Entanglement Entropy of Critical Spin Liquids

Tarun Grover

with Frank Zhang, Ashvin Vishwanath (UC Berkeley)

Our aim: To calculate Renyi Entropy for critical spin liquids.

Why bother?

Demotivators:

- Gapless spin-liquids correspond to strongly interacting gauge-matter theories $\Rightarrow$ very difficult to analyze.
- Entanglement entropy is a highly non-local operator and hence difficult to calculate.
- Experimental relevance?

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- We have candidate variational wave-functions for ground state of gapless spin-liquids that may provide *non-perturbative* access to interesting properties.
- Precisely because entanglement entropy is a highly non-local operator and hence may capture non-local physics of 'quantum order'.
- Experiments $\Rightarrow$ we study candidate spin-liquid wavefunctions for triangular lattice organic spin-liquids. Conjectured to have emergent Fermi surface.

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Oshikawa-Hastings argument:
Mott insulators with odd number of electrons per unit cell that do not break any symmetry have low lying excitations → topological degenracy OR critical phase.

Slave particles and deconfinement

- $$H = J \sum_{<ij>} S_i.S_j + ... \quad (1)$$
- $$\vec{S} = \frac{f^\dagger \vec{\sigma} f}{2} \quad (2)$$
  with the constraint $f^\dagger f = 1$.
- $U(1)$ redundancy $\Rightarrow$ gauge fields coupled to spinons f.

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Slave particle matter-gauge theory

- Full gauge-matter action

$$S = \sum_{<xx'>} \left(\overline{f}_{\vec{x}\sigma} f_{\vec{x'}\sigma} e^{ia_{xx'}} + c.c.\right) + \sum_{\Box} cos(\vec{\nabla} \times \vec{a}) \qquad (3)$$

- If deconfinement $\Rightarrow$ Non-compact gauge field

$$S = \int_{k,\omega} \overline{f}_{k\sigma}(-i\omega + \epsilon_k + \mu) f_{k\sigma} + k^2/e^2 |a(k,\omega)|^2 + j(k,\omega) a(k,\omega)$$

- Theory difficult to analyze directly for $D = 2+1$ and $SU(2)$ symmetry. Lee, Nagaosa, Halperin, Read, Senthil, Mross, Sung-sik Lee, Reizer ...

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From slave particles to projected wavefunctions

- Consider the 'mean-field' Hamiltonian in the weak-coupling (≡ deconfined) limit:

$$H = \sum_{<ij>} f_i^\dagger f_j + h.c. \quad (4)$$

with $f^\dagger f = 1$.

- Ground state wavefunction

$$|\Psi\rangle_{var} \equiv |\Psi\rangle_{PFL} = \prod_i (1 - n_{i\uparrow} n_{i\downarrow}) \left(\prod_{\vec{k}\sigma}' f^\dagger_{\vec{k}\sigma} \right) |0\rangle \quad (5)$$

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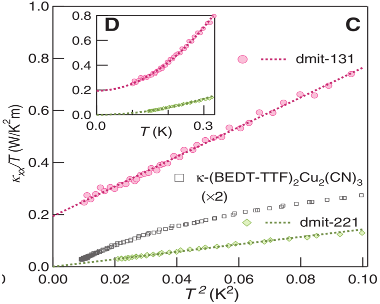
Insulator with metallic thermal transport and specific heat



- Material: $EtMe_3Sb[Pd(dmit)_2]_2$ Yamashita et al, Science 2010.
- $\kappa/T(T \to 0)$ extrapolates to non-zero value as $T \to 0$.

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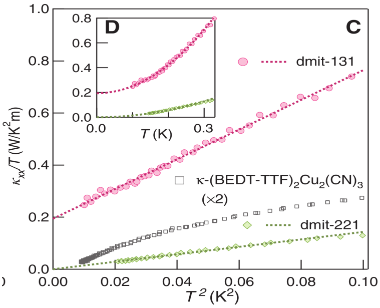

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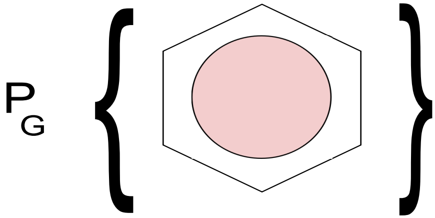
Projected wave-functions and triangular lattice spin-liquids



where $P_G = \prod_i (1 - n_{i\uparrow} n_{i\downarrow})$ Motrunich 2005

Do projected wave-functions have the correct entanglement properties to faithfully capture physics of critical spin-liquids?

