Self-similar expansion of turbulent Bose-Einstein condensates

Alexander L. Fetter

Stanford University

KITP, November 2011

1. Introduction to quantum turbulence
   - regular vortex arrays
   - turbulent vortex arrays in superfluids
2. Production of turbulent BECs
3. Observation of self-similar expansion
4. Dynamical model: uniform vorticity based on Thomas-Fermi approximation
5. Variational Lagrangian model: random anisotropic turbulent vorticity

Much of this talk is based on recent preprint arXiv:1103.2039v2 (to be published in Journal of Low Temperature Physics)

"Self-similar Expansion of the Density Profile in a Turbulent Bose-Einstein Condensate," M. Caracanhas, A. L. Fetter, S. R. Muniz, M. K. F. Margalães, G. Roati, G. Bagnato, and V. S. Bagnato

work done in collaboration with V. S. Bagnato and his group,
Instituto de Física de São Carlos,
Universidade de São Paulo, Brazil

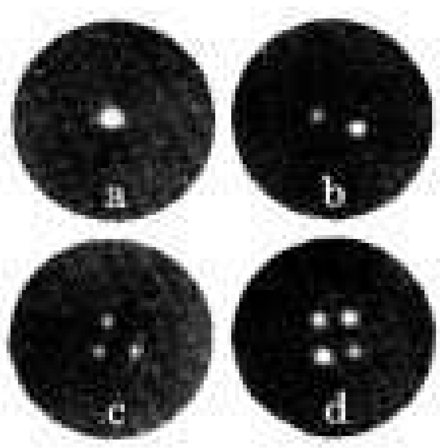
and with G. Roati
LENS and Dipartimento di Fisica
Università di Firenze and INFM-CNR, Italy

1. Introduction to quantum turbulence

Recall behavior of superfluid ^{4}He at zero temperature

- superfluid velocity $\boldsymbol{v}_s$ is irrotational (Landau 1941)
- hence $\boldsymbol{\nabla} \times \boldsymbol{v}_s$ vanishes
- but Onsager (1947) suggested superfluid circulation is quantized for any closed contour $\mathcal{C}$:
  $\kappa_s \equiv \oint_{\mathcal{C}} d\boldsymbol{l} \cdot \boldsymbol{v}_s =$ integer $\times\, 2\pi\hbar/M$ where M is atomic mass
- Feynman (1955) resolved apparent paradox with introduction of quantized superfluid vortex
- superfluid $\boldsymbol{v}_s$ is related to phase S of effective one-body wave function: $\boldsymbol{v}_s = (\hbar/M)\boldsymbol{\nabla} S$
- for single vortex along z axis, take $S =$ polar angle ϕ, which immediately gives Onsager's quantized circulation

- many experiments in period 1950s-1970s demonstrated the existence of quantized vortices in superfluid ^{4}He
- one common approach relied on rotation, where straight parallel vortices provided relevant angular momentum
- eventually took photographs of small regular arrays of straight vortices (Yarmchuk, Gordon, and Packard, 1979 saw up to ~ 11 vortices)



- note late date 1979 for this experiment, compared to Hall and Vinen's first experiments on quantized circulation and vortices more than 20 years earlier ($\sim$1956-61)
- Packard's experiments were very difficult and took many years for success
- visualization of vortices in ^{4}He relied on trapping of electron bubbles on vortex cores (maximum $\sim 10^3$ ions per cm)
- apply strong voltage pulse to extract charges that travel through vapor above liquid and make image on a phosphorescent screen
- thermal fluctuations of vortices scale like T, but normal fluid density ρ_n scales like T^4
- as $T \to 0$, fluctuations predominate
- added small fraction ^{3}He that provided necessary viscous drag (but not too much to avoid scattering of electrons in vapor phase)

Another whole class of experiments focused on tangled vortices

- in an early study, Vinen (1957) used heat currents to create a random tangled array of quantized vortices
- Feynman (1955) suggested that such a random configuration was effectively a kind of quantum turbulence
- experiments show that mean vortex line density is proportional to a power of the external heat current
- use attenuation of second sound to measure vortex line density
- in second sound, the normal fluid and superfluid oscillate against each other, out of phase
- since vortices move with the superfluid, they scatter the excitations of the normal fluid, damping the coherent relative motion of the second sound

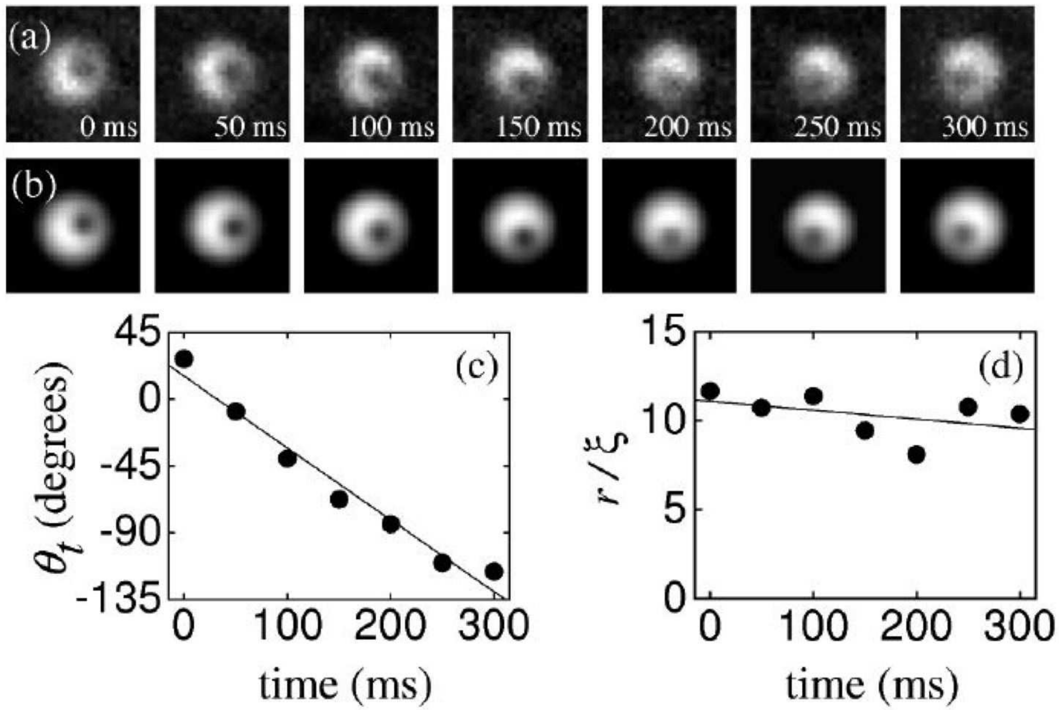
Compare situation for dilute trapped Bose-Einstein condensate (BEC)

- these systems are dilute and can be described with order parameter (equivalently a condensate wave function) Ψ
- here, $\Psi = |\Psi|\, e^{iS}$ provides the phase S in Feynman's expression $\boldsymbol{v}_s = (\hbar/M)\boldsymbol{\nabla} S$
- note that here M is typically much larger than for ^{4}He (common trapped gases are ^{23}Na or ^{87}Rb)
- non-rotating trapped condensates without vortices are well described by Gross-Pitaevskii equation (a nonlinear Schrödinger equation)
- specifically, the predicted frequencies of collective monopole and quadrupole modes agreed with experiment within a few %

How to create vortices in trapped BEC?

- first experiment (Cornell, JILA 1999) used mixture of two hyperfine states with $|m_F| = 1$ that are coupled by near resonant electromagnetic radiation
- this coupling converts two independent $U(1)$ systems into a single $SU(2)$ system that acts like spin-1/2 system
- while coupling is on, stir the condensate to add angular momentum
- turn off coupling and end up with vortex in one component surrounding non-rotating core of other component
- can control fraction of each component and can visualize each one separately because of slightly different resonant frequencies
- non-rotating core is typically large $\sim$ 10 μm and can be visualized with visible light $\lambda \sim 0.5$ μm)

JILA group used this capability to study precession of these two component vortices in harmonic traps

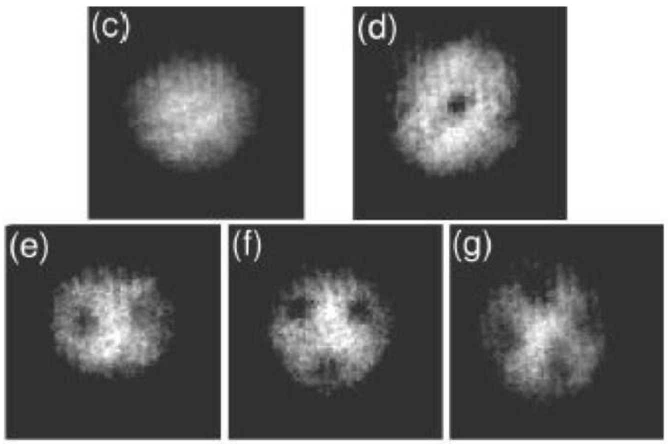


- top row are experimental figures (at 50 ms intervals)
- second row are smoothed figures (used Thomas-Fermi density profile)
- left graph shows precession angle over 300 ms

More direct approach to vortex creation is to stir a one-component condensate (Dalibard, ENS, Paris, 2000)

- use magnetic trap and stir with off-center toggled laser beam at frequency $\Omega/2\pi \sim 200$ Hz
- rotate cigar-shaped condensate around its symmetry axis
- here the resulting vortex cores are $\sim 0.5\ \mu$m, too small to visualize with visible light
- hence need to turn off the trap and expand the condensate
- vortex cores also expand and appear as holes in expanding condensate
- condensate expends rapidly in tightly confined (here radial) direction, becoming pancake shaped

These ENS experiments yielded pictures of small vortex arrays like those seen earlier in superfluid ^{4}He (up to $\sim$ 11 vortices)



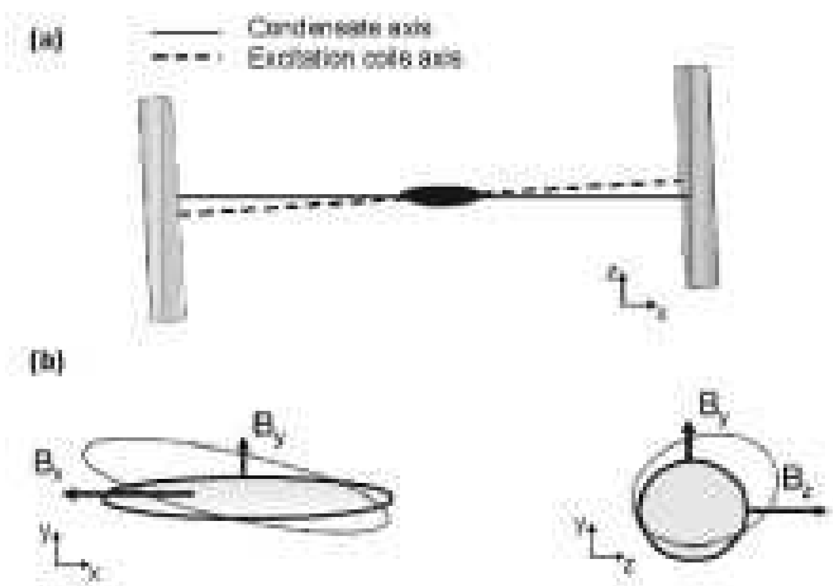
- these experiments were much less complicated than those for ^{4}He
- later experiments (MIT and JILA) produced much larger triangular arrays (up to $\sim$ 130 vortices)

2. Production of turbulent trapped BECs

Experimental studies by Bagnato's group in São Carlos, Brazil [Henn *et al.*, PRA **79**, 043618 (2009) and PRL **103**, 045301 (2009)]

Start with ^{87}Rb cigar-shaped condensate in magnetic trap with $\sim 10^5$ atoms

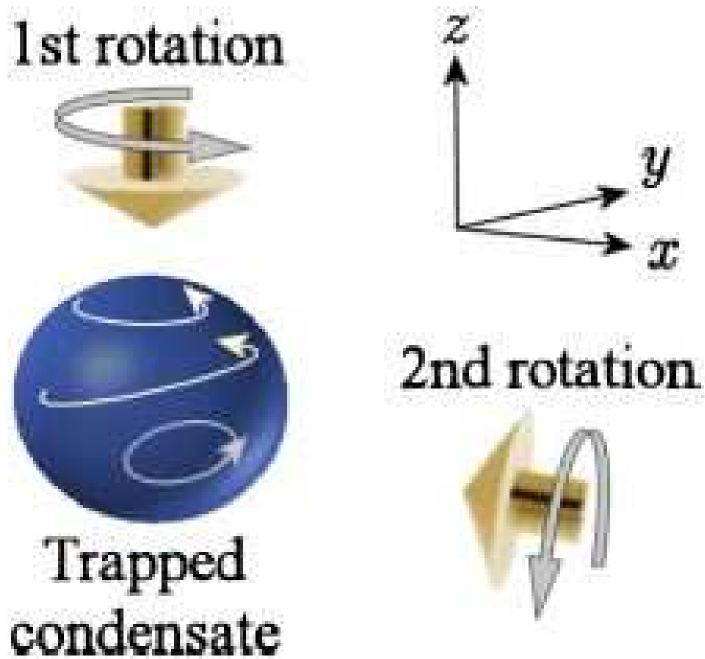
- then apply oscillating magnetic field aligned nearly but not exactly along symmetry axis
- this field has components along each of the principal axes of the condensate
- it explicitly breaks the rotational symmetry
- leads to displacement of center of mass, rotation of condensate, and shape deformation
- together, these oscillatory effects produce tangled vortices for sufficiently strong applied fields and sufficient duration



- (a) shows basic structure, with solid line a symmetry axis and dashed line as axis of oscillatory magnetic field
- (b) shows expected motion of condensate in xy and yz planes

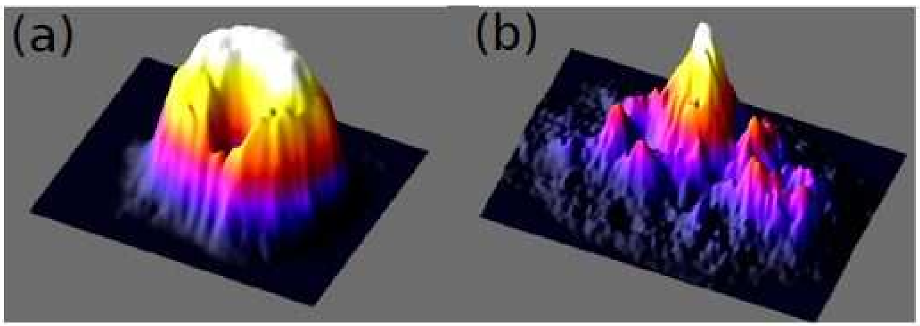
This oscillatory magnetic field is similar to that in a theoretical study by Kobayashi and Tsubota (2007)

They use sequential rotations about two perpendicular axes (first z and then additionally x)



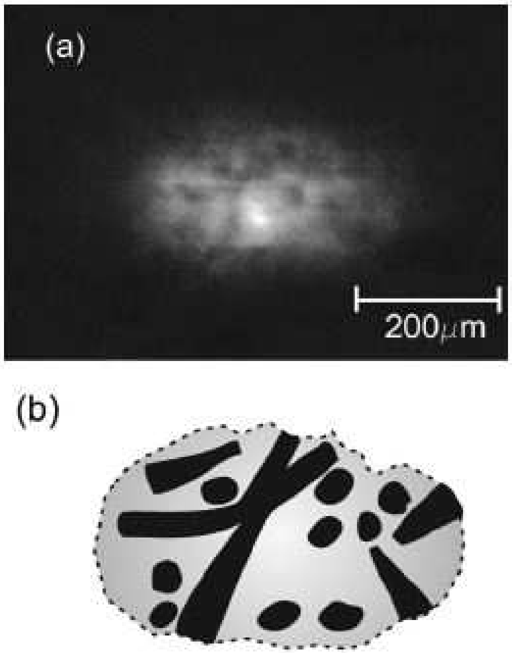
For small amplitude of oscillatory field, experiments find scissors mode and various other shape modes of condensate (monopole and quadrupole)

- for larger amplitude, find vortex creation is quasi-regular pattern (but not reproducible from shot to shot)



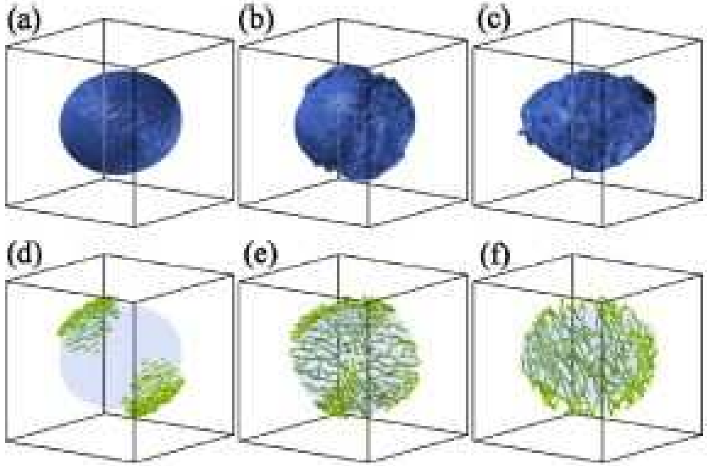
- (a) shows density distribution for a single straight vortex with empty core in center of condensate (asymmetry due to gravity)
- for still larger amplitude and longer exposure, (b) shows what looks like vortex tangle in the expanded images (quantum turbulence)
- note present creation scheme is very different from Vinen's heat flow in superfluid ^{4}He
- nevertheless, final vortex tangle is presumed to be similar in both cases

- more detailed experimental studies seek to understand the vortex tangle



- (a) shows typical experimental image of expanded condensate with vortex tangle
- (b) is schematic diagram of inferred vortex distribution

This vortex tangle is quite similar to theoretical pictures of Kobayashi and Tsubota (2007)



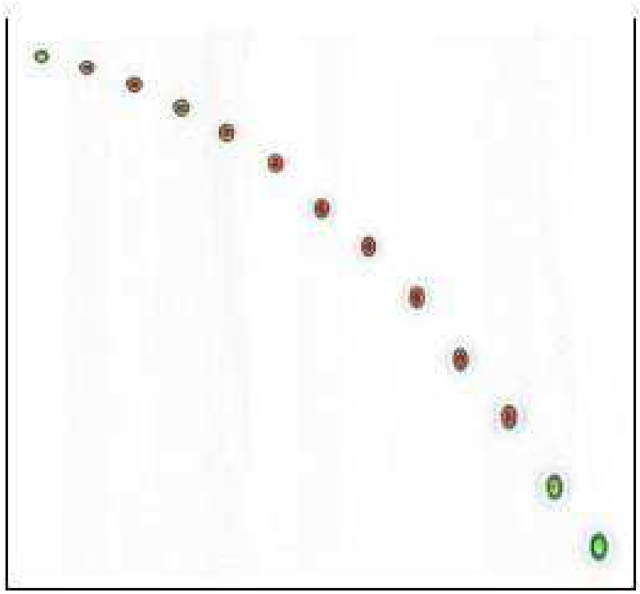
- here (a)-(c) shows altered shape of originally nearly spherical condensate under influence to two successive rotations (z followed by additionally x) at dimensionless times 10, 50, and 300
- (d)-(f) show the growth of the vortex tangle inside the Thomas-Fermi radius at same dimensionless times

3. Observation of self-similar expansion in turbulent condensates

For nonrotating nonspherical condensate, turning off trap leads to expansion with "reversal of aspect ratio"

- in general, tightly confined direction expands most rapidly, so initial cigar expands to a disk, and initial disk expands to elongated cigar shape
- in weakly interacting BEC, this reflects the large kinetic energy associated with tight confinement
- interaction effects enhance this behavior, because the associated effective interaction pressure is proportional to local density $gn(\boldsymbol{r})$
- resulting force $-\boldsymbol{\nabla} gn(\boldsymbol{r})$ acts most strongly in tightly confined direction where gradient is large
- note that this quantum behavior differs from that in a thermal cloud, where temperature fixes the initial momentum, leading to isotropic expansion

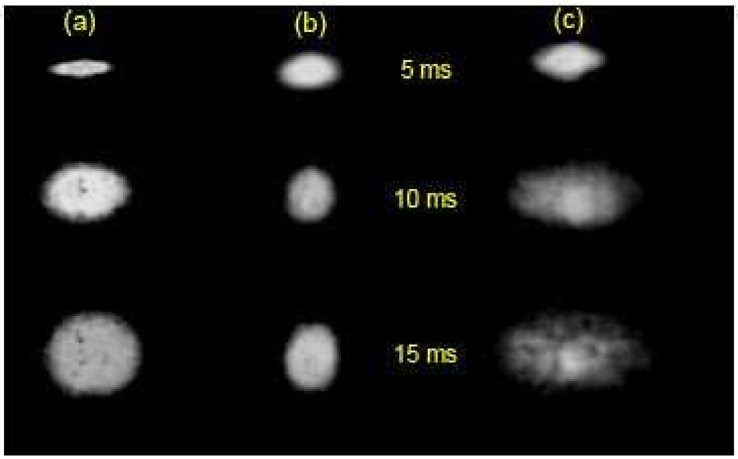
- this reversal of aspect ratio is readily seen in following example (a montage of images after condensate is released from trap and falls under gravity)



- initial condensate is elliptical with elongation along horizontal axis (too small to resolve fully)
- under expansion, condensate first becomes circular and then elongates vertically
- already mentioned this behavior in connection with ENS experiments on creation of quantized vortices in initial cigar-shaped condensate

Behavior of turbulent condensates is very different

- experiment by Bagnato's group shows roughly self-similar expansion with no reversal of aspect ratio



- (a) first column shows expansion of thermal cloud tending to unity
- (b) second column shows usual anisotropic expansion of nonrotating BEC
- (c) third column shows self-similar expansion of condensate in turbulent regime (note extra expansion leading to larger image)
- this behavior must arise from the vortex tangle, since regular vortex arrays generally show reversal of aspect ratio

4. Theoretical model based on Thomas-Fermi approximation

Recall Euler equation for an ideal (nonviscous) fluid

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\, \boldsymbol{v} + \frac{1}{\rho} \boldsymbol{\nabla} p = \boldsymbol{f},$$

where $\boldsymbol{v}$ is the local velocity, p is the pressure, ρ is the mass density, and $\boldsymbol{f}$ is the force density (such as gravity)

- this equation is simply Newton's law applied to a particular element of moving fluid (Eulerian picture)
- first two terms are the convective or hydrodynamic derivative

$$\frac{d\boldsymbol{v}}{dt} = \frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\, \boldsymbol{v}$$

- familiar vector identities lead to equivalent form

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{\nabla}\left(\tfrac{1}{2}v^2\right) + \frac{1}{\rho}\boldsymbol{\nabla} p = \boldsymbol{f} + \boldsymbol{v} \times (\boldsymbol{\nabla} \times \boldsymbol{v})$$

- for an incompressible irrotational fluid with $\boldsymbol{\nabla} \times \boldsymbol{v} = 0$, the velocity may be expressed by a velocity potential $\boldsymbol{v} = \boldsymbol{\nabla}\Phi$
- for an external potential with $\boldsymbol{f} = -\boldsymbol{\nabla} V$, this immediately yields Bernoulli's equation

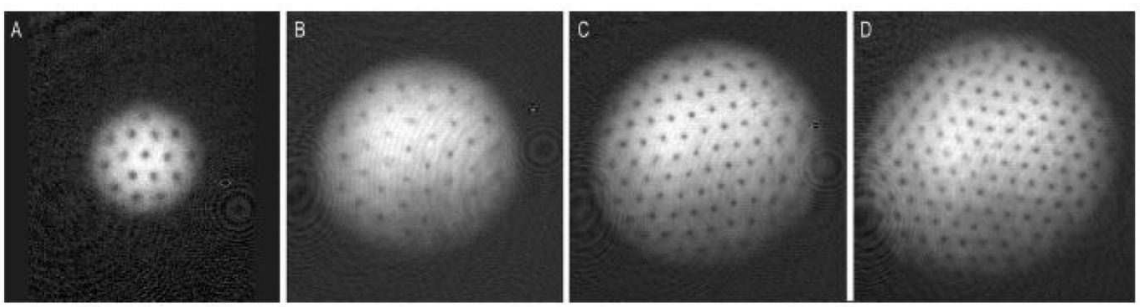
$$\boldsymbol{\nabla}\left(\tfrac{1}{2}v^2 + \frac{p}{\rho} + V + \frac{\partial \Phi}{\partial t}\right) = 0$$

- in presence of vorticity, need to retain term $\boldsymbol{v} \times (\boldsymbol{\nabla} \times \boldsymbol{v})$
- for solid-body rotation with angular velocity $\boldsymbol{\Omega}$, velocity is $\boldsymbol{v} = \boldsymbol{\Omega} \times \boldsymbol{r}$ and $\boldsymbol{\nabla} \times \boldsymbol{v} = 2\boldsymbol{\Omega}$, so this term becomes $2\boldsymbol{v} \times \boldsymbol{\Omega}$
- in this latter situation, cannot use velocity potential and need full form of dynamical equation

How does these considerations apply to turbulent BECs?

Recall Feynman's suggestion (1955) that the areal vortex density in a rotating superfluid should be $n_v = 2\Omega/\kappa = M\Omega/(\pi\hbar)$, where $\kappa = 2\pi\hbar/M$ is the quantum of circulation

- this result follows by assuming that the mean vorticity $n_v\kappa$ should be the classical value for solid-body rotation $\boldsymbol{\nabla} \times \boldsymbol{v} = 2\Omega$
- experiments on rotating BECs with vortex lattices confirm this Feynman relation with considerable accuracy



- MIT experiment with up to ~ 130 vortices

Return to full time-dependent GP equation in hydrodynamic form for n and $\boldsymbol{v}$

- first is conservation of particles

$$\frac{\partial n}{\partial t} + \boldsymbol{\nabla} \cdot (n\boldsymbol{v}) = 0$$

- second is usual dynamical equation, now written in terms of hydrodynamic (convective) derivative $d\boldsymbol{v}/dt = \partial\boldsymbol{v}/\partial t + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\,\boldsymbol{v}$

$$M\frac{d\boldsymbol{v}}{dt} + \boldsymbol{\nabla}\left(V_{\rm tr} + gn - \mu\right) = 0,$$

  which is just Newton's law for the superfluid BEC

- here $V_{\rm tr}$ is confining trap potential, gn describes the repulsive interactions with $g = 4\pi\hbar^2 a/M > 0$, $a \sim$ a few nm is the s-wave scattering length, and μ is chemical potential

- here use Thomas-Fermi picture to ignore "quantum pressure" that involves $\hbar^2$ and derivative of density

- note that these equations are written in the *laboratory* frame of reference

- as for ideal classical fluid, expand convective term $(\boldsymbol{v} \cdot \boldsymbol{\nabla})\, \boldsymbol{v} = \boldsymbol{\nabla}(\frac{1}{2}v^2) - \boldsymbol{v} \times (\boldsymbol{\nabla} \times \boldsymbol{v})$
- for dense vortex array, approximate mean vorticity as $\langle \boldsymbol{\nabla} \times \boldsymbol{v} \rangle \approx 2\boldsymbol{\Omega}$, where Ω follows from Feynman's relation $n_v = M\Omega/(\pi\hbar)$
- in this way, obtain the new dynamical equation for the superfluid in presence of distributed mean vorticity

$$M\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{\nabla}\left(\tfrac{1}{2}Mv^2 + V_{\mathrm{tr}} + gn - \mu\right) + 2M\boldsymbol{\Omega} \times \boldsymbol{v} = 0$$

- this approach was used by Sedrakian and Wasserman (2001) and by Cozzini and Stringari (2003)
- gives good description of quadrupole modes in rapidly rotating condensate, including periodic formation of "stripes" in vortex lattice (JILA, 2002)

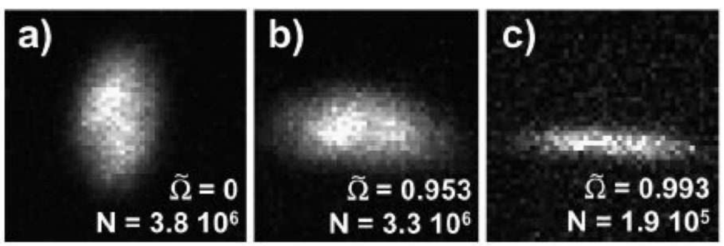
- apply to axisymmetric condensate rotating about symmetry (z) axis with angular velocity Ω

- in equilibrium, mean velocity in laboratory frame $\langle \boldsymbol{v}_s \rangle \approx \boldsymbol{\Omega} \times \boldsymbol{r}$ leads to familiar Thomas-Fermi condensate radii that now depend on Ω

$$R_\perp(\Omega)^2 = \frac{2\mu(\Omega)}{M(\omega_\perp^2 - \Omega^2)} \quad \text{and} \quad R_z(\Omega)^2 = \frac{2\mu(\Omega)}{M\omega_z^2},$$

where $\mu(\Omega) = \mu(0)\left(1 - \Omega^2/\omega_\perp^2\right)^{2/5}$, with $\mu(0)$ for a nonrotating condensate

- aspect ratio $R_z(\Omega)/R_\perp(\Omega) = \sqrt{\omega_\perp^2 - \Omega^2}/\omega_z$ now depends on Ω



- provides good diagnostic method to infer angular velocity (JILA)

Now apply these ideas to expansion of the condensate when trap is turned off

- goal is to describe free expansion of a BEC containing fluctuating angular momentum along all different directions (a vortex tangle)
- as first attempt, consider simpler situation with angular momentum along each principal axis of *axisymmetric* condensate
- focus on two limiting cases to develop some intuition about the dynamics
- parametrize the density as follows (this is the usual Thomas-Fermi model)

$$n(\boldsymbol{r}, t) = n_0(t)\left(1 - \frac{x^2}{R_x(t)^2} - \frac{y^2}{R_y(t)^2} - \frac{z^2}{R_z(t)^2}\right)$$

- $n_0(t)$ is given by usual Thomas-Fermi normalization condition

$$n_0(t) = \frac{15N}{8\pi\, R_x(t)R_y(t)R_z(t)}$$

First assume uniform vortex array along the symmetry axis $\hat{\boldsymbol{z}}$

- here $\boldsymbol{\Omega}$ is along symmetry axis ($\hat{\boldsymbol{z}}$)
- assume velocity has both irrotational part caused by the expansion and solid-body part from the uniform vortex array

$$\boldsymbol{v} = \underbrace{\tfrac{1}{2}\boldsymbol{\nabla}\left[b_x(t)x^2 + b_y(t)y^2 + b_z(t)z^2\right]}_{\text{irrotational expansion flow}} + \underbrace{\boldsymbol{\Omega}\times\boldsymbol{r}}_{\text{rotational flow}}$$

- substitute these expressions into the dynamical equations assuming that trap potential vanishes for $t > 0$

- find coupled second-order differential equations for evolution of condensate radii

$$\ddot{R}_x = \frac{15N\hbar^2 a}{M^2}\frac{1}{R_x^2 R_y R_z} + \left(\frac{N_v \hbar}{M}\right)^2 \frac{1}{R_x R_y^2}$$

with similar equation for R_y, where N_v is total number of vortices (this number is conserved), and

$$\ddot{R}_z = \frac{15N\hbar^2 a}{M^2}\frac{1}{R_z^2 R_x R_y}$$

- note role of repulsive interaction ($a > 0$) in expanding in all three directions
- initial R_z is large, so axial expansion ($\propto R_z^{-2}$) is small, leading to reversal of aspect ratio for usual condensate

- repeat transverse equations

$$\ddot{R}_x = \frac{15N\hbar^2 a}{M^2}\frac{1}{R_x^2 R_y R_z} + \left(\frac{N_v \hbar}{M}\right)^2 \frac{1}{R_x R_y^2}$$

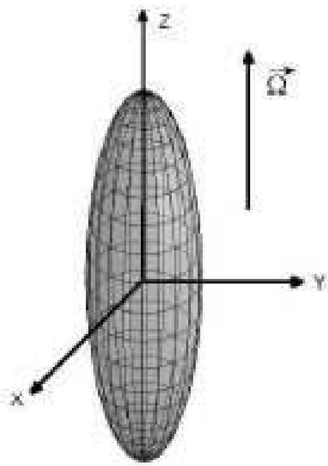
  with similar equation for R_y, where N_v is total number of vortices (this number is conserved)

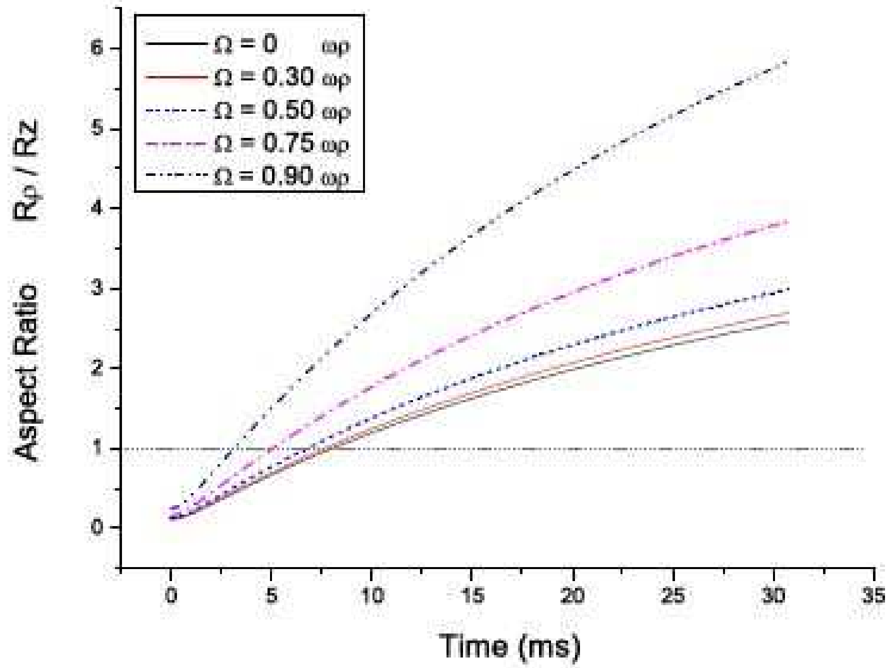
- note directions perpendicular to $\mathbf{\Omega}$ (namely those along x and y) experience *additional* outward force proportional to N_v^2 (and hence proportional to initial Ω^2)

- vortex lines experience effective mutual repulsive interactions (like magnetic field lines in a plasma)

- for large times, interaction terms ($\propto a$) are smaller than rotation terms ($\propto N_v^2$) by one factor of R^{-1}

- numerical studies yield the time-dependent aspect ratio (condensate here remains axisymmetric)





- left side shows typical geometry of condensate with $\mathbf{\Omega}$ along symmetry axis
- right side shows evolution of aspect ratio $R_\perp/R_z$ for various initial vortex densities
- as anticipated, presence of axial vorticity (along z) enhances the radial expansion and hence the aspect ratio during expansion

Next assume uniform vortex array perpendicular to the symmetry axis

- specifically, assume $\boldsymbol{\Omega} = \Omega\hat{\boldsymbol{x}}$
- now rotating condensate is *not symmetric* about rotation axis
- this rotating asymmetry induces an additional irrotational flow induced by the moving boundary
- induced velocity potential is proportional to Ωyz; corresponding induced irrotational flow is proportional to $\boldsymbol{\nabla}(\Omega yz)$
- hence generalize previous assumption for fluid velocity:

$$\boldsymbol{v} = \underbrace{\tfrac{1}{2}\boldsymbol{\nabla}\left[b_x(t)x^2 + b_y(t)y^2 + b_z(t)z^2\right]}_{\text{irrot. expansion flow}} + \underbrace{\boldsymbol{\Omega}\times\boldsymbol{r}}_{\text{rotational flow}} + \underbrace{\alpha(t)\,\boldsymbol{\nabla}(\Omega\,yz)}_{\text{induced irrot. flow}}$$

with

$$\alpha(t) = \frac{R_y^2(t) - R_z^2(t)}{R_y^2(t) + R_z^2(t)}$$

- now find different coupled second-order differential equations for evolution of condensate radii

$$\ddot{R}_x = \frac{15N\hbar^2 a}{M^2}\frac{1}{R_x^2 R_y R_z},$$

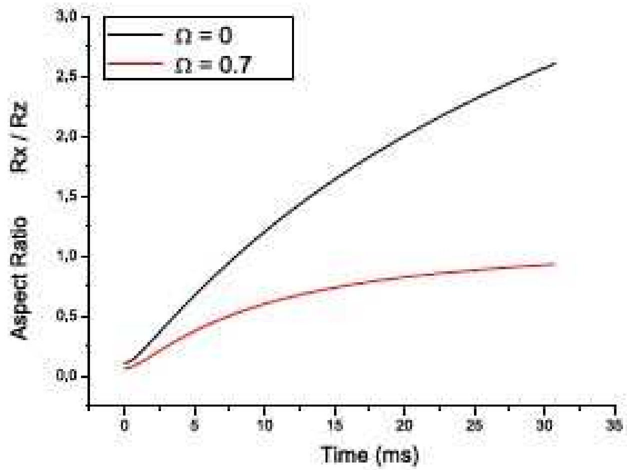
$$\ddot{R}_y = \frac{15N\hbar^2 a}{M^2}\frac{1}{R_y^2 R_z R_x} + 4\left(\frac{N_v\hbar}{M}\right)^2 \frac{R_y}{(R_y^2 + R_z^2)^2},$$

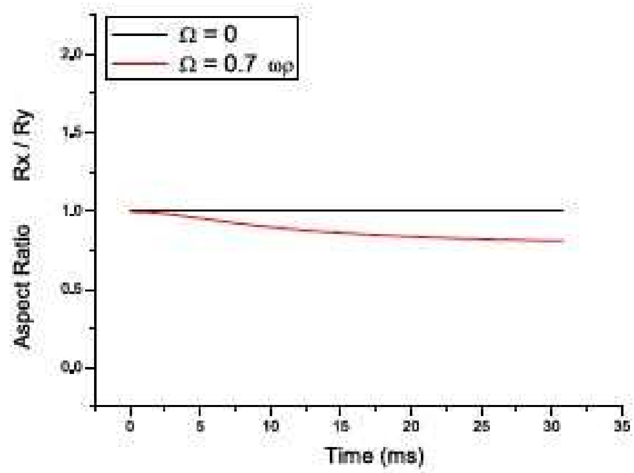
and

$$\ddot{R}_z = \frac{15N\hbar^2 a}{M^2}\frac{1}{R_z^2 R_x R_y} + 4\left(\frac{N_v\hbar}{M}\right)^2 \frac{R_z}{(R_y^2 + R_z^2)^2}$$

- so far, numerical studies ignore the time dependence of initial conditions
- note presence of extra rotation-induced expansion along y and z, but not along x

- evaluate evolution of aspect ratios R_x/R_z and R_x/R_y shown below for $\Omega = 0$ (nonrotating) and for $\Omega = 0.7\omega_\perp$





- note that Ω dramatically reduces growth of first aspect ratio
  1. for $\Omega = 0$, recover usual reversal of aspect ratio
  2. for large Ω, aspect ratio R_x/R_z saturates near unity
- second figure shows that expanding condensate remains nearly axisymmetric

Variational Lagrangian approach: anisotropic random vorticity

Use variational functional

$$\mathcal{L} = \int d^3r \left[i\frac{\hbar}{2}\left(\Psi^* \frac{\partial \Psi}{\partial t} - \Psi \frac{\partial \Psi^*}{\partial t}\right) - \mathcal{E}[\Psi]\right],$$

for a trial function Ψ that depends on various variational parameters.

- here the energy density has the Gross-Pitaevski form

  $$\mathcal{E}[\Psi] = \frac{\hbar^2 |\nabla \Psi|^2}{2M} + V_{\rm tr}|\Psi|^2 + \frac{1}{2} g |\Psi|^4,$$

  where $g = 4\pi\hbar^2 a/M$, with a the s-wave scattering length, and $V_{\rm tr}(\boldsymbol{r})$ is the trap potential (which is turned off at $t = 0$).

- generalize Thomas-Fermi Ψ to include expansion velocity and vortex velocity.

- time-dependent terms and interaction energy lead to simple integrals

- subtle point is evaluation of vortex contribution
- ignore slow translational motion of vortex tangle
- each vortex has circulating velocity field $\boldsymbol{v} = (\hbar/Mr)\hat{\boldsymbol{\phi}}$ around the local vortex axis
- a single vortex line has kinetic energy per unit length $(\pi\hbar^2 n/M)\ln(r_0/\xi)$, where n is local number density away from core, r_0 is inter-vortex distance and ξ is core radius
- assume a length L of turbulent vortex lines per unit volume
- integral over condensate volume yields approximate vortex energy

$$E_v \approx \frac{N\pi\hbar^2}{M} L \ln\left(\frac{1}{L^{1/2}\xi}\right),$$

with $L^{-1/2}$ as the approximate inter-vortex separation.

How does vortex line length per unit volume L depend on condensate dimensions?

- assume a total number N_v of vortices in condensate
- for isotropic turbulence, L should scale like $\sqrt{R_x^2 + R_y^2 + R_z^2}/(R_x R_y R_z)$ (I thank G. Baym for discussions on this point)
- geometry of experiments suggests that turbulence in preferentially in xy plane
- hence introduce anisotropy parameter θ and assume

$$L \approx N_v \frac{\sqrt{\sin^2\theta \left(R_x^2 + R_y^2\right) + \cos^2\theta\, R_z^2}}{R_x R_y R_z}$$

- $\theta = \pi/4$ is isotropic case, and $\theta \approx \pi/2$ describes turbulence in xy plane

Lagrangian approach eventually yields dynamical equations for the expansion radii

$$\frac{2}{7} M \ddot{R}_\perp = -\frac{\partial U}{\partial R_\perp} \quad \text{and} \quad \frac{1}{7} M \ddot{R}_z = -\frac{\partial U}{\partial R_z}$$

- here, U is an effective potential

$$U(R_\perp, R_z) = \underbrace{\frac{15}{7} \frac{\hbar^2 N a}{M R_\perp^2 R_z}}_{\text{interactions}} + \underbrace{\frac{\pi \hbar^2 N_v}{M} \frac{\sqrt{2 \sin^2\theta\, R_\perp^2 + \cos^2\theta\, R_z^2}}{R_\perp^2 R_z} \ln\left(\frac{1}{L^{1/2}\xi}\right)}_{\text{turbulent vortices}}$$

  with separate terms arising from the repulsive interactions and from the turbulent vortices

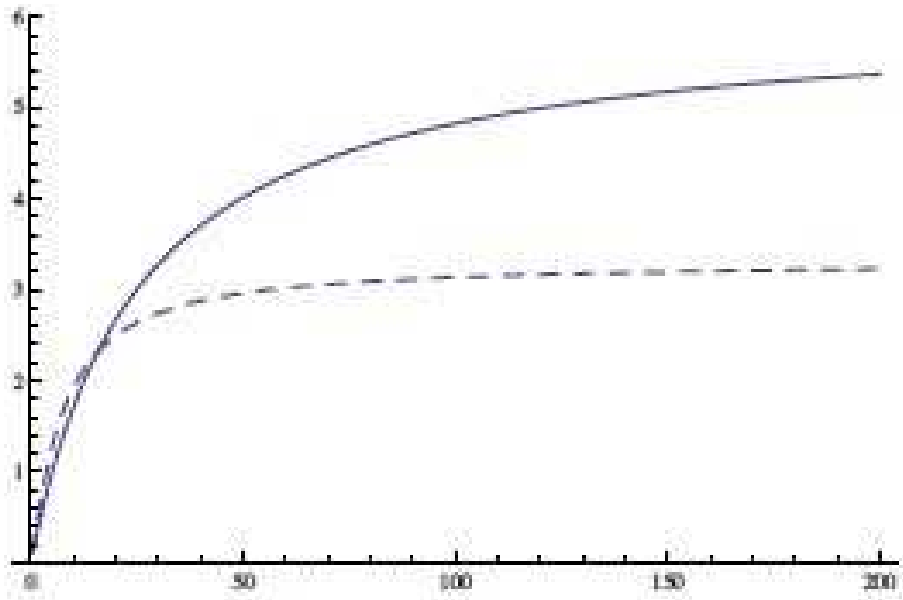
- interaction term is of order R^{-3} and dominates for short time
- vortex term is of order R^{-2} and dominates for large time
- turbulent vortex term here contains only N_v, in contrast to N_v^2 for uniform vortex array (because of random cancellation)

Typical experimental values for Brazil trap are:

- number of ^{87}Rb atoms $N = 2 \times 10^5$
- $\omega_\perp = 2\pi \times 207$ Hz and $\omega_z = 2\pi \times 23$ Hz
- leads to geometric mean angular frequency $\omega_0 = \left(\omega_\perp^2 \omega_z\right)^{1/3} = 625\ \mathrm{s}^{-1}$
- mean bare trap size is $d_0 = \sqrt{\hbar / M\omega_0} = 1.08\ \mu$m (this ignores repulsion)
- use ω_0^{-1} and d_0 as units of time and distance
- initial condensate radii in these units are $R_\perp = 3.30$ and $R_z = 29.7$ (cigar shaped)
- initial vortex core size is small $\xi = 0.146$
- assume number of vortices $N_v \approx 100$

Integrate resulting equations of motion for $0 \le t \le 200$ (this is about 0.3 s)

- need to assume anisotropy parameter $\theta \approx \pi/2$ (random vorticity in xy plane) since otherwise vorticity in z direction enhances radial expansion
- condensate expands radially initially but saturates with aspect ratio $R_\perp/R_z \approx 2.5$ (experiment suggests this ratio is of order one)



- solid line is vortex-free condensate and dashed line is turbulent condensate
- if number of vortices is $N_v \approx 200$, then aspect ratio saturates near 2

Comments, Discussion, and Conclusions

Here, explain qualitatively the anomalous self-similar expansion of turbulent condensate

- first use macroscopic toy model based on rotational dynamics with uniform distributed vorticity aligned perpendicular to symmetry axis of condensate
- based on experimental geometry that couples applied excitation preferentially to axes perpendicular to symmetry axis
- leads to significant reduction of aspect-ratio inversion
- second use Lagrangan variational method to include random turbulent vorticity (preferentially in xy plane)
- both models lead to roughly self-similar expansion of turbulent condensate, but details are not quantitatively correct

Acknowledgments

I thank Vanderlei Bagnato for terrific hospitality during visits to São Carlos (Brazil) and Mônica Caracanhas for valuable discussions and many numerical studies of expanding condensates. Many discussions with Gordon Baym in Aspen helped understand how to include the random vorticity in the Lagrangian formalism.