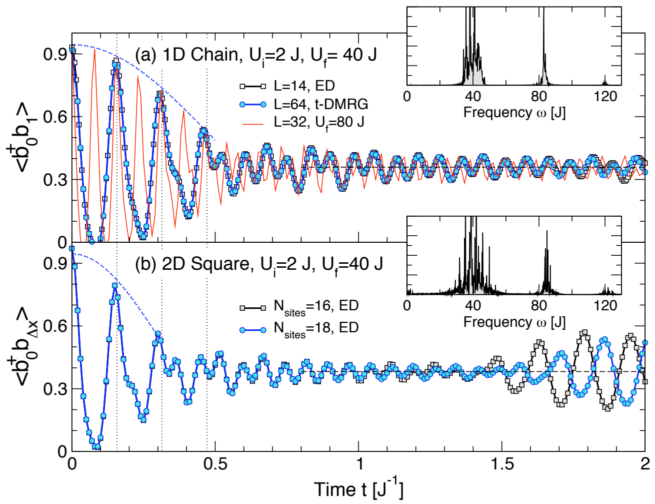

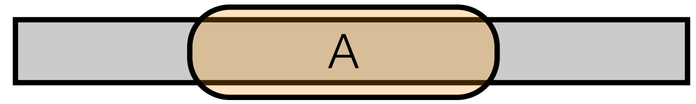

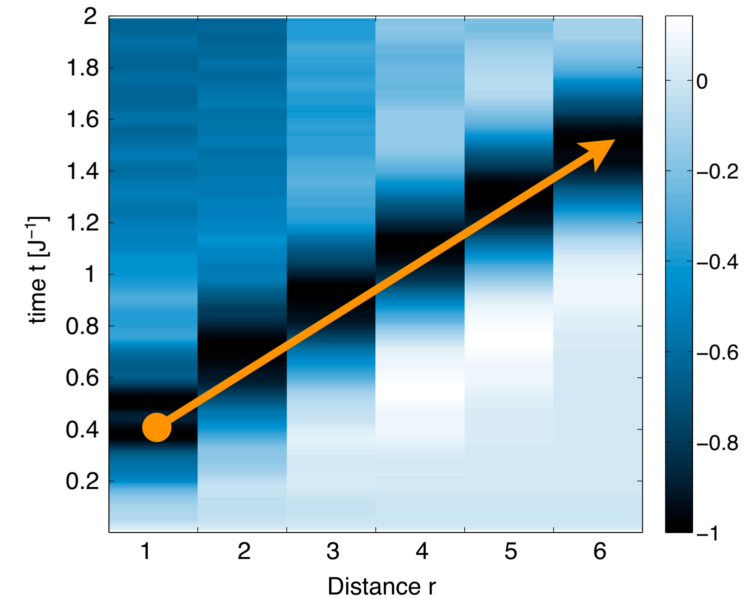

Numerical Simulations of Quenches in Bose Hubbard models

Andreas Läuchli,
"New states of quantum matter"
MPI für Physik komplexer Systeme - Dresden

http://www.pks.mpg.de/~aml

C. Kollath, AML, E. Altman, PRL 2007
AML and C. Kollath, JSTAT 2008
G. Biroli, C. Kollath, and AML, arXiv:0907.3731

Disentangling Quantum Many-Body Systems, KITP, November 1, 2010

Fermi Pasta Ulam Tsingou Paradox (1953)

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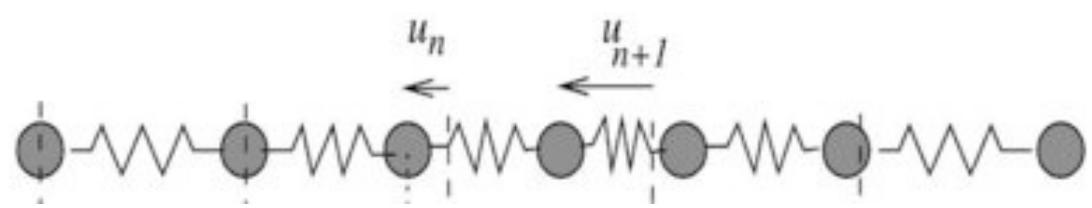
- A chain of harmonic oscillators, coupled with nonlinear couplings



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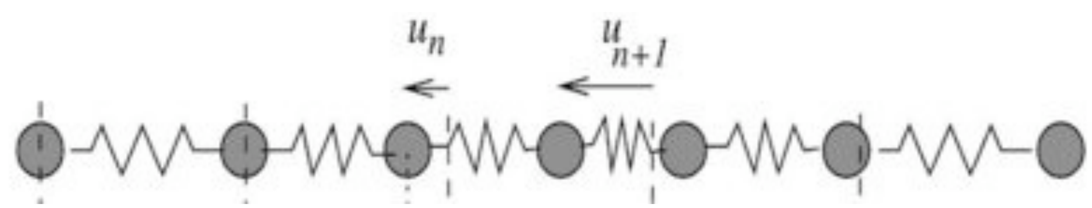

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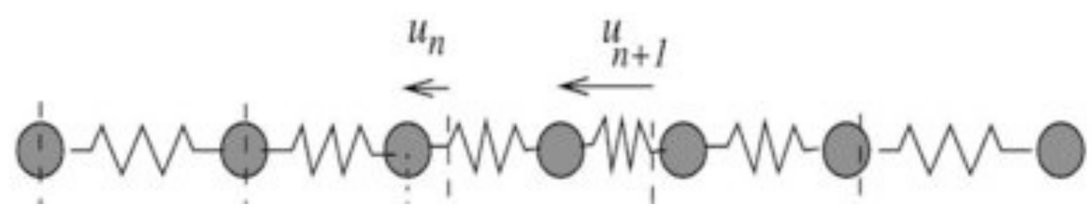

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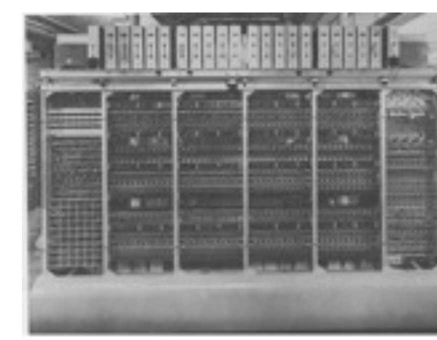

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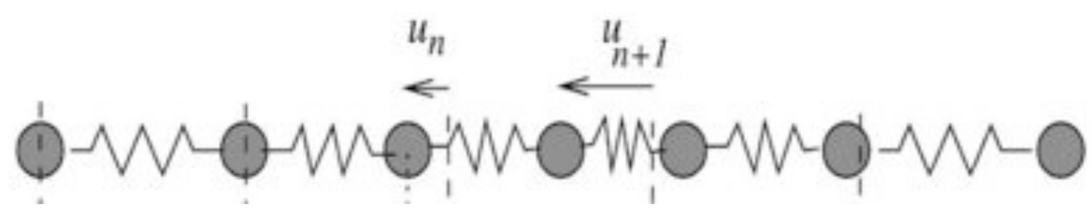

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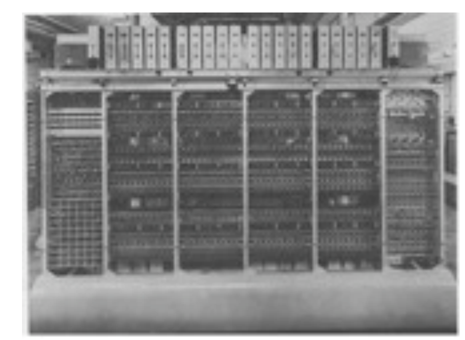

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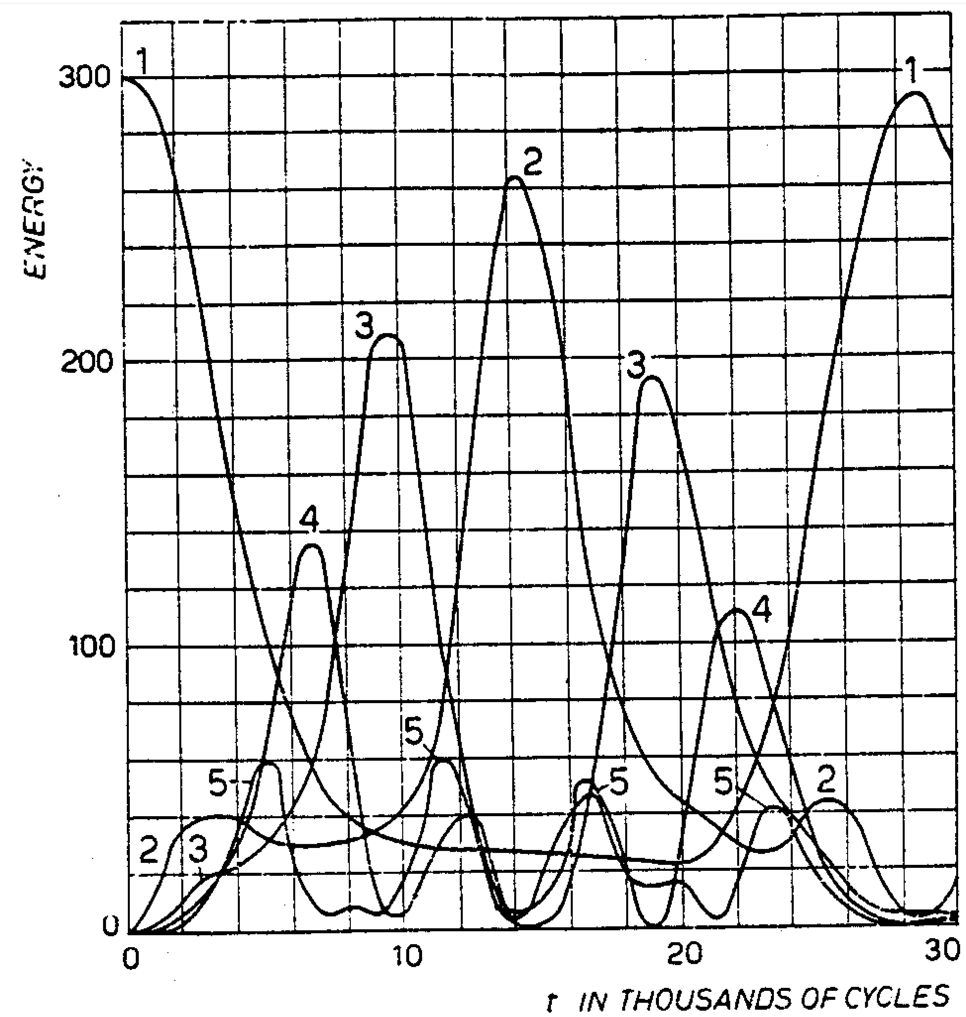
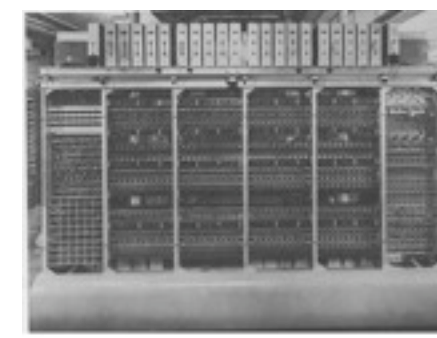


Fig. 1. – The quantity plotted is the energy (kinetic plus potential in each of the first five modes). The units for energy are arbitrary. $N = 32$; $\alpha = 1/4$; $\delta t^2 = 1/8$. The initial form of the string was a single sine wave. The higher modes never exceeded in energy 20 of our units. About 30,000 computation cycles were calculated.

- A nonintegrabl … um



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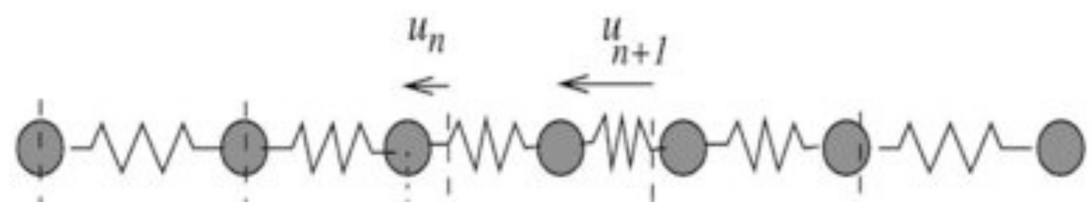

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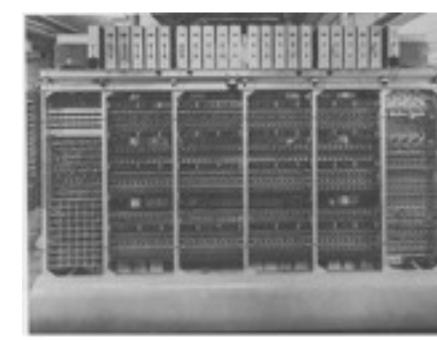

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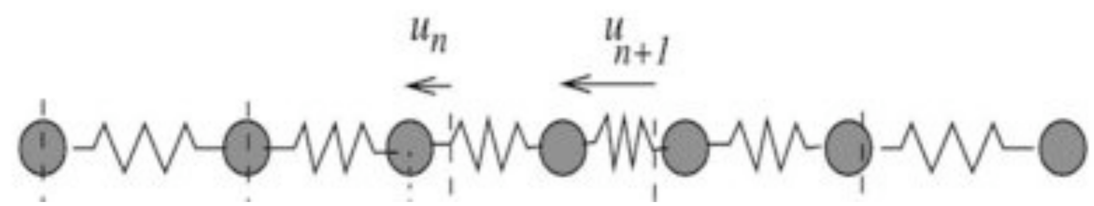

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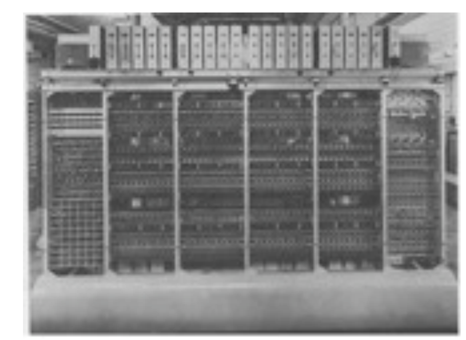

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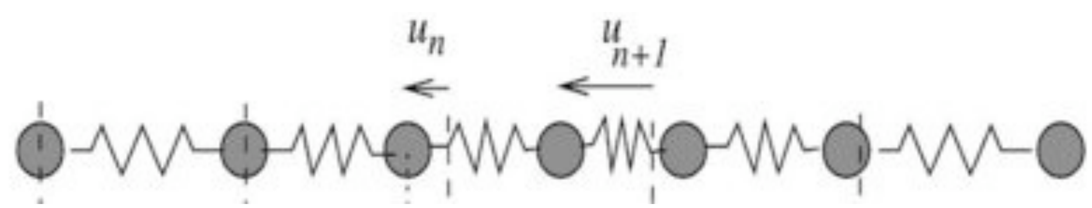

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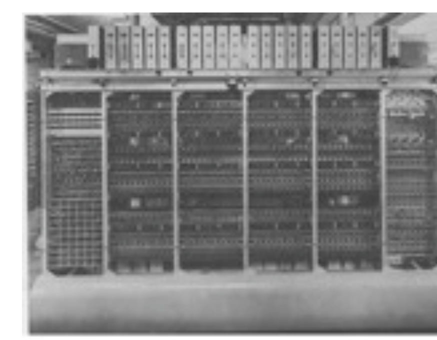

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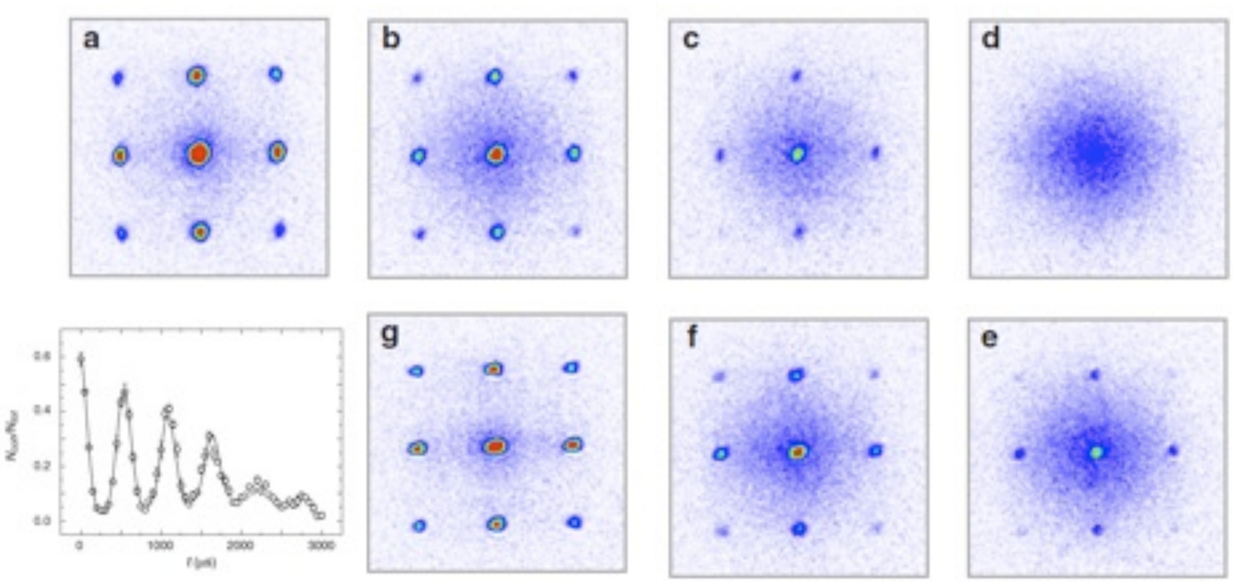
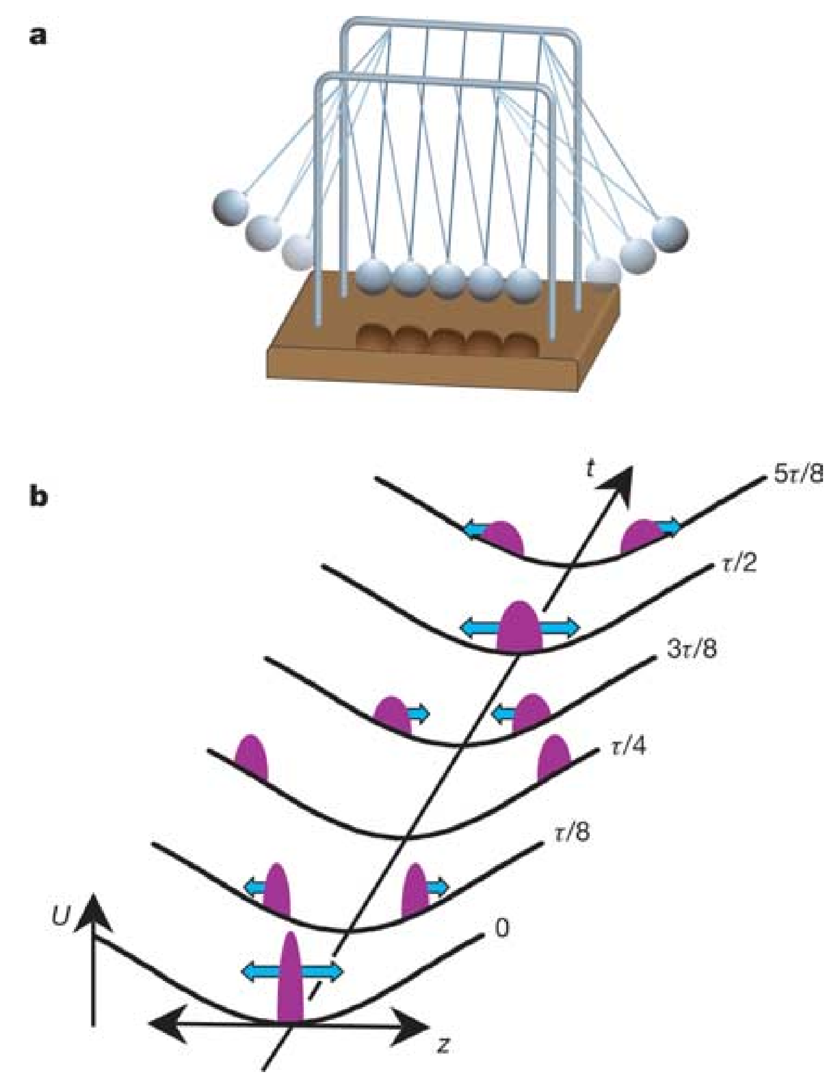
- Seminal work leading KAM Theory etc, ...
- Quantum world ?

Quantum World: Two experimental examples

- Collapse and Revival of a superfluid
- Quantum Newton's craddle



Greiner et al, Nature 2002



Kinoshita et al, Nature 2006

Outline

- Experimental Motivation: Ultracold bosons in an optical lattice
- Short time behavior
  - Light-cone effect:
    spreading of correlations
    entanglement entropy
- Long time behavior
  - Properties of steady state ?
  - Is the steady state already “thermal” ?

