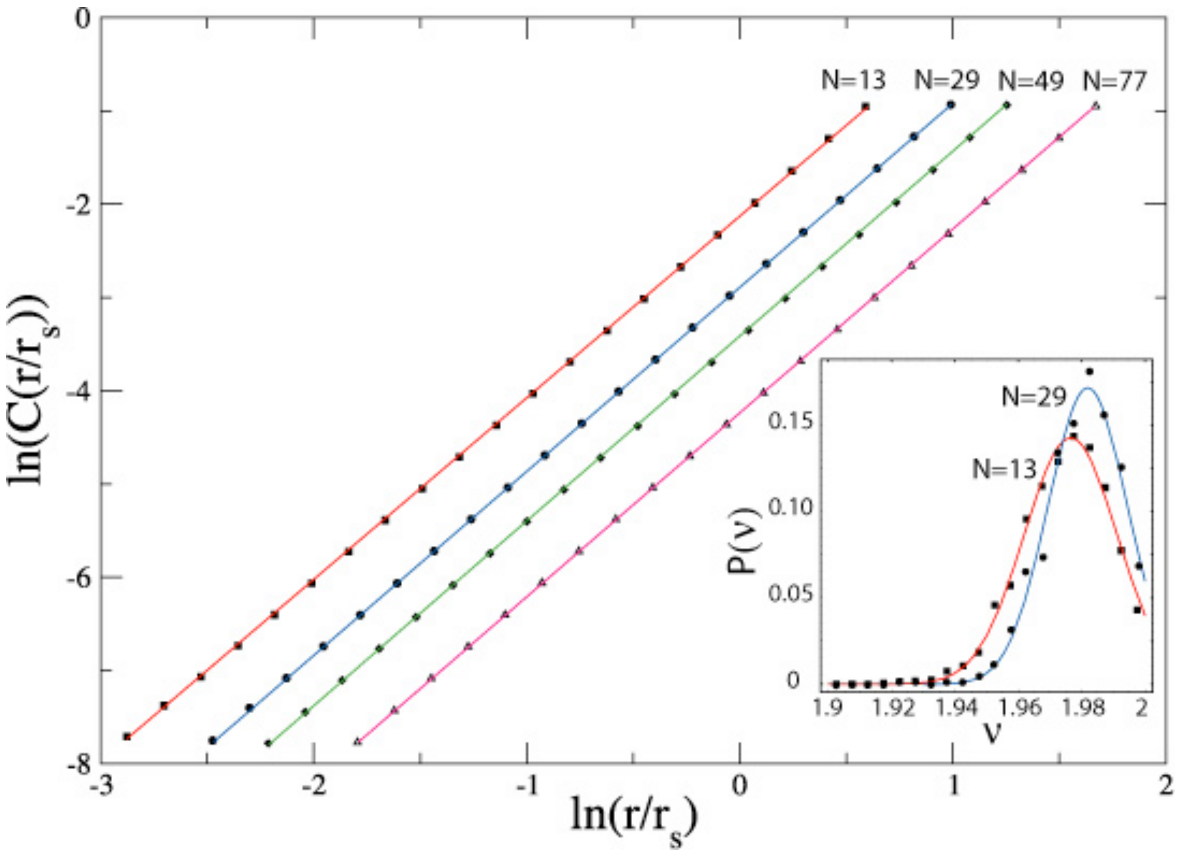
$$C(r) \sim r^\nu, \quad \nu \le \dim_H$$

Inequality very tight, relative error below 1%

Grassberger & Procaccia, PRL (1983)

