Harmony of Scattering Amplitudes: From Quantum Chromodynamics to Gravity

KITP Colloquium
December 10, 2008
Zvi Bern, UCLA
Scattering particles is fundamental to our ability to unravel microscopic laws of nature.

Imminent arrival of the LHC raises importance of scattering amplitudes.

Here we discuss some theoretical developments on scattering in QCD, gravity and supersymmetric gauge theory.
Outline

Will outline new developments in understanding scattering amplitudes. Surprising harmony.

1. Scattering from Feynman diagrams. Obscures harmony.
4. Applications:
   — LHC Physics.
   — Reexamination of divergences in gravity theories.
Every graduate student in particle theory learns how to calculate scattering amplitudes via Feynman diagrams.

\[ \frac{i}{k^2 + i\epsilon} \]

In principle this is a complete solution for small coupling

In practice not so easy:
- Proper way to calculate in QCD? Asymptotic freedom, many scales, strong coupling, infrared safety, non-perturbative contributions, etc.
- Beyond the very simplest processes an explosion of complexity.
- Completely obscures the beauty and harmony.
Example of difficulty

Consider a tensor integral:

\[
\int \frac{d^{4-2\epsilon} \ell}{(2\pi)^{4-\epsilon}} \frac{\ell^\mu \ell^\nu \ell^p \ell^\lambda}{\ell^2 (\ell - k_1)^2 (\ell - k_1 - k_2)^2 (\ell + k_4)^2}
\]

Note: this is trivial on modern computer. Non-trivial for larger numbers of external particles.

Evaluate this integral via Passarino-Veltman reduction. Result is …
Result of performing the integration

Calculations explode for larger numbers of particles or loops. Clearly, there should be a better way
In 1948 Schwinger computed anomalous magnetic moment of the electron.

60 years later typical examples:

\[ pp \to W, Z + 2 \text{ jets} \]

\[ pp \to VVV \quad V = Z, W \]

No complete 6-point cross-section calculations in QCD, though serious progress described in this talk.

Two-loops: \( e^+e^- \to 3 \text{ jets state of the art.} \)
Why are Feynman diagrams clumsy for high-loop or high-multiplicity processes?

- Vertices and propagators involve unphysical gauge-dependent off-shell states. An important origin of the complexity.

\[ \int \frac{d^3 \vec{p} \, dE}{(2\pi)^4} \]

Individual Feynman diagrams unphysical

Einstein’s relation between momentum and energy violated in the loops. **Unphysical states! Not gauge invariant.**

- All steps should be in terms of gauge invariant on-shell physical states. **On-shell formalism.**
**Heisenberg**

Feynman diagram loops violate on shellness because they encode the uncertainty principle.

\[ \Delta E \Delta t \geq \frac{\hbar}{2} \]

\[ E^2 - \vec{p}^2 \neq m^2 \]

\[ \int \frac{d^3\vec{p}}{(2\pi)^4} dE \]

You can create new particles even with insufficient energy as long as you destroy them quickly enough.

**Theorem:** Off-shellness or energy conservation violation is essential for getting the correct answer.

It looks like an on-shell formalism will fail to capture everything.
Favorite Theorem: Theorems in physics are bound to be misleading or have major loopholes.

What’s the loophole here?

• Want to reconstruct the complete amplitude using only on-shell physical information.
• Keep particles on-shell in intermediate steps of calculation, not in final results.
On-Shell Recursion for Tree Amplitudes

Consider amplitude under complex shifts of the momenta

\[ p_1^{\mu}(z) = p_1^{\mu} - zq^{\mu} \quad p_n^{\mu}(z) = p_n^{\mu} + zq^{\mu} \quad q^2 = 0, \ p \cdot q = 0 \]

\( (p_i^{\mu}(z))^2 = 0 \)

complex momenta

If \( A(z) \to 0, \ z \to \infty \) \( A(z) \) is amplitude with shifted momenta

\[ \oint_{C_{\infty}} \frac{A(z)}{z} \, dz = 0 \ \Rightarrow \ A(z = 0) = -\sum_{\alpha} \text{Res}_{\alpha} \frac{A(z)}{z} \]

\[ A(z) = \sum_{\alpha} \frac{c_{\alpha}}{z - z_{\alpha}} \]

on-shell amplitude

Sum over residues gives the on-shell recursion relation

Poles in \( z \) come from kinematic poles in amplitude.