Basic definitions

- Wavefunction, $|\phi\rangle$: trace out B to get a density matrix on A: $\rho_A = \mathrm{Tr}_B|\phi\rangle\langle\phi|$.
- Renyi entropies:

$$S_n = \frac{1}{1-n}\log(\mathrm{Tr}\rho_A^n) \tag{6}$$

- $S_1 = S_{vN} = \mathrm{Tr}\rho_A \log\rho_A$
- This talk: S_2

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Usefulness of S_2

- Numerically easier to calculate than S_{vN}.
- $S \geq S_2$
- S_2 and S share:
  -Violation of area law for CFT in 1D and for free fermions in any D.
  -Non-zero topological entanglement/Renyi entropy for gapped 2D topologically ordered phases.
  Wilczek, Holzhey, Larsen, Cardy, Calabrese, Casini, Huerta, Swingle, Flammia, Hamma, Hughes, Wen ...

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Entanglement entropy as a non-local probe

- For topological ordered phases,

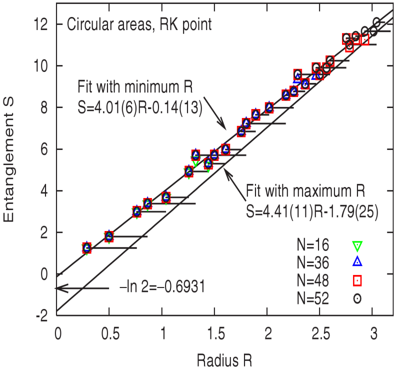
$$S_n = \alpha L - log(D) \tag{7}$$

where

$$D = \sqrt{\sum_i {d_i}^2} \tag{8}$$

d_i = quantum dimension of i'th quasiparticle.

Example: Entanglement entropy as a non-local probe



Furukawa, Misguich, More recent work: Roger Melko et al. (Unpublished)

Calculation of Renyi entropy in Monte Carlo

$$
\begin{aligned}
e^{-S_2} &= \mathrm{tr}\rho_A^2 = \sum_{a,a'} \rho_A(a,a')\rho_A(a',a) \\
&= \sum_{a,a',b,b'} \phi^*(a,b)\phi(a',b)\phi^*(a',b')\phi(a,b')
\end{aligned}
$$

Calculation of Renyi entropy in Monte Carlo

$\Rightarrow$

$$e^{-S_2} = \frac{\langle \Phi | Swap_A | \Phi \rangle}{\langle \Phi | \Phi \rangle} \tag{9}$$

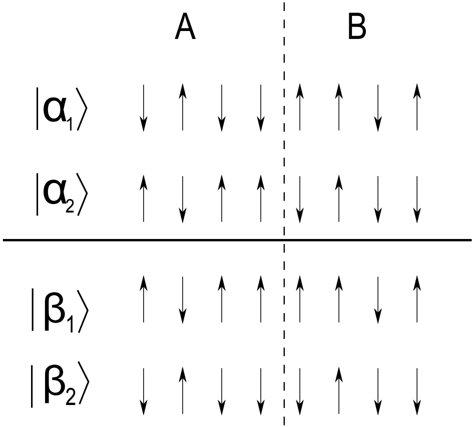
where

$$Swap_A |a, b\rangle |a', b'\rangle = |a', b\rangle |a, b'\rangle$$

$$|\Phi\rangle = |\phi\rangle \otimes |\phi\rangle$$

Swap operator

$$\mathrm{Swap}_A |\alpha_1\rangle \otimes |\alpha_2\rangle =$$



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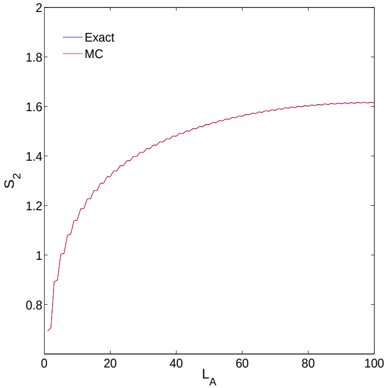
\begin{aligned}
\langle Swap_A \rangle &= \frac{\sum_{\alpha_1,\alpha_2} \langle \phi|\alpha_1\rangle \langle \beta_1|\phi\rangle \langle \phi|\alpha_2\rangle \langle \beta_2|\phi\rangle}{|\langle \phi|\phi\rangle|^2} \\
&= \sum_{\alpha_1\alpha_2} \frac{|\langle \phi|\alpha_1\rangle|^2 \, |\langle \phi|\alpha_2\rangle|^2}{\langle \phi|\phi\rangle \langle \phi|\phi\rangle} \cdot \frac{\langle \beta_1|\phi\rangle \langle \beta_2|\phi\rangle}{\langle \alpha_1|\phi\rangle \langle \alpha_2|\phi\rangle} \\
&= \sum_{\alpha_1\alpha_2} \rho_{\alpha_1}\rho_{\alpha_2} f(\alpha_1,\alpha_2)
\end{aligned}
$$