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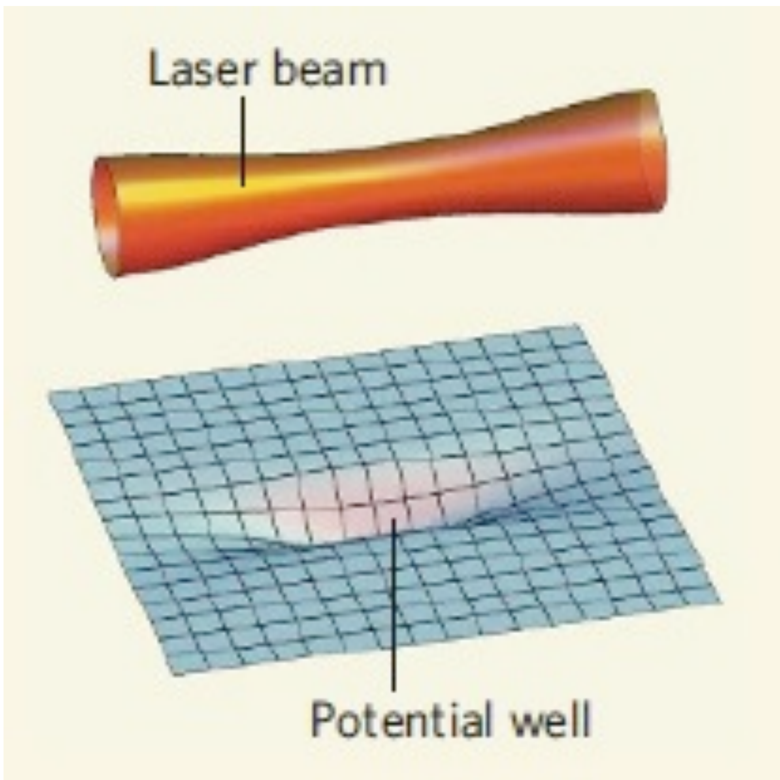
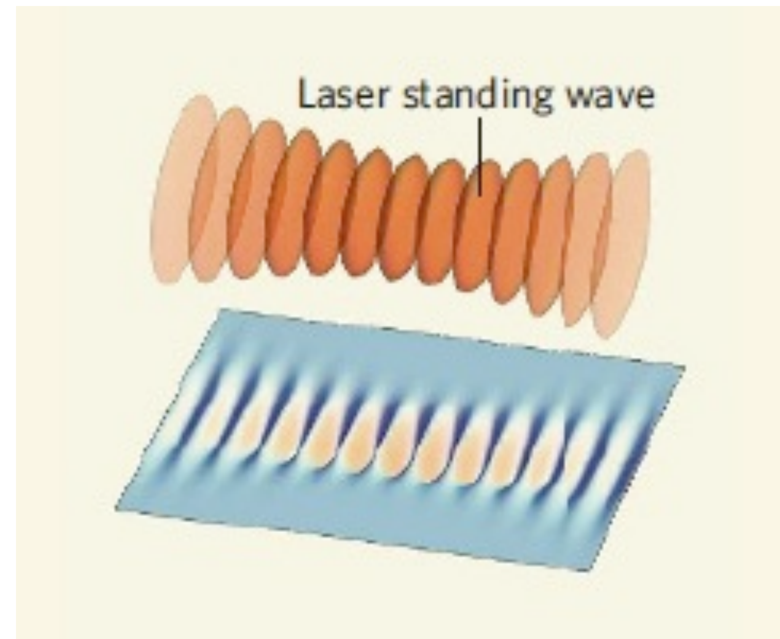
Optical lattices and Hamiltonian

Greiner and Fölling, Nature 2008





- Bosonic atoms: tunneling between wells and interaction within the same well:

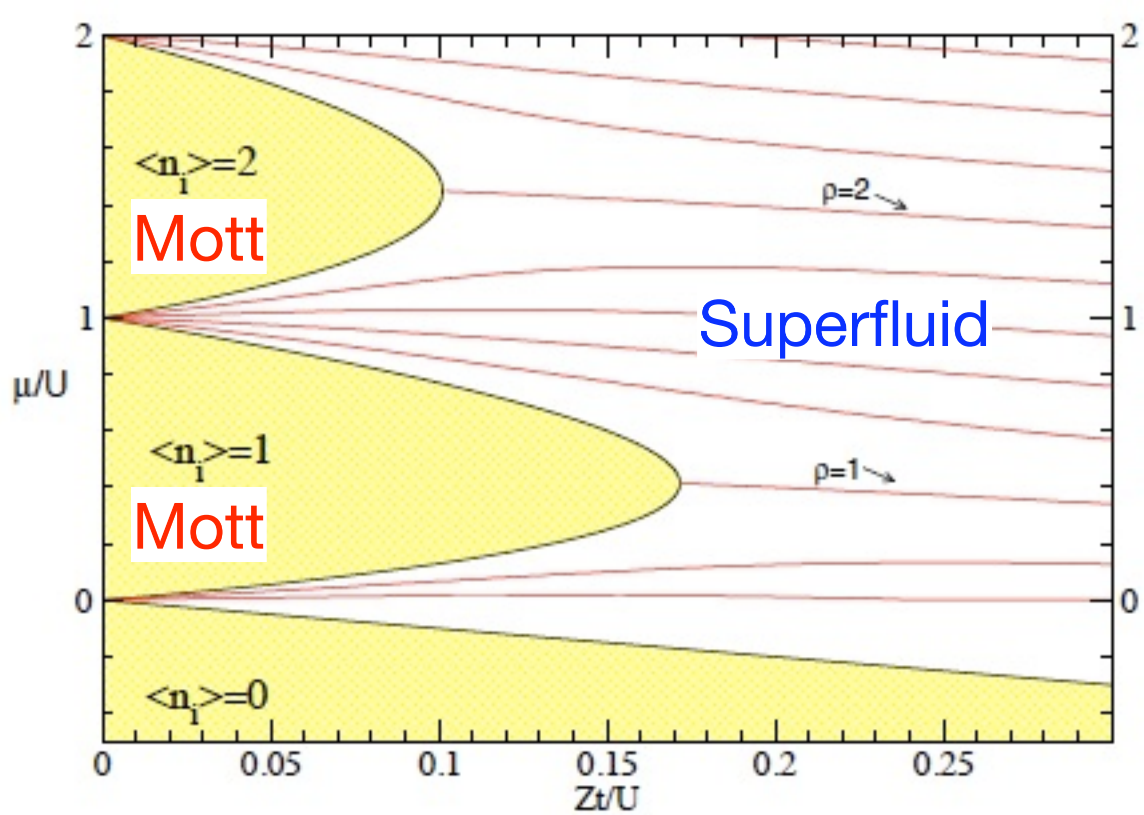
$$H = -J \sum_{\langle i,j \rangle} (b_i^\dagger b_j + h.c.) + \frac{U}{2} \sum_i n_i (n_i - 1)$$

M.P.A. Fisher et al, PRB ‘89
D. Jacksch et al. PRL ‘98

Bose-Hubbard model

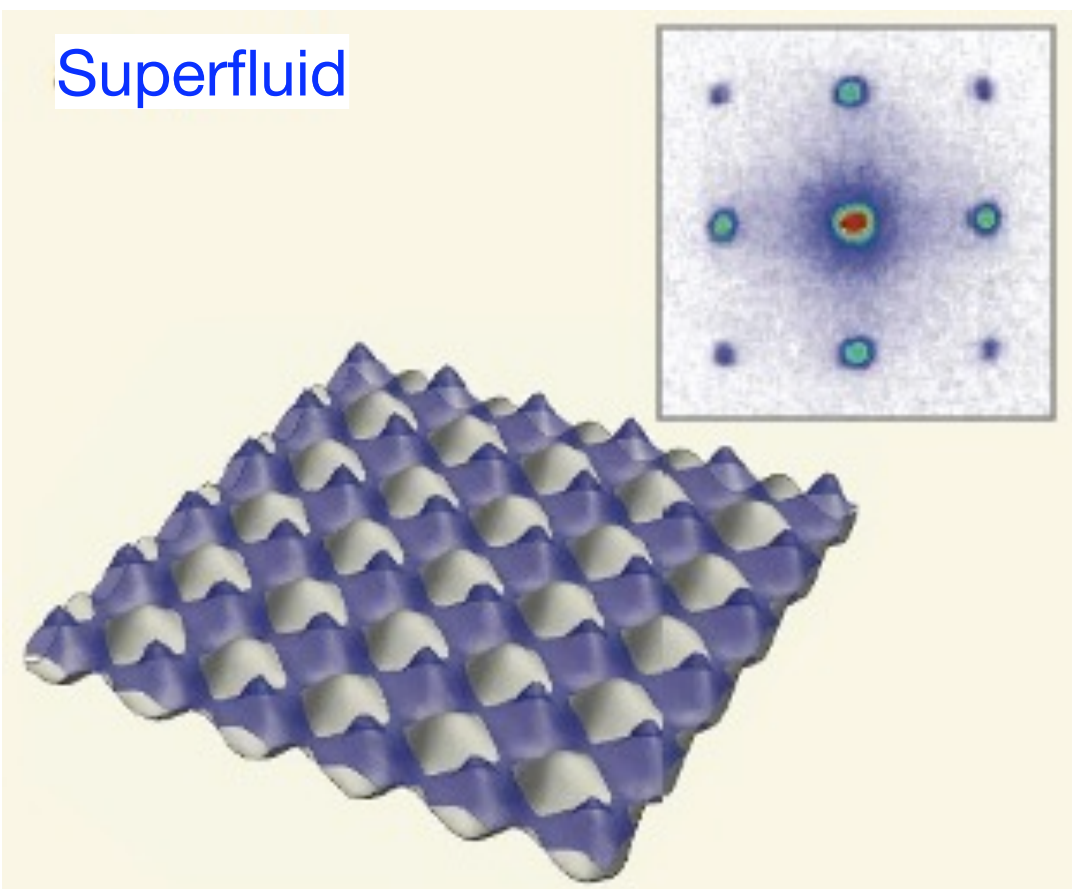
$$H = -J \sum_{\langle i,j \rangle} (b_i^\dagger b_j + h.c.) + \frac{U}{2} \sum_i n_i(n_i - 1)$$

- Transition from Superfluid at large J/U
  to Mott Insulator at small J/U (Integer filling)

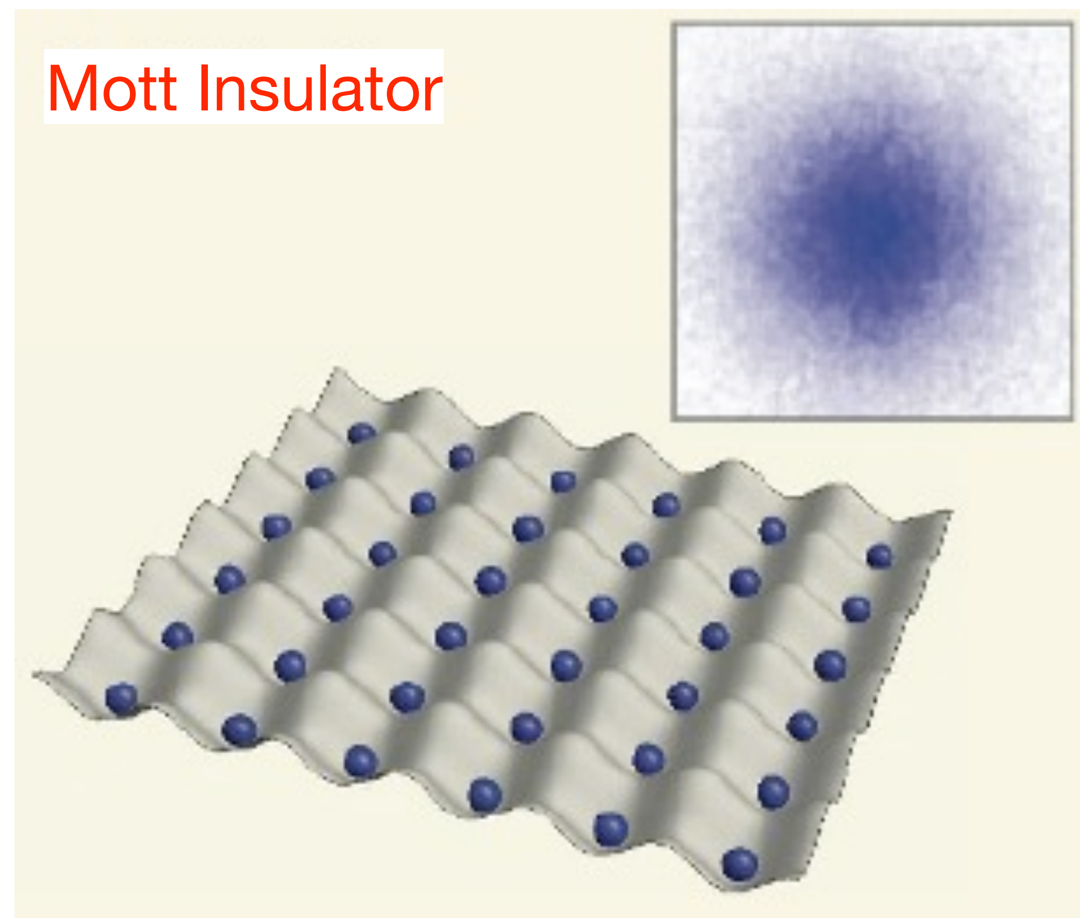


Superfluid versus Mott-Insulator

- Quantum states of bosons and their fingerprint in “time-of-flight” images



$$\langle b_i^\dagger b_j \rangle \longrightarrow \langle b_i^\dagger \rangle \langle b_j \rangle$$

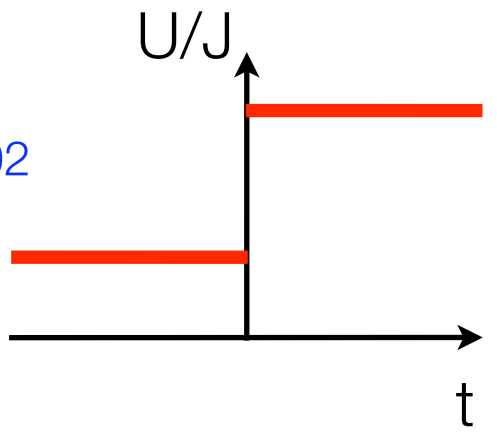


$$\langle b_i^\dagger b_j \rangle \longrightarrow 0$$

Greiner and Fölling, Nature 2008

Quench from the Superfluid to Mott Insulator

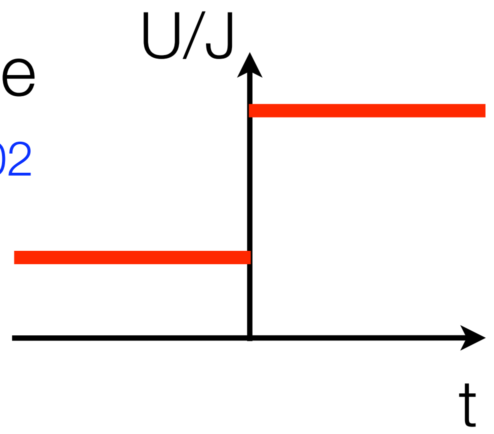
Greiner et al, Nature 2002



Quench from the Superfluid to Mott Insulator

- Sudden increase of the interaction strength from small to large value

Greiner et al, Nature 2002



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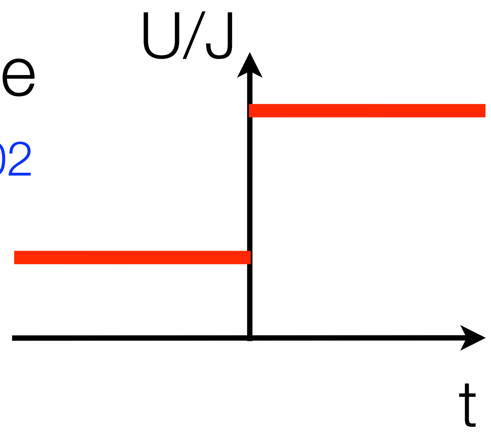
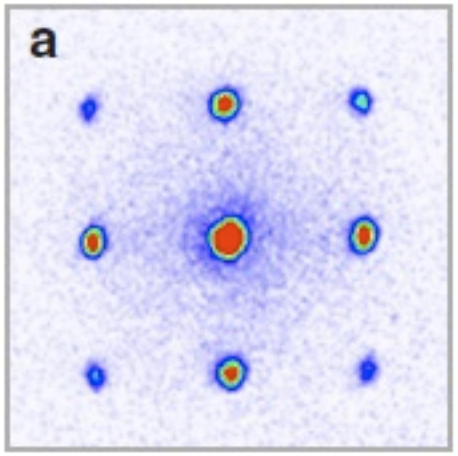


Greiner et al, Nature 2002

- Time dependence of time-of-flight images:



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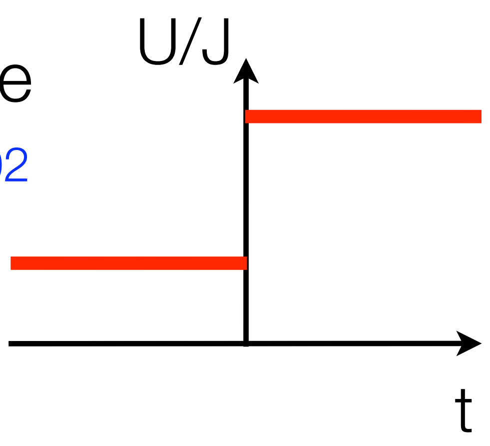

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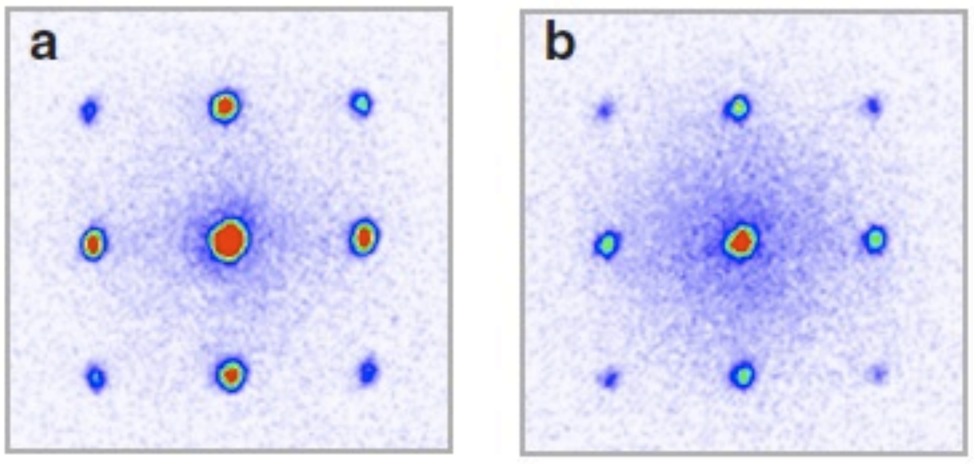

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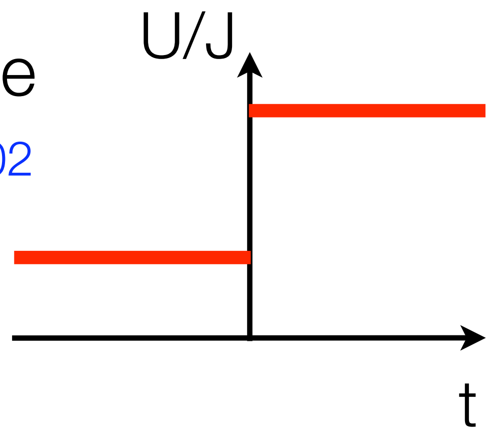

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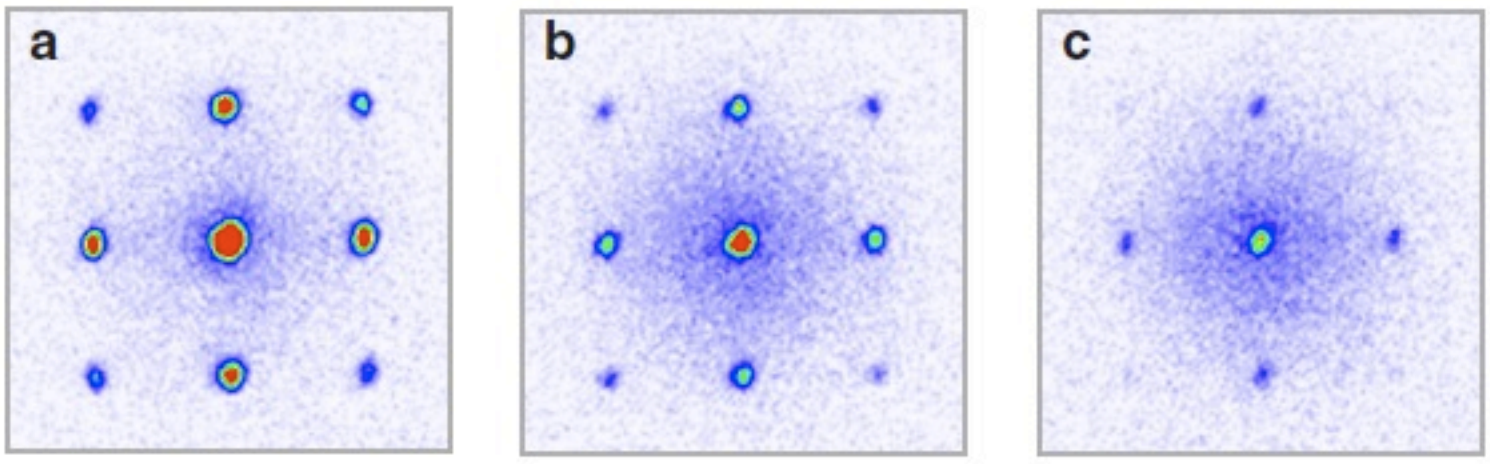