Same construction works in gravity

Brandhuber, Travaglini, Spence; Cachazo, Svrcek; Benincasa, Boucher-Veronneau, Cachazo; Arkani-Hamed and Kaplan, Hall
Modern Unitarity Method

Two-particle cut:

Three-particle cut:

Generalized unitarity as a practical tool.

Systematic assembly of complete amplitudes from cuts for any number of particles or loops.

Different cuts merged to give an expression with correct cuts in all channels.

Reproduces Feynman diagrams except intermediate steps of calculation based on physical quantities not unphysical ones.
Method of Maximal Cuts

A refinement of unitarity method for constructing complete higher-loop amplitudes in any theory is “Method of Maximal Cuts”. Systematic construction in any theory.

To construct the amplitude we use cuts with maximum number of on-shell propagators:

Then systematically release cut conditions to obtain contact terms:

Related to leading singularity method.

Cachazo and Skinner; Cachazo, Spradlin, Volovich; Spradlin, Volovich, Wen
Examples of Harmony
Gravity vs Gauge Theory

Consider the gravity Lagrangian

\[ L_{\text{gravity}} = \frac{2}{\kappa^2} \sqrt{-g} R \]

\[ \kappa^2 = 32\pi G_{\text{Newton}} \]

\[ g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu} \]

Infinite number of complicated interactions

Compare to Yang-Mills Lagrangian on which QCD is based

\[ L_{\text{YM}} = \frac{1}{g^2} F^2 \]

Gravity seems so much more complicated than gauge theory.

Does not look harmonious!
Three Vertices

Three gluon vertex:

\[ V_{3}^{abc}_{\mu\nu\sigma} = -gf^{abc}(\eta_{\mu\nu}(k_1 - k_2)_{\rho} + \eta_{\nu\rho}(k_1 - k_2)_{\mu} + \eta_{\rho\mu}(k_1 - k_2)_{\nu}) \]

Three graviton vertex:

\[ G^{3\mu_{\alpha,\nu_{\beta,\sigma,\gamma}}(k_1, k_2, k_3)} = \]

\[ \text{sym}[-\frac{1}{2}P_3(k_1 \cdot k_2 \eta_{\mu\alpha} \eta_{\nu\beta} \eta_{\sigma\gamma}) - \frac{1}{2}P_6(k_1 \nu k_1 \beta \eta_{\mu\alpha} \eta_{\sigma\gamma}) + \frac{1}{2}P_3(k_1 \cdot k_2 \eta_{\mu\nu} \eta_{\alpha\beta} \eta_{\sigma\gamma}) + P_6(k_1 \cdot k_2 \eta_{\mu\nu} \eta_{\alpha\gamma}) + 2P_3(k_1 \nu k_1 \gamma \eta_{\mu\alpha} \eta_{\beta\gamma}) - P_3(k_1 \beta k_2 \mu \eta_{\alpha\nu} \eta_{\sigma\gamma}) + P_6(k_1 \sigma k_2 \gamma \eta_{\mu\nu} \eta_{\alpha\beta}) + 2P_6(k_1 \nu k_2 \gamma \eta_{\beta\mu} \eta_{\alpha\sigma}) + 2P_3(k_1 \nu k_2 \mu \eta_{\beta\sigma} \eta_{\gamma\alpha}) - 2P_3(k_1 \cdot k_2 \eta_{\alpha\nu} \eta_{\beta\sigma} \eta_{\gamma\mu})] \]

About 100 terms in three vertex

Naïve conclusion: Gravity is a nasty mess.
Simplicity of Gravity Amplitudes

On-shell three vertices contains all information:

\[ k_i^2 = 0 \]

\[ -gf^{abc}(\eta_{\mu\nu}(k_1 - k_2)_\rho + \text{cyclic}) \]

Gauge theory:

\[ \begin{array}{c}
\begin{array}{c}
g_2 \end{array} \\
\begin{array}{c}
\begin{array}{c}
2_b \\
1_\mu \\
3_c \\
a \end{array}
\end{array}
\end{array} \]

Gravity:

\[ i\kappa(\eta_{\mu\nu}(k_1 - k_1)_\rho + \text{cyclic}) \times (\eta_{\alpha\beta}(k_1 - k_2)_\gamma + \text{cyclic}) \]

Any gravity scattering amplitude constructible solely from on-shell 3 vertex.