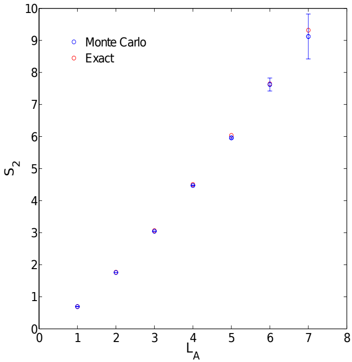
Hastings et al (2010)

Free fermions in 1D



Almost exact answer!

Free fermions in 2D



- $L_{total} = 18, L_A \leq 7$.
- Very good agreement, can detect area-law violation!

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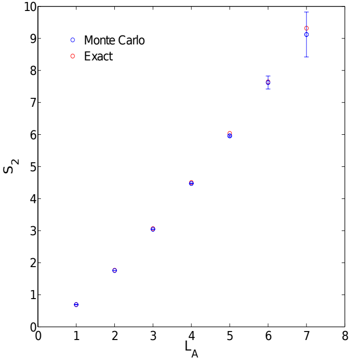

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Projected Fermi sea

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- Does projected state has the same symmetries?

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Warm-up: Projected Fermi Sea in One dimension

- Before projection: total central charge $= 2$.
- After projection: exact ground state of Haldane-Shastry model Luttinger liquid with central charge $= 1$.

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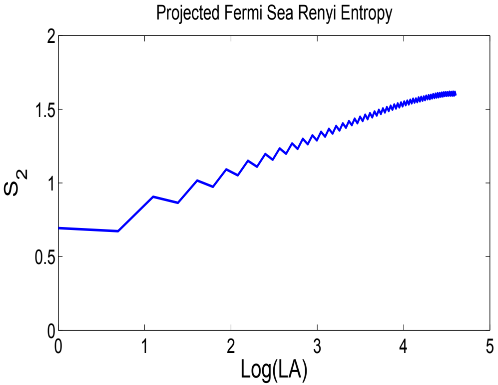
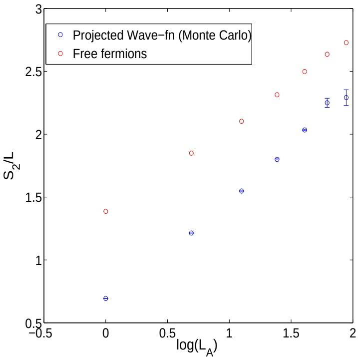


Consistent with $c = 1$.

Projected Fermi sea on triangular lattice



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- *Area-law violation in a bosonic wavefunction!* $\Rightarrow$ fits $L_A log L_A$ scaling.

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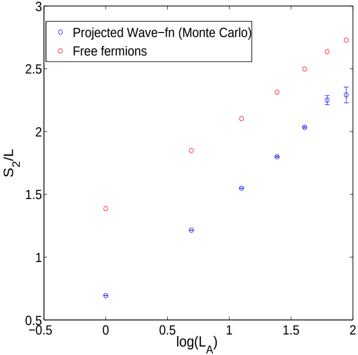

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Projected Fermi sea on triangular lattice

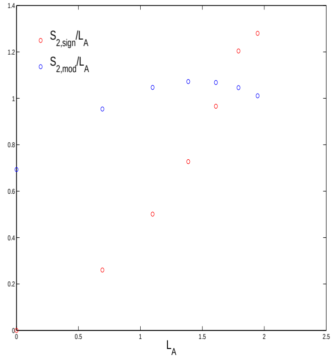
- Can write $S_2 = S_{2,sign} + S_{2,mod}$ where

$$e^{-S_{2,mod}} = \sum_{\alpha_1 \alpha_2} \rho_{\alpha_1} \rho_{\alpha_2} |f(\alpha_1, \alpha_2)| \qquad (10)$$

and

$$S_{2,sign} = S_2 - S_{2,mod} \qquad (11)$$

Projected triangular: $S_{2,sign}$ Vs $S_{2,mod}$



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- Conjecture: holds for fermi surfaces in general.