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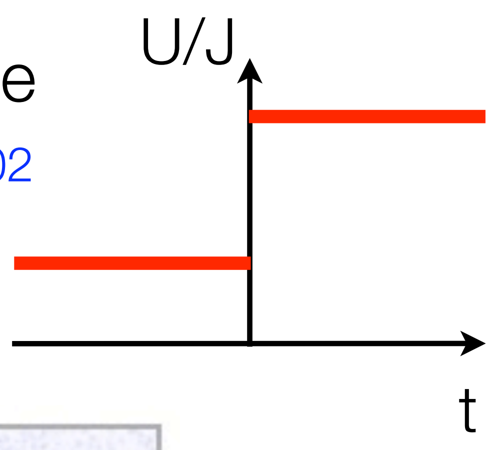

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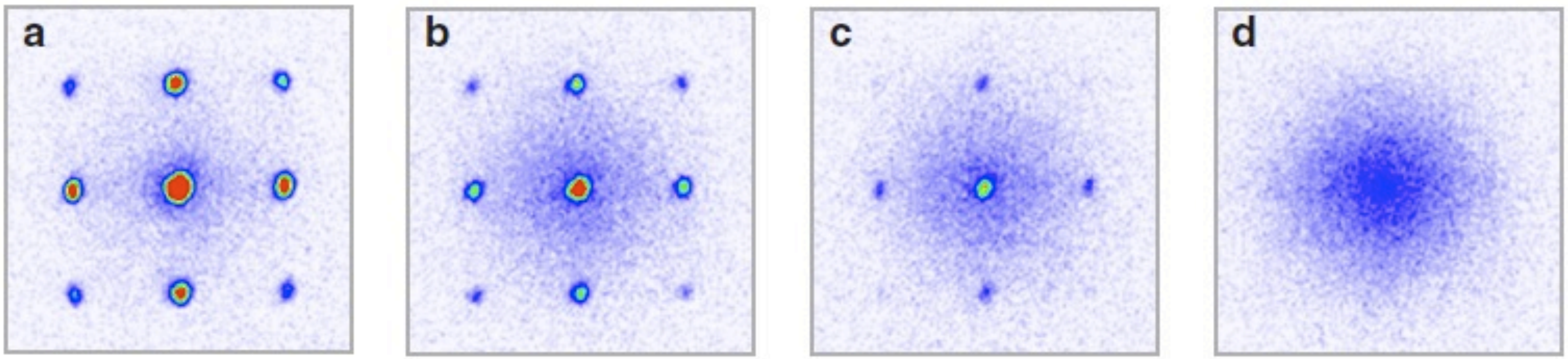

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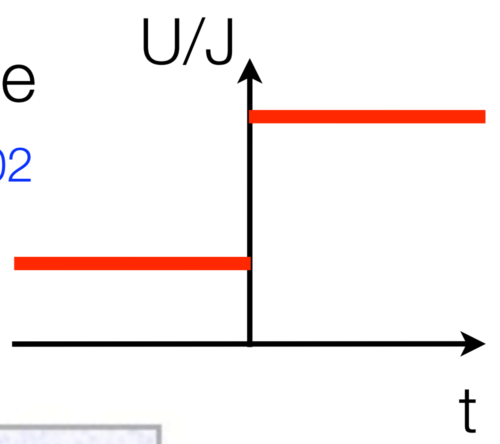

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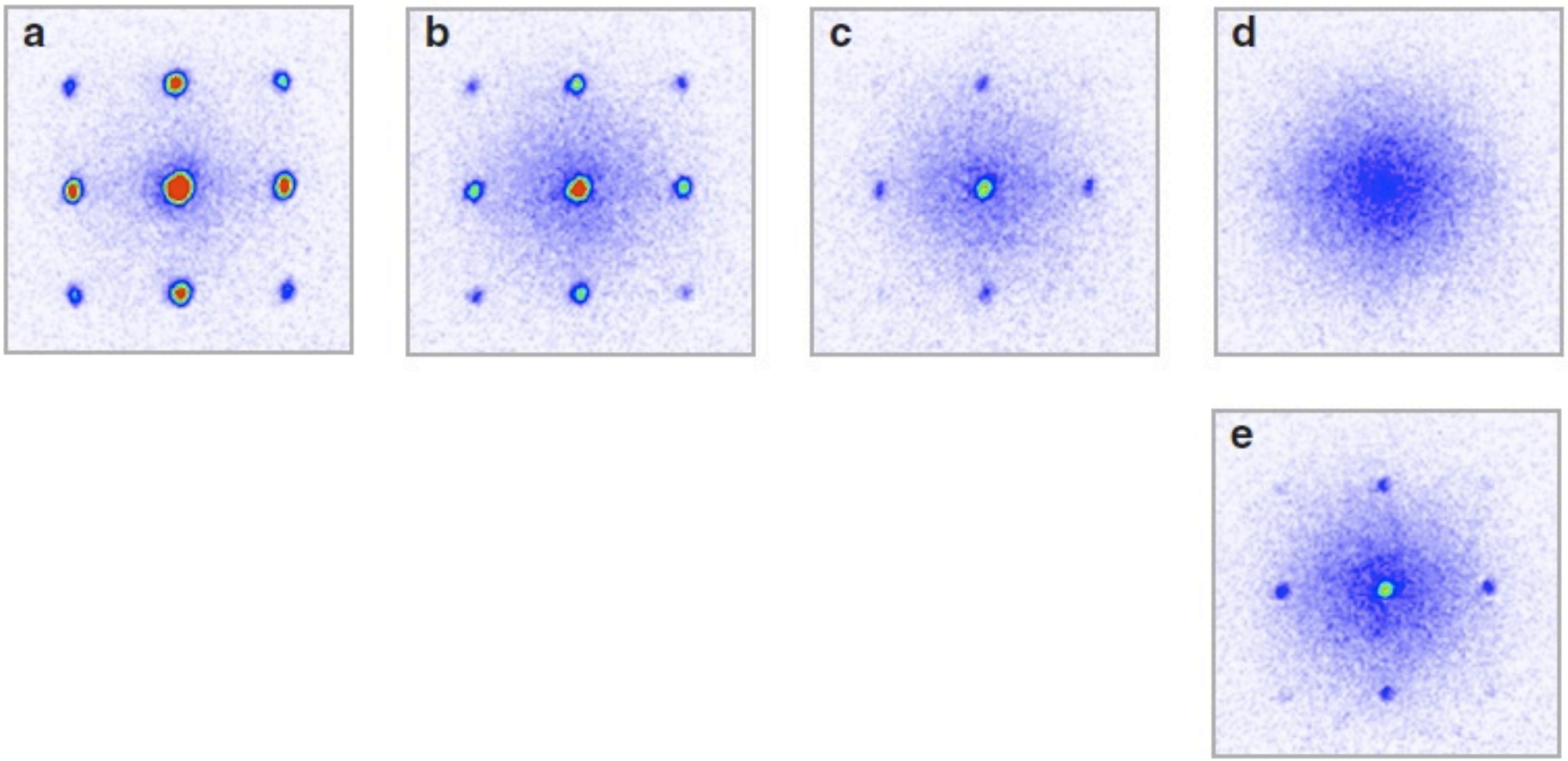

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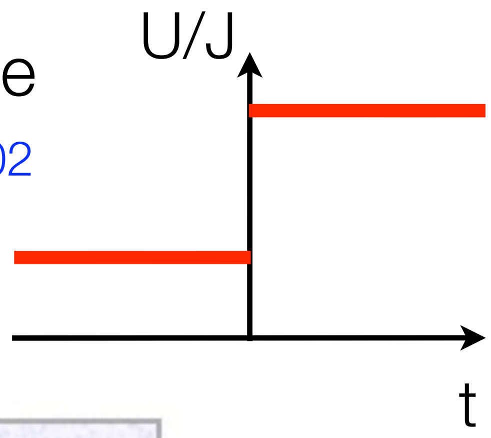

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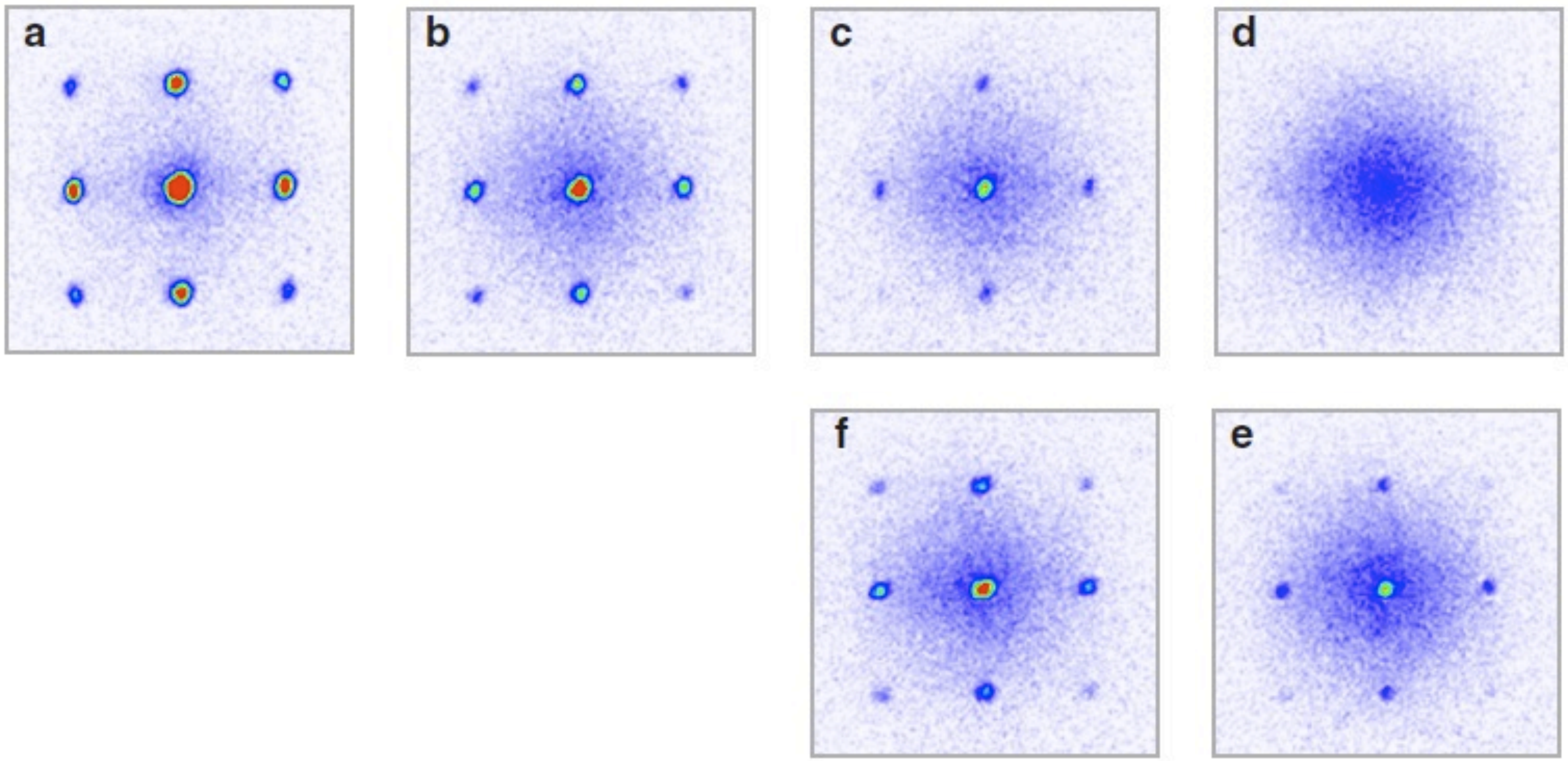

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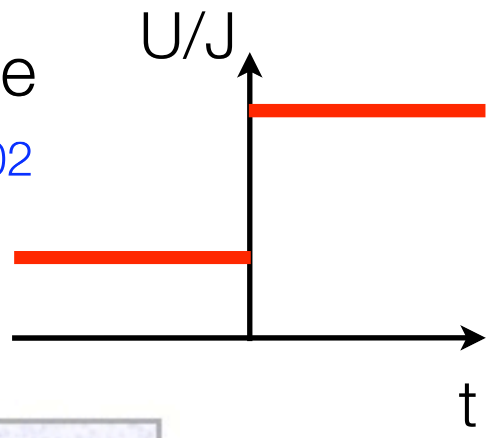

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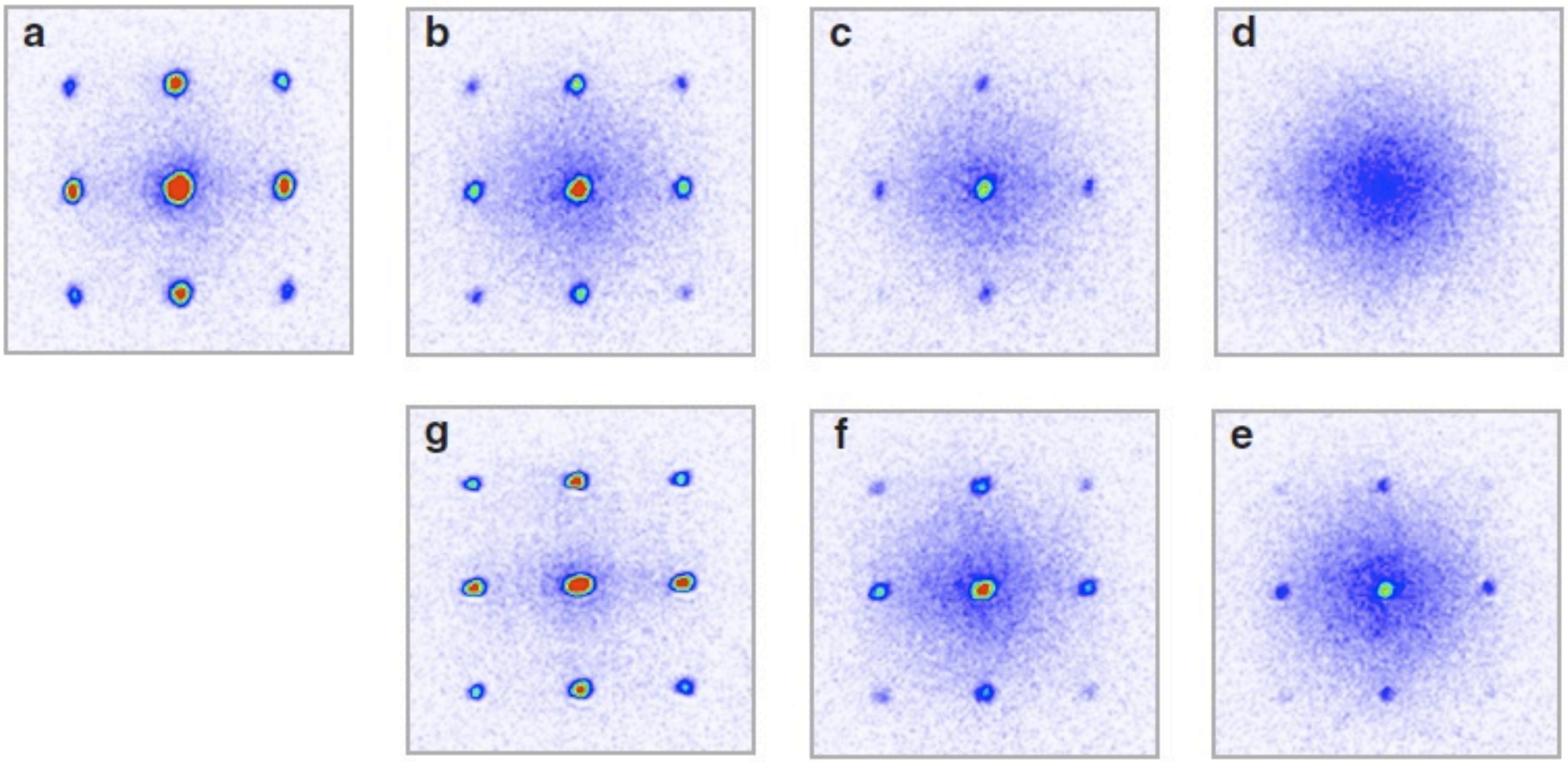

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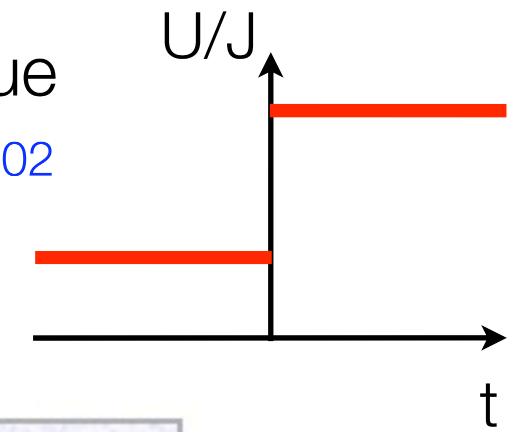

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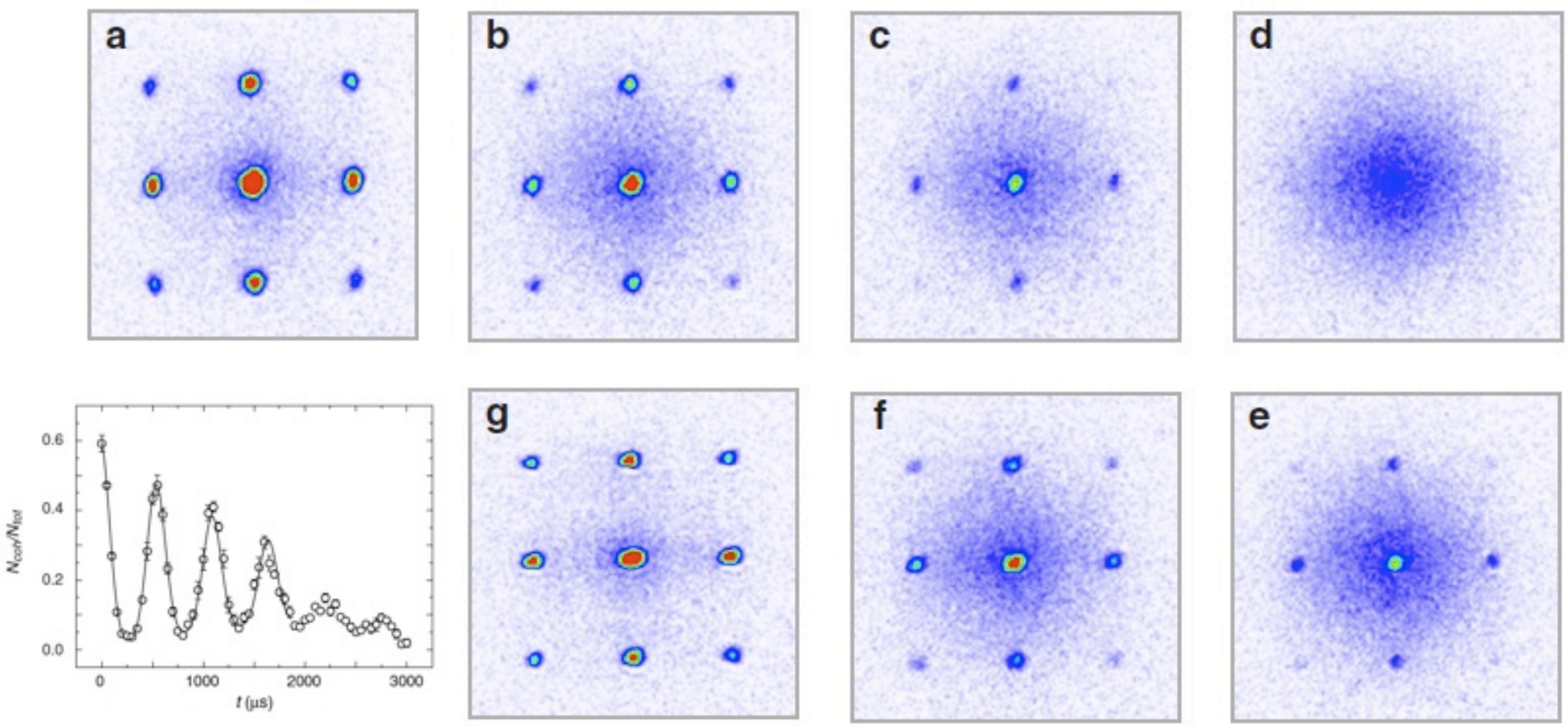

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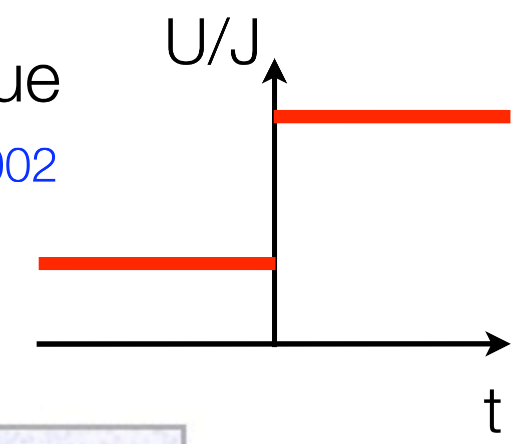