- BCFW on-shell recursion for tree amplitudes.
  Britto, Cachazo, Feng and Witten; Brandhuber, Travaglini, Spence; Cachazo, Svrcek; Benincasa, Boucher-Veronneau, Cachazo; Arkani-Hamed and Kaplan, Hall

- Unitarity method for loops.
  ZB, Dixon, Dunbar and Kosower; ZB, Dixon, Kosower; Britto, Cachazo, Feng; ZB, Morgan; Buchbinder and Cachazo; ZB, Carrasco, Johansson, Kosower; Cachzo and Skinner.
Consider the gravity Lagrangian

\[ L_{\text{gravity}} = \frac{2}{\kappa^2} \sqrt{-g} \, R \]

\[ \kappa^2 = 32\pi \, G_{\text{Newton}} \]

\[ g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu} \]

Infinite number of irrelevant interactions!

Compare to Yang-Mills Lagrangian

\[ L_{\text{YM}} = \frac{1}{g^2} F^2 \]

Gravity seems so much more complicated than gauge theory.

Does not look harmonious!
Even more remarkable relation between gauge and gravity amplitudes.

At *tree level* Kawai, Lewellen and Tye derived a relationship between closed and open string amplitudes. In field theory limit, relationship is between gravity and gauge theory

\[ M_4^{\text{tree}}(1, 2, 3, 4) = s_{12} A_4^{\text{tree}}(1, 2, 3, 4) A_4^{\text{tree}}(1, 2, 4, 3), \]

\[ s_{ij} = (k_i + k_j)^2 \]

\[ M_5^{\text{tree}}(1, 2, 3, 4, 5) = s_{12} s_{34} A_5^{\text{tree}}(1, 2, 3, 4, 5) A_5^{\text{tree}}(2, 1, 4, 3, 5) \]

\[ + s_{13} s_{24} A_5^{\text{tree}}(1, 3, 2, 4, 5) A_5^{\text{tree}}(3, 1, 4, 2, 5) \]

where we have stripped all coupling constants

\[ A_4^{\text{tree}} = g^2 \sum_{\text{non-cyclic}} \text{Tr}(T^{a_1} T^{a_2} T^{a_3} T^{a_4}) A_4^{\text{tree}}(1, 2, 3, 4) \]

Holds very generally. See review: gr-qc/0206071

Progress in gauge theory can be imported into gravity theories
Gravity and Gauge Theory Amplitudes

Berends, Giele, Kuijf; ZB, De Freitas, Wong

\[ M^\text{tree}_4(1_h^-, 2_h^-, 3_h^+, 4_h^+) = \left( \frac{\kappa}{2} \right)^2 s_{12} A^\text{tree}_4(1_g^-, 2_g^-, 3_g^+, 4_g^+) \times A^\text{tree}_4(1_g^-, 2_g^-, 4_g^+, 3_g^+) \]

\[ = \left( \frac{\kappa}{2} \right)^2 s_{12} \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} \times \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 24 \rangle \langle 43 \rangle \langle 31 \rangle} \]

\[ \langle jl \rangle = \langle k_j^- | k_l^+ \rangle = \frac{1}{2} \bar{u}(k_j)(1+\gamma_5)u(k_l) = \sqrt{2k_j \cdot k_l} e^{i\phi} \]

- Agrees with result starting from Einstein Lagrangian
- Holds very generally for gravity theories.
Harmony of Color and Kinematics

ZB, Carrasco, Johansson

Color factors based on a Lie algebra: $[T^a, T^b] = i f^{abc} T^c$

Jacobi identity

$$[[T^a, T^b], T^c] + [[T^b, T^c], T^a] + [[T^c, T^a], T^b] = 0$$

Use $1 = s/s = t/t = u/u$ to assign 4-point diagram to others.

$$A_4^{\text{tree}} = g^2 \left( \frac{n_s c_s}{s} + \frac{n_t c_t}{t} + \frac{n_u c_u}{u} \right)$$

Color and kinematics are singing same tune!
Harmony of Color and Kinematics

At higher points similar structure:

\[ A_{5\text{tree}} = \sum_{i=1}^{15} \frac{c_i n_i}{D_i} \]

\[ c_3 \equiv f^{a_3 a_4 b} f^{b a_5 c} f^{c a_1 a_2} , \quad c_5 \equiv f^{a_3 a_4 b} f^{b a_2 c} f^{c a_1 a_5} , \quad c_8 \equiv f^{a_3 a_4 b} f^{b a_1 c} f^{c a_2 a_5} \]

\[ c_3 - c_5 + c_8 = 0 \iff n_3 - n_5 + n_8 = 0 \]

Claim: We can always find a rearrangement so color and kinematics satisfy the same Jacobi constraint equations.

