# Horizon entropy, higher curvature, and spacetime equations of state

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## Outline

1. Einstein equation of state
2. Higher curvatures and black hole entropy
3. Equation of state with higher curvatures
4. Lessons?

# QFT & Thermo

From relativistic QFT:

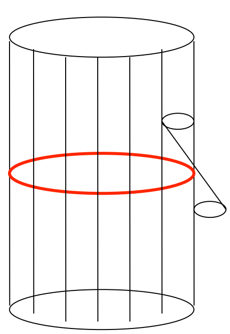
- In a Rindler wedge the vacuum is thermal at $T = \hbar/2\pi$ wrt the boost Hamiltonian, and has (entanglement) entropy $S = \alpha A$, with $\alpha = \infty$.
- The entropy scales with the area because the entanglement is dominated by vacuum correlations which diverge at short distances.

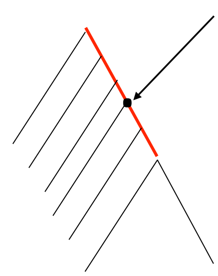
From thermodynamics:

- The Clausius relation $dS = dQ/T$ gives the entropy increase when heat $dQ$ enters a thermal bath at temperature $T$.

# Local causal horizon

Stationary black hole horizon






Local horizon

Boundary of the past of the red line (2-surface)

## Spacetime Thermodynamics

- *Assume* a horizon entropy S = α A with universal α.
- *Assume* the Clausius relation dS = dQ/T holds for all LCH's, with dQ the (local) boost energy flux across the horizon.
- Raychaudhuri focusing eqn then implies causal structure of spacetime must respond via

$$\alpha R_{ab} = \frac{2\pi}{\hbar} T_{ab} + \Phi g_{ab}$$

## Einstein equation of state

Local energy conservation $\nabla^a T_{ab} = 0$

and Bianchi identity $\nabla^a R_{ab} = \frac{1}{2}\nabla_b R$

imply $\Phi/\alpha = \frac{1}{2}R + \Lambda$

hence we obtain Einstein's equation

$$R_{ab} - \tfrac{1}{2}R g_{ab} - \Lambda g_{ab} = \frac{2\pi}{\hbar\alpha} T_{ab} \qquad G_N = \frac{1}{4\hbar\alpha}$$

1. More entanglement → weaker gravity (spacetime more rigid).
2. $1/G_N$ always tracks the net thermodynamic entropy

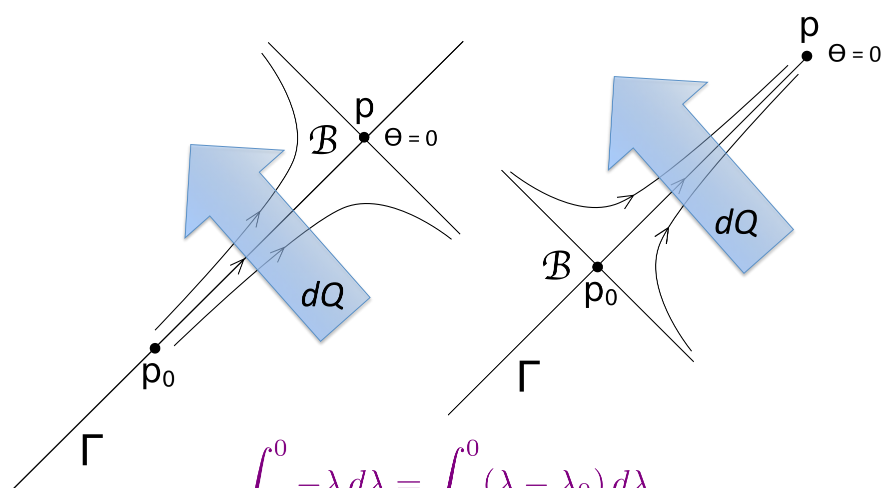
*(worries about multiple species, nonminimal coupling, gauge fields and gravitons notwithstanding)*