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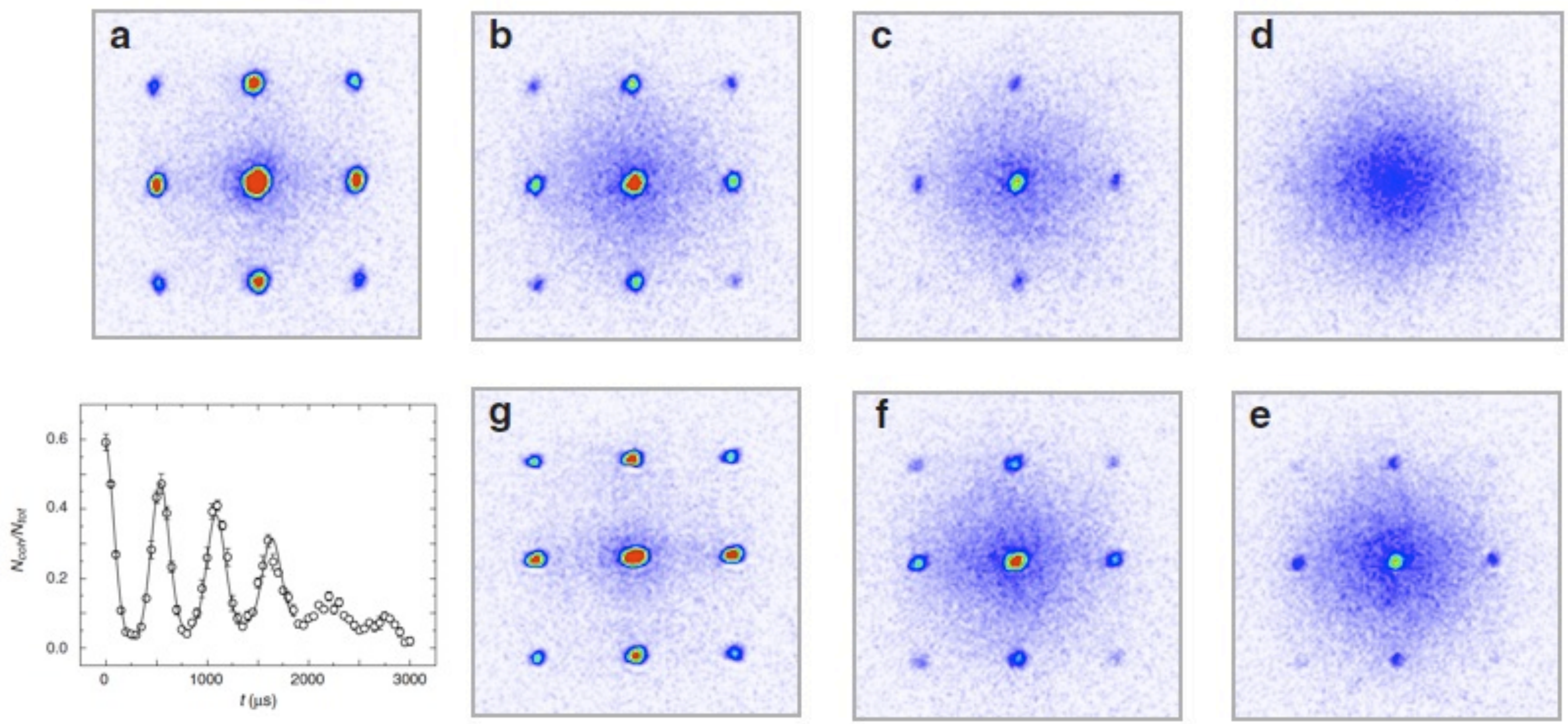

- Collapse and Revival ! Suppressed at longer times.

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- Time evolution is performed numerically using Exact Diagonalization and t-DMRG

Exact Diagonalization
Real-Time Dynamics

- It is expensive to obtain the full propagator $\exp[-itH]$

- Krylov methods exist to approximate the propagator for a given state $|\psi(0)\rangle$
  One can get the time propagated state $|\psi(t)\rangle$ with only $|v\rangle = H|v\rangle$operations.

$$
\begin{aligned}
|\phi'\rangle &= H|\phi_n\rangle - \beta_n|\phi_{n-1}\rangle\,, \\
\alpha_n &= \langle\phi_n|\phi'\rangle\,, \\
|\phi''\rangle &= |\phi'\rangle - \alpha_n|\phi_n\rangle\,, \\
\beta_{n+1} &= ||\phi''|| = \sqrt{\langle\phi''|\phi''\rangle}\,, \\
|\phi_{n+1}\rangle &= |\phi''\rangle/\beta_{n+1}\,,
\end{aligned}
\qquad
\tilde{H}_N = \begin{bmatrix}
\alpha_0 & \beta_1 & 0 & \cdots\cdots & & 0 \\
\beta_1 & \alpha_1 & \beta_2 & 0 & \cdots & 0 \\
0 & \beta_2 & \alpha_2 & \beta_3 & 0 & 0 \\
 & & \ddots & \ddots & \ddots & \\
0 & \cdot\cdot & 0 & \beta_{N-2} & \alpha_{N-2} & \beta_{N-1} \\
0 & \cdots\cdots & & 0 & \beta_{N-1} & \alpha_{N-1}
\end{bmatrix}
$$

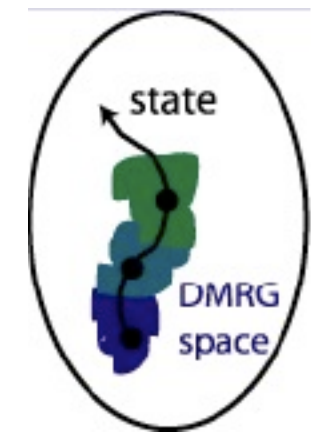
- Calculate matrix exponential of H_N, instead of the full Hamiltonian

Park & Light, J. Chem. Phys 1986

- Time evolution of quantum systems with up to 10^8 degrees of freedom (dim H)

Numerical Methods: tDMRG

- t-DMRG (large systems L=100++, but relatively short times)
  - Adaptive control of the optimal Hilbert space as time evolves.



G. Vidal PRL '03
A. Daley et al., JSTAT '04
S. White & A. Feiguin, PRL '04

  - Maximum time depends on entanglement growth.

  - More refined approaches can reach larger times (however still require exponential resources):
    - Light-cone MPS
    - Heisenberg picture

M. Hastings, J. Math. Phys. 2009
M.C. Bañuls, *et al.*, PRL 2009
M.C. Bañuls, *et al.*, arXiv:1007:3957

M. Hartmann *et al.*, PRL 2009

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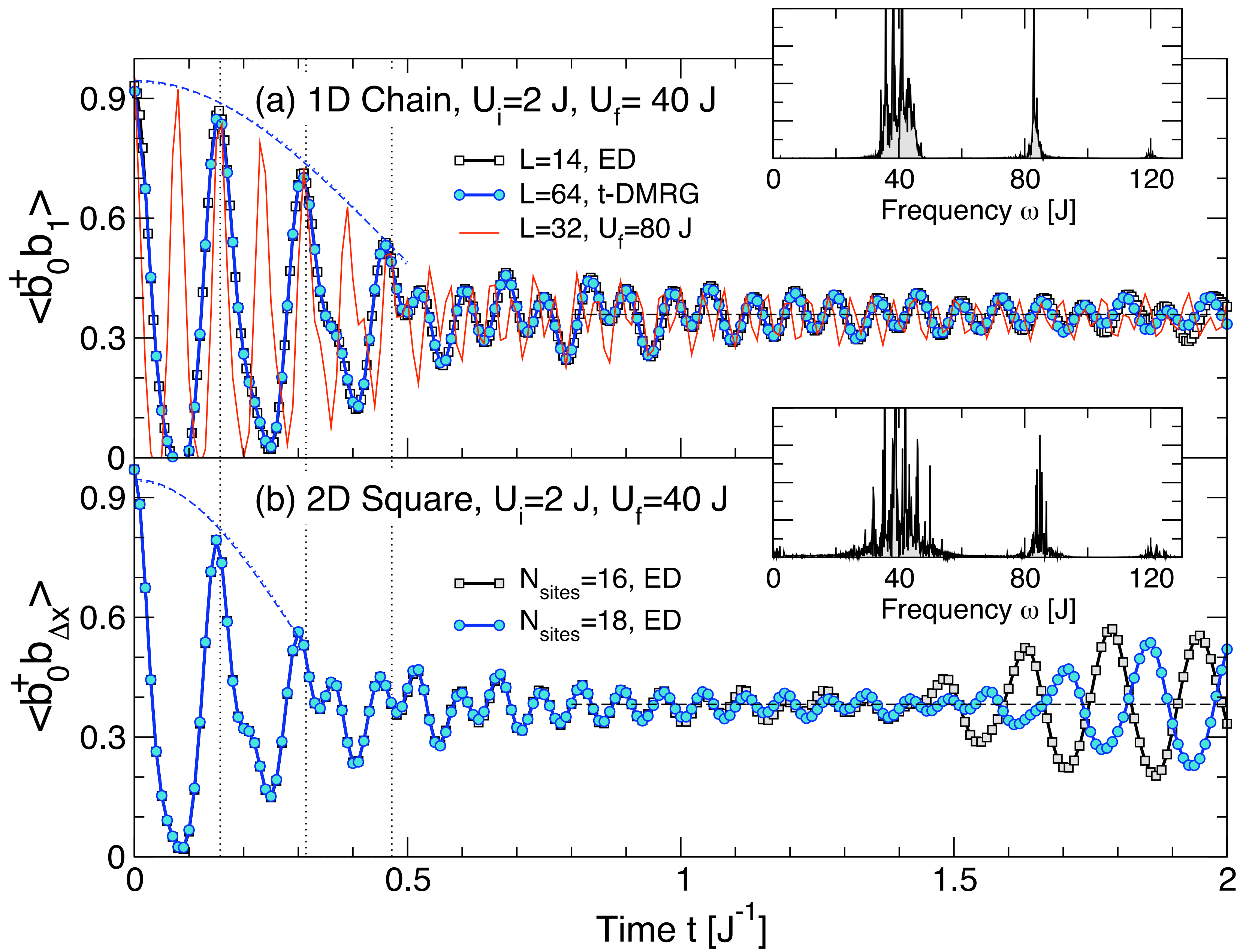
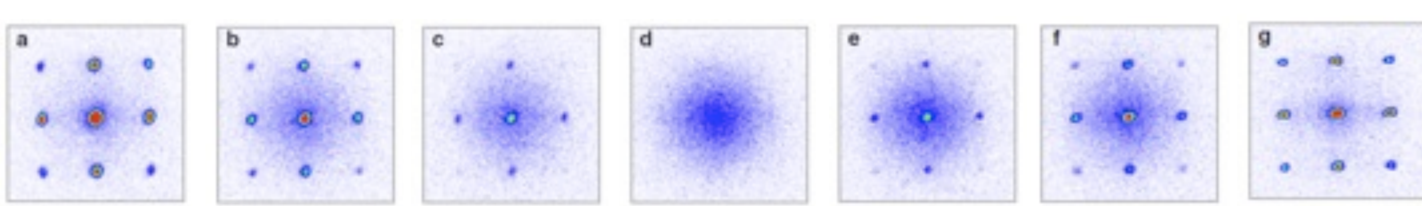
Superfluid correlation

- Pronounced collapse and revival. Relaxation to steady state at later times
- Collapse and revival controlled by U
- Relaxation faster for larger bandwidth (1D/2D)

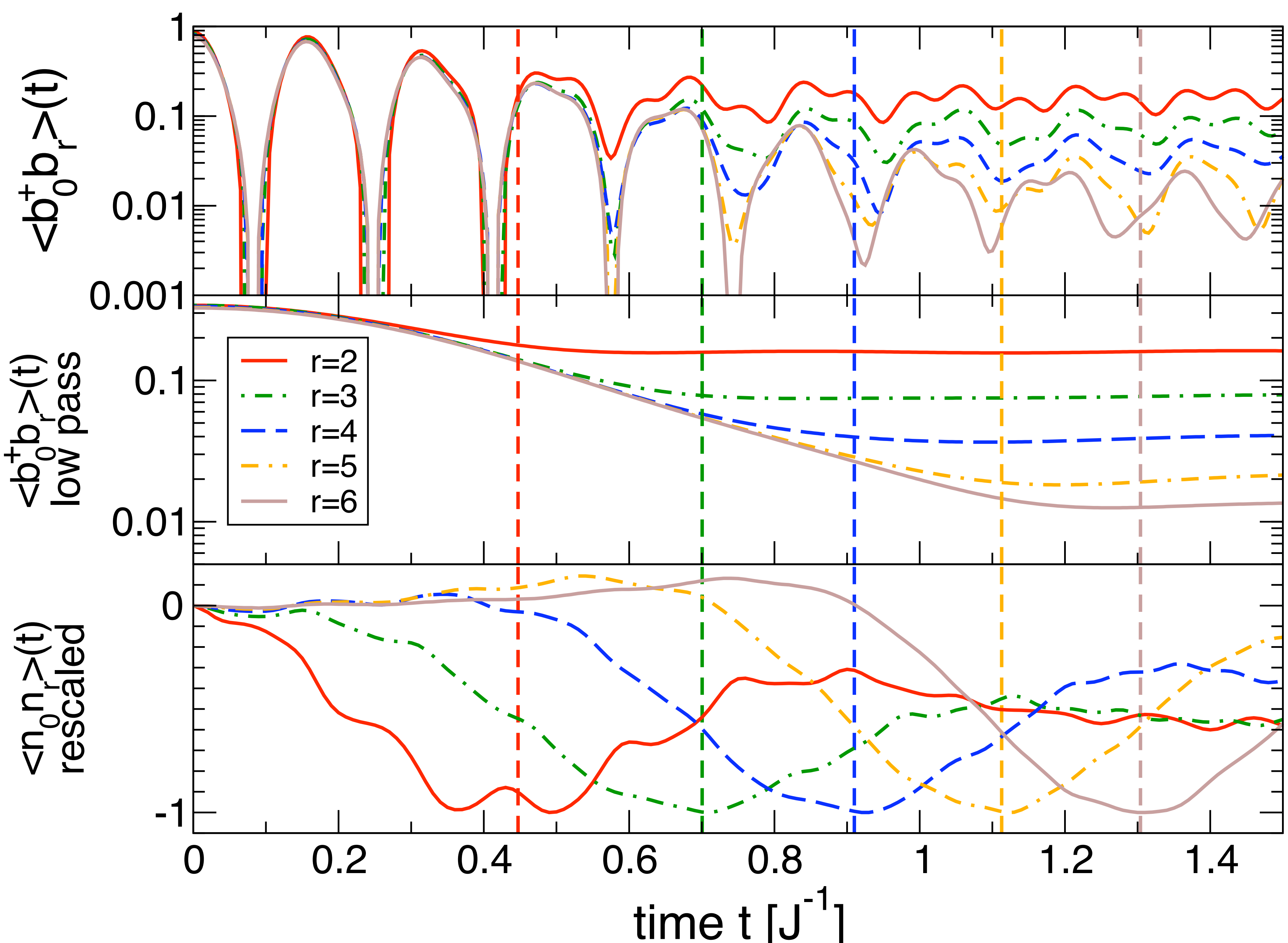




C. Kollath, AML, E. Altman, PRL 2007

Space-time dependence of correlators

- Correlation functions at different spatial separations:



AML and Kollath, JSTAT '08

Space-time dependence of correlators

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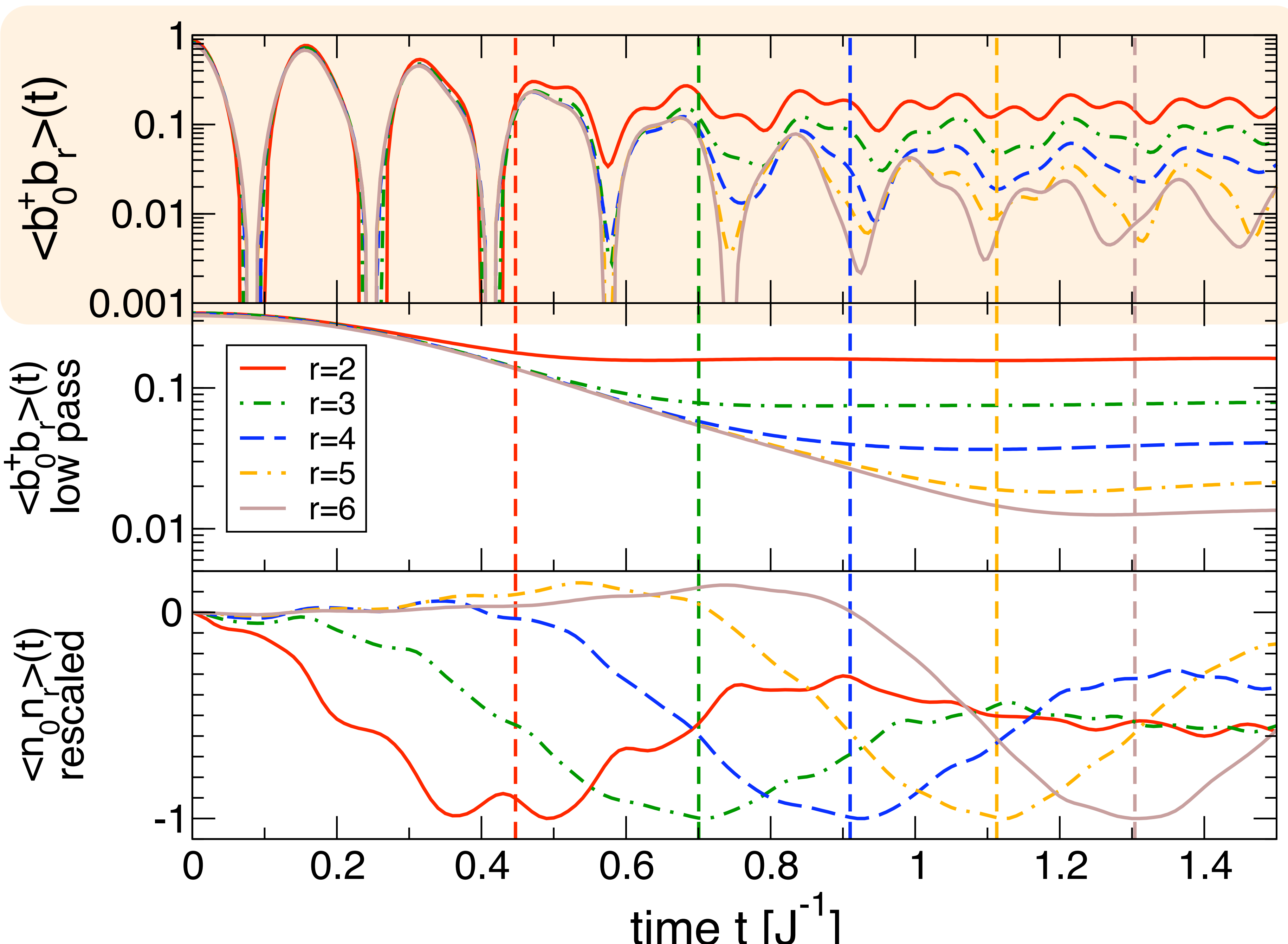

AML and Kollath, JSTAT ‘08

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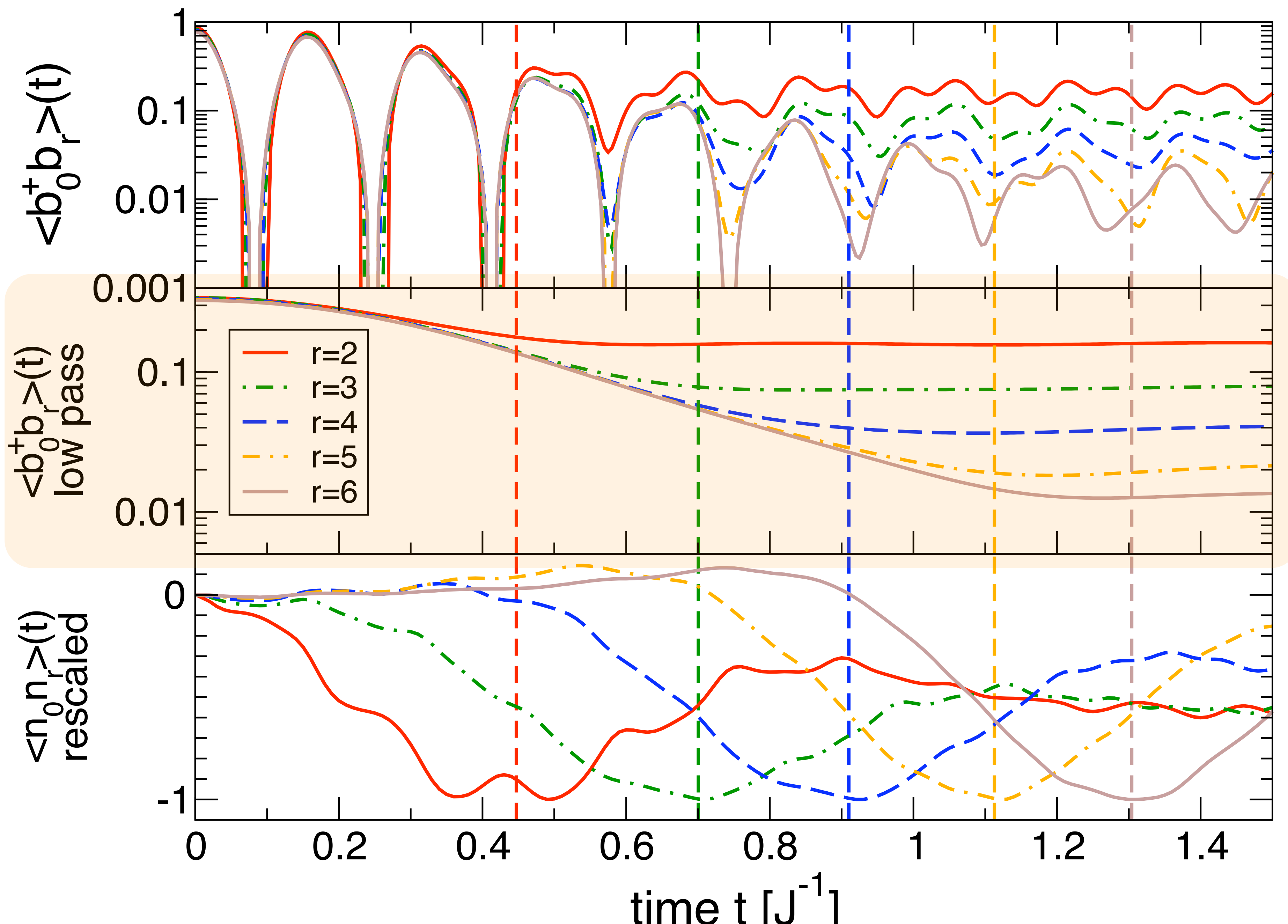