- Color and kinematics sing same tune!
- Nontrivial constraints on amplitudes.
Higher-Point Gravity and Gauge Theory

QCD: \[ A_n^{\text{tree}} = i g^{n-2} \sum_i \frac{c_i n_i}{D_i} \]
sum over diagrams with only 3 vertices

Einstein Gravity: \[ M_n^{\text{tree}} = i \kappa^{n-2} \sum_i \frac{n_i^2}{D_i} \]

Claim: This is equivalent to KLT relations

Gravity and QCD kinematic numerators sing same tune!

Cries out for a unified description of the sort given by string theory.

ZB, Carrasco, Johansson
Applications to LHC Physics
Early ATLAS TDR studies using PYTHIA overly optimistic.

- **ALPGEN** is based on LO matrix elements and much better at modeling hard jets.

- What will disagreement between ALPGEN and LHC data mean for this plot? Need NLO QCD to properly answer this.

We need \( pp \rightarrow Z + 4 \text{ jets at NLO} \)

No complete 6-point NLO cross-section calculations in QCD, though serious progress described in this talk.
Example of Typical NLO Improvements

$W + 2$ jets at the Tevatron

Note disagreement

leading order + parton showering

NLO does better, smallest theoretical uncertainty

Want similar studies at the LHC also with extra jets.
**Experimenter’s NLO Wish List**

<table>
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<th>Diboson</th>
<th>Triboson</th>
<th>Heavy flavour</th>
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<td>$WWW + \leq 3j$</td>
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<td>$W + bb + \leq 3j$</td>
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<td>$Z\gamma + \leq 3j$</td>
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Five-particle processes under good control with Feynman diagram based approaches.

Better ways needed to go beyond this.
Approaches for higher points

• Traditional or numerical Feynman approaches.
  Anastasiou, Andersen, Binoth, Ciccolini; Czakon, Daleo, Denner, Dittmaier, Ellis; Heinrich, Karg, Kauer; Giele, Glover, Guffanti, Lazopoulos, Melnikov, Nagy, Pilon, Roth, Passarino, Petriello, Sanguinetti, Schubert; Smillie, Soper, Reiter, Veltman, Wieders, Zanderighi, and many more.

• On-shell methods: unitarity method, on-shell recursion
  Anastasiou, Badger, Bedford, Berger, Bern, Bernicot, Brandhuber, Britto, Buchbinder, Cachazo, Del Duca, Dixon, Dunbar, Ellis, Feng, Febres Cordero, Forde, Giele, Glover, Guillet, Ita, Kilgore, Kosower, Kunszt; Mastrolia; Maitre, Melnikov, Spence, Travaglini; Ossola, Papadopoulos, Pittau, Risager, Yang; Zanderighi, etc

• Most physics results have been from Feynman diagrams.
  — two notable exceptions $pp \rightarrow W + 2 \text{ jets}$ and $pp \rightarrow VVV$
  ZB, Kosower, Dixon, Weinzierl; Ossola, Papadopoulos, Pittau

• Most people working on this are instead now pursuing on-shell methods because of demonstrated excellent scaling with number of external particles. See recent LoopFest conference.
  http://electron.physics.buffalo.edu/loopfest7
BlackHat: An automated implementation of on-shell methods for one-loop amplitudes

BlackHat is an automated C++ package for numerically computing one-loop matrix elements with 6 or more external particles.

- Input is numerical on-shell tree-level amplitudes.
- Output is numerical on-shell one-loop amplitudes.

BlackHat incorporates ideas discussed above to achieve the speed and stability required for LHC phenomenology at NLO.

Two other similar packages under construction

— CutTools  Ossola, Papadopoulos, Pittau
— Rocket  Ellis, Giele, Kunszt, Melnikov, Zanderighi
Extremely mild scaling with number of legs

Scaling with number of legs

Berger, ZB, Dixon, Febres Cordero, Forde, Ita, Kosower, Maitre

Extremely mild scaling with number of legs

2.33 GHz Xeon

relative precision = \log_{10}\left(\frac{|A_{\text{num}} - A_{\text{ref}}|}{|A_{\text{ref}}|}\right)

amusing count for 8 gluons

+ 3,017,489 Feynman diagrams

vector bosons under control

More to be done to get physics
Applications to AdS/CFT
Since ‘t Hooft’s paper thirty years ago on the planar limit of QCD we have dreamed of solving QCD in this limit. This is too hard. $N = 4$ sYM is much more promising.