*(Also, cf. poster of work by Donnelly and Wall showing that 1-loop renormalization of $1/G_N$ by gauge fields is not negative in 2d, and perhaps higher d, after all.)*

# Remarks on Einstein equation of state

- If α = ∞ then dS = dQ/T cannot be satisfied for general stress tensors. Clausius relation not apply to infinite entropy?
- Gravity itself provides a Lorentz invariant cutoff: entanglement at a scale L is "cloaked" by horizon fluctuations if $L^2 < \hbar G(L)$. (TJ, 1204.6349)
- Shear included via $dS = dQ/T + dS_i$, → shear viscosity/α = ħ/4π (Eling, Guedens, TJ, gr-qc/0602001)
- Move bifurcation surface to the past?

# Local Killing flow




$$\int_{\lambda_0}^{0} -\lambda \, d\lambda = \int_{\lambda_0}^{0} (\lambda - \lambda_0) \, d\lambda$$

# Higher curvatures & horizon entropy

For *stationary* horizons, Wald entropy for L = R + aR$^2$ + … is

$$S_{BH} = \frac{A}{4\hbar G_N} + \text{curvature terms} = \frac{2\pi}{\hbar}\oint_\Sigma Q^{ab}[\hat{\chi}]N_{ab}dA$$

$$Q^{ab}[\xi] = W^{abc}\xi_c + X^{abcd}\nabla_c\xi_d$$

Noether potential for horizon-generating Killing vector

For $L = L[g_{ab}, R_{abcd}]$, can choose $X^{abcd} = \frac{\partial L}{\partial R_{abcd}}$
and $W^{abc} = 2\nabla_d X^{abcd}$ (Lopes Cardoso, de Wit, Mohaupt: hep-th/9904005)

*Dependence on Killing vector field disappears on stationary Killing horizon.*

# Higher curvatures & horizon entropy

Remarks:

1. 2$^{nd}$ law *not* established, except for (*a*) perturbations, where it follows from the 1$^{st}$ law and the null energy condition, and (*b*) *f(R)* theories.

2. Causal structure of such theories is generally *not* the metric light cone … so what is a dynamical "horizon"?

3. A *curious example*: pure curvature$^2$ theories have black holes with *zero* entropy (Solodukhin, 1203.2961). *Is it because such theories have no consistent quantization with a stable vacuum?*

# Higher curvature equation of state - I

Idea: adopt new horizon entropy density $s(g_{ab}, R_{abcd}, N_{ab})$, impose *dS = dQ/T* on all LCH's, infer field equation.

Problem 1: $\delta S \sim \nabla R_{abcd} \cdot \Delta\lambda, \text{ whereas } \frac{\delta Q}{T} \sim (\Delta\lambda)^2$

Problem 2*:* Can't use Raychaudhuri eqn…

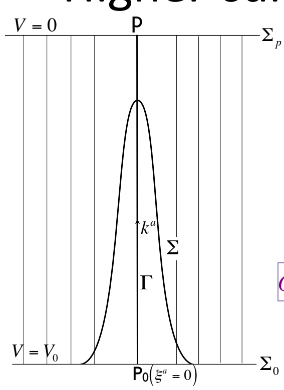
*Non-equilibrium* solution for $s(g_{ab}, R)$ case: choose nonzero expansion $\Theta \neq 0$ at p to cancel the offending term, and allow for bulk viscosity. Works *only* for this special case. (Eling, Guedens, TJ, gr-qc/0602001)

*Other approaches*:

Elizalde and Silva (0804.3721) – based on Iyer-Wald "boost invariant" proposal

Brustein and Hadad (0903.0823)
Parikh and Sarkar (0903.1176)
Padmanabhan (0903.1254)
Guedens, TJ, Sarkar (1112.6215)
-- based on "Noether potential"