Fractal dimension of the nodal surface

Calculate the correlation integral $C(r) \sim r^{\nu}$ on random d=2 dimensional cuts



$$Nd - 1 < \quad D_H = N\nu_d \quad < Nd$$

$$d - \frac{1}{N} < \quad \nu_d \quad < d$$

$N = 13:$ $\quad \nu = 1.976 \pm 0.012$

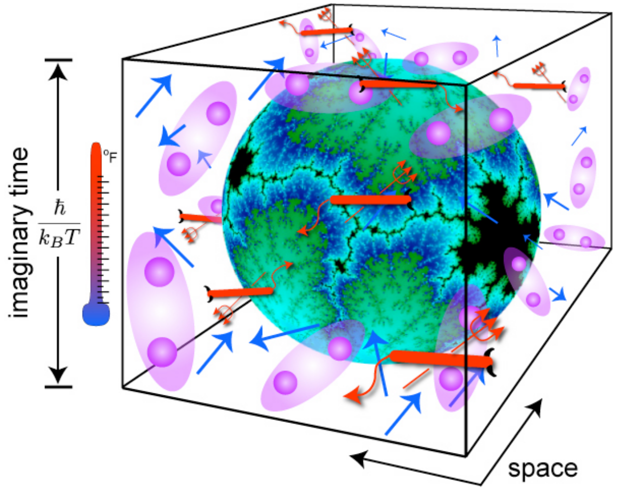
$\quad \rightarrow D_H = 25 + (0.69 \pm 0.16)$

$N = 29:$ $\quad \nu = 1.982 \pm 0.008$

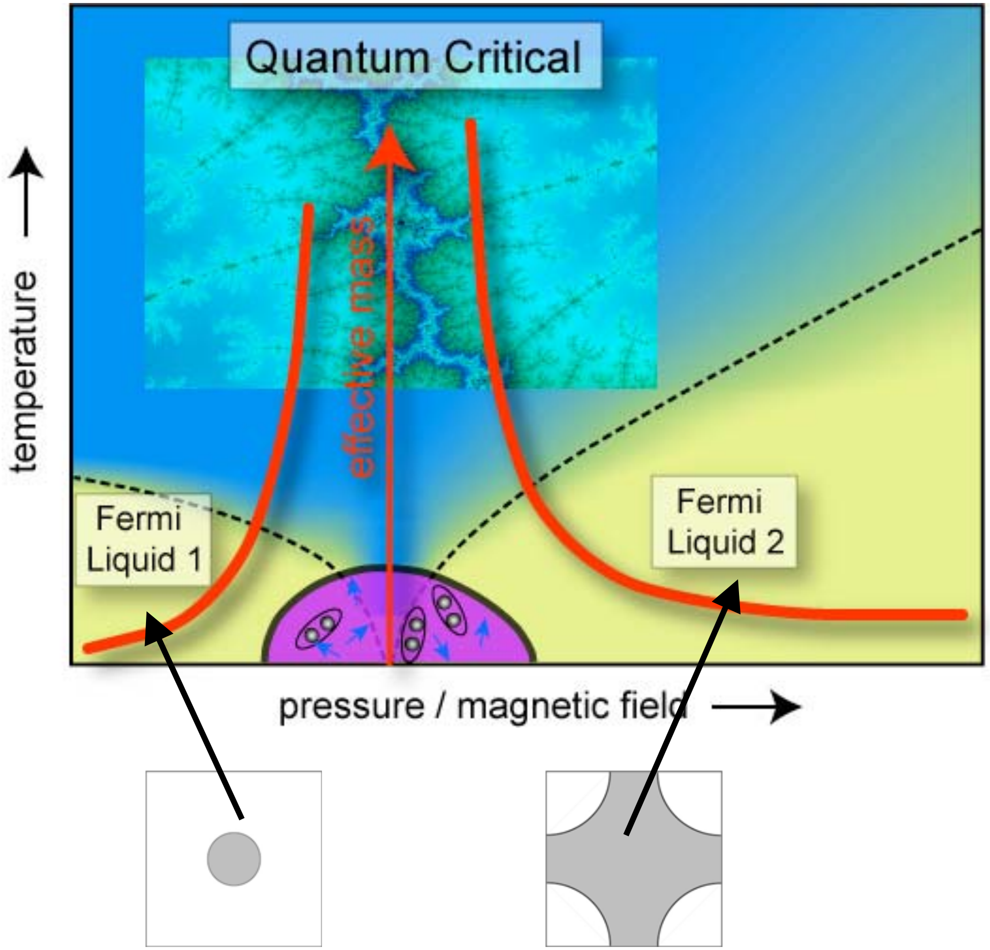
$\quad \rightarrow D_H = 57 + (0.48 \pm 0.23)$

Backflow turns nodal surface into a fractal !!!

Fermionic quantum phase transitions in the heavy fermion metals



JZ, Science 319, 1205 (2008)



Paschen et al., Nature (2004)

Turning on the backflow

Nodal surface has to become fractal !!!

Try backflow wave functions

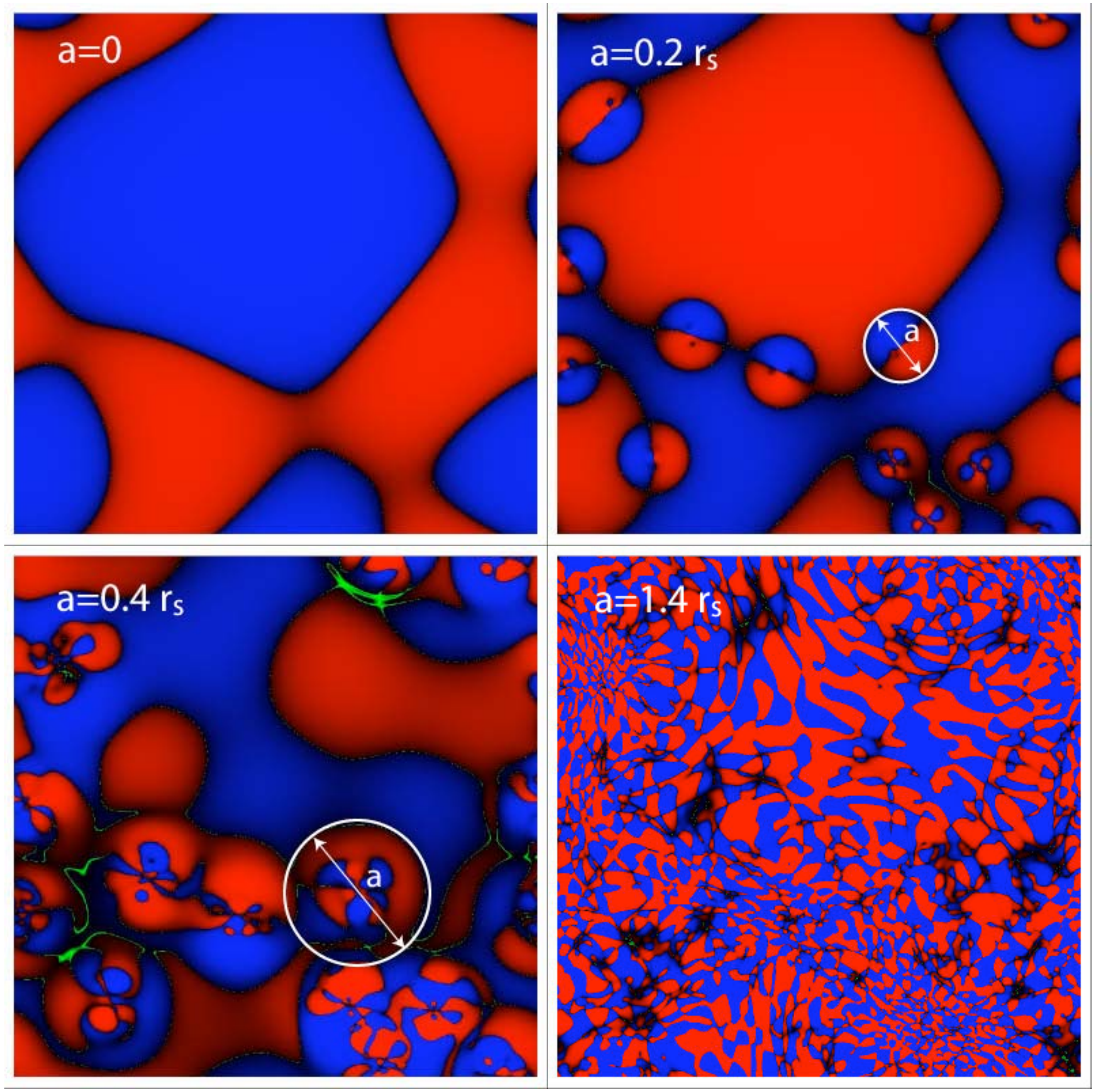
$$\psi_{bf}(\mathbf{R}) \sim \text{Det}\left(e^{i\mathbf{k}_i\tilde{\mathbf{r}}_j}\right)_{ij}$$

$$\tilde{\mathbf{r}}_j = \mathbf{r}_j + \sum_{l(\neq j)} \eta(r_{jl})(\mathbf{r}_j - \mathbf{r}_l)$$