AML and Kollath, JSTAT '08

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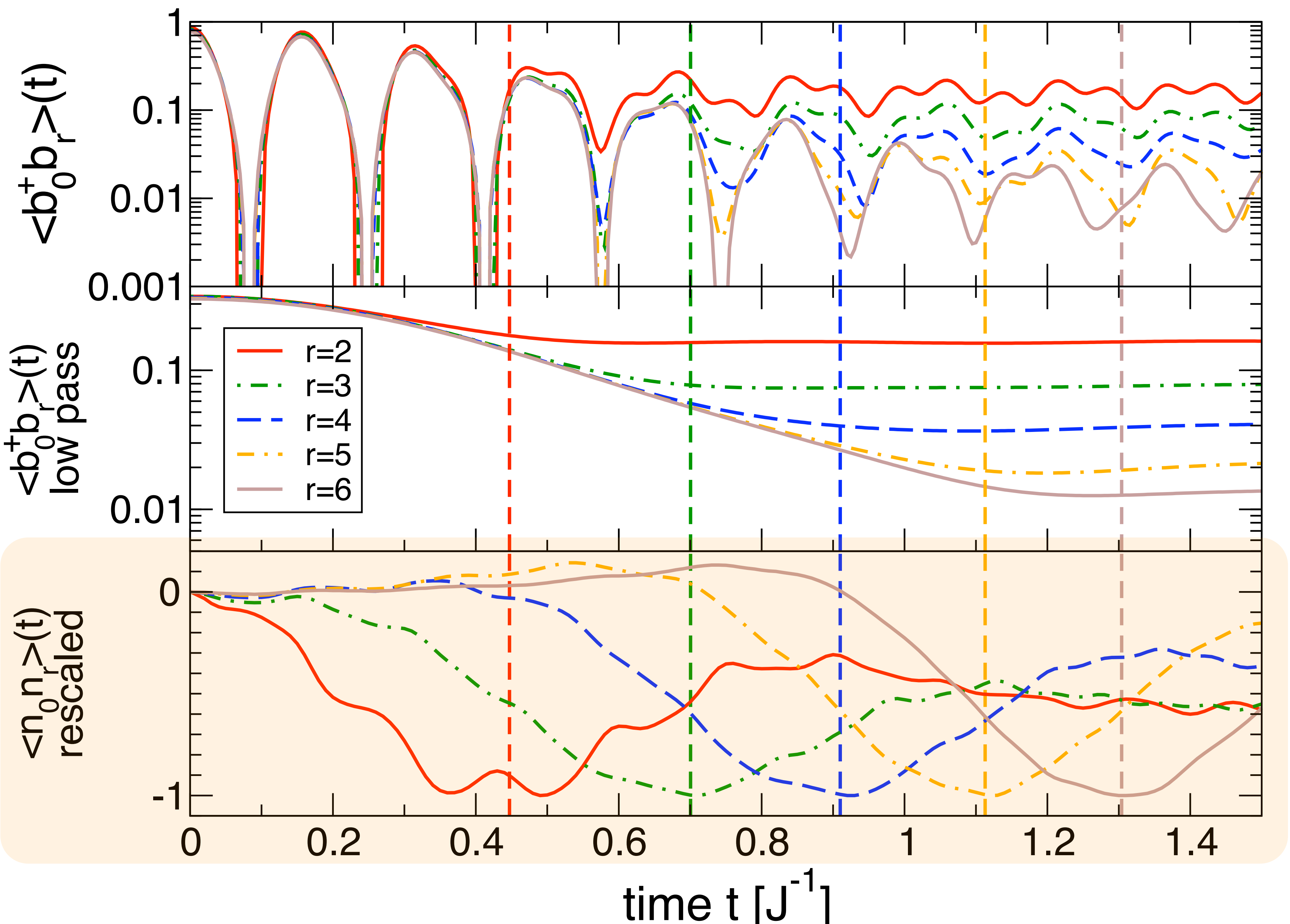
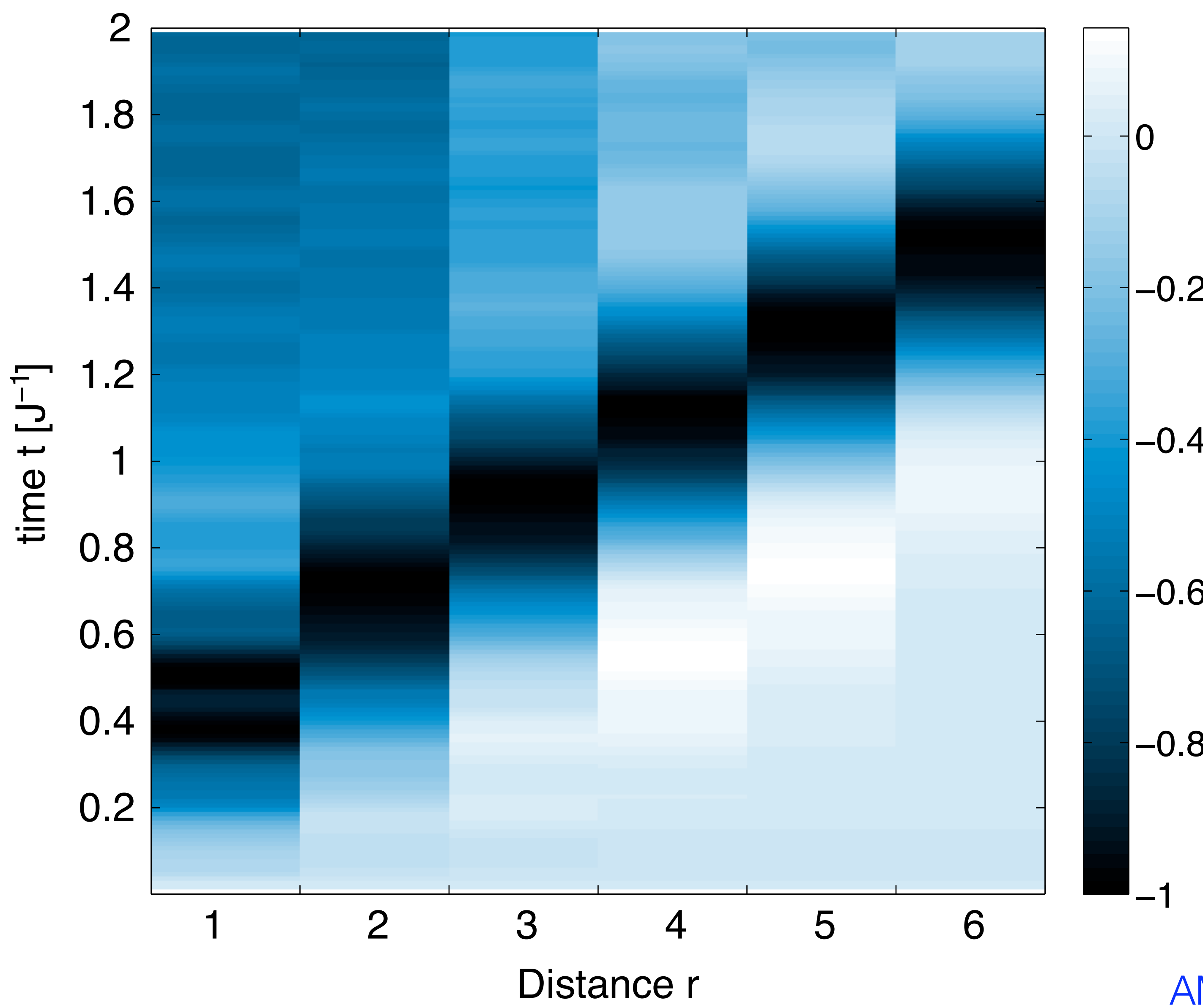


AML and Kollath, JSTAT '08

Lightcone / Horizon effect

Lieb and Robinson CMP ‘72

- More distant sites see correlation signal pass at later times. Linear relation.



AML and Kollath, JSTAT ‘08

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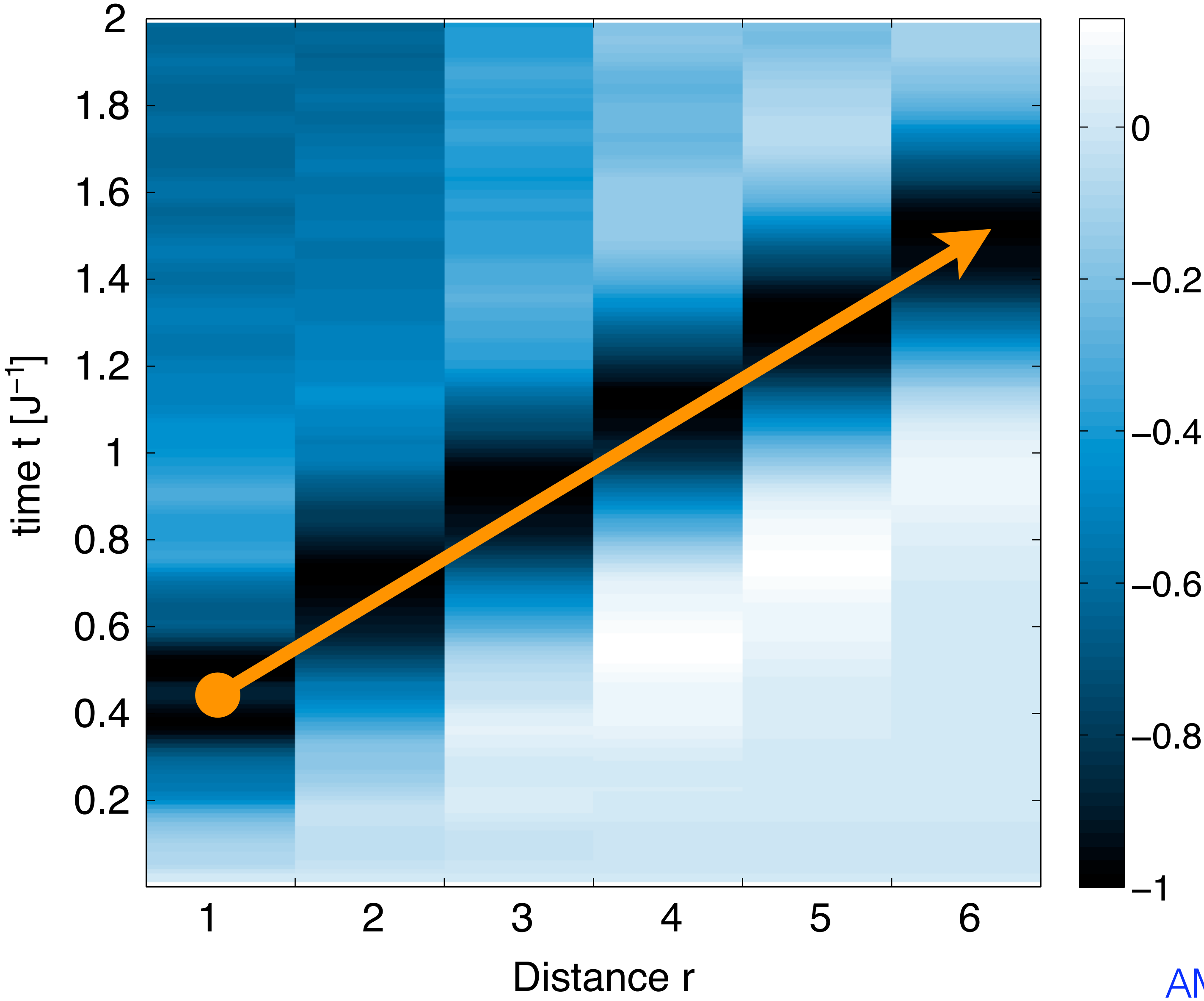
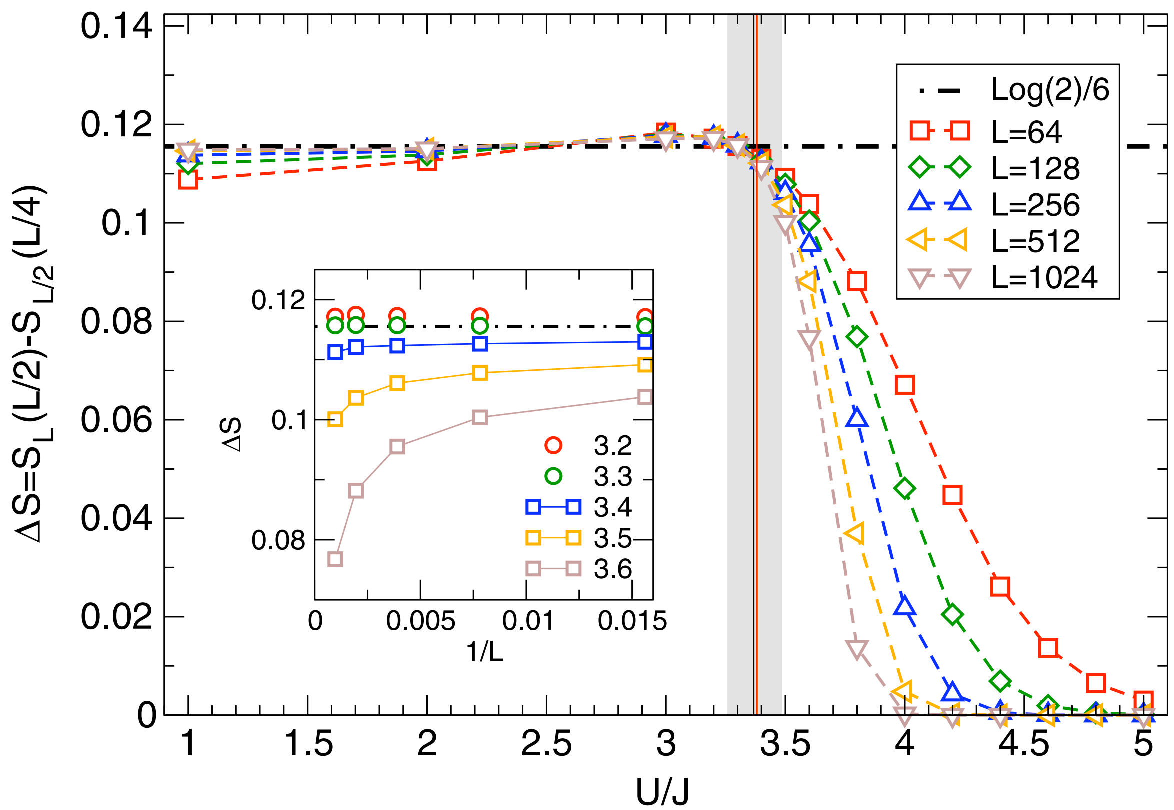


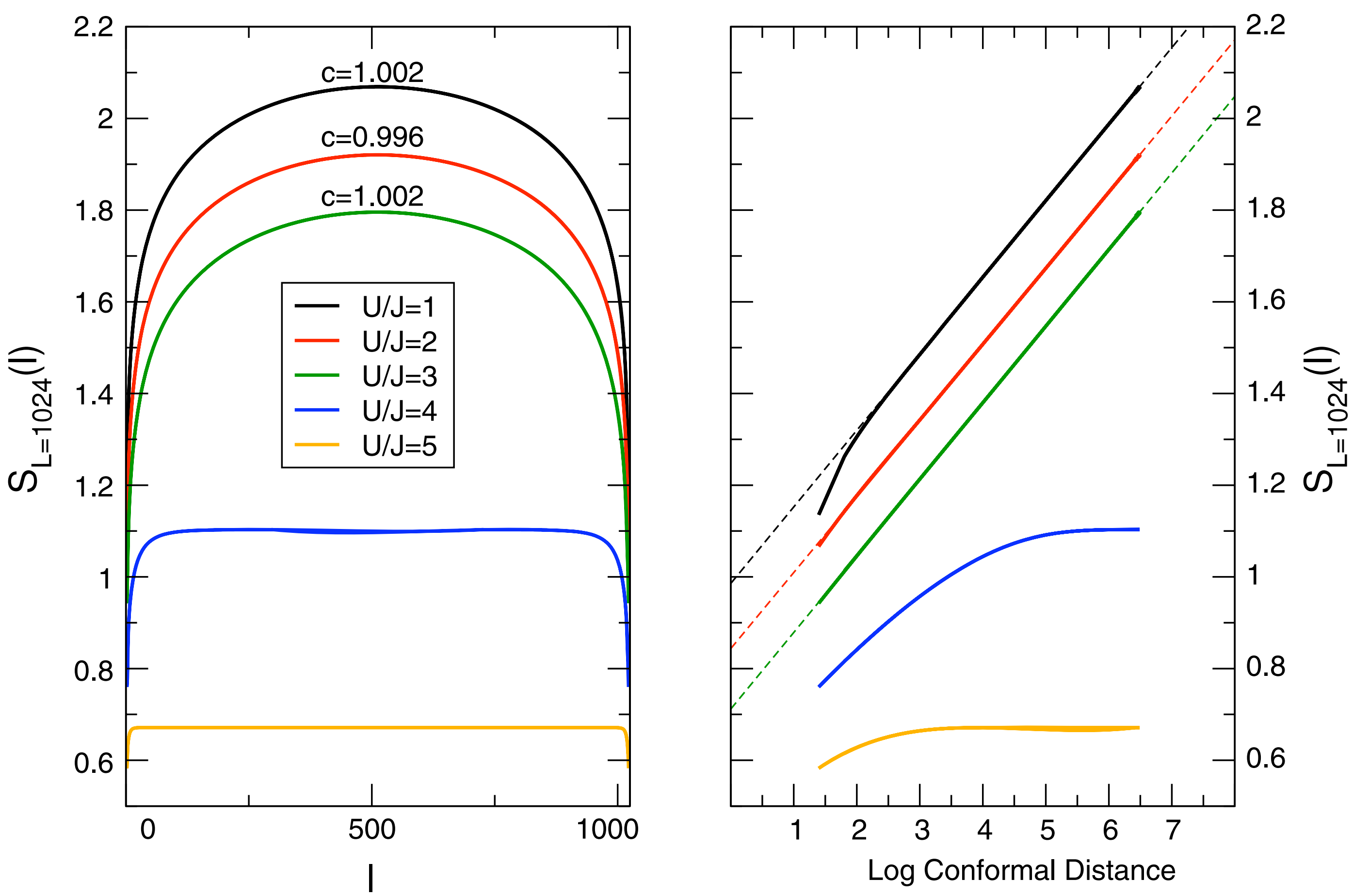
AML and Kollath, JSTAT '08

von Neumann Entanglement Entropy (first static)

- Saturation of vN EE in gapped phase
- Logarithmic divergence in critical phase (single component c=1)

$$S_A = \mathrm{Tr}_A[-\rho_A \log \rho_A]$$



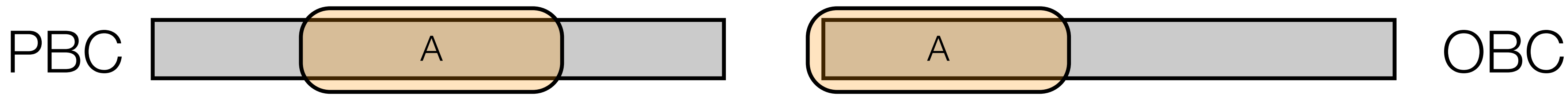


- Entropy increase upon system doubling reveals the phase transition quite accurately.