# Higher curvature equation of state - II




$$S = \int_\Sigma s^{ab} d\Sigma_{ab}$$

$$\delta S = 2\int_H \nabla_b s^{ab} dH_a$$

$$\delta Q = \int_H T^{ab}\xi_a dH_b$$

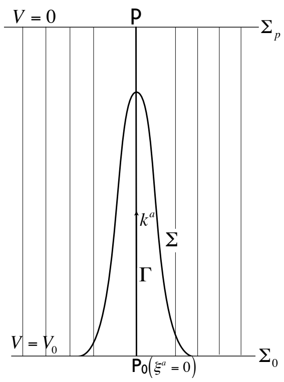
$$\delta S = \delta Q/T \text{ for all } p,\, \Sigma \text{ as } p_0 \to p \implies$$

$$\frac{\hbar}{\pi}\nabla_b s^{ab}\, k_a = T^{ab}\xi_b k_a + O(x^2)$$

Assume Noetheresque form for entropy wrt approximate local Killing vector:

$$s^{ab} = \frac{2\pi}{\hbar}\left(W^{abc}\xi_c + X^{abcd}\nabla_{[c}\xi_{d]}\right)$$

# Local Killing vector




There exists an approximate Killing vector satisfying:

$$
\begin{aligned}
\xi^\alpha|_\Gamma &= (V - V_0)k^\alpha \\
\nabla_\alpha \xi_\beta|_{p_0} &= (k_\alpha l_\beta - l_\alpha k_\beta)|_{p_0} \\
\nabla_{(\alpha}\xi_{\beta)} &= O(x^2) \\
\nabla_\alpha \nabla_\beta \xi_\gamma|_\Gamma &= \left(R^\delta{}_{\alpha\beta\gamma}\xi_\delta\right)\big|_\Gamma
\end{aligned}
$$

# Higher curvature equation of state - III

Using the properties of the local Killing vector and the narrowness of the slice, the Clausius relation implies

$$W^{arb} = \nabla_s \left(X^{sarb} + X^{sbra} + X^{srba}\right)$$

$$R^{(a}{}_{rst}X^{b)rst} + 2\nabla_r \nabla_s X^{(a|s|b)r} + \Phi g^{ab} = \tfrac{1}{2}T^{ab}$$

Conservation of the stress tensor then implies

$$\nabla^a \Phi = -\nabla_b \left(R^{(a}{}_{rst}X^{b)rst} + 2\nabla_r \nabla_s X^{(a|s|b)r}\right)$$

This imposes an integrability condition on X, which can be satisfied for

$$X^{abcd} = \frac{\partial L}{\partial R_{abcd}}, \text{ with } L = L(g_{ab}, R_{abcd})$$

Then the Clausius relation implies the field equation corresponding to this L.

We have not proved this is the *only* way to satisfy the integrability condition.
It doesn't seem to work if derivatives of Riemann are included in L…

# What's not to like…

- Entropy, and entropy changes between two local slices, depends on the arbitrary choice of bifurcation point for the local Killing vector
- In the GR case, the entropy change is not the area change for generic horizon slices.

# *Should* it make sense?

No! It seems the *local* thermodynamic analysis can only capture the leading order, area term in the entropy:

The local Killing vector, and therefore the heat flux, is ambiguous at order $(L_{patch}/L_{curv})^2$, where $L_{patch}$ = scale of the horizon patch.

If the entropy is $\propto (A + L_1^2 \int R)$, the curvature correction is of order $(L_1/L_{curv})^2$.

To capture the correction unambiguously, need $L_{patch} < L_1$.

If $L_1 = L_{planck}$, this patch is too small for the analysis to be justified.

If $L_1 = L_{string} > L_{planck}$, it is still too small. For *any* $L_1$, it is probably too small!

# What about the "virial expansion"?

$$\frac{p}{T} = \frac{\partial S}{\partial V} = \frac{N}{V} + B_2(T)\left(\frac{N}{V}\right)^2 + \ldots$$

small expansion parameter:
particle size or interaction range
intermolecular distance

leading order simple because of
Independent particles; corrections
allow for small correlations.

By contrast, the leading order horizon entropy is simple (proportional to area) because of underlying complexity, summarized by $G_N$. Curvature corrections represent a small deformation of that complexity, not the addition of a small physical effect.