Baron and Cat

- Cats rotate without angular momentum
- Why can’t the Baron displace without momentum?
- Motion as Berry’s phase
- Swimming in curved space
- Motion by commutation

What I have learned from:

Oded Kenneth, Amos Ori, Jack Wisdom, Frank Wilczek, J.P. Eckmann, O. Raz, O. Gat, B. Gutkin, and D. Oaknin
Rotation by commutation

Stretch

Shear

\[ Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]

symmetric

\[ X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]

symmetric

\[ [X, Z] = 2\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \leftarrow Rotation \]

anti–symmetric
Rigid body motions: Abelian Gauge theory

Rotations: Uniquely defined
Translations: Depend on choice of gauge (=fiducial point)

- **A**: Instantaneously stationary
- **C**: Center of mass velocity
- **B**: Maximal speed

You can’t avoid getting dirty by going faster
Deformable body motion: Non-Abelian Gauge theory

Need fiducial frame to fix translations and rotations:

Translations and rotations: gauge invariant after a full stroke

An analog of: $\oint A \cdot d\ell = \int B \cdot dS$

- Vector pot.
- flux
Newton’s Law

**Lex I**

Corpus omne perseverare in statu suo quiescendi vel Movendi uniformiter in directum, nisi quatenus a viribus impressis cogitur statum illum mutare.

*A body perseveres being at rest or moving uniformly, except insofar as it is compelled to change its state by force impressed.*

**Modern formulation**

*The center-of-mass of the system will remain at rest in the absence of external forces*
Ambiguous center of Mass

Center of mass: A Euclidean notion
Order irrelevant:
Medians in a (Euclidean) triangle intersect at 2/3

Order matters
Medians in a spherical triangles do not intersect at 2/3 and in general manifolds they even need not intersect
Swimming on a sphere

Swimming triangle on a sphere

J. Wisdom; Animation: O Raz
Swimming on a negatively curved surface

Swimming triangle on a surface with negative curvature

Animation: O Raz
Swimming as holonomy

Swimming with periodic strokes

Closed path in shape space
Control: Shape space

**Shape space**

- **Triangle**

- **$C^{60}$**

- Degrees of freedom $180 = 60 \times 3$

- Generic spaces: Shapes locked

$(a, b, c) \in Cone \subset \mathbb{R}^3$

Mutual distances

\[
\binom{60}{2} = 1770
\]

Lucky if u fit anywhere
Homogeneous and Isotropic spaces

Swimming with periodic strokes:
Possible only in homogeneous and isotropic spaces
Killing fields

Definition (Killing equation)
Motion (vector field) inducing no strain

Example (Euclidean plane)
translations: $\xi_x = (1, 0), \quad \xi_y = (0, 1)$, rotations: $\xi_\theta = (-y, x)$
Conservation laws as Controls

Total (conserved) momentum associated to $\xi$

$$P_\xi = \sum m_n \dot{x}_n \cdot \xi(x_n)$$

$P_\xi = 0$ linear constraint on $dx_n$

$$0 = \sum m_n \xi(x_n) \cdot dx_n$$

$$dx_n = d(\text{rigid motion}) + d(\text{deformation})$$

Control equation ODE:

$$d(\text{rigid body}) = M(\text{shape, position}) \ d(\text{deformation})$$
Cats spin and Euclidean Barons lie

Spatial part of Holonomy: $\nabla \times \xi$

- Euclidean Cats spin:
  $\nabla \times \xi_\theta = \nabla \times (-y, x) = \partial_x x - \partial_y (-y) = 2$
- Euclidean Barons lie: $\nabla \times \xi_x = \nabla \times (1, 0) = 0$
- On curved manifold $\xi_{l;i;j} = -R_{lij} \xi^k$

Swimming in curved space

$$\delta x = 8 R \left( \sum_{n} \frac{m_n x_n y_n^2}{M} \right) dA$$

Small $\implies$ Slow (Kenneth)
Parking: Exercise in commutation

\[(\text{drive, steer}) = \text{park}\]

With tight space \(\text{drive} \approx \text{steer} \approx \epsilon: \) number of maneuvers \(\epsilon^{-2}\)
Purcell swimmer

- Micro-swimmers: dominated by friction
- No inertia
- Swimming is geometric

Animation: O. Raz
Quantum Swimmers

- Swimming in Fermi sea
- Swimmer controls scattering matrix $S$
- Adiabatic swimming is geometric

Quantum swimmer:

At $T = 0$ and in 1-D: $\text{distance} = \text{Integer} \times \frac{\lambda f}{2}$.

Boris Gutkin, David Oaknin
Summary

Swimming by commutation

- Geometric
- Intriguing
- Effective for large strokes