AML and Kollath, JSTAT '08

Entanglement time evolution

- von Neumann entropy of a block A consisting of l sites $S_A = \mathrm{Tr}_A[-\rho_A \log \rho_A]$



AML and Kollath, JSTAT '08

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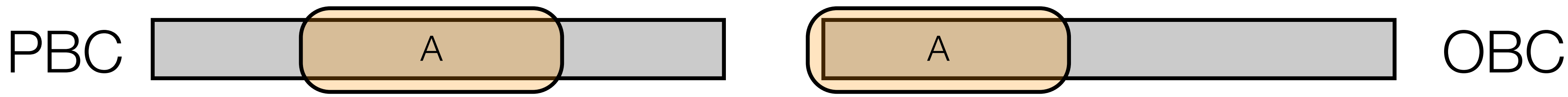

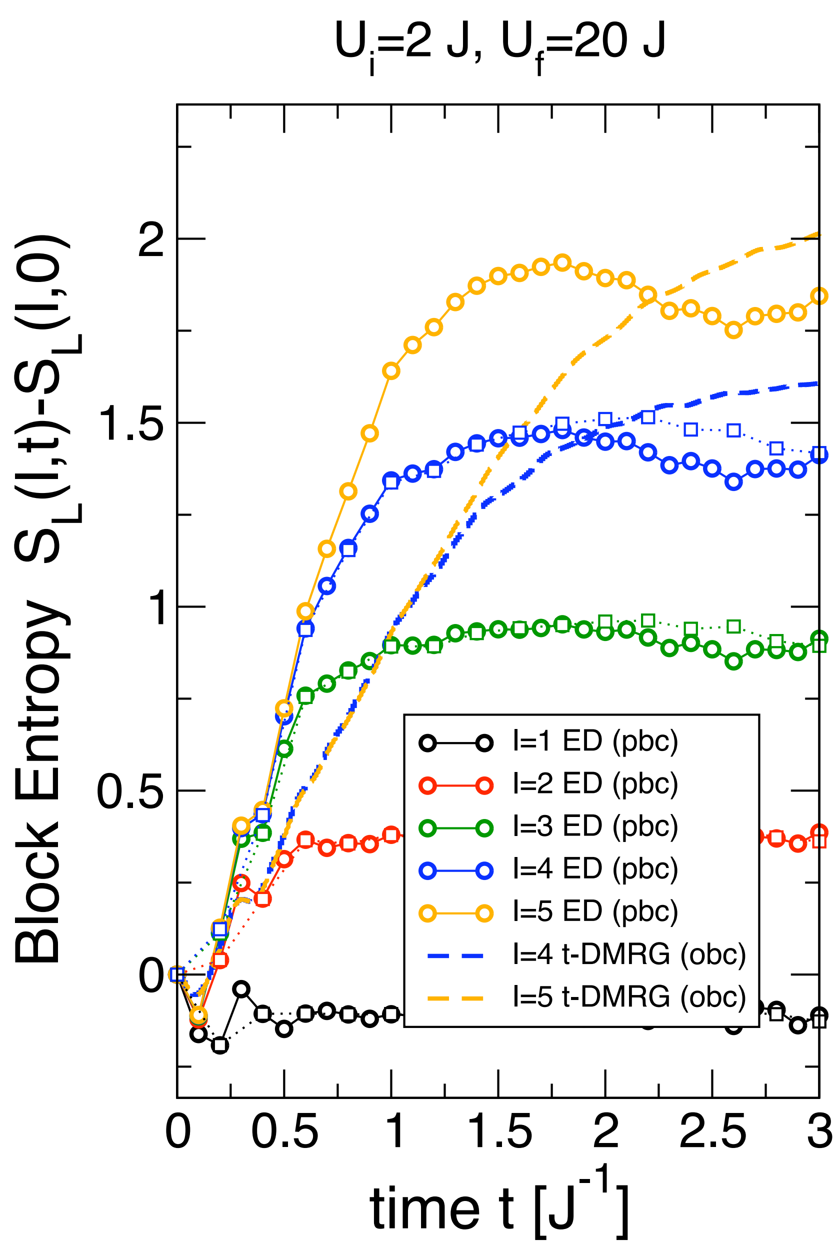

AML and Kollath, JSTAT '08

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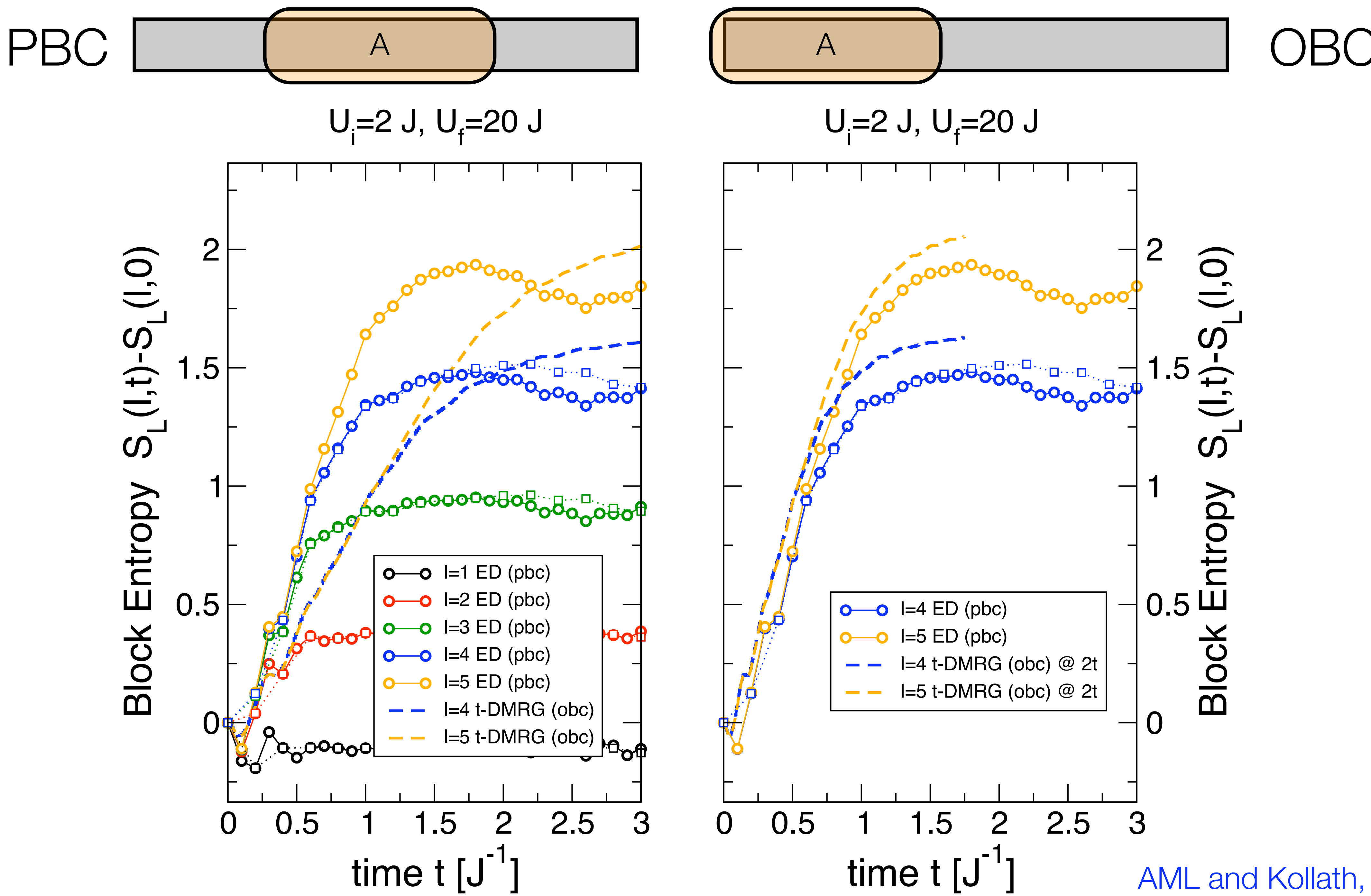
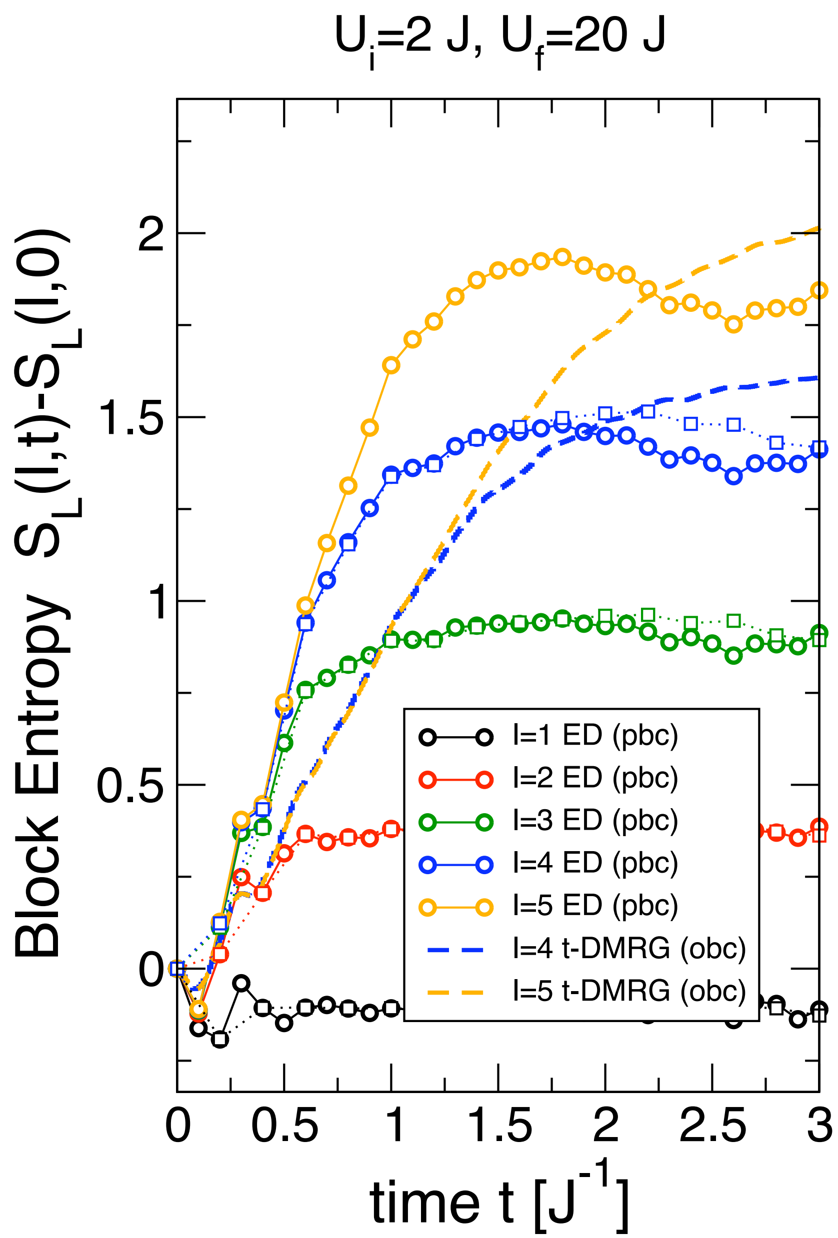


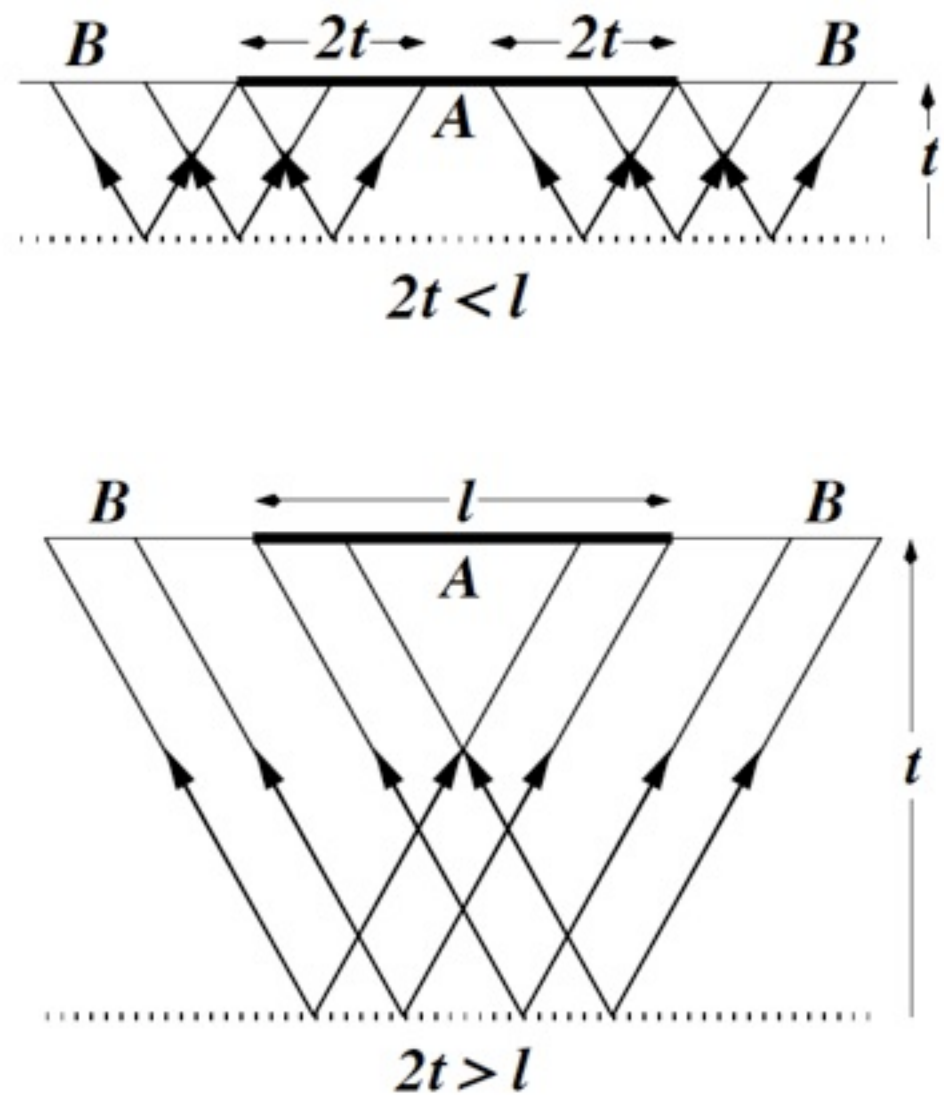
AML and Kollath, JSTAT '08

Entanglement time evolution

- Linear growth in time first, then saturation to value proportional to l



AML and Kollath, JSTAT '08



Calabrese and Cardy, JSTAT '05
G. De Chiara *et al.*, JSTAT '06

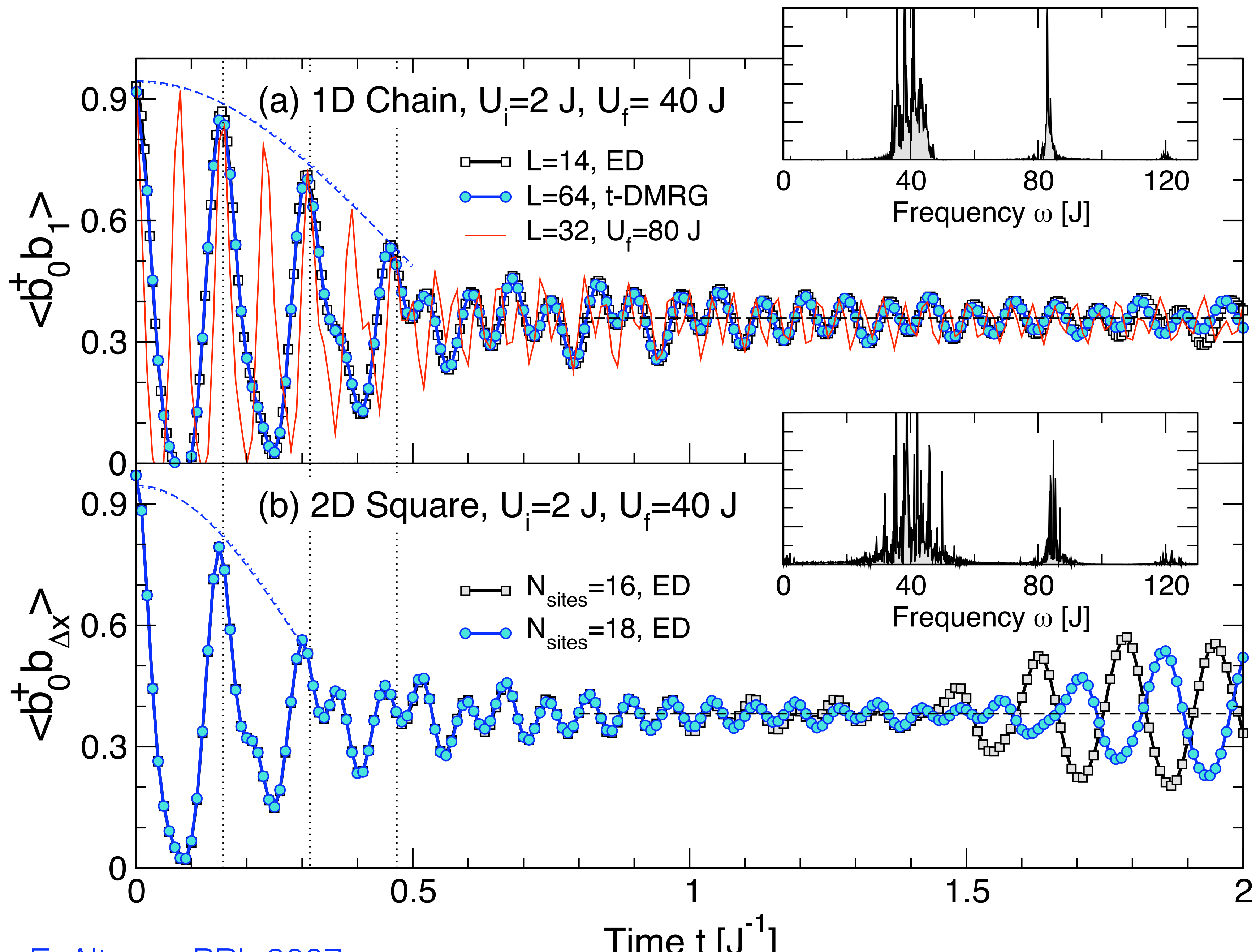
Related systems: Barmettler *et al.*, PRA 2008, Manmana *et al*. PRB 2009

Outline

- Experimental Motivation: Ultracold bosons in an optical lattice
- Short time behavior
  - Light-cone effect:
    spreading of correlations
    entanglement entropy
- Long time behavior
  - Properties of steady state ?
  - Is the steady state already “thermal” ?

