Quantum Black Holes, Localization & Mock Modular Forms

(*BITS of BRANES from BLACK HOLES*)

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Bits , Branes, & Black Holes
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References

- A. D., João Gomes, Sameer Murthy
  arXiv:1111.1161
  arXiv:1012.0265
- A. D., Sameer Murthy, Don Zagier
  arXiv:1206.nnnn
- A. D., João Gomes, Sameer Murthy, Ashoke Sen
  arXiv:1009.3226

- A new application of localization techniques in *gravitational* theories to reduce functional integral over string fields on $AdS_2 \times S^2$ to ordinary integrals.
- Wall-crossing and Mock Modular Forms.
- Define and compute a supersymmetric Index from black hole entropy.

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Two Related Motivations

Entropy of black holes remains one of the most important and precise clues about the microstructure of quantum gravity.

Can we compute exact quantum entropy of black holes including all corrections both microscopically and macroscopically?

Holography has emerged as one of the central concepts regarding the degrees of freedom of quantum gravity.

Can we find simple example of AdS/CFT holgraphy where we might be able to ‘prove’ it exactly?

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Black Hole Entropy

Bekenstein [72]; Hawking[75]

For a BPS black hole with charge vector (q, p), *for large charges*, the leading Bekenestein- Hawking entropy precisely matches the logarithm of the degeneracy of the corresponding quantum microstates

$$d(q,p) \sim \exp\left[\frac{A(q,p)}{4}\right] + \ldots \qquad (q, p >> 1)$$

Strominger & Vafa [96]

This beautiful approximate agreement raises two important questions:

- What exact formula is this an approximation to?
- Can we systematically compute corrections to both sides of this formula, perturbatively and nonperturbatively in $1/q$ and may be even exactly for arbitrary finite values of the charges?

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Finite Size Effects or Going Beyond Thermodynamics

- We do not know which phase of string theory might correspond to the real world. For such a theory under construction, a useful strategy is to focus on *universal* properties that must hold in all phases. One universal requirement for a quantum theory of gravity is that in *any* phase of the theory that admits a black hole, it must be possible to interpret black hole as a statistical ensemble of quantum states.
- Finite size corrections to the entropy, unlike the leading area formula, depend on the details of the phase, and provide a sensitive probe of short distance degrees of freedom of quantum gravity.

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- Near horizon geometry of a BPS black hole is $AdS_2 \times S^2$.
- Using holography, a quantum generalization of Wald entropy is given in terms of a Wilson line expectation value

$$W(q,p) = \left\langle \exp\left[-i\, q_I \int_0^{2\pi} A^I \, d\theta \right] \right\rangle_{\mathrm{AdS}_2}^{\mathrm{finite}} \qquad I = 0, \ldots n_v .$$

This gives a precise quantum version of the equation we want to prove

$$\mathbf{d(q,p) = W(q,p)}$$

Our goal will be to compute both sides and compare.

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AdS_2 Functional Integral

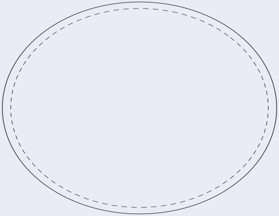

Figure: Wilson line inserted at the boundary with a cutoff at $r = r_0$.

$$ds^2(r^2 - 1)d\theta^2 + \frac{dr^2}{r^2 - 1} \qquad 1 \leq r < r_0$$

Localization in Supergravity

- A formal functional integral over spacetime string fields in AdS_2. One can integrate out massive fields to get a functional integral over supergravity fields. Even so, it seems almost impossible to tackle.
- *One of our main results is evaluation of a functional integral in supergravity by 'localizing' onto finite-dimensional manifold in field space. of instanton solutions*

$\mathcal{N} = 2$ supergravity coupled to n_v vector multiplets

Vector multiplet: vector field A^I_μ, complex scalar X^I, $SU(2)$ triplet of auxiliary fields Y^I_{ij}, fermions Ω^I_i. Here i in doublet.

$$\mathbf{X}^I = \left(X^I, \Omega^I_i, A^I_\mu, Y^I_{ij} \right) \qquad I = 0, \ldots, n_v \, .$$

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Localizing Instanton Solution

$$X^I = X^I_* + \frac{C^I}{r}\,, \qquad \bar{X}^I = \bar{X}^I_* + \frac{C^I}{r}$$

$$Y^{I1}_1 = -Y^{I2}_2 = \frac{2C^I}{r^2}\,, \qquad f^I_{\mu\nu} = 0\,.$$

Solves a major piece of the problem by identifying the off-shell field configurations onto which the functional integral localizes. This instanton is *universal* and does not depend on the physical action.

Scalar fields are very off-shell far away in field space from the classical attractor values X^I_* and *auxiliary fields* get nontrivial position dependence. Gravity multiplet not excited. Gupta & Murthy [12].

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Renormalized Action for Chiral Physical Actions

- A chiral physical action is described by a prepotential F which is a function of the scalar superfields. We substitute the above solution and can extract the finite piece.
- After a tedious algebra, one obtains a remarkably simple form for the renormalized action S_{ren} as a function of $\{C^I\}$.

$$\mathcal{S}_{ren}(\phi, q, p) = -\pi q_I \phi^I + \mathcal{F}(\phi, p)$$

with $\phi^I := e^I_* + 2C^I$ and $\mathcal{F}$ given by

$$\mathcal{F}(\phi, p) = -2\pi i \left[F\Big(\frac{\phi^I + ip^I}{2}\Big) - \bar{F}\Big(\frac{\phi^I - ip^I}{2}\Big) \right] ,$$

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Integration Measure

The line element on ϕ-space is

$$d\Sigma^2 = M_{IJ}\,\delta\phi^I\delta\phi^J$$

with the metric

$$M_{IJ} = K_{IJ} - \frac{1}{4}\frac{\partial K}{\partial \phi^I}\frac{\partial K}{\partial \phi^J}$$

given in terms of the Kähler potential

$$e^{-K} := -i(X^I\bar{F}_I - \bar{X}^I F_I)$$

The functional integral has collpased to an ordinary integral

$$\int \prod_{I=0}^{n_v} d\phi^I \sqrt{\det(M)}\; e^{S_{ren}(\phi)}\,.$$

For $N=2$ chiral truncation of $N=8$ the classical prepotential is quantum exact

$$F(X) = \frac{X^1 X^a X^b C_{ab}}{X^0} \qquad a = 1, \ldots 6\,.$$

It turns out one can even evaluate the finite-dimensional integral to obtain

$$W_1(\Delta) \;=\; (-1)^{\Delta+1}\, 2\pi \Big(\frac{\pi}{\Delta}\Big)^{7/2} I_{\frac{7}{2}}\big(\pi\sqrt{\Delta}\big)\;.$$

where $\Delta = q^2p^2 - (p\cdot q)^2$ is the U-duality invariant and

$$I_\rho(z) = \frac{1}{2\pi i}\int_{\epsilon - i\infty}^{\epsilon+i\infty} \frac{d\sigma}{\sigma^{\rho+1}} \exp[\sigma + \frac{z^2}{4\sigma}]$$

is the Bessel function of first kind of index ρ.

Note that the contour is parallel to imaginary axis and not real axis.
Related to the analytic continuation of the conformal factor of the metric.

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Example: Large dyonic black holes of Type II on T^6

The degeneracy $d(\Delta)$ depends only on the duality invariant $\Delta = q^2p^2 - (q \cdot p)^2$. and is given in terms of the Fourier coefficients of

$$F(\tau, z) = \frac{\vartheta_1^2(\tau, z)}{\eta^6(\tau)} \, .$$

$$\vartheta_1(\tau, z) = q^{\frac{1}{8}}(y^{\frac{1}{2}} - y^{-\frac{1}{2}}) \prod_{n=1}^{\infty}(1-q^n)(1-yq^n)(1-y^{-1}q^n) \, ,$$

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n) \, .$$

with $q := e^{2\pi i\tau}$ and $y := e^{2\pi iz}$. Moore, Maldacena, Strominger [99]

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Table: Comparison of the microscopic degeneracy $d(\Delta)$ with $W_1(\Delta)$ and the exponential of the Wald entropy.

Δ	3	4	7	8	11	12
$d(\Delta)$	8	12	39	56	152	208
$W_1(\Delta)$	7.972	12.201	38.986	55.721	152.041	208.455
$\exp(\pi\sqrt{\Delta})$	230.765	535.492	4071.93	7228.35	33506	53252

The area of the horizon goes as $4\pi\sqrt{\Delta}$ in Planck units. Already for $\Delta = 12$ this area would be 50, and one might expect that the Wald entropy would be a good approximation. Not true! The discrepancy between the degeneracy and the exponential of the Wald entropy arises entirely from integration over massless fields.

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The $d(\Delta)$ admits an **exact** expansion. Rademacher

$$d(\Delta) = \sum_{c=1}^{\infty} d_c$$

$$d_c(\Delta) = (-1)^{\Delta+1}\, 2\pi \Big(\frac{\pi}{\Delta}\Big)^{7/2}\, I_{\frac{7}{2}}\Big(\frac{\pi\sqrt{\Delta}}{c}\Big)\, \frac{1}{c^{9/2}} K_c(\Delta)\,.$$

The sum $K_c(\Delta)$ is a discrete version of the Bessel function

$$K_c(\Delta) := e^{5\pi i/4} \sum_{\substack{-c\le d<0;\\ (d,c)=1}} e^{2\pi i \frac{d}{c}(\Delta/4)}\, M(\gamma_{c,d})^{-1}_{\ell 1}\, e^{2\pi i \frac{a}{c}(-1/4)}$$

- *An* **exact** *expansion and not just an asymptotic expansion. Because of* **localization**, *it is meaningful to consider subleading exponentials.*
- It is **guaranteed** to add up to an integer but only after adding all terms and not at any finite order even though it converges very fast.

Nonperturbative contributions from orbifolds

- Consider Z_c orbifolds of the disk which implies $0 \leq \theta < 2\pi/c$. But by a coordinate trasnformation $\tilde{\theta} = c\theta$ and $\tilde{r} = r/c$ we get the same asymptotic metric
$$ds^2 \sim \tilde{r}^2 d\tilde{\theta}^2 + \frac{d\tilde{r}^2}{\tilde{r}^2}$$
Hence there are more localizing instantons but with an action reduced by a factor of c. This correctly reproduces the Bessel function with a reduced factor of c in the argument.
- If we accompany by a shift in a charge lattice then one also picks up a phase from the Wilson line exactly as in the sum K_c.

It seems possible therefore to reproduce the integer exactly.

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Caveats and Open Problems

- Implement localization without using N=2 truncation of N=8
  Black hole not charged under truncated gauge fields
- Show that D-terms do not contribute.
  Near horizon has enhanced supersymmetry
- Show that hypers do not contribute.
  Hypers are flat directions of Wald entropy.
- Show that the orbifold phases reproduce the Kloosterman sum.
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On a philosophical note,

- The functional integral of quantum string theory near black hole horizons appears to have the ingredients to reproduce an integer — the bits of the branes. It appears to be an exact dual description with its own rules of computation rather than an emergent description.
- That the bulk can 'see' this integrality may be relevant for information retrieval because a necessary requirement for information retrieval is that gravity sees the 'discreteness' of quantum states.
- Conversely, we can use the data from microscopic bits to learn about the nonperturbative rules of the functional integral of quantum gravity. It is useful to have explicit answers to compare with.

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Ramanujan's example

In Ramanujan's famous last letter to Hardy in 1920, he gives 17 examples of mock theta functions, without giving any complete definition of this term. A typical example (Ramanujan's second mock theta function of "order 7" — a notion that he also does not define) is

$$\begin{aligned} \mathcal{F}_7(\tau) &= -q^{-25/168} \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1-q^n)\cdots(1-q^{2n-1})} \\ &= -q^{143/168}\left(1+q+q^2+2q^3+\cdots\right) . \end{aligned}$$

Hints of modularity such as Cardy behavior of Fourier coefficients but not quite modular! Despite much work, this fascinating 'hidden' modular symmetry remained mysterious until the thesis of Zwegers [2005].

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Mock Modular Form and its Shadow Zwegers[05], Zagier [07]

A mock modular form $h(\tau)$ of weight k is the first of the pair (h, g)

1. $g(\tau)$ is a modular form of weight $2-k$,
2. the sum $\widehat{h} = h + g^*$, of h is modular with weight k with

$$g^*(\tau, \bar{\tau}) = \left(\frac{i}{2\pi}\right)^{k-1} \int_{-\bar{\tau}}^{\infty} (z+\tau)^{-k} \overline{g(-\bar{z})}\, dz \,.$$

Then g is called the *shadow* of h and $\widehat{h}$ is called *modular completion* of h which obeys a 'holomorphic anomaly' equation

$$(4\pi\tau_2)^k \, \frac{\partial \widehat{h}(\tau)}{\partial \bar{\tau}} = -2\pi i \, \overline{g(\tau)}.$$

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"My dream is that I will live to see the day when our young physicists, struggling to bring the predictions of superstring theory into correspondence with the facts of nature, will be led to enlarge their analytic machinery to include not only theta-functions but mock theta-functions... But before this can happen, the purely mathematical exploration of the mock-modular forms and their mock-symmetries must be carried a great deal further."

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Mock Modular Forms and Quantum Black Holes

A. D., Sameer Murthy, Don Zagier [2012]

A summary of results

- Quantum degeneracies of single-centered black holes in $N = 4$ theories are given by Fourier coefficients of a mock modular form.
- Mock modularity is a consequence of *wall-crossing in spacetime* and *noncompactness* of the microscopic SCFT.

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- Conceptually, for AdS_2 and AdS_3 holography.
- Practically, for developing a Rademacher type expansion.

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Quantum Black Holes and Meromorphic Jacobi Forms

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$$\psi_m(\tau, z) = \frac{\eta^6(\tau)}{\theta_1^2(\tau, z)} \, \frac{1}{\eta^{24}(\tau)} \, \chi_{m+1}(\tau, z)$$

for $m = p^2/2$ with τ and z as chemical potentials for $q^2/2$ and $p \cdot q$.
Dijkgraaf, Verlinde, Verlinde [96]; Gaiotto, Yin, Strominger [06]; David, Sen[07]

Meromorphy and Moduli Dependence

- This is meromorphic with a double pole at $z = 0$.
- Degeneracies depend on the contour. This problem becomes a feature if the contour is chosen to depend on the moduli appropriately. Dabholkar, Gaiotto, Nampuri [07], Sen[07], Cheng, Verlinde [07]

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Wall-crossing and Multi-centered black holes

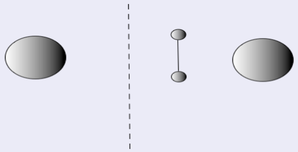

Figure: On the left of the wall there are only single-centered black holes but on the right of the wall there are both single-centered and multi-centered black holes.

Contours, Poles, and Walls

- Contour depends upon moduli.
- Pole-crossing corresponds to wall-crossing.
- Residue at the pole gives the jump in degeneracy upon wall-crossing.

How to isolate the degeneracies of single-centered black holes?

Under modular transformation $z \to z/c\tau + d$, the contour shifts. As a result, Fourier coefficients no longer have nice modular properties.

Modular symmetry is lost. How to restore modular symmetry?

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Decomposition Theorem

There is a unique decomposition of the counting function:

$$\psi_m(\tau, z) = \psi_m^F(\tau, z) + \psi_m^P(\tau, z)\,,$$

such that

- $\psi_m^P(\tau, z)$ has the same pole structure in z as $\psi_m(\tau, z)$:

$$\psi_m^{\mathrm{P}} := \frac{p_{24}(m+1)}{\eta^{24}(\tau)} \sum_{s\in\mathbb{Z}} \frac{q^{ms^2+s} y^{2ms+1}}{(1-q^s y)^2}\,,$$

- $\psi_m^F(\tau, z)$ has no poles.

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Mock modularity and holomorphic anomaly

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Mock modularity and noncompactness

For a compact SCFT, the ellptic genus counts right-moving ground states and left-moving excitations. Hence is holomorphic. *This can fail for a noncompact SCFT because of a continuum of right-moving states.*

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Physical Interpretation and AdS_2 Holography

Both pieces in the decomposition have a natural physical interpretation.

- ψ_m^{P} is the counting function of multi-centered black holes
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This enables us to cleanly isolate the contribution of single-centered black holes at the *microscopic* level.

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- Modular transformations are global diffeomorphisms of boundary torus. Hence, restoring modular symmetry is essential for holography.
- Elliptic transformations are large gauge transformations.

It is natural to identify the completion $\widehat{\psi}^F_m$ with the generalized elliptic genus of the SCFT dual to a single-centered AdS_3 geometry discussed in de Boer, Denef, El-Showk, Messamah, Van den Bleeken [10]

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Payoff for Number Theory

- Our decomposition theorem was partially motivated by the notion of 'attractor contour' in black hole physics.
- Using our results, the infinite family of meromorphic black hole counting functions $\{\psi_m\}$ and another related family furnish an infinite list of examples of mock modular forms.
- This list contains many known mock modular forms including the mock theta functions of Ramanujan, the generating function of Hurwitz-Kronecker class numbers, the mock modular form conjecturally related to the Mathieu group M_{24}, *as well as an infinite number of new examples.*

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Supersymmetric Index from Black Hole Entropy

K	Limit	$\log d_{macro}$
$K3$	n large	$2\pi\sqrt{Q_1 Q_5\left(n - \frac{J^2}{4Q_1Q_5}\right)}$
$K3$	Q_1 large	$2\pi\sqrt{Q_5(n+3)\left(Q_1 - \frac{J^2}{Q_5(n-1)}\right)}$
T^4	n large	$2\pi\sqrt{Q_1 Q_5\left(n - \frac{J^2}{4Q_1Q_5}\right)}$

Table: Three charge Black holes for Type-IIB on $K \times S^1$.

A. D., João Gomes, Sameer Murthy, Ashoke Sen

Computed by defining a macroscopic spacetime index using the fact that for a black hole degeneracy equals index.

These results are **exact** *including all corrections. Only one charge becomes large but all other charges can take arbitrarily small.*

Exact results without Localization

- On the microscopic side, Q_1 large is not the usual Cardy limit but in fact is the Anti-Cardy Limit (fixed L_0 with c large). Hence usual asymptotics does not work. One requires a clever use of 4d-5d lift (Gaiotto, Strominger, Yin [09]) and duality symmetries of 4d theory (Castro, Murthy [09]) .
- On the microscopic side, one requires a clever use of anomaly inflow (Kraus, Larsen [07]), determination of Chern-Simons coefficients using scaling arguments, and a definition of spacetime supersymmetric index using black hole entropy.
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