- Special theory because of AdS/CFT correspondence:
- Maximally supersymmetric — boson/fermion symmetry
- Simplicity both at strong and weak coupling.

Remarkable relation

scattering at strong coupling in $N = 4$ sYM $\leftrightarrow$ classical string theory in AdS space

To make this link need to evaluate $N=4$ super-Yang-Mills amplitudes to all loop orders. Seems impossible even with modern methods.
The planar four-point two-loop amplitude undergoes fantastic simplification.

\[ -st A_{4}^{\text{tree}} \left\{ \begin{array}{c} s \cr 3 \end{array} \right. \left( \begin{array}{c} 4 \cr 2 \end{array} \right) + \left. \begin{array}{c} t \cr 3 \end{array} \right. \left( \begin{array}{c} 4 \cr 2 \end{array} \right) \right\} \]

\[ M_{4}^{2-\text{loop}}(s, t) = \frac{1}{2} \left( M_{4}^{1-\text{loop}}(s, t) \right)^2 + f(\epsilon) M_{4}^{1-\text{loop}}(s, t) \big|_{\epsilon \rightarrow 2\epsilon} - \frac{1}{2} \zeta_2^2 \]

\[ D = 4 - 2\epsilon \]

\[ f(\epsilon) = -\zeta_2 - \zeta_3 \epsilon - \zeta_4 \epsilon^2 \]

\[ M_{4}^{\text{loop}} = A_{4}^{\text{loop}} / A_{4}^{\text{tree}} \]

\[ ZB, Rozowsky, Yan \]

\[ Anastasiou, ZB, Dixon, Kosower \]

Two-loop four-point planar amplitude is an iteration of one-loop amplitude.

Three loop satisfies similar iteration relation. Rather nontrivial.

\[ ZB, Dixon, Smirnov \]
All-Loop Generalization

Why not be bold and guess scattering amplitudes for all loop and all legs, at least for simple helicity configurations?

\[ \mathcal{A}_n = A_{n \text{tree}} A_{n \text{divergent}} \exp \left[ \gamma_K F_{n \text{1-loop}} + C \right] \]

“BDS conjecture”

\[ F_{4 \text{1-loop}} = \frac{1}{2} \ln^2 \left( \frac{t}{s} \right) + 4 \zeta_2 \]

- To make this guess used strong constraint constraints from analytic properties of amplitudes.

Gives a definite prediction for all values of coupling given BES integral equation for the cusp anomalous dimension.

Anastasiou, ZB, Dixon, Kosower
ZB, Dixon and Smirnov

Beisert, Eden, Staudacher
Alday and Maldacena Strong Coupling

For MHV amplitudes:

\[ \mathcal{A}_4 = \mathcal{A}_4^{\text{tree}} \mathcal{A}_4^{\text{divergent}} \exp \left( \frac{1}{4} \gamma K F_4^{\text{1-loop}} + C \right) \]

In a beautiful paper Alday and Maldacena confirmed the conjecture for 4 gluons at strong coupling from an AdS string theory computation. Minimal surface calculation—like a soap bubble.

- **Identification of new symmetry:** “dual conformal symmetry”
- **Link to integrability.** Infinite number of conserved charges

Drummond, Henn, Korchemsky, Sokatchev; Berkovits and Maldacena; Beisert, Ricci, Tseytlin, Wolf Brandhuber, Heslop, Travaglini

Unfortunately, trouble at 6 and higher points.

Alday and Maldacena; ZB, Dixon, Kosower, Roiban, Spradlin, Vergu, Volovich
Applications to Quantum Gravity
Is a UV finite theory of gravity possible?

\[ \kappa = \sqrt{32\pi G_N} \quad \text{Dimensionful coupling} \]

\[ \kappa p^\mu p^\nu \]

Gravity:
\[ \int \prod_{i=1}^{L} \frac{dp_i^D}{(2\pi)^D} \frac{(\kappa p_i^\mu p_j^\nu)}{\text{propagators}} \]

Gauge theory:
\[ \int \prod_{i=1}^{L} \frac{d^D p_i}{(2\pi)^D} \frac{(g p_j^\nu)}{\text{propagators}} \]

Extra powers of loop momenta in numerator means integrals are badly behaved in the UV.

Much more sophisticated power counting in supersymmetric theories but this is the basic idea.

Reasons to focus on \( N = 8 \) maximal supergravity:

• With more susy suspect better UV properties. \( \text{Cremmer and Julia} \)
• High symmetry implies technical simplicity—may even be “simplest” quantum field theory \( \text{Arkani-Hamed, Cachazo, Kaplan} \)
Finiteness of Point-Like Gravity Theory?

We are interested in UV finiteness of $N = 8$ supergravity because it would imply a new symmetry or non-trivial dynamical mechanism.

The discovery of either would have a fundamental impact on our understanding of gravity.

• Here we only focus on order-by-order UV finiteness.
• Non-perturbative issues and viable models of Nature are *not* the goal for now.
If certain patterns that emerge should persist in the higher orders of perturbation theory, then ... \( N = 8 \) supergravity in four dimensions would have ultraviolet divergences starting at three loops.  

Green, Schwarz, Brink, (1982)

Unfortunately, in the absence of further mechanisms for cancellation, the analogous \( N = 8 \ D = 4 \) supergravity theory would seem set to diverge at the three-loop order.  

Howe, Stelle (1984)

The idea that all supergravity theories diverge at 3 loops has been widely accepted wisdom for over 20 years

There are a number of very good reasons to reanalyze this.  
Non-trivial one-loop cancellations: no triangle & bubble integrals

ZB, Dixon, Perelstein, Rozowsky; ZB, Bjerrum-Bohr, Dunbar; Dunbar, Ita, Perkins, Risager; Green, Vanhove, Russo; Bjerrum-Bohr Vanhove; Arkani-Hamed, Cachazo, Kaplan

Unitarity method implies higher-loop cancellations.  

ZB, Dixon, Roiban
Suppose we wanted to check superspace power counting proposal of 3 loop divergence.

If we attack this directly get $\sim 10^{20}$ terms in diagram. The algebraic explosion is a reason why this hasn’t been evaluated using Feynman diagrams.

Counted number of terms in one diagram from expanding vertices and propagators, not number of diagrams or the algebraic explosion trying to reduce to integral basis.
Complete Three Loop Result

ZB, Carrasco, Dixon, Johansson, Kosower, Roiban; hep-th/0702112
ZB, Carrasco, Dixon, Johansson, Roiban; arXiv:0808.4112 [hep-th]

Obtained via maximal cut method:

\[ \tau_{ij} = 2k_i \cdot k_j \]

Three-loop is not only ultraviolet finite it is "superfinite"—cancellations beyond those needed for finiteness!
Some $N=4$ YM contributions:

\[ s_{12}^2 s_{45}^2 \]

50 distinct planar and non-planar diagrammatic topologies

$N = 4$ super-Yang-Mills case is complete!
$N = 8$ supergravity case still in progress.

Four-loops will teach us a lot – bottles of wine to be exchanged:
1. Direct challenge to simplest superspace explanations.
2. Proof of finiteness will likely need insights gathered from this calculation.

ZB, Carrasco, Dixon, Johansson, Roiban
We can probe infinite loop orders by looking at a limited class of cuts.

Probes reveal superfiniteness: finite for \( D < \frac{6}{L} + 4 \)

Not a proof of finiteness because you would need to check all cuts.

Improved behavior can be traced back to good behavior of tree-level amplitudes under large complex shifts of momenta.
Summary

• On-shell methods offer a powerful alternative to Feynman diagrams.
• Remarkable structures in scattering amplitudes:
  — color $\leftrightarrow$ kinematics.
  — gravity $\sim (\text{gauge theory})^2$
• Remarkable harmony between gravity and gauge theory scattering amplitudes.

Applications:
• NLO QCD for the LHC: Amplitudes under control, physics on its way.
• $N = 4$ Supersymmetric gauge theory: New venue opened for studying AdS/CFT.
• Quantum gravity: Is a point-like UV finite theory possible? New evidence suggests it is but proof is a challenge.
Some amusement

YouTube: Search “Big Bang DMV”, first hit
Extra
Basic Strategy

$N = 4$ Super-Yang-Mills Tree Amplitudes $\rightarrow$ KLT

$N = 8$ Supergravity Tree Amplitudes $\rightarrow$ Unitarity

$N = 8$ Supergravity Loop Amplitudes $\rightarrow$ Divergences

- Kawai-Lewellen-Tye relations: sum of products of gauge theory tree amplitudes gives gravity tree amplitudes.
- Modern unitarity method: efficient formalism for perturbatively quantizing gauge and gravity theories. Loop amplitudes from tree amplitudes.

Key features of this approach:

- Gravity calculations mapped into much simpler gauge theory calculations.
- Only on-shell states appear.

ZB, Dixon, Dunbar, Kosower (1994)
Full Three-Loop Calculation

Need following cuts:

For cut (g) have:

\[ \sum_{N=8 \text{ states}} M_{4}^{\text{tree}}(1, 2, l_3, l_1) \times M_{5}^{\text{tree}}(-l_1, -l_3, q_3, q_2, q_1) \times M_{5}^{\text{tree}}(3, 4, -q_1, -q_2, -q_3) \]

Use KLT

\[ M_{4}^{\text{tree}}(1, 2, l_3, l_1) = -i s_{12} A_{4}^{\text{tree}}(1, 2, l_3, l_1) A_{4}^{\text{tree}}(2, 1, l_3, l_1) \]

\[ M_{5}^{\text{tree}}(-l_1, -l_3, q_3, q_2, q_1) = i s_{l_1 q_1} s_{l_3 q_3} A_{5}^{\text{tree}}(-l_1, -l_3, q_3, q_2, q_1) A_{5}^{\text{tree}}(-l_1, q_1, q_3, -l_3, q_2) + \{l_1 \leftrightarrow l_3\} , \]

\( N = 8 \) supergravity cuts are sums of products of \( N = 4 \) super-Yang-Mills cuts

ZB, Carrasco, Dixon, Johansson, Kosower, Roiban
Same power count as $N=4$ super-Yang-Mills

UV behavior unknown

from feeding 2 and 3 loop calculations into iterated cuts.

No triangle property with unitarity bootstrap

ZB, Dixon, Roiban

explicit 2 and 3 loop computations

terms

4 loop calculation in progress.
Some Developments

• Generalized cuts – used to produce one-loop matrix elements for $pp \rightarrow W, Z + 2$ jets. Used in MCFM

• Realization of the remarkable power of complex momenta in generalized cuts. Inspired by Witten’s twistor string paper. Very important.

• $D$-dimensional unitarity to capture rational pieces of loops.

• On-shell recursion for loops (based on BCFW)

• Efficient on-shell reduction of integrals.
Where are the $N = 8$ Divergences?

Depends on who you ask and when you ask.

3 loops: Conventional superspace power counting.

5 loops: Partial analysis of unitarity cuts.  
If harmonic superspace with $N = 6$ susy manifest exists

6 loops:  
If harmonic superspace with $N = 7$ susy manifest exists

7 loops:  
If a superspace with $N = 8$ susy manifest were to exist.

8 loops: Explicit identification of potential susy invariant counterterm with full non-linear susy.

9 loops: Assume Berkovits’ superstring non-renormalization theorems can be naively carried over to $N = 8$ supergravity.  
Also need to extrapolate.  
Superspace gets here with additional speculations.

Note: none of these are based on demonstrating a divergence.  They are based on arguing susy protection runs out after some point.
Origin of Cancellations?

There does not appear to be a supersymmetry explanation for all-loop cancellations.

If it is *not* supersymmetry what might it be?

\[
k_1^{\mu} \to k_1^{\mu} + \frac{z}{2} \langle k_1^- | \gamma^{\mu} | k_2^- \rangle \\
k_2^{\mu} \to k_2^{\mu} - \frac{z}{2} \langle k_1^- | \gamma^{\mu} | k_2^- \rangle,
\]

\[A_{\text{tree}}(z) \to 0 \quad z \to \infty\]

This property useful for constructing BCFW recursion relations for gravity.

Bedford, Brandhuber, Spence, Travaglini; Cachazo, Svrcek; Benincasa, Boucher-Veronneau, Cachazo; Arkani-Hamed, Kaplan; Hall

This same property appears to be directly related to the novel cancellations observed in the loops.

ZB, Carrasco, Forde, Ita, Johansson; Arkani-Hamed, Cachazo, Kaplan

Can we prove perturbative finiteness of \( N = 8 \) supergravity?

Time will tell...