WALL-CROSSING, FIELD THEORY, AND 5-BRANES

WORK DONE WITH
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KITP MINICONFERENCE ON
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OUTLINE

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1. INTRODUCTION

THIS TALK IS ABOUT THE BPS SPECTRUM OF $\mathcal{N}=2$, $D=4$ FIELD THEORIES.

THE BPS SPECTRUM OF THE THEORY ON $\mathbb{R}^4$ IS A "PIECEWISE CONSTANT" FUNCTION OF THE BOUNDARY CONDITIONS AT ∞ OF VECTORMULTIPLET SCALARS.

RECENTLY THERE HAS BEEN SOME PROGRESS IN UNDERSTANDING PRECISELY <u>HOW</u> THE SPECTRUM DEPENDS ON BOUNDARY CONDITIONS.

THESE ARE CALLED WALL-CROSSING FORMULAE (WCF). THIS TALK WILL GIVE A PHYSICAL INTERPRETATION AND PROOF OF A FAMOUS WCF OF KONTSEVICH + SOIBELMAN.

2. REVIEW $N=2, D=4$ WALL CROSSING

CONSIDER A THEORY ON $\mathbb{R}^4$ WITH $N=2$ SUPERPOINCARE' SYMMETRY

LET $\mathcal{H}$ BE THE ONE-PARTICLE HILBERT SPACE.

AS A REPRESENTATION OF THE $N=2$ SUPERPOINCARE' ALGEBRA $\mathcal{S}$, $\mathcal{H}$ DEPENDS ON THE BOUNDARY VALUES OF FIELDS AT ∞.

THESE BOUNDARY CONDITIONS ARE VALUED IN THE MODULI SPACE OF VACUA: $\mathcal{B}$

FOR $u \in \mathcal{B}$, WRITE $\mathcal{H}_u$.

FOR ALL $u \in \mathcal{B}$ THERE IS AN UNBROKEN ABELIAN GAUGE SYMMETRY OF RANK r, SO $\mathcal{H}$ IS GRADED BY THE SYMPLECTIC LATTICE Γ OF ELEC. + MAG. CHARGES. (OF RANK $2r$).

$$\mathcal{H}_u = \bigoplus_{\gamma \in \Gamma} \mathcal{H}_{\gamma,u}$$

ON EACH SUBSPACE $\mathcal{H}_{\gamma,u}$ THE CENTRAL CHARGE OPERATOR $Z \in \Delta$ IS A SCALAR.

DENOTE THE VALUE $Z_\gamma(u)$

RECALL THE $\mathcal{N}=2, D=4$ SUSY ALGEBRA

$$\mathcal{S} = \mathcal{S}_0 \oplus \mathcal{S}_1$$

$$\mathcal{S}_0 = \left(\mathrm{Spin}(1,3) \ltimes \mathbb{R}^4\right) \oplus u(2) \oplus \mathbb{R}$$

$$M_{\mu\nu} \qquad P_\mu \qquad\qquad Z$$

$$\mathcal{S}_1 = \left[\mathrm{Spinor} \otimes \mathbb{C}^2\right]_{\mathbb{R}}$$

$$Q_{\alpha I}, \ \bar{Q}^{\dot{\alpha} I}$$

$$\{Q_{\alpha I}, \bar{Q}_{\dot{\beta}}{}^{J}\} = 2 P_\mu \sigma^\mu_{\alpha\dot{\beta}} \delta_I{}^J$$

$$\{Q_{\alpha I}, Q_{\beta J}\} = 2 Z \epsilon_{\alpha\beta} \epsilon_{IJ}$$

UNITARY IRREPS SATISFY BPS BOUND:

$$E \geq |Z|$$

DEF. $\mathcal{H}^{BPS}_{\gamma,u}$ = SUBSPACE SATURATING THE BPS BOUND.

ON THIS SUBSPACE $E = |Z_\gamma(u)|$

SOME BPS PARTICLES CAN BE VIEWED AS BOUNDSTATES OF OTHERS

[Cecotti, Fendley, Intriligator, Vafa ; Seiberg & Witten]

$Z_\gamma(u)$ IS LINEAR IN $\gamma = \gamma_1 + \gamma_2$ SO

$$E(u) = |Z_\gamma(u)| - \left(|Z_{\gamma_1}(u)| + |Z_{\gamma_2}(u)|\right) \leq 0$$

$\Longrightarrow$ DECAY ONLY HAPPENS ALONG WALLS OF MARGINAL STABILITY:

$$MS(\gamma_1, \gamma_2) := \{ u \mid Z_{\gamma_1}(u) / Z_{\gamma_2}(u) \in \mathbb{R}_+ \}$$

WCF: BE MORE QUANTITATIVE ABOUT "HOW MANY" STATES DECAY

DEFINE THE BPS INDEX

$$\Omega^{Ph}(\gamma;u) := -\frac{1}{2}\mathrm{Tr}_{\mathcal{H}^{BPS}_{\gamma,u}}(2J_3)^2(-1)^{2J_3}$$

DENEF & MOORE GAVE FORMULAE FOR $\Delta\Omega$ FOR DECAYS $\gamma \to \gamma_1 + \gamma_2$ WHERE AT LEAST ONE OF γ_1, γ_2 ARE PRIMITIVE.

THE DERIVATION IS BASED ON DENEF'S MULTICENTERED SOLUTIONS OF $\mathcal{N}=2$ SUGRA AND QUIVER QUANTUM MECHANICS

THE METHODS ARE DIFFICULT TO USE WHEN <u>BOTH</u> γ_1, γ_2 ARE NON-PRIMITIVE

KONTSEVICH & SOIBELMAN PROPOSED A REMARKABLE W.C.F. FOR AN INDEX $\Omega^{DT}(\gamma;u)$,

"GENERALIZED DONALDSON-THOMAS INVARIANT OF A CALABI-YAU 3-FOLD"

WE EXPECT THAT

$$\Omega(\gamma;u) = \Omega^{Ph}(\gamma;u) = \Omega^{DT}(\gamma;u)$$

SO THE KS WCF APPLIES TO PHYSICAL BPS DEGENERACIES.

THEIR FORMULA APPLIES TO ALL DECAYS $\gamma \to \gamma_1 + \gamma_2$.

THIS TALK PROVES THE KS WCF FOR $\Omega^{Ph}(\gamma;u)$ IN $\mathcal{N}=2$ FIELD THEORIES

SUMMARY OF PP. 2–9 :

- $\mathcal{N} = 2$, $D = 4$ FIELD THEORY

- UNBROKEN RANK r ABELIAN GAUGE THEORY IN THE IR $\Rightarrow$

Γ = SYMPLECTIC LATTICE OF RANK $2r$.

- $Z \in \mathrm{Hom}(\Gamma, \mathbb{C})$ <u>CENTRAL</u> <u>CHARGE</u>

DEPENDS ON $u \in \mathcal{B}$:= MODULI OF VACUA

- $\mathcal{H}^{BPS}_{\gamma, u} = \{ \psi \mid E = |Z_\gamma(u)| \}$

$\Omega(\gamma; u) = \mathrm{Tr}_{\mathcal{H}} (-1)^F$; PIECEWISE CONST.