Relaxation

- Properties of the steady state after the relaxation



C. Kollath, AML, E. Altman, PRL 2007

Relaxation

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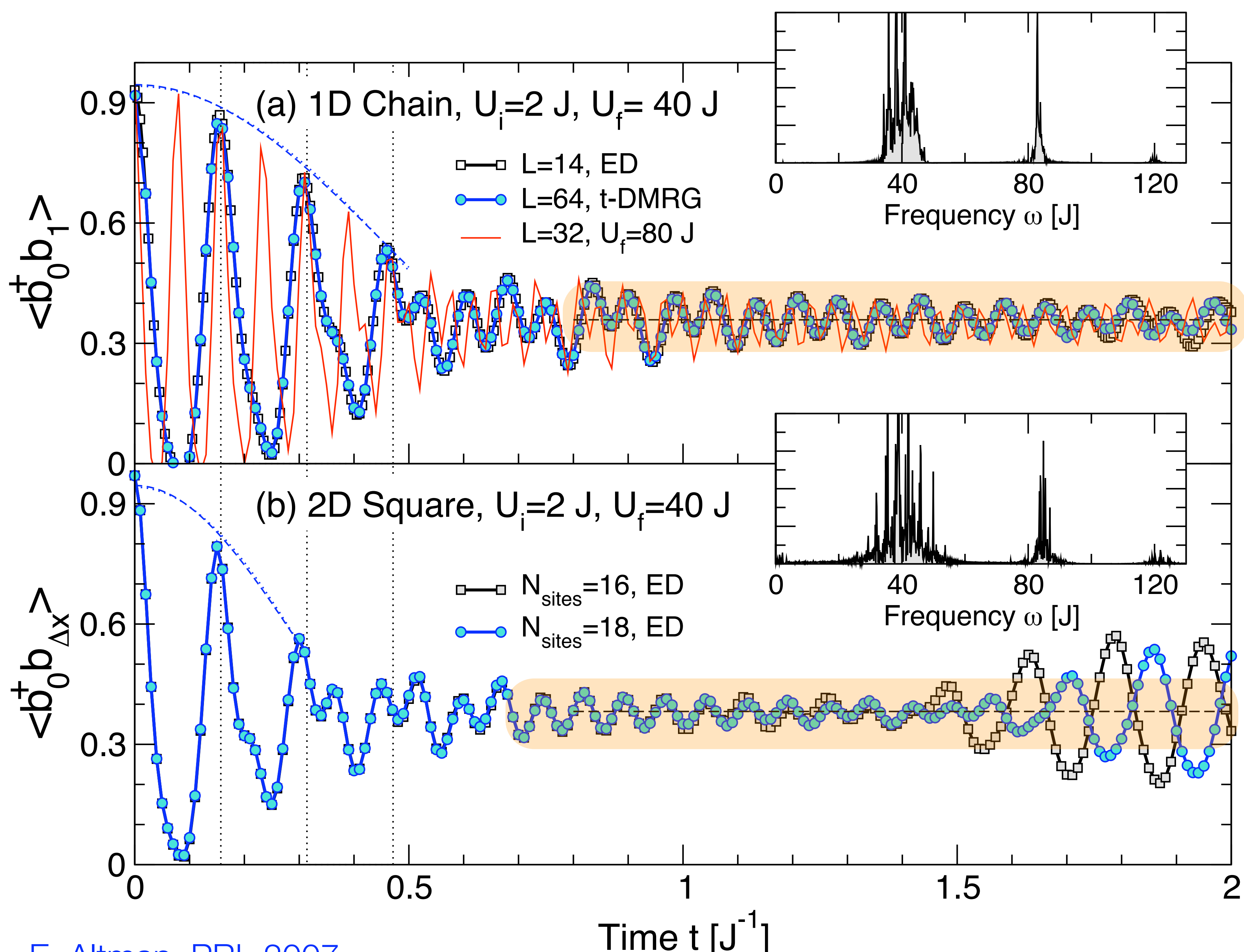
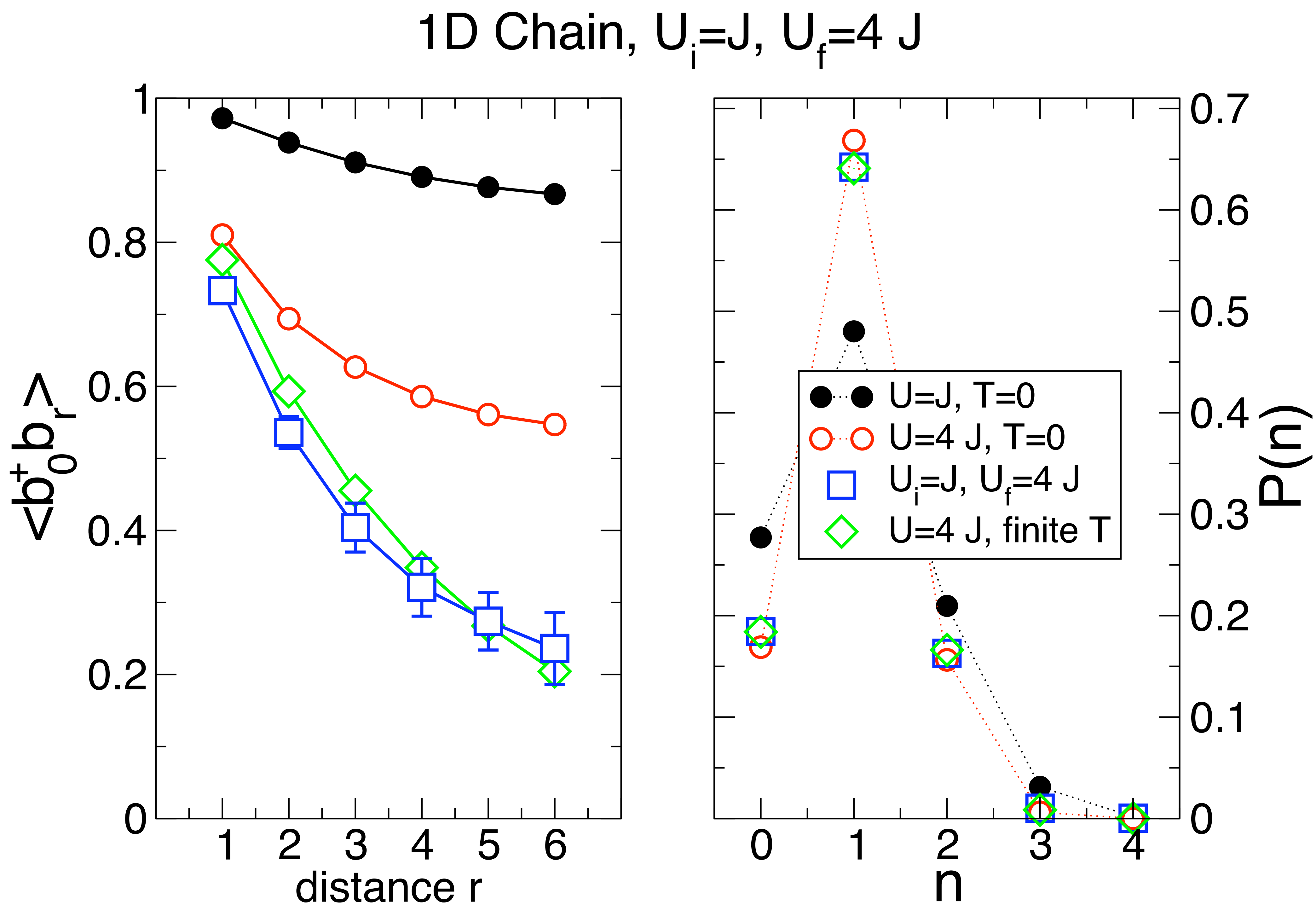


C. Kollath, AML, E. Altman, PRL 2007

Weak Quench: Looks “thermal”



C. Kollath, AML, E. Altman, PRL 2007

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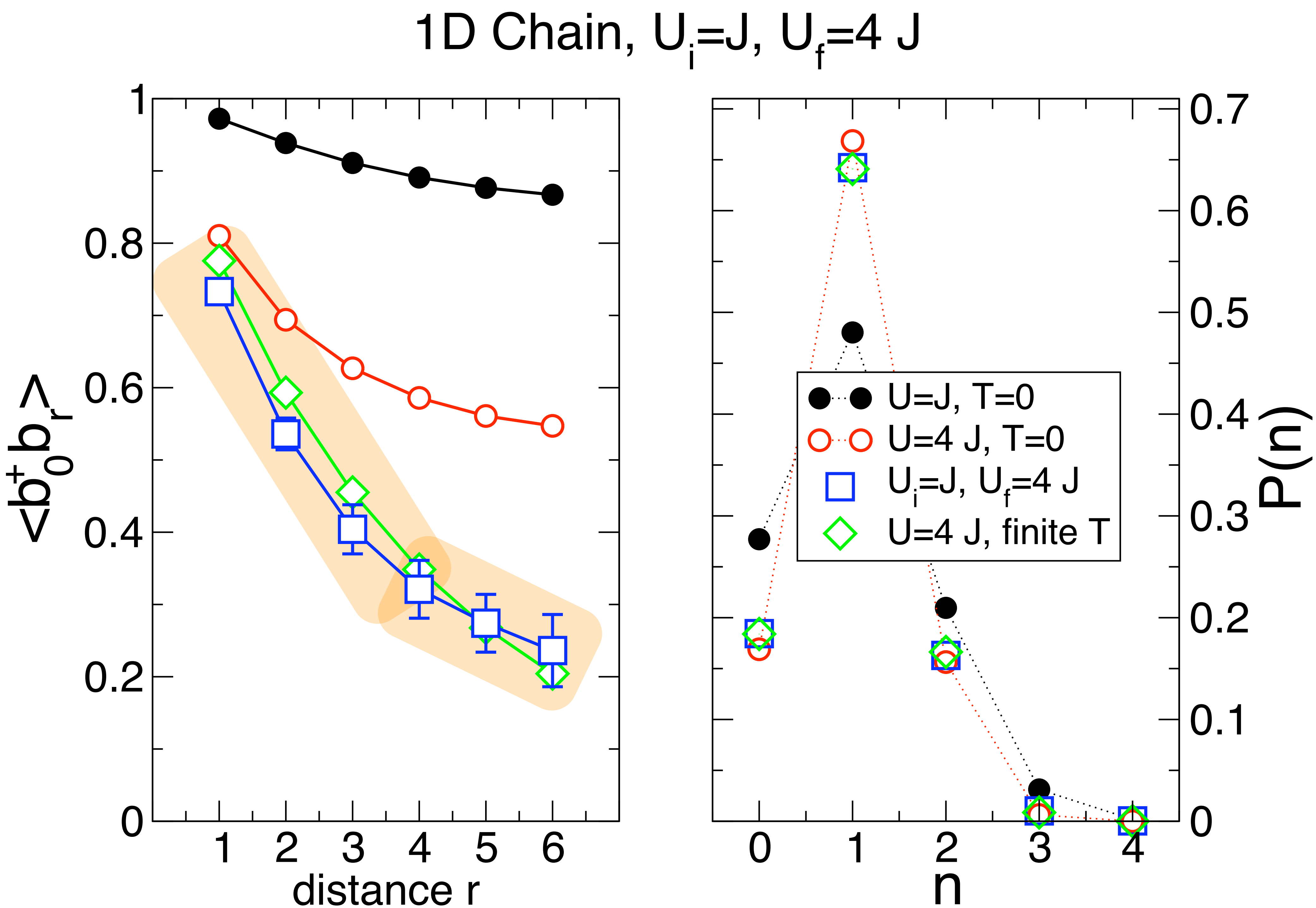
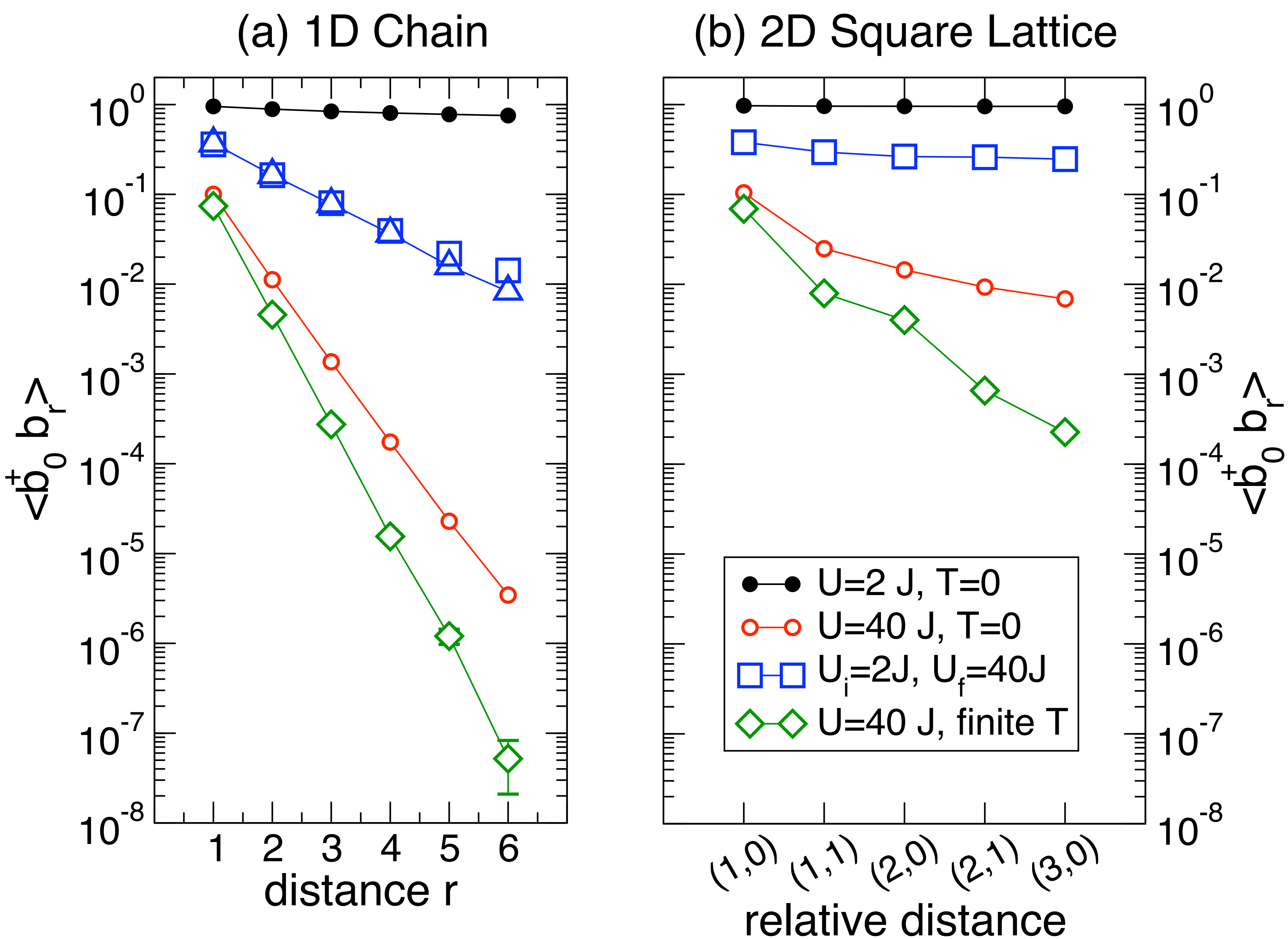


C. Kollath, AML, E. Altman, PRL 2007

Deep Quench: non-thermal / memory effect ?



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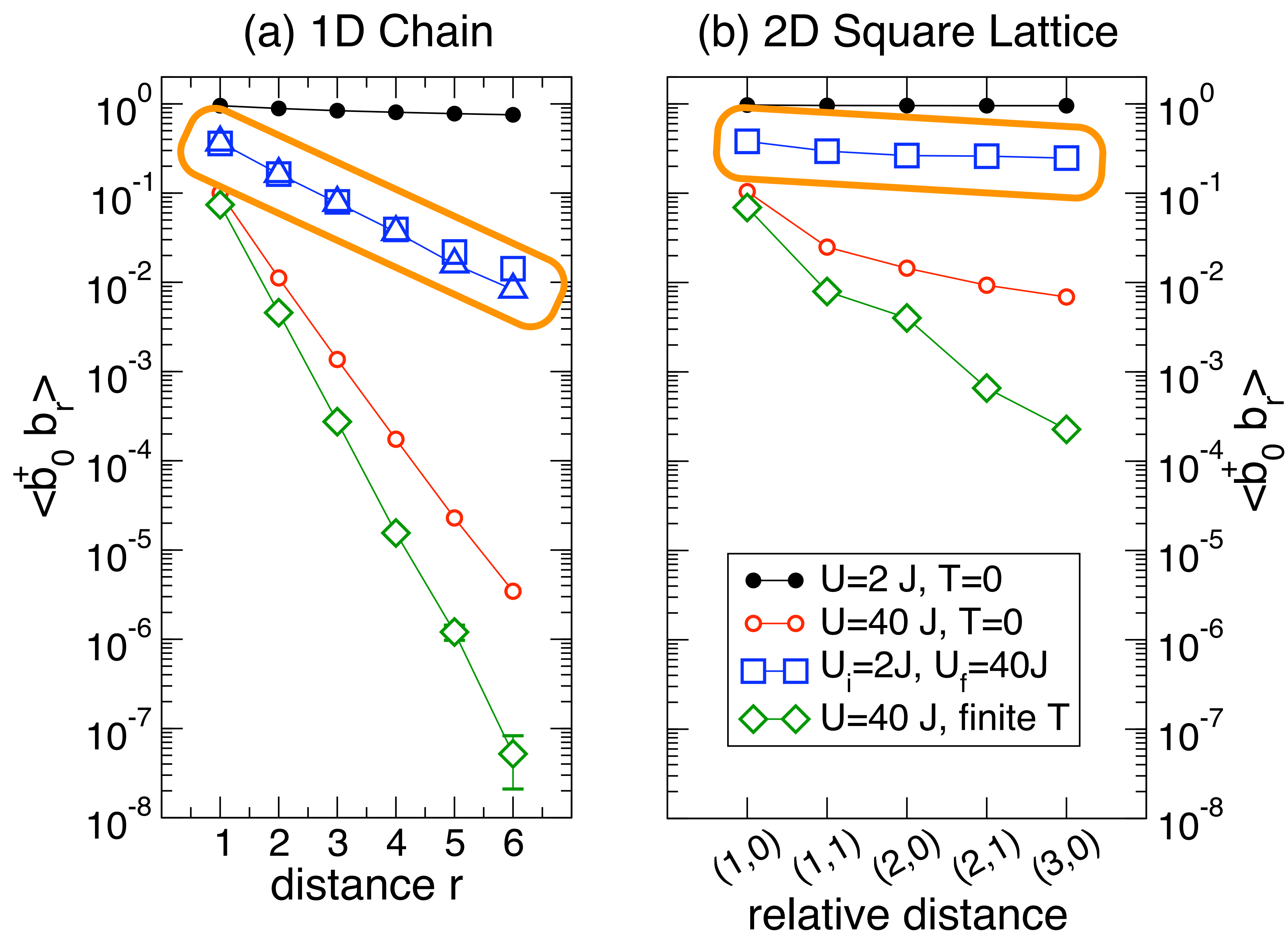
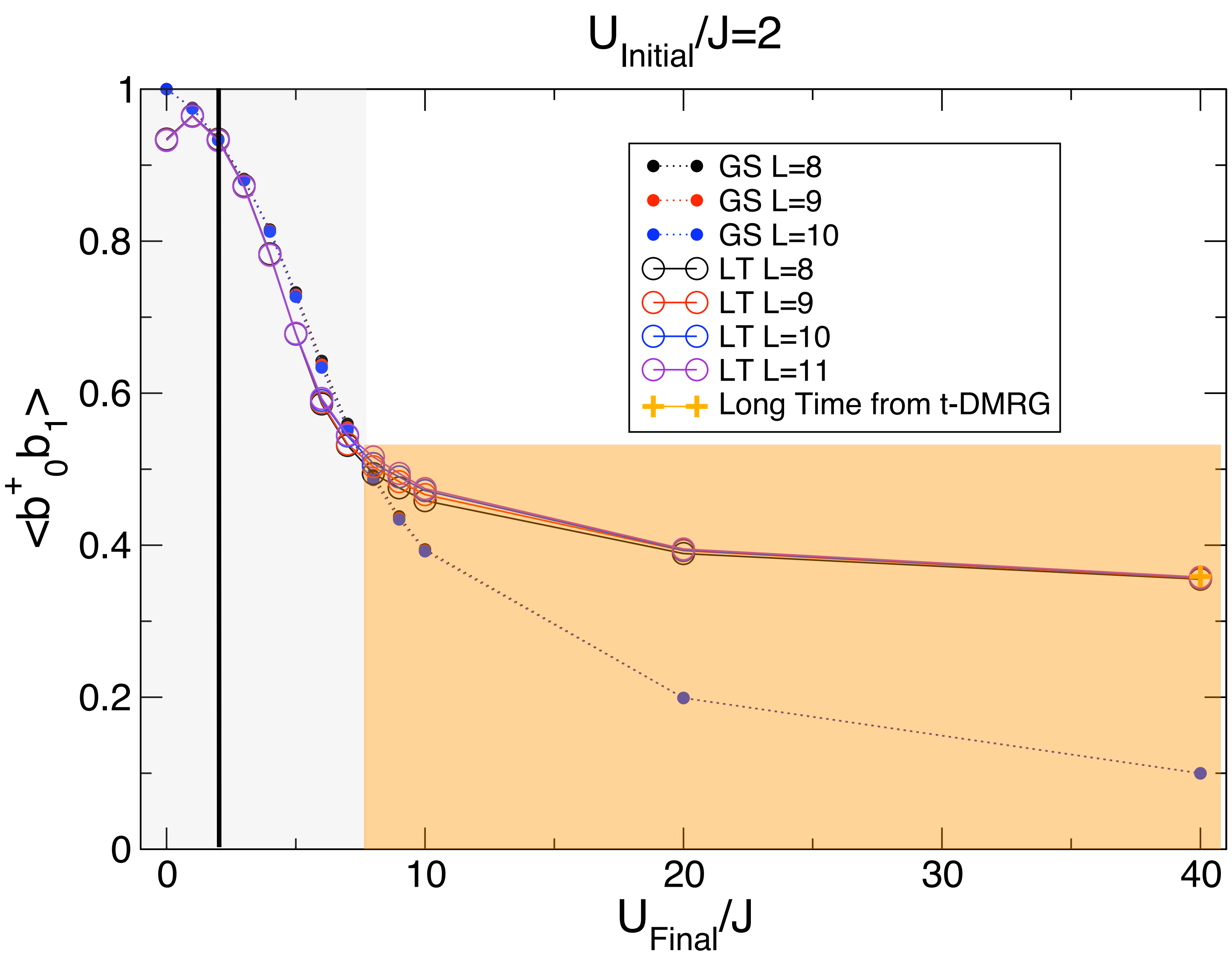


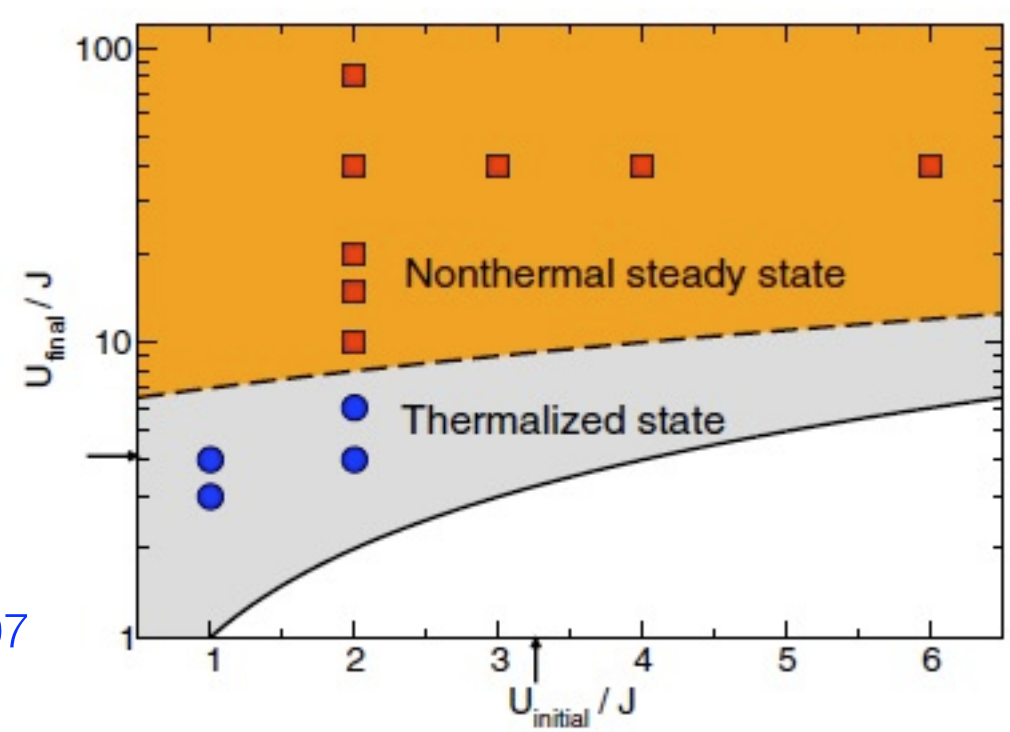
C. Kollath, AML, E. Altman, PRL 2007

Quench depth dependence



C. Kollath, AML, E. Altman, PRL 2007

- Appearance of a seemingly non-thermal steady state at large U_{final}
- Why is this ?
  (Non)-Integrability ?
  atypical eigenstates targeted ?



Back to the roots

- At $t > 0$ $|\psi(t)\rangle = e^{-iHt}|\psi_0\rangle$, and long time averages are given by:

$$\langle \mathcal{O} \rangle_{\infty} \underset{T \to \infty}{=} \frac{1}{T}\int_0^T dt \langle \psi(t)|\mathcal{O}|\psi(t)\rangle = \frac{1}{T}\int_0^T dt \sum_{\alpha,\beta} c_\alpha c_\beta^* e^{-it(E_\alpha - E_\beta)}\langle \beta|\mathcal{O}|\alpha\rangle$$

$$c_\alpha = \langle \alpha|\psi_0\rangle$$

$$\langle \mathcal{O} \rangle_{\infty} = \sum_\alpha |c_\alpha|^2 \langle \alpha|\mathcal{O}|\alpha\rangle = Tr[\rho_D \mathcal{O}]$$

$$\rho_D = \sum_\alpha |c_\alpha|^2 |\alpha\rangle\langle\alpha|$$

- Thermalization:
  von Neumann 1929

$$\sum_\alpha |c_\alpha|^2 \langle \alpha|\mathcal{O}|\alpha\rangle = \langle \mathcal{O} \rangle_{E,N}$$

dependent *independent*

on the initial state

Eigenstate thermalization hypothesis (ETH)

- Deutsch (91), Srednicki (94)

The expectation value $\langle\alpha|\mathcal{O}|\alpha\rangle$ *of a few body observable in an eigenstate* $|\alpha\rangle$ *of a large interacting many body system equals the thermal micro-canonical average at the intensive energy* E_α/N.

- Rigol, Dunjiko, Olshanii, Nature 2008

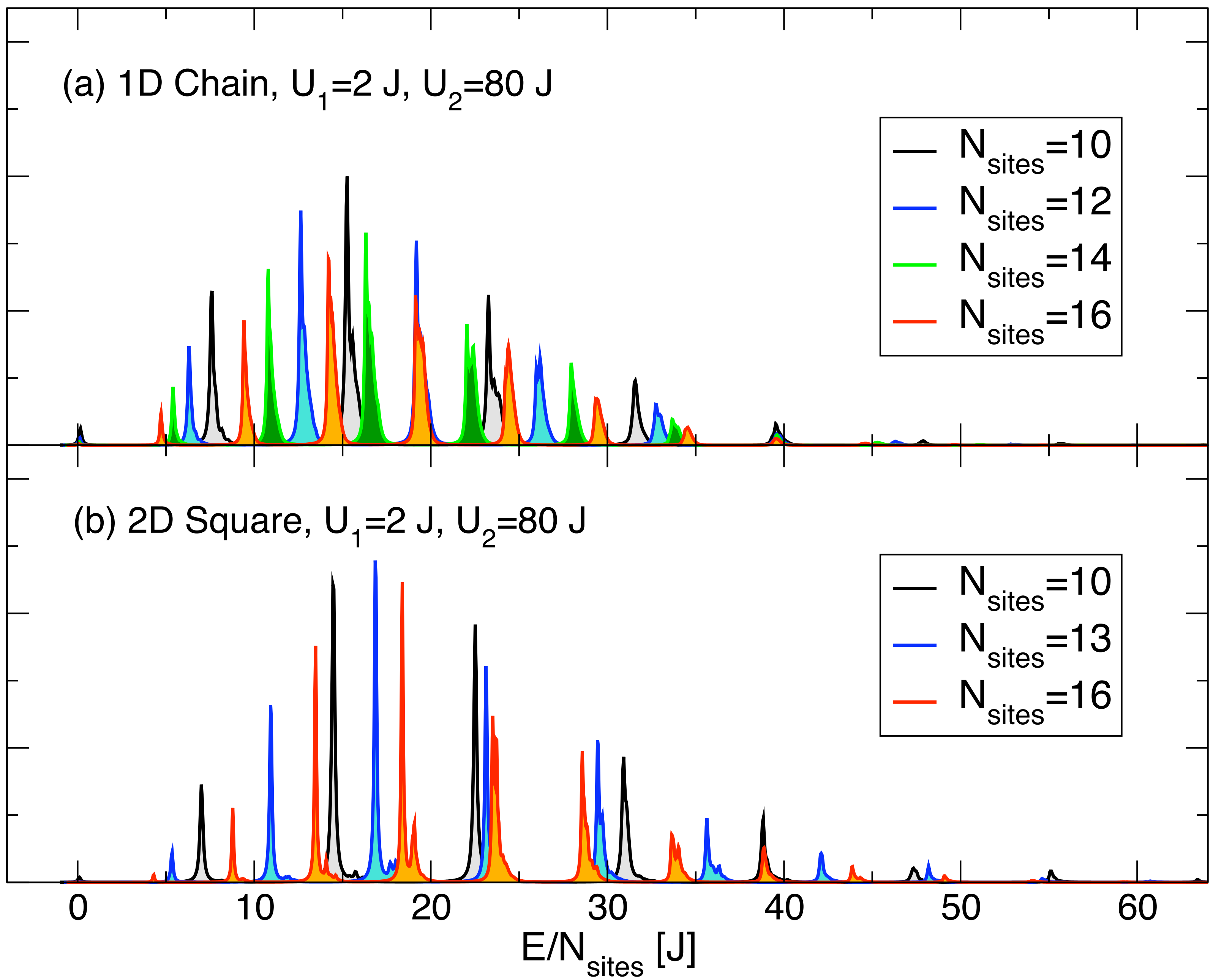
In the diagonal ensemble the intensive energy does not fluctuate and is equal (by definition) to the micro-canonical one
$\left(\sum (E_\alpha/N)^2 |c_\alpha|^2 - \left(\sum (E_\alpha/N)|c_\alpha|^2 \right)^2 \to 0 \quad for \quad N \to \infty \right)$

- Thermalization: $$\sum_\alpha |c_\alpha|^2 \langle\alpha|\mathcal{O}|\alpha\rangle = \sum_\alpha |c_\alpha|^2 \langle O\rangle_{E_\alpha,N} = \langle O\rangle_{E,N}$$

Quench: Overlap with U_f eigenbasis

- Extensive amount of energy pumped into the system through the quench

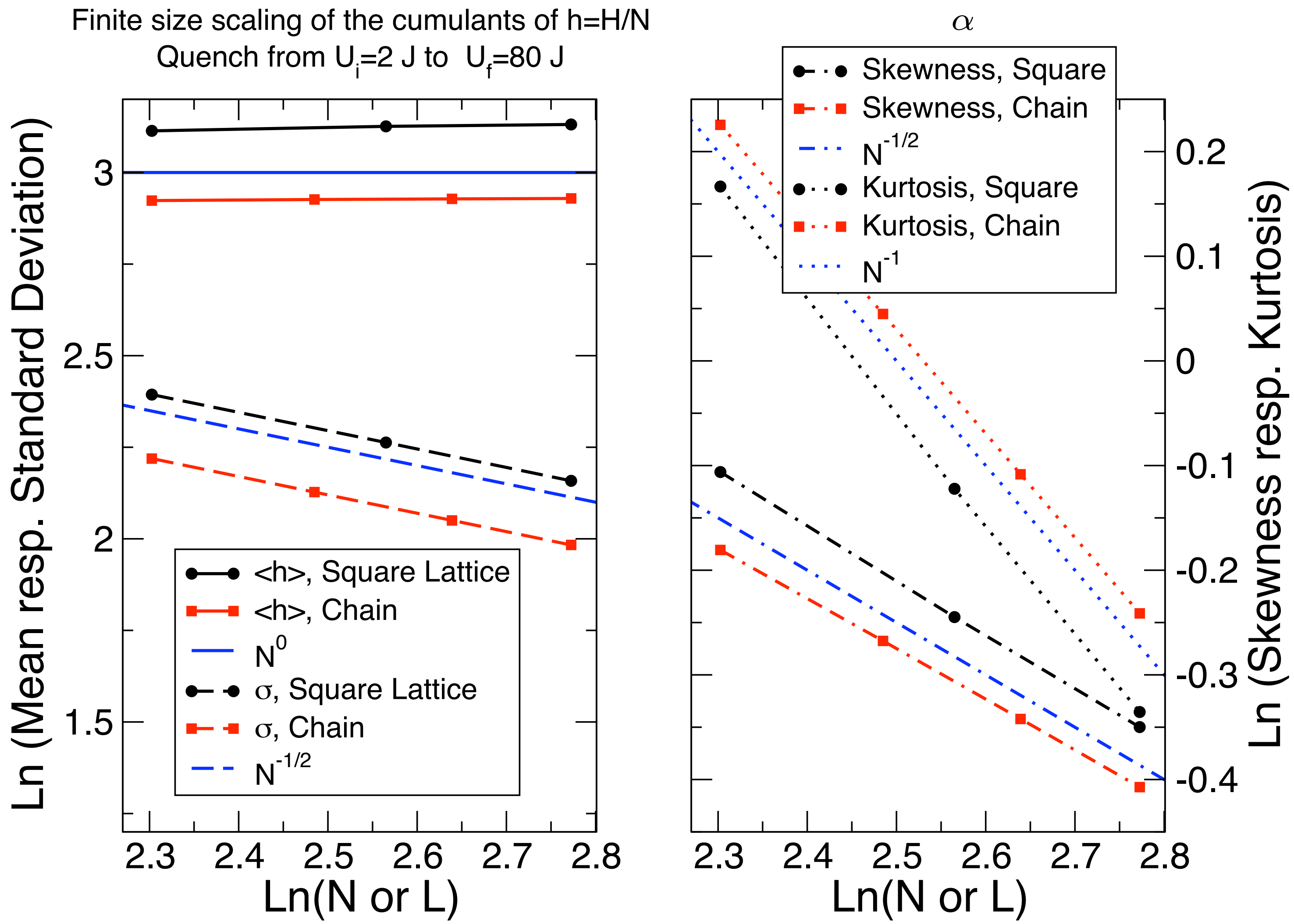
$$f(\omega) = \sum_{\alpha} |c_\alpha|^2 \times \delta(N\omega - E_\alpha)$$



Quench: Cumulants of overlaps

- Cumulants of Quench distribution: low cumulants approach Gaussian values (as in CLT)

$$f(\omega) = \sum_{\alpha} |c_\alpha|^2 \times \delta(N\omega - E_\alpha)$$



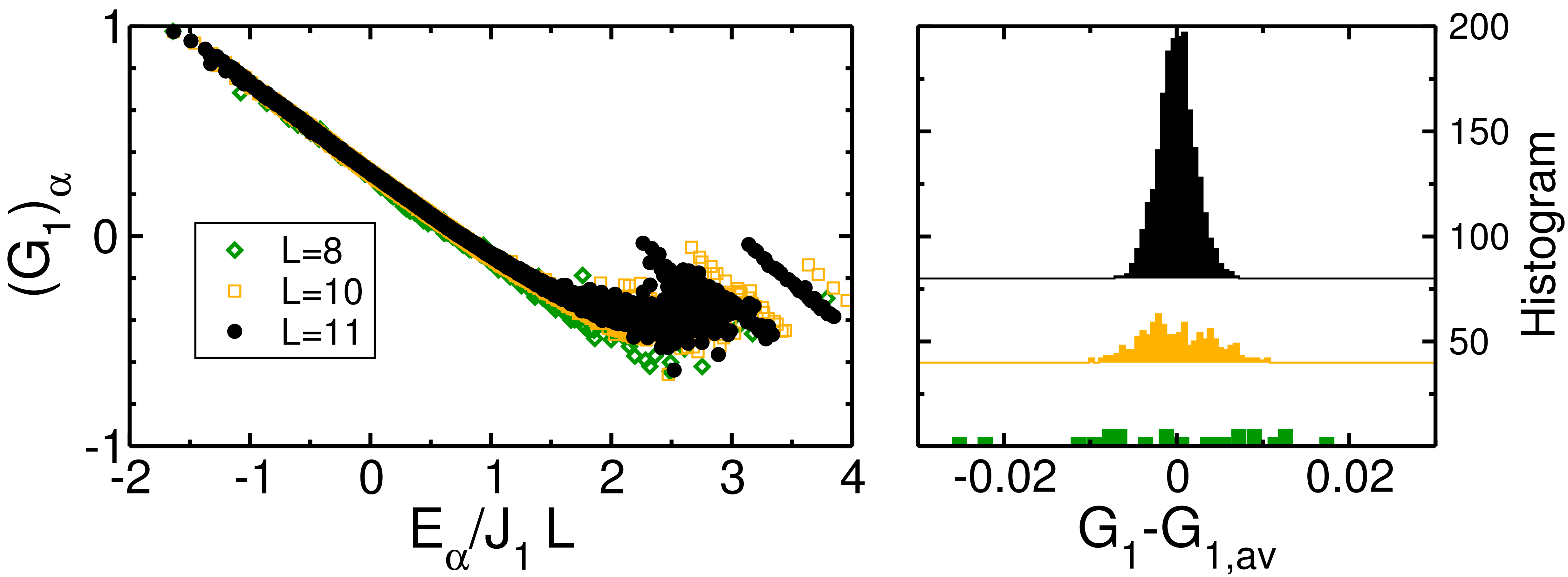
see also S. Manmana *et al.*, PRL 2007; M. Rigol *et al.*, Nature 2008

- What do very high moments do ? Does some structure survive ?

ETH: G_1 of eigenstates at U/J=1

- at small U rare states seem to absent already for small system sizes.

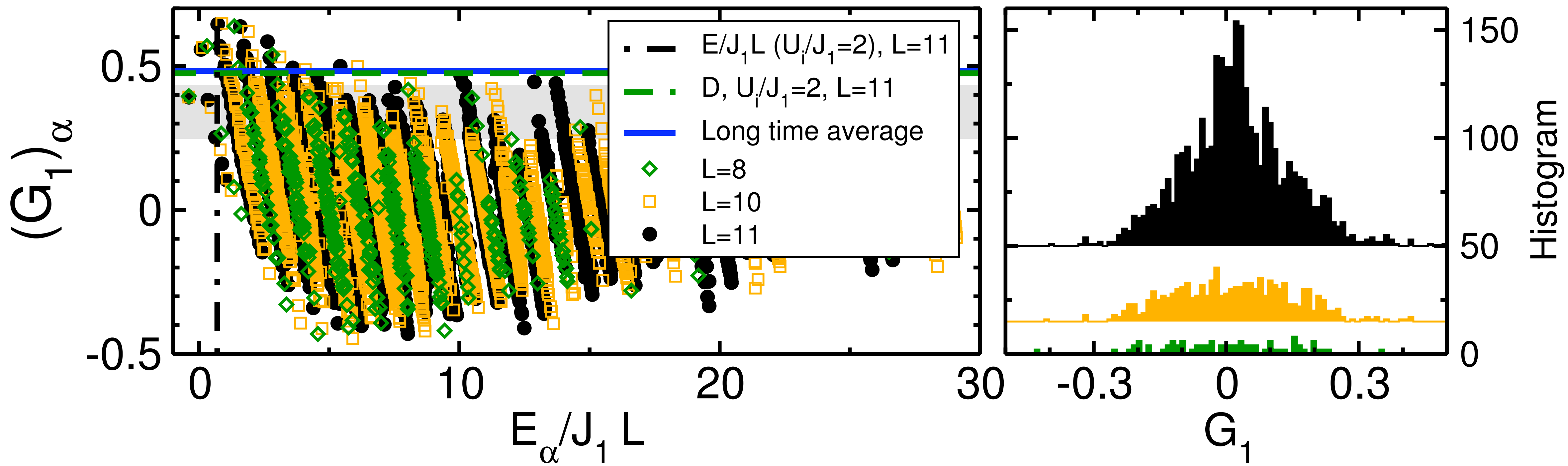
$$(\mathcal{G}_1)_\alpha = \langle \alpha | b_j^\dagger b_{j+1} | \alpha \rangle$$

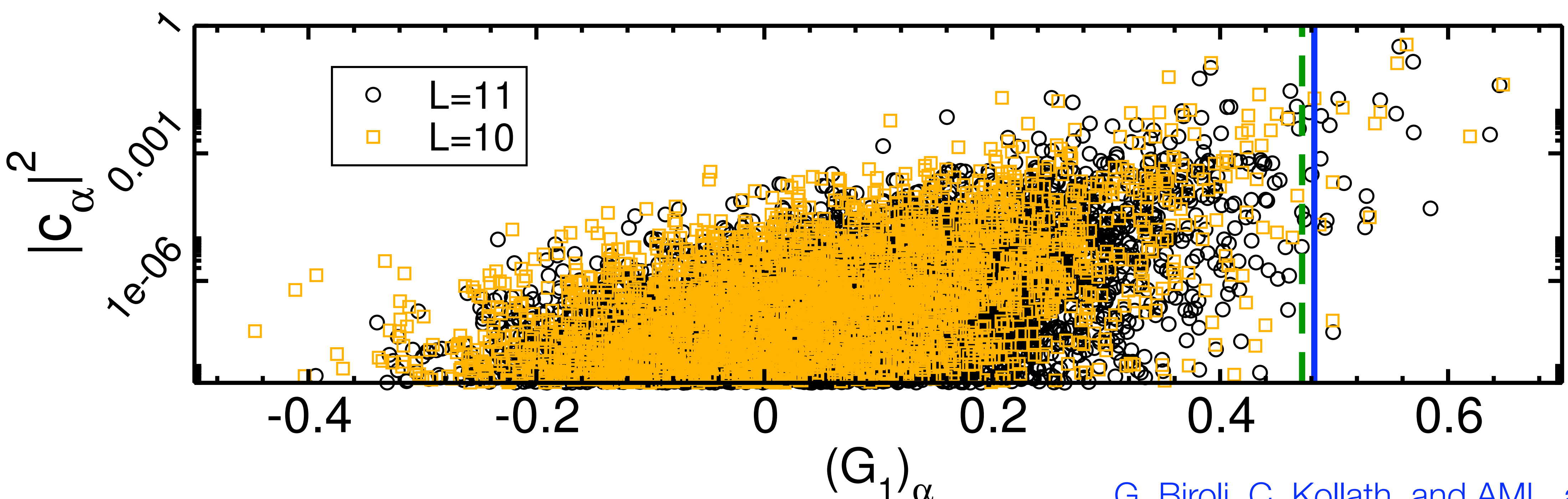


G. Biroli, C. Kollath, and AML, arXiv:0907.3731
M. Rigol *et al.*, Nature 2008

ETH ?: G_1 of eigenstates at U/J=10

- Full diagonalization results for small systems $(\mathcal{G}_1)_\alpha = \langle \alpha | b_j^\dagger b_{j+1} | \alpha \rangle$





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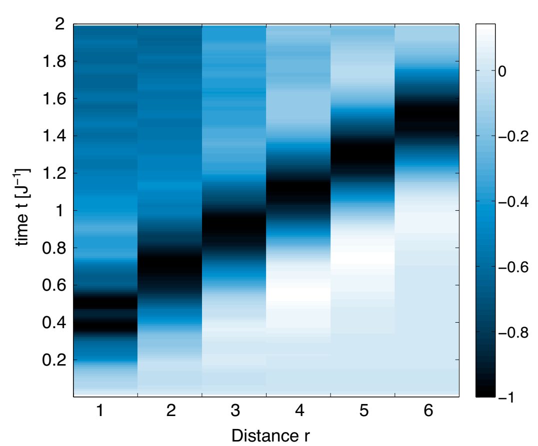
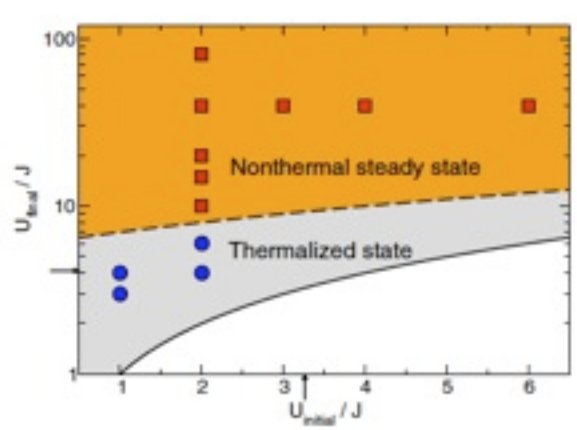
Conclusions



- “Causal” horizon evidenced in the spreading of correlations and entanglement growth
- Nice agreement with theoretical predictions



- Small quenches in this model lead to “thermal” state w.r.t. the observables we checked.
- Deep quenches lead to apparently non-thermal steady state. Importance of rare states for finite systems
- Experimentally relevant for ultracold atomic systems, which are typically much smaller than Avogadro’s number.

Thank you !