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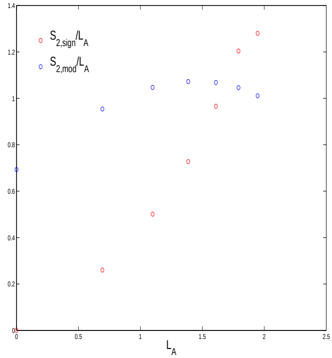

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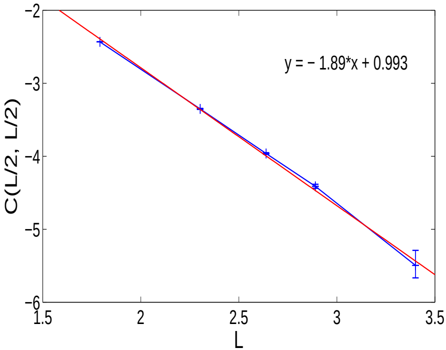
Projected Fermi sea on π-flux square lattice

Spin-spin correlations:



$\Rightarrow$ Algebraically decaying correlations!

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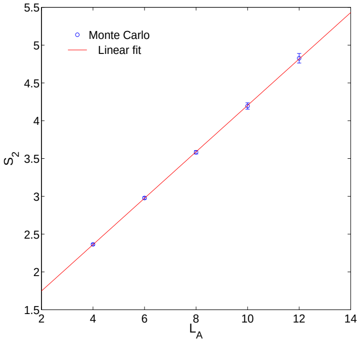

- Near perfect area law $\Rightarrow$ consistent with ‘nodal Dirac liquid’.
- Can extract ‘universal constant’.
- Prediction for QED-3(?).

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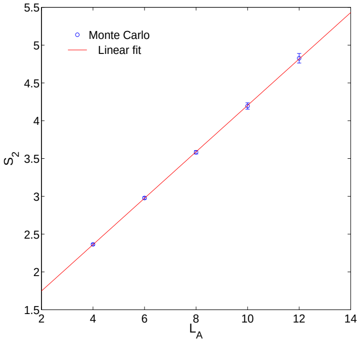

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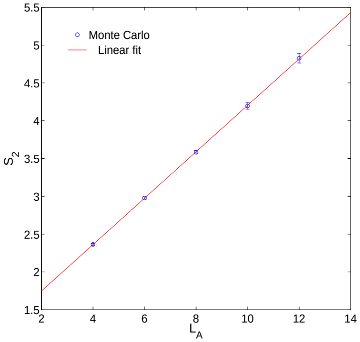

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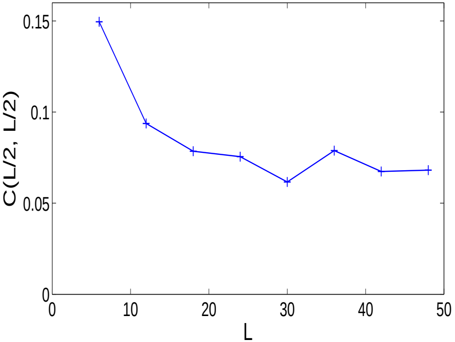

⇒ Magnetically ordered!

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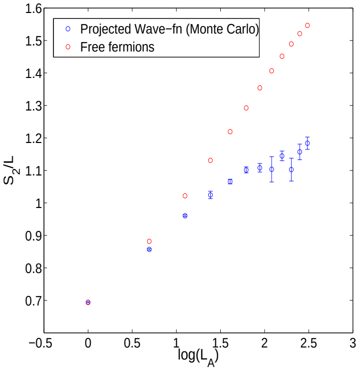

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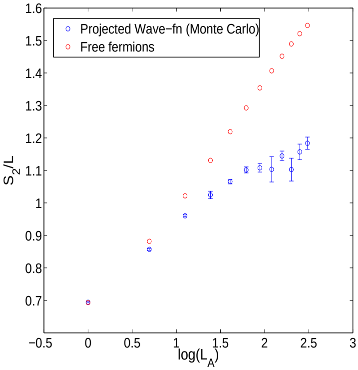

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Ongoing work

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- Renyi entropy of partially projected Fermi sea ($\equiv$ correlated Fermi liquid).

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Acknowledgements

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