Velocity Distributions in Granular and Active Suspensions

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Wealth of Applications

technical

food processing

random close packing
Chaikin et al. 2004

and in nature

ring of Saturn
nonequilibrium model system

- grain of sand (diameter $d$) at room temperature: $\frac{k_B T}{mgd} \sim 10^{-12}$
  
  $T$ defined by heatbath (environment) is completely irrelevant thermodynamics and equil. statistical mechanics not applicable
... of Fundamental Interest

**nonequilibrium** model system

- $T$ defined by heatbath (environment) is completely irrelevant
- interactions between macroscopic bodies are dissipative
  energy is lost in collision of two grains
nonequilibrium model system

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- interactions between macroscopic bodies are dissipative
- decay of an initially agitated state: $E_{kin} = \frac{m}{2} \sum_{i=1}^{N} v_i^2$

How does the granular fluid cool down?

of particular interest for dilute systems → structure formation
nonequilibrium model system

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- interactions between macroscopic bodies are dissipative
- decay of an initially agitated state: $E_{\text{kin}} = \frac{m}{2} \sum_{i=1}^{N} v_i^2$
  $\rightarrow$ spontaneous structure formation
- Grains left to themselves settle into static packing
  How dense can grains be packed?
  Jamming transition as a function of packing fraction
nonequilibrium model system

- $T$ defined by heatbath (environment) is completely irrelevant
- interactions between macroscopic bodies are dissipative
- decay of an initially agitated state: $E_{kin} = \frac{m}{2} \sum_{i=1}^{N} v_i^2$ → spontaneous structure formation
- Grains left to themselves settle into static packing → jamming
- Dynamics due to driving shear: rheology of granular particles with and without friction gravity, e.g. flow on an inclined plane, out of a hopper ... fluidized beds: pumping air or fluid through a granular bed
- stationary state; no detailed balance, no fluctuation dissipation theorem,...
Experiments on Fluidized Beds

binary mixture of spherical particles in a sieve; driven by uniform upflow of air

Abate and Durian Phys.Rev. E74,031308, 2006

mean square displacement: development of a plateau as volume fraction increases
Velocity Distributions in Experiment

vibrated granular medium
van Zon, Swinney et al. PRE 2004

swimming bacteria
(bacillus subtilis)
Sokolov, Aranson et al. PNAS 2010
**in vitro cells**

Czirok, Vicsek et al. PRL 1998
Simple Model

hard spheres in a fluid with viscous drag $\gamma$:

\[
\partial_t v_i = -\gamma v_i + \frac{\Delta v_i}{\Delta t} \bigg|_{\text{coll}} + \frac{\Delta v_i}{\Delta t} \bigg|_{\text{Dr}}
\]

collisions: $(v_i - v_j)n = -\epsilon(v'_i - v'_j)n$

elastic collisions $\epsilon = 1$
incomplete normal restitution: $\epsilon < 1$

discrete kicks: $u$ with frequency $f_{Dr}$
crude approximation to run-and-tumble behavior of bacteria; in time interval $\Delta t = 1/f_{Dr}$ particle is accelerated, subsequently randomized by surrounding fluid and interactions with other particles

stationary state: $2m\gamma < v^2 > = m < u^2 > f_{Dr}$ elastic case

3 parameters: $\gamma$, $f_{Dr}$ and volume fraction $\Phi$
Event Driven Simulations

Ballistic motion
idea: in between collision (or kicks) particles move freely

\[ r_i(t) - r_j(t) \equiv r_{i,j}(t) = r_{i,j}(t_0) + v_{i,j}(t_0)(t - t_0) \] (1)

compute time \( t_{coll} \) for next collision to happen \( R_i + R_j = |r_{i,j}(t_{coll})| \)

including viscous drag

\[ r_{i,j}(t) = r_{i,j}(t_0) + v_{i,j}(t_0) \frac{1 - e^{-\gamma(t-t_0)}}{\gamma} \] (2)

collision time known from ballistic simulation, replace:

\[ (t_{coll} - t_0) \rightarrow (1 - e^{-\gamma(t_{coll}-t_0)})/\gamma \] (3)

allows to simulate several millions of particles
important parameter $\beta := \gamma / f_{Dr}; \Phi = 0.35$
Velocity Distributions in Stationary State

\[ \beta := \gamma / f_{Dr}; \; \Phi = 0.35 \]

- Gaussian only in the limit \( \beta \to 0 \)
- overpopulated at small \( \nu \), velocities decay with rate \( \gamma \) singularity for \( \beta \to \infty \)
- overpopulated at large \( \nu \) due to kicks
- independent of volume fraction \( 0.05 \leq \Phi \leq 0.4 \)
- depends only on ratio \( \beta := \gamma / f_{Dr} \)
Single-Particle-Model

\[ f(v) = \int_{-\infty}^{\infty} dx \ p_k(v - x) \ f(xe^\beta) \ e^\beta \]

\[ \beta = \gamma/f_{Dr} \rightarrow 0: \text{recover Maxwell-Boltzmann distribution} \]
\[ \beta \gg 1: \text{nontrivial distribution} \]
solve by iteration; fast convergence
good agreement with simulations
Single-Particle-Model

\[ f(v) = \int_{-\infty}^{\infty} dx \ p_k(v - x) \ f(x e^\beta) \ e^{\beta} \]

\( \beta \gg 1: \)

\[ f(v) \approx \begin{cases} 
\frac{e^\beta - 1}{2 \sqrt{\pi} \beta^3} & |v| \ll e^{-\beta} \\
\frac{1}{2\beta|v|} & e^{-\beta} \ll \frac{|v|}{\sqrt{2\beta}} \ll 1 \\
\frac{1}{\sqrt{\pi|\beta|}} \frac{1}{v^2} e^{-v^2/4\beta} & \frac{|v|}{\sqrt{2\beta}} \gg 1 
\end{cases} \]

\( \beta = 5 \) (lower), \( \beta = 10 \) (upper), 
\( \Phi = 0.35 \)
Conclusion and Generalisations

Largely universal velocity distributions depending only on $\beta = \gamma / f_{Dr}$, independent of volume fraction, particle interactions

$\beta \to 0$: Maxwell- Boltzmann is recovered

$\beta \gg 1$: Distribution is divergent for small $\nu$, falls off as $1/\nu$ for intermediate $\nu$ and is Gaussian for the largest $\nu$

- distribution of kick amplitudes and waiting times in between kicks
- particles with orientation, include rotational degrees of freedom
- anisotropic particles, simplest case: needles