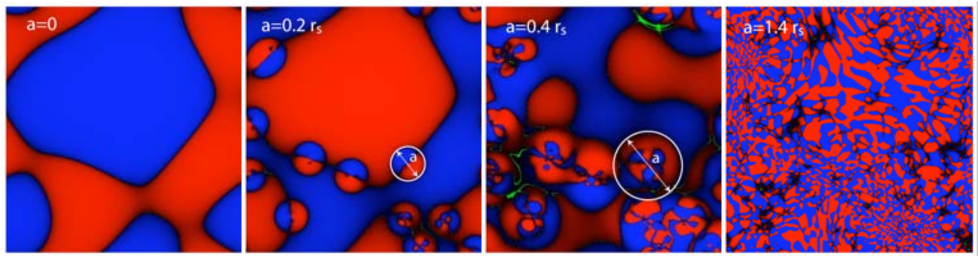
$$\eta(r) = \frac{a^3}{r^3 + r_0^3}$$

Collective (hydrodynamic) regime:

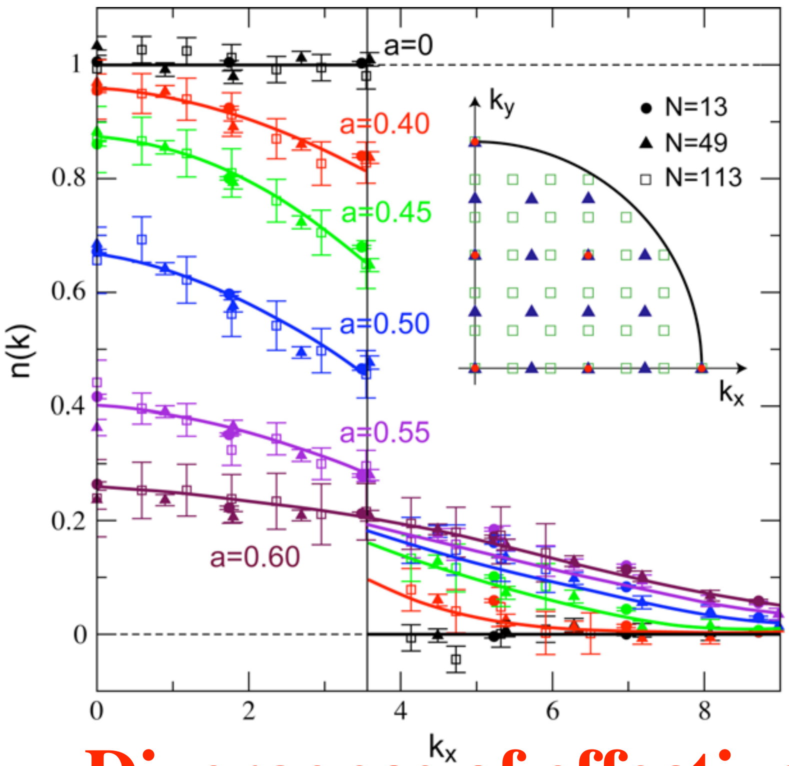
$$a \gg r_s$$

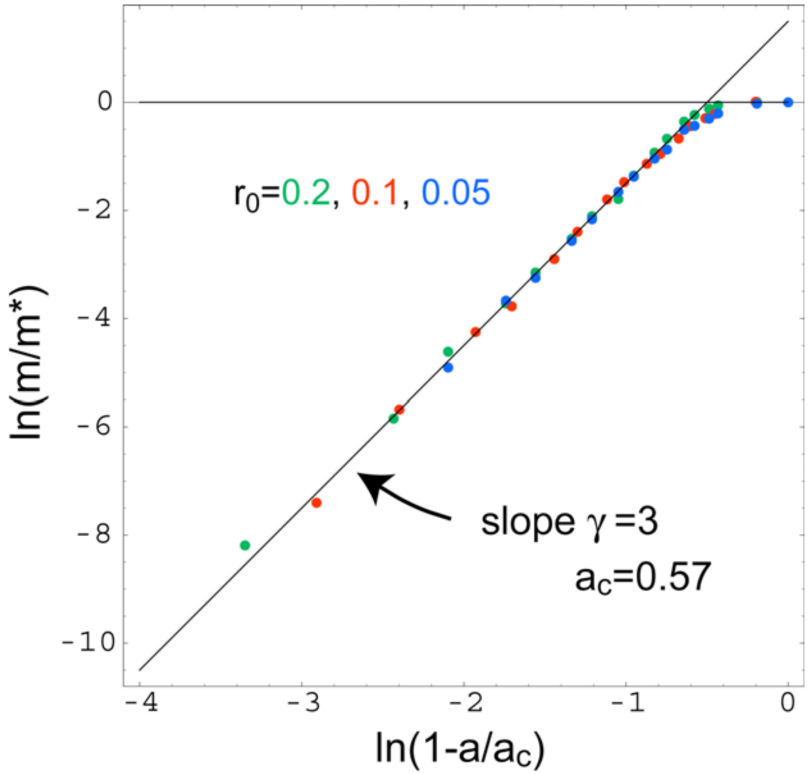


MC calculation of n(k)



$$\frac{m}{m^*} \propto \left(1 - \frac{a}{a_c}\right)^3$$





Divergence of effective mass as $a \rightarrow a_c$

The fixed point Hamiltonian

$$\psi_{bf}(\mathbf{R}) \sim \mathrm{Det}\left(e^{i\mathbf{k}_i\tilde{\mathbf{r}}_j}\right)_{ij}$$
$$\tilde{\mathbf{r}}_j = \mathbf{r}_j + \sum_{l(\neq j)} \eta(r_{jl})(\mathbf{r}_j - \mathbf{r}_l)$$

=> $\left|k_1,\ldots,k_N\right\rangle_{bf} = \int_{q_1,\ldots,q_N} \Gamma_{q_1,\ldots,q_N} \left|k_1+q_1,\ldots,k_N+q_N\right\rangle_{bare}$

turns singular at the QPT.

It is the ground state of a Fermi-gas of backflow particles: $H = \sum_k \varepsilon_k \hat{c}_k^+ \hat{c}_k$

Expressed in bare particles: $H \propto \sum_k \varepsilon_k c_k^+ c_k + \sum_{N=2}^{\infty} \left(\frac{a}{r_s}\right)^N \sum_{\{kq\}} f(\{k,q\})\left(c_{\ldots}^+ c_{\ldots}\right)^N$

- At the critical point $a \to r_s$ the fixed point Hamiltonian reveals a divergence in N where N refers to N-body interaction!

- No symmetry change, criticality is entirely of 'statistical' nature (information in nodal surface)!

Where is the Fermi-surface?

Fractality originating in the non-locality of the nodes:

$$\rho_F = Det\left(\exp\left[ik_i(r_j + \sum_j \eta(r_{ij})(r_i - r_j))\right]\right) = 0$$

=> Dynamics becomes conformal (vanishing of renormalized Fermi energy).

But Fermi-surface information is also globally wired into the nodes through backflow particle gas: $|k| < k_F$!

Conjecture: there have to be singularities at the 'remnant' Fermi surface that are not conflicting with scale invariant dynamics because of the local-global dichotomy.

In conclusion …

Fermions at finite density: the fermion signs are wrecking established mathematical machinery, but it leaves room for BIG surprises.

AdS/CFT has started to show it muscles: the emergent Fermi-liquid (Koenraad), the singular-, marginal - … Fermi liquids (Hong).

Does a critical Fermion liquid need a Fermi-surface? The scale invariant backflow state suggests that the Fermi-surface is wired into the global structure of the nodal surface that is locally fractal.

Outlook

AdS/CFT is a very rich mathematical machine: rich enough to literally describe the fermion side of condensed matter?

- **Magnetic fields:** **Landau quantization of fermions? Fractional quantum Hall??**
- **Bosonic (chiral) phase transitions: what happens with the coexisting Fermions?**
- **Hartnoll's 'hairy black hole': is the superconductivity BCS like?**
- **Total currents versus fermionic currents: linear resistivity??**
- **Fermionic pair susceptibilities: BCS mechanism for fermionic quantum critical states??**
- **Fermions in the non-relativistic AdS/CFT ??**

In conclusion …

Fermions at finite density: the fermion signs are wrecking established mathematical machinery, but it leaves room for BIG surprises.

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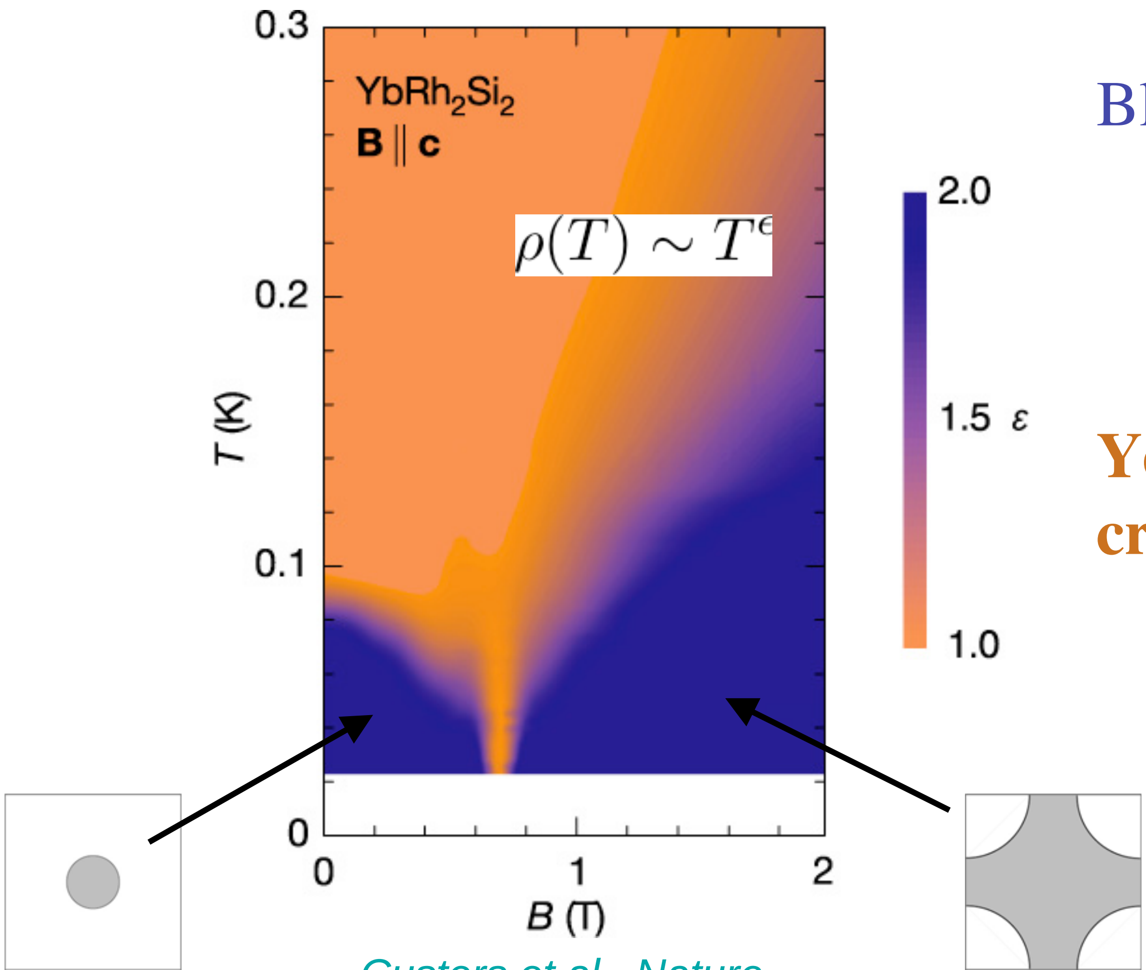
Quantum critical transport in heavy fermion systems



Custers et al., Nature (2003)

Blue = Fermi liquid:

$$\rho \propto T^2$$

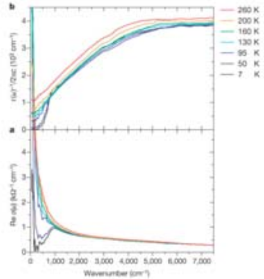
Yellow= quantum critical regime:

$$\rho \propto T$$

Critical Cuprates are Planckian Dissipators

van der Marel, JZ, … Nature 2004:

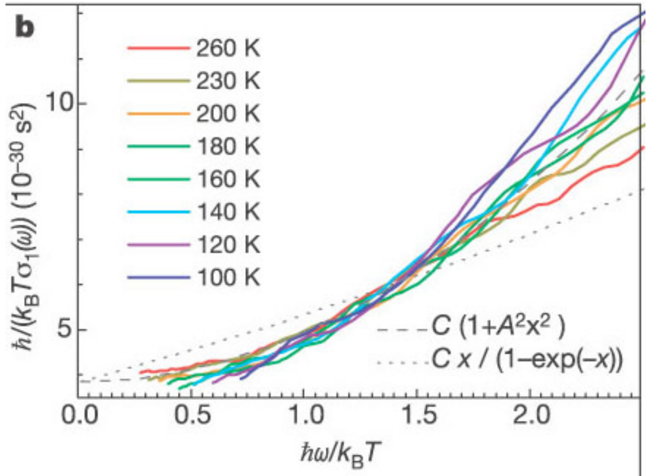
Optical conductivity QC cuprates



Frequency less than temperature:

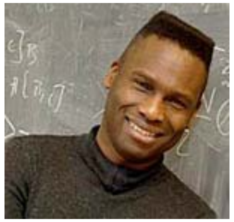
$$\sigma_1(\omega,T) = \frac{1}{4\pi}\frac{\omega_{pr}^2\tau_r}{1+\omega^2\tau_r^2}, \quad \tau_r = A\frac{\hbar}{k_B T}$$

$$\Rightarrow \quad [\frac{\hbar}{k_B T\sigma_1}] = const.(1+A^2[\frac{\hbar\omega}{k_B T}]^2)$$



A= 0.7: the normal state of optimallly doped cuprates is a Planckian dissipator!

Why the Wilsonian renormalization group fails …

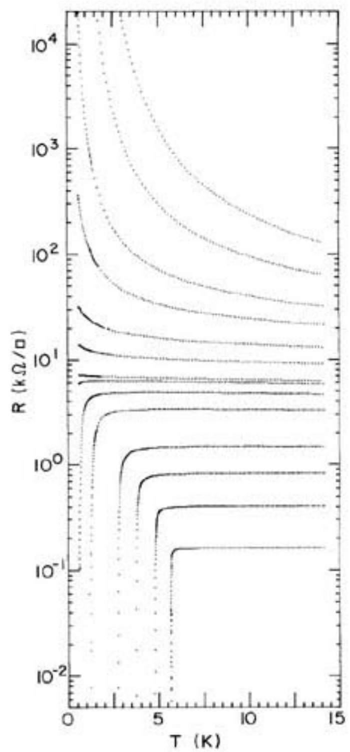


Phillips

Chamon

PRL 95, 107002 (2005)

Superconductor-insulator QCP



(a) **Charge conservation** ('hydrodynamics') imposes engineering scaling dimensions on current

(b) Scale invariance: assume **one** diverging length scale.

$$\sigma(\omega,T) = \frac{e^2}{\hbar}\left(\frac{k_B T}{\hbar c}\right)^{\frac{d-2}{z}} \Sigma\left(\frac{\hbar\omega}{k_B T}\right) \quad \Rightarrow \sigma_{DC}(T) = \frac{e^2}{\hbar}\Sigma(0)\left(\frac{k_B T}{\hbar c}\right)^{\frac{d-2}{z}}$$

$$\sigma_{DC} \propto \frac{1}{T} ??$$

d =2 or 3 implies that z < 0 !?

Empty

Empty