- BPS BOUNDSTATES OF BPS STATES OF CHARGES γ_1, γ_2 CAN DECAY AS u CROSSES

$$MS(\gamma_1, \gamma_2) := \{ u \mid Z_{\gamma_1}(u) / Z_{\gamma_2}(u) \in \mathbb{R}_+ \}$$

- THIS TALK PROVES THE KS WCF FOR FIELD THEORIES.

3. THE KONTSEVICH–SOIBELMAN FORMULA

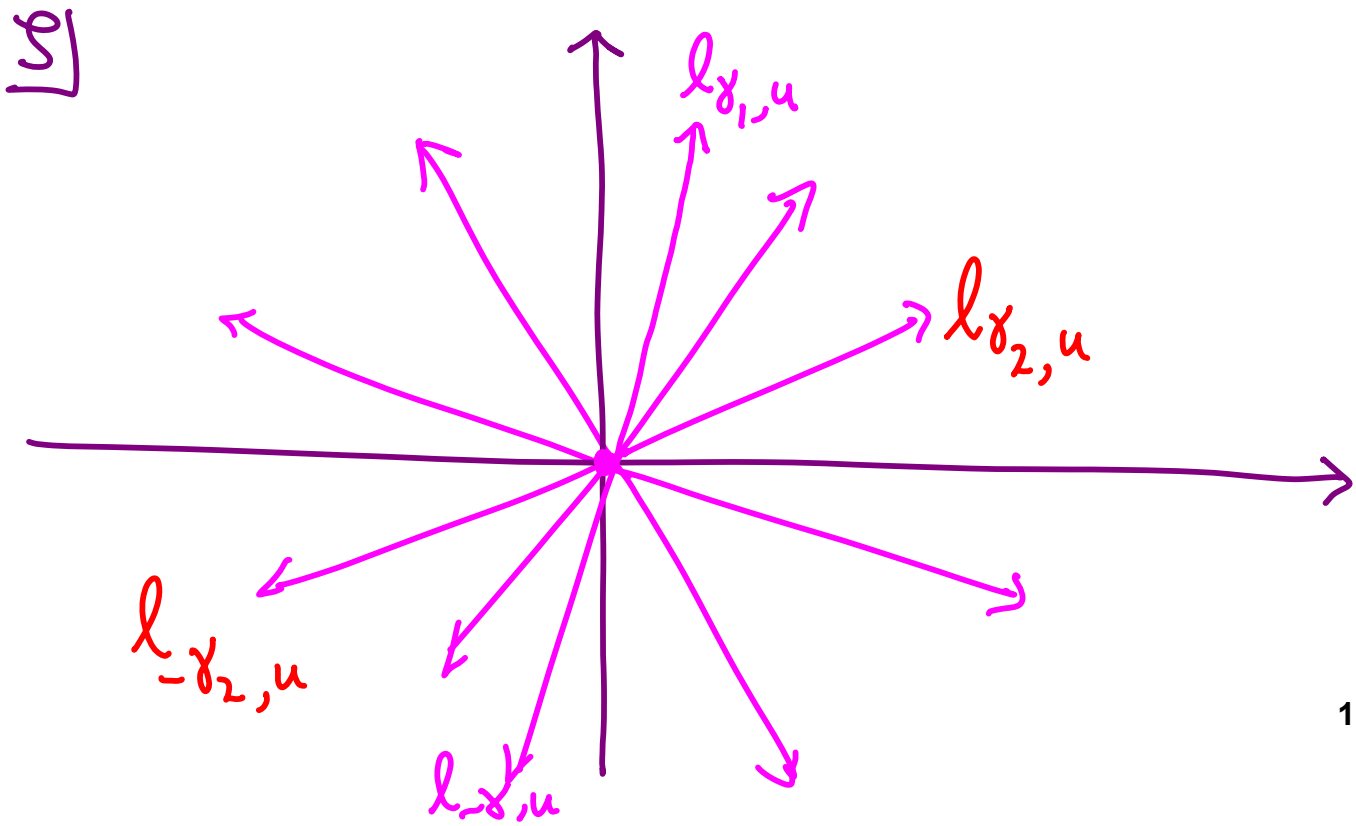
DATA:

1. SYMPLECTIC LATTICE Γ
2. CENTRAL CHARGES $Z_\gamma(u)$, $u \in \mathcal{B}$
3. PIECEWISE CONSTANT $\Omega(\gamma; u) \in \mathbb{Z}$

FIRST INGREDIENT: BPS RAYS:

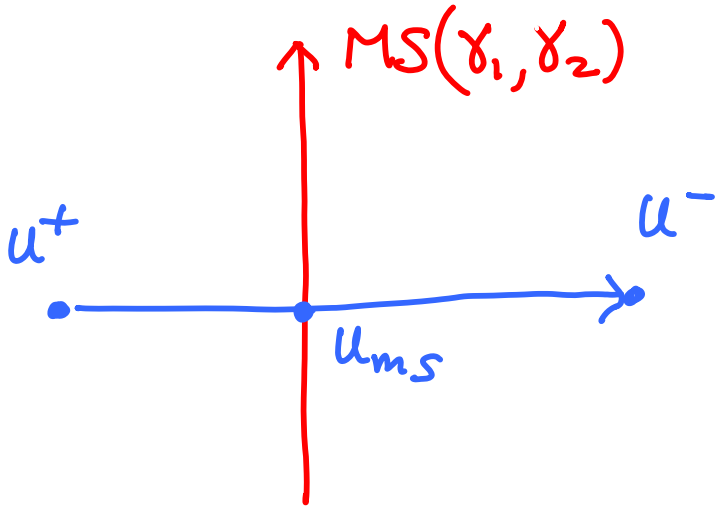
FOR $u \in \mathcal{B}$, $\gamma \in \Gamma$

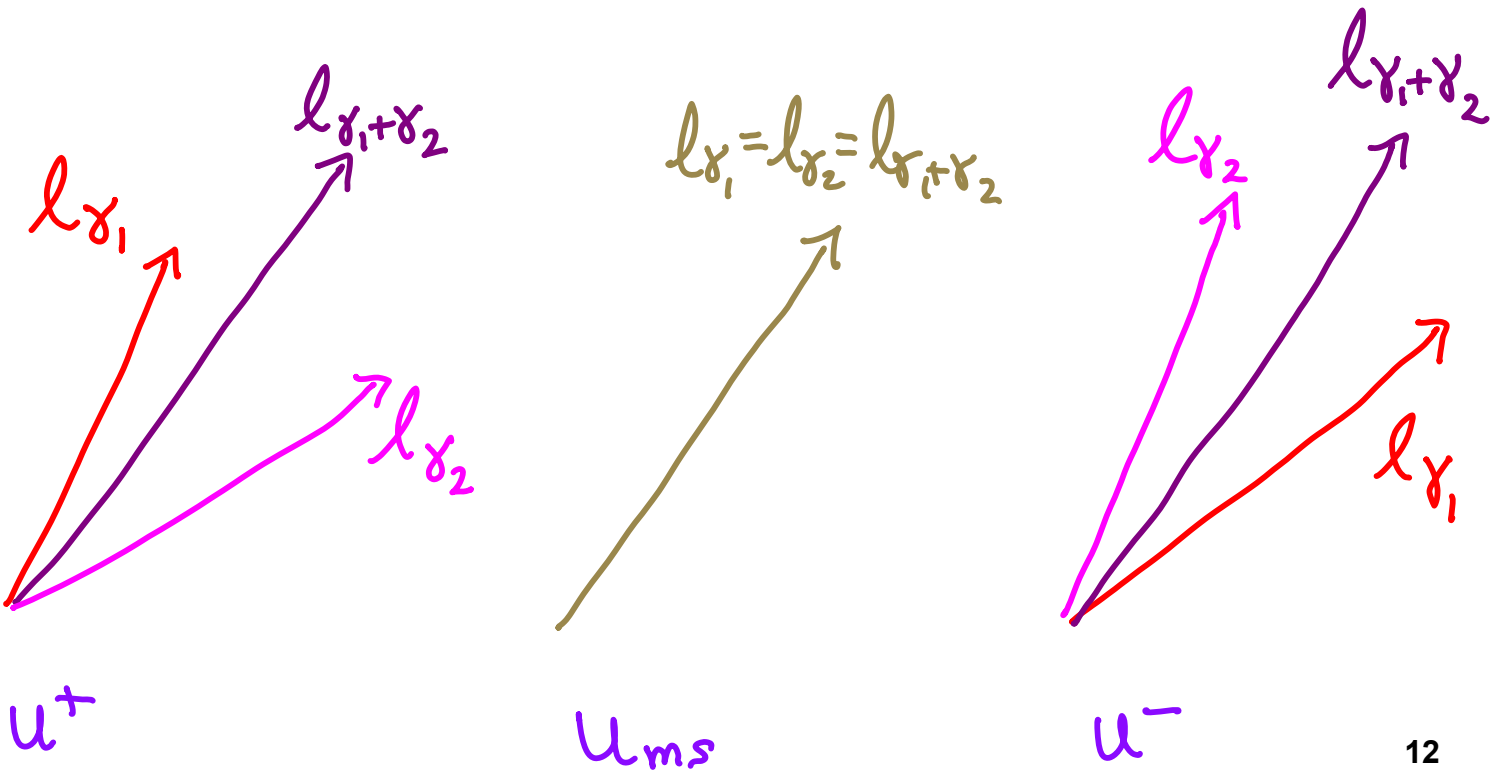
$$\ell_{\gamma,u} := Z_\gamma(u)\mathbb{R}_- = \{\zeta \mid \zeta/Z_\gamma(u) \in \mathbb{R}_-\}$$



AS u VARIES THE SLOPES OF THE BPS RAYS VARY

AS u CROSSES A WALL $MS(\gamma_1, \gamma_2)$ BPS RAYS WILL COALESCE





SECOND INGREDIENT: A SYMPLECTIC TORUS:

- INTRODUCE THE COMPLEX TORUS

$$T = \Gamma^* \otimes_{\mathbb{Z}} \mathbb{C}^* \cong \underbrace{\mathbb{C}^* \times \cdots\cdots \times \mathbb{C}^*}_{2r}$$

$\gamma \in \Gamma \implies$ FUNCTION $X_\gamma : T \to \mathbb{C}^*$

"HOLOMORPHIC FOURIER MODES"

CHOOSING A BASIS γ_i FOR $\Gamma \implies$

$$(e^{\theta_1}, \cdots, e^{\theta_{2r}}) \in T, \quad \theta_i \in \mathbb{C}$$

$$X_\gamma = \exp[\gamma \cdot \theta]$$

- HOLOMORPHIC SYMPLECTIC FORM:

$$\varpi^T := \frac{1}{2}\epsilon^{ij}\frac{dX_{\gamma_i}}{X_{\gamma_i}} \wedge \frac{dX_{\gamma_j}}{X_{\gamma_j}} \qquad \epsilon_{ij} = \langle \gamma_i, \gamma_j \rangle$$

- FOR EACH $\gamma \in \Gamma$ DEFINE A SYMPLECTOMORPHISM:

$$K_\gamma : X_{\gamma'} \longrightarrow X_{\gamma'}\left(1 - \sigma(\gamma) X_\gamma\right)^{\langle \gamma', \gamma \rangle}$$

$$X_{\gamma'} \rightarrow X_{\gamma'} \exp\left[\langle \gamma, \gamma' \rangle \log\left(1 - \sigma(\gamma) X_\gamma\right)\right]$$

$$\sigma(\gamma) = \pm 1$$

ABOUT THE SIGN :

K & S INTRODUCE A LIE ALGEBRA

$$[e_{\gamma_1}, e_{\gamma_2}] = (-1)^{\langle \gamma_1, \gamma_2 \rangle} \langle \gamma_1, \gamma_2 \rangle e_{\gamma_1 + \gamma_2}$$

DEFINE A GROUP ELEMENT

$$U_\gamma := \exp\left(\sum_{n=1}^{\infty} \frac{e_{n\gamma}}{n^2} \right)$$

AND WORK WITH U_γ INSTEAD OF K_γ

CHOOSE A QUADRATIC REFINEMENT

$$\frac{\sigma(\gamma_1 + \gamma_2)}{\sigma(\gamma_1)\sigma(\gamma_2)} = (-1)^{\langle \gamma_1, \gamma_2 \rangle}$$

$\sigma(\gamma) e_\gamma$ GENERATE THE LIE ALGEBRA OF SYMPLECTIC VECTOR FIELDS.

EXAMPLE

$$r=1 \implies \Gamma \cong \mathbb{Z} \oplus \mathbb{Z}$$

$$\langle (a,b),(a',b') \rangle = ab' - a'b$$

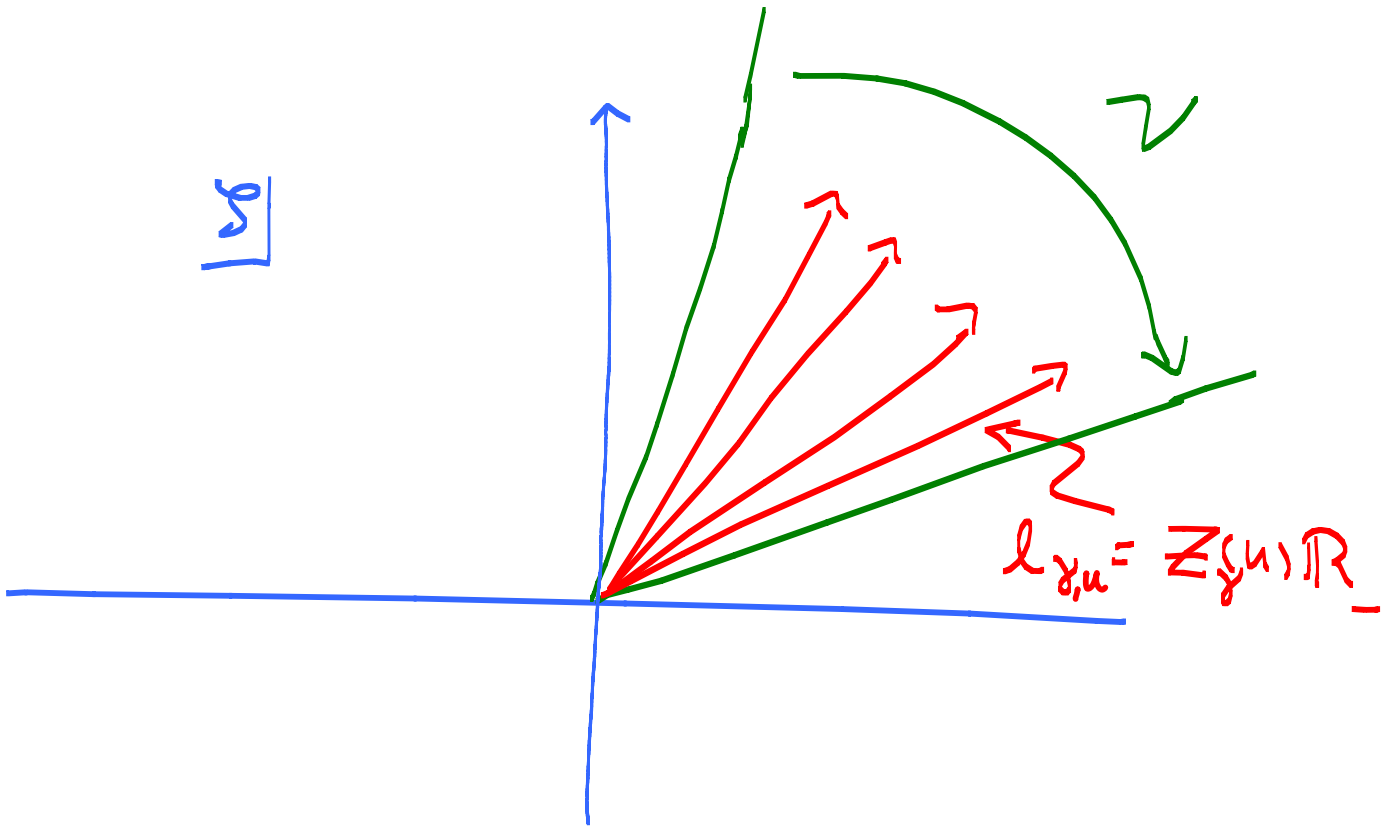
$$T \cong \mathbb{C}^* \times \mathbb{C}^*$$

$$x = X_{1,0} \qquad y = X_{0,1}$$

$$\overline{\omega}^T = \frac{dx}{x} \wedge \frac{dy}{y}$$

$$K_{a,b} : \begin{cases} x \longrightarrow x\left(1-(-1)^{ab}x^a y^b\right)^b \\ y \longrightarrow y\left(1-(-1)^{ab}x^a y^b\right)^{-a} \end{cases}$$

THIRD INGREDIENT: CONVEX CONE $\mathcal{V}$



FOR EACH BPS RAY DEFINE

$$S_\gamma := \prod_{\gamma' \parallel \gamma} K_{\gamma'}^{\Omega(\gamma';u)}$$

AND THEN DEFINE:

$$\mathcal{A}_{\mathcal{V}} := \overrightarrow{\prod_{\ell_{\gamma,u} \subset \mathcal{V}}} S_\gamma$$

$$\mathcal{A}_{\mathcal{V}} := \overrightarrow{\prod_{\ell_\gamma \subset \mathcal{V}}} S_\gamma = \overrightarrow{\prod_{-z_\gamma \in \mathcal{V}}} K_\gamma^{\Omega(\gamma;u)}$$

THE PRODUCT IS TAKEN OVER THE RAYS IN THE <u>CLOCKWISE ORDER</u> (DECREASING SLOPE)

$\mathcal{A}_{\mathcal{V}}$ DEPENDS ON u IN TWO WAYS

1. THE ORDERING OF FACTORS DEPENDS ON u

2. THE $\Omega(\gamma;u)$ DEPEND ON u ...

THE KS FORMULA STATES THAT

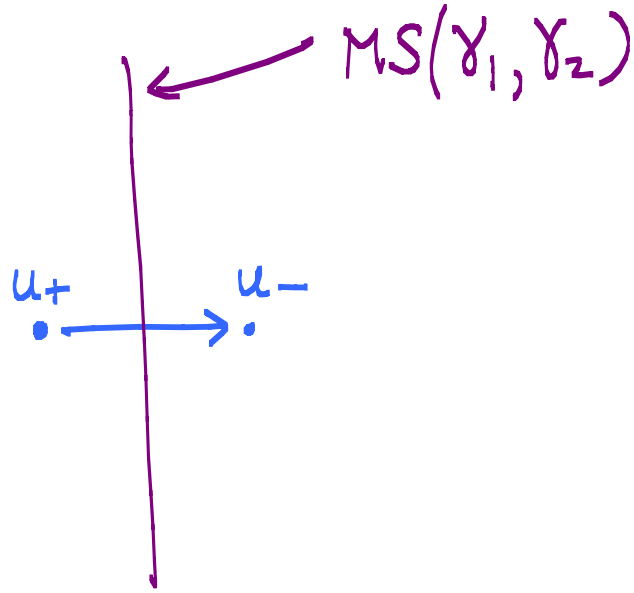
NEVERTHELES,

$$\mathcal{A}_{\mathcal{V}} = \overrightarrow{\prod_{\ell_\gamma \subset \mathcal{V}}} K_\gamma^{\Omega(\gamma;u)}$$

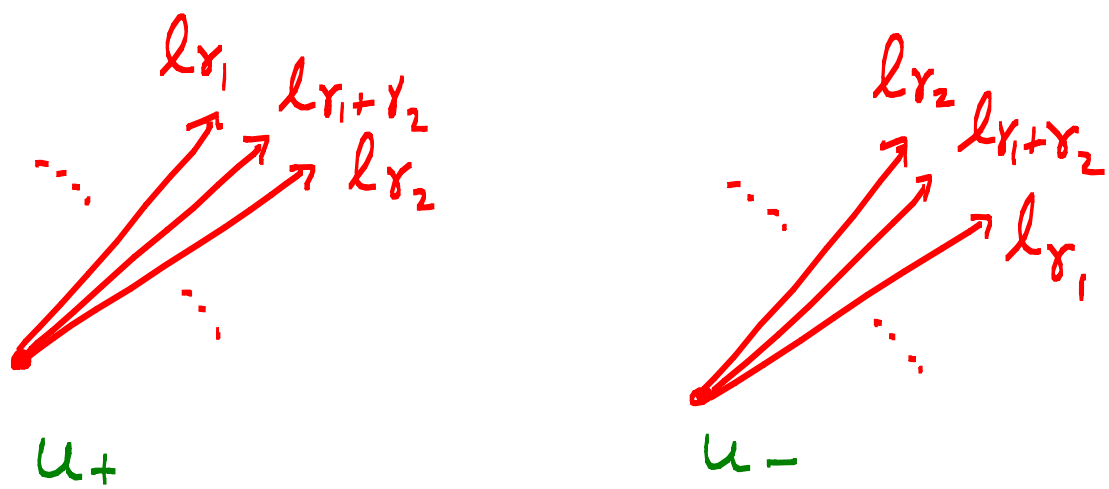
IS <u>CONSTANT IN u</u> AS LONG AS NO BPS RAY ENTERS OR LEAVES THE SECTOR $\mathcal{V}$.

THIS IS A WALL-CROSSING FORMULA ...

AS u CROSSES A WALL



$$\ell_\gamma = Z_\gamma(u) \cdot \mathbb{R}_-$$ ROTATES



$\Longrightarrow$ EXCHANGE ORDER IN

$$\mathcal{A}_\nu = \overrightarrow{\prod_{\ell_\gamma \subset \nu}} K_\gamma^{\Omega(\gamma;u)}$$

$\Omega(\gamma;u)$ MAKES A COMPENSATING CHANGE

4. COMPACTIFICATION OF $N=2, D=4$ FIELD THEORIES ON S^1

A. SEIBERG-WITTEN SOLUTION

G - COMPACT S.S. GAUGE GROUP, RANK $= r$

$\Longrightarrow$ $D=4, N=2$ FIELD THEORY

(CAN ALSO INCLUDE HM'S)

$$\mathcal{B} = (\mathfrak{g}_{\mathbb{C}})^G \cong \mathbb{C}^r : \quad \begin{aligned} u_2 &= \langle \mathrm{Tr}\, \Phi^2 \rangle \\ u_3 &= \langle \mathrm{Tr}\, \Phi^3 \rangle \\ &\vdots \end{aligned}$$

S & W GAVE FORMULAE FOR

- $Z_\gamma(u)$
- LOW ENERGY ABELIAN GAUGE THRY.

IN TERMS OF

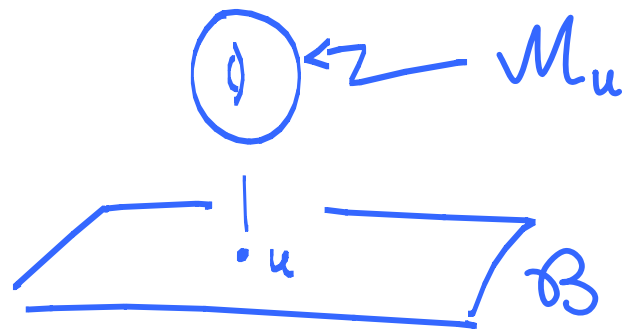
<u>SPECIAL</u> <u>KÄHLER</u> <u>GEOMETRY</u>

REVIEW SPECIAL KÄHLER GEOM:

c.f. D. FREED, hep-th/9712042

VIEW Γ AS A LOCAL SYSTEM OVER $\mathcal{B}$

$$\begin{tikzcd} \mathcal{M}_u \arrow[r] \arrow[d] & \mathcal{M} \arrow[d] \\ u \arrow[r, hook] & \mathcal{B} \end{tikzcd}$$



$$\mathcal{M}_u = \Gamma_u^* \otimes_{\mathbb{Z}} \left(\mathbb{R}/2\pi\mathbb{Z}\right) \cong U(1)^{2r}$$

IN REGIONS OF $\mathcal{B}$ CHOOSE A DUALITY FRAME:

$$\Gamma = \Gamma_{el} \oplus \Gamma_{mag}, \quad \Gamma_{mag} = \Gamma_{el}^*$$

$$= \mathrm{Span}\{\alpha^I\} \oplus \mathrm{Span}\{\beta_I\}$$

$$\langle \alpha^I, \alpha^J \rangle = \langle \beta_I, \beta_J \rangle = 0 \qquad \langle \alpha^I, \beta_J \rangle = \delta^I_J$$

CHOOSING A DUALITY FRAME:

1. CENTRAL CHARGE FUNCTION

$$Z_\gamma(u) = a \cdot \gamma_{el} + a_D \cdot \gamma_{mg}$$

$$a^I = Z_{\alpha_I}(u) \qquad a_{D,I} = Z_{\beta^I}(u)$$

2. $a^I = Z_{\alpha_I}(u) \qquad I = 1, \dots, r$

LOCAL COORD'S ON $\mathcal{B}$

3. $$\tau_{IJ} = \frac{\partial a_{D,J}}{\partial a^I} = \tau_{JI}, \quad \mathrm{Im}\,\tau > 0$$

$\Rightarrow \mathcal{M}_u$ IS AN ABELIAN VARIETY

4. LOW ENERGY LAGRANGIAN:

$$\mathcal{L} = \frac{-1}{4\pi} \mathrm{Im}\,\tau_{IJ} \left(da^I * d\bar{a}^J + F^I * F^J \right)$$

$$+ \frac{i}{4\pi} \mathrm{Re}\,\tau_{IJ} \; F^I \wedge F^J$$

S & W CONSTRUCT AN EXPLICIT FAMILY OF RIEMANN SURFACES Σ_u, TOGETHER WITH A MEROMORPHIC (1,0)-FORM λ_u ON Σ_u

- $\Gamma_u = H_1(\overline{\Sigma}_u, \mathbb{Z})$
- $Z_\gamma(u) = \oint_\gamma \lambda_u \qquad \gamma \in \Gamma_u$
- $\mathcal{M}_u = \mathrm{JAC}(\overline{\Sigma}_u)$

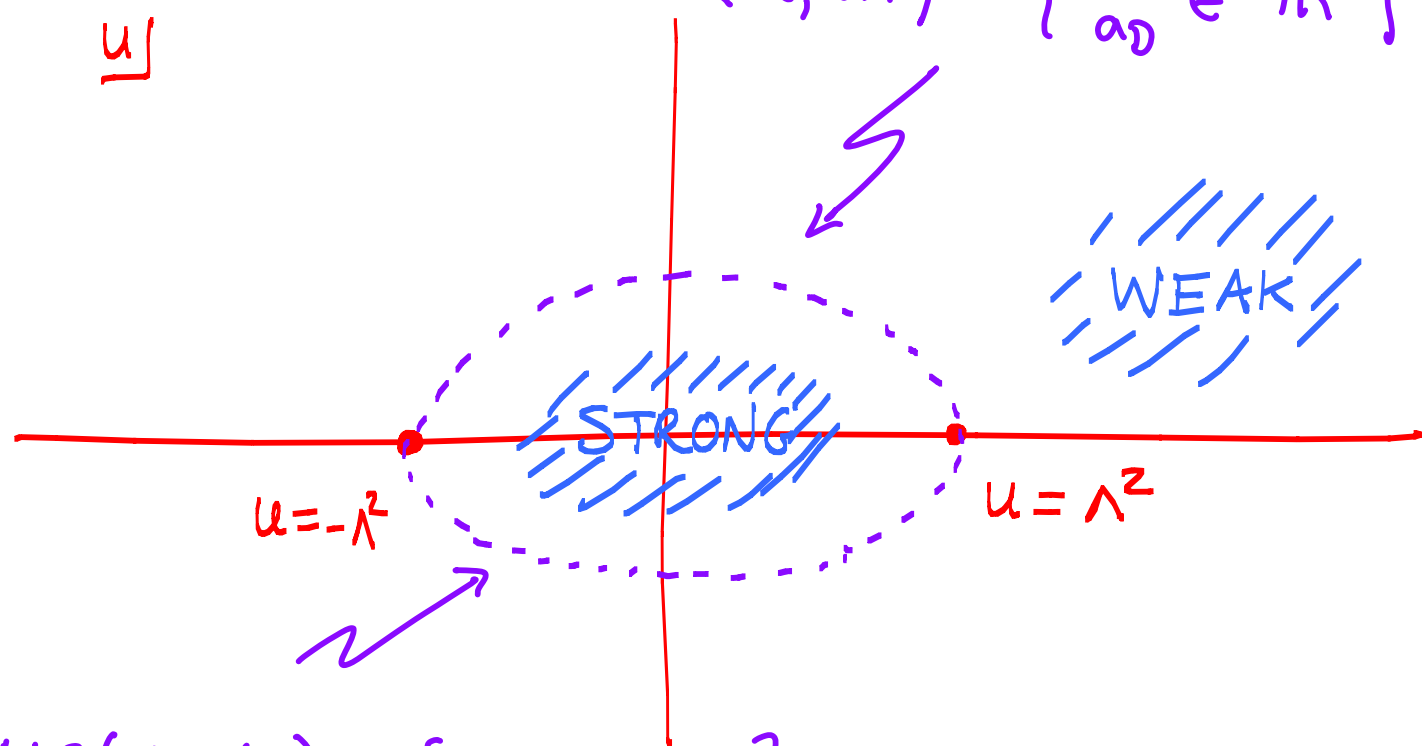
BASIC EXAMPLE: $G = SU(2)$

$$\Sigma_u : \quad t + \frac{\Lambda^4}{t} = v^2 - 2u$$

$$\lambda_u = v \frac{dt}{t}$$

$$a = \oint_\alpha \lambda_u \qquad a_D = \oint_\beta \lambda_u$$

$$MS(\gamma_e, \gamma_m) = \left\{ \frac{a}{a_D} \in \mathbb{R} \right\}$$



$$MS(-\gamma_e, \gamma_m) = \left\{ -\frac{a}{a_D} \in \mathbb{R} \right\}$$

SPECTRUM:

$$\mathcal{H}^{BPS}_{WEAK} = \bigoplus_{n\in\mathbb{Z}} HM(2n,1) \oplus VM(2,0)$$

$$\oplus \text{ CONJUGATE}$$

$$\mathcal{H}^{BPS}_{STRONG} = HM(2,-1) \oplus HM(0,1)$$

$$\oplus \text{ CONJUGATE}$$

[Bilal & Ferrari]

KS IDENTITY:

$$K_{2,-1}K_{0,1} = K_{0,1}K_{2,1}K_{4,1}\cdots K_{2,0}^{-2}\cdots K_{6,-1}K_{4,-1}K_{2,-1}$$

IT IS TRUE !!!

REMARK: ADDING MASSIVE FLAVORS GENERALIZES THE W.C.F.

$$\Gamma \longrightarrow \Gamma \oplus \Gamma^f$$
$$\gamma \longrightarrow \gamma + \gamma^f$$

Introduce constants $\log \mu \in (\Gamma^f)^* \otimes \mathbb{C}^*$

$$X_{\gamma, \gamma_f} = X_\gamma \cdot \prod_a (\mu^a)^{\gamma_a^f}$$

$$K_{\gamma, \gamma_f} : X_{\gamma'} \to X_{\gamma'} \left(1 - \sigma(\gamma) X_{\gamma, \gamma_f}\right)^{\langle \gamma', \gamma \rangle}$$

WCF AGAIN HOLDS WITH

$$Z_{\gamma, \gamma_f}(u) = Z_\gamma(u) + m^a \gamma_a^f$$

B. COMPACTIFY ON A CIRCLE.

- NOW CONSIDER THE THEORY ON $\mathbb{R}^3 \times S^1_R$.

- LOW ENERGY THEORY IS A 3D σ-MODEL: $\mathbb{R}^3 \longrightarrow \mathcal{M}$

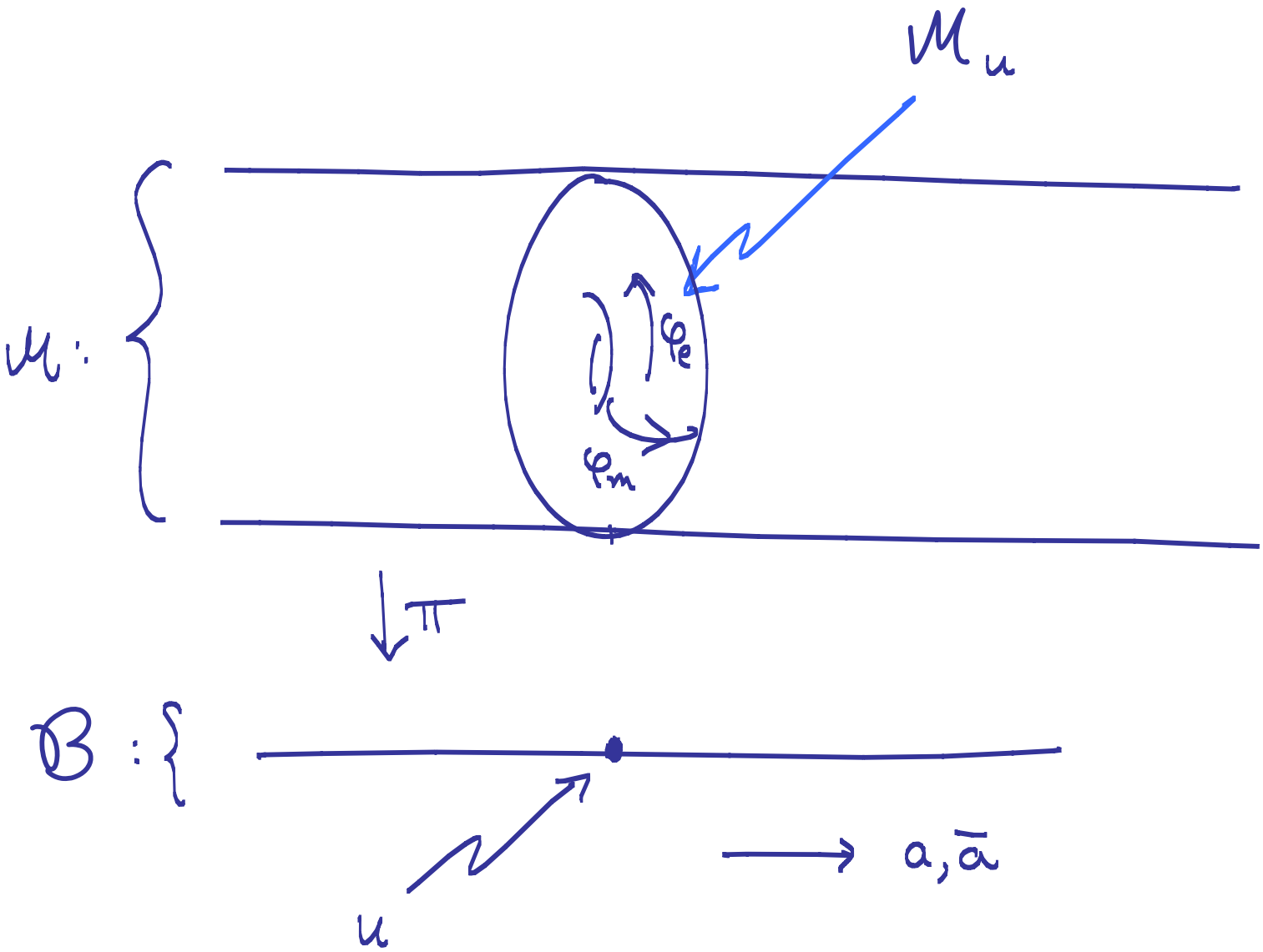
$$a^I(\vec{x}, x^4) \longrightarrow a^I(\vec{x})$$

$$\left.\begin{aligned} \varphi_e^I &= \int_{S^1} A_3^I \, dx^4 \\ \varphi_{m,I} &= \int_{S^1} (A_{D,3})_I \, dx^4 \end{aligned}\right\} \text{ PERIODIC!}$$

TOPOLOGICALLY $\mathcal{M}$ IS A TORUS FIBRATION OVER $\mathcal{B}$:

IT IS EXACTLY $\mathcal{M} = T^* \otimes \mathbb{R}/2\pi\mathbb{Z}$

THAT APPEARED ABOVE:



- SUPERSYMMETRY $\Longrightarrow$

$\mathcal{M}$ MUST CARRY A HYPERKÄHLER METRIC

LET US TRY TO DESCRIBE IT

THE SEMI-FLAT METRIC

LEADING $R \to \infty$ APROXIMATION:

USE DIMENSIONAL REDUCTION

+ DUALIZATION OF 3D GAUGE FIELD:

$$\mathcal{L}^{(3)} = -\frac{R}{2} \operatorname{Im}\tau_{IJ}\, da^I * d\bar{a}^J$$

$$- \frac{1}{8\pi^2 R} (\operatorname{Im}\tau)^{-1,IJ}\, dz_I * d\bar{z}_J$$

$$dz_I = d\varphi_{m,I} - \tau_{IJ}\, d\varphi_e^J$$

THIS DEFINES THE SEMIFLAT METRIC

$$g^{sf} = R(\operatorname{Im}\tau)|da|^2 + \frac{1}{4\pi^2 R}(\operatorname{Im}\tau)^{-1}|dz|^2$$

C. THE KEY IDEA

- THE METRIC g^{sf} RECEIVES QUANTUM CORRECTIONS FROM BPS PARTICLE WORLD-LINES WRAPPING S^1.

- THEREFORE THE QUANTUM CORRECTIONS DEPEND ON THE BPS SPECTRUM.

- THE TRUE METRIC g SHOULD BE A SMOOTH METRIC ON $\mathcal{M}$ AWAY FROM THE LOCUS IN $\mathcal{B}$ WHERE BPS PARTICLES BECOME $M=0$.

- SMOOTHNESS OF g ACROSS WALLS OF M.S. IMPLIES A WCF.

CLAIM: IT IS THE KS WCF.

5. TWISTOR SPACE APPROACH

WE WILL USE HITCHIN'S THEOREM: KNOWING $(\mathcal{M}, g)$ IS EQUIVALENT TO KNOWING TWISTOR SPACE $Z := \mathcal{M} \times \mathbb{CP}^1$ AS A HOLOMORPHIC MANIFOLD.

<u>THEOREM</u>: IF $(\mathcal{M}, g)$ IS HK OF DIMENSION $4r$ THEN:

1. $\exists$ HOLO. FIBRATION

$$p : Z \longrightarrow \mathbb{CP}^1$$

$$\mathcal{M}^\zeta = p^{-1}(\zeta) = \mathcal{M} \text{ IN COMPLEX STRUCTURE } \zeta$$

2. $\exists$ HOLOMORPHIC SECTION

$$\varpi \text{ OF } \Omega^2_{Z/\mathbb{CP}^1} \otimes \mathcal{O}(2)$$

$\varpi_\zeta := \varpi|_{\mathcal{M}^\zeta}$ = HOLOMORPHIC SYMPLECTIC FORM ON $\mathcal{M}^\zeta$

3. $\exists$ ANTI-HOLOMORPHIC $\sigma : Z \to Z$

COVERING $\zeta \to -1/\bar{\zeta}$

4. $\forall x \in \mathcal{M}$, $\exists$ HOLOMORPHIC SECTION

$S_x : \mathbb{CP}^1 \longrightarrow Z$ WITH NORMAL BUNDLE $\mathcal{O}(1)^{\oplus 2r}$

CONVERSELY,

GIVEN 1, 2, 3, 4 ONE CAN RECONSTRUCT THE METRIC:

FOR $\zeta \in \mathbb{C}^*$:

$$\varpi_\zeta = -\frac{i}{2\zeta}\omega_+ + \omega_3 - \frac{i}{2}\zeta\,\omega_-$$

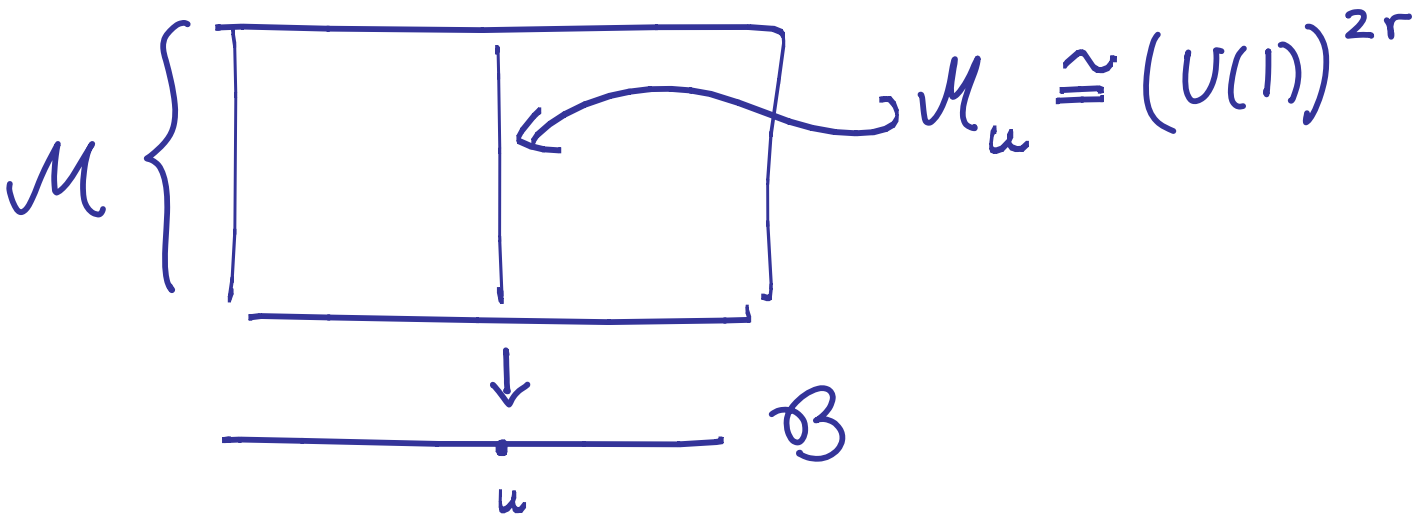
KÄHLER FORM

$$\omega_+ = \omega_1 + i\,\omega_2$$

OUR STRATEGY IS TO CONSTRUCT ϖ_ζ EXPLICITLY FOR $\mathcal{M}^5$ USING A "NICE" SET OF HOLOMORPHIC FUNCTIONS ON TWISTOR SPACE:

$$\chi_\gamma \,, \quad \gamma \in \Gamma$$

USE THE TORUS FIBRATION OF $\mathcal{M}$:



FOR $\zeta \neq 0, \infty$ $\mathcal{M}_u$ IS NOT HOLOMORPHIC

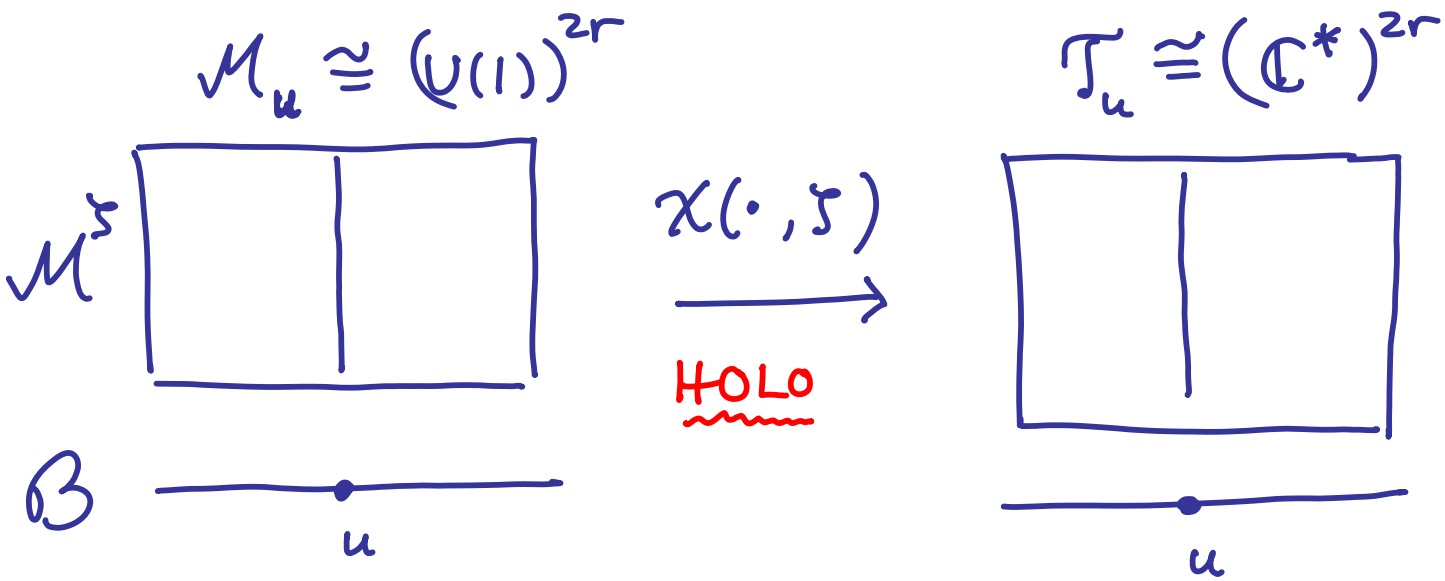
BUT CONSIDER $\mathcal{T} := \Gamma^* \otimes \mathbb{C}^*$

- $\mathcal{T}$ HAS A FIXED COMPLEX STRUCTURE WITH HOLOMORPHIC FIBERS $\cong (\mathbb{C}^*)^{2r}$
- $\mathcal{T}$ HAS HOLOMORPHIC FUNCTIONS X_γ
- $\mathcal{T}$ HAS A FIBERWISE HOLOMORPHIC SYMPLECTIC FORM:

$$\varpi^{\mathcal{T}} := \frac{1}{2}\epsilon^{ij} \frac{dX_{\gamma_i}}{X_{\gamma_i}} \wedge \frac{dX_{\gamma_j}}{X_{\gamma_j}}$$

WE SEARCH FOR A HOLOMORPHIC MAP

$$\mathcal{X} : Z \longrightarrow \mathcal{T} = \Gamma^* \otimes \mathbb{C}^*$$



SO THAT: $\varpi_\zeta = \mathcal{X}(\cdot, \zeta)^* (\varpi^T)$

IF WE DEFINE $\mathcal{X}_\gamma := \mathcal{X}^* (X_\gamma)$

$$\Rightarrow \quad \varpi_\zeta = \frac{1}{2} \epsilon^{ij} \frac{d\mathcal{X}_{\gamma_i}}{\mathcal{X}_{\gamma_i}} \wedge \frac{d\mathcal{X}_{\gamma_j}}{\mathcal{X}_{\gamma_j}}$$

WE CAN VIEW

$$\varpi_\zeta = \frac{1}{2}\epsilon^{ij}\,\frac{d\mathcal{X}_{\gamma_i}}{\mathcal{X}_{\gamma_i}} \wedge \frac{d\mathcal{X}_{\gamma_j}}{\mathcal{X}_{\gamma_j}}$$

IN TWO WAYS:

- KNOW THE METRIC $\Rightarrow$ CONSTRUCT $\mathcal{X}_\gamma$

$\rightarrow$ DO THIS FOR THE SEMIFLAT METRIC AND FIRST QUANTUM CORRECTION

- ULTIMATELY, WE DEFINE THE $\mathcal{X}_\gamma$ AND USE THEM TO DEFINE ϖ_ζ (AND HENCE THE HK METRIC)

EXAMPLE: SEMI-FLAT LIMIT

WE KNOW g^{sf} $\Longrightarrow$ COMPUTE

$$\varpi_\zeta = -\frac{i}{2\zeta}\omega_+ + \omega_3 - \frac{i}{2}\zeta\,\omega_-$$

$$= \frac{i}{8\pi\zeta}\, da^I \wedge dz_I + \cdots + \frac{i\zeta}{8\pi}\, \overline{da^I \wedge dz_I}$$

FIND $\mathcal{X}_{\gamma_i}$ SO THAT:

$$\varpi_\zeta = \frac{1}{8\pi^2 R}\, \epsilon^{ij}\, \frac{d\mathcal{X}_{\gamma_i}}{\mathcal{X}_{\gamma_i}} \wedge \frac{d\mathcal{X}_{\gamma_j}}{\mathcal{X}_{\gamma_j}}$$

SOLUTION: DEFINE $\theta_\gamma : \Gamma^* \otimes \mathbb{R}/2\pi\mathbb{Z} \longrightarrow \mathbb{R}/2\pi\mathbb{Z}$

$$\boxed{\mathcal{X}_\gamma^{sf} = \exp\left[\pi R \zeta^{-1} Z_\gamma + i\,\theta_\gamma + \pi R \zeta\, \bar{Z}_\gamma\right]}$$

[A. NEITZKE & B. PIOLINE]

- LEADING APPXT. TO $\mathcal{X}_\gamma$ FOR $R \to \infty$
- NO Q.C.'S FROM BPS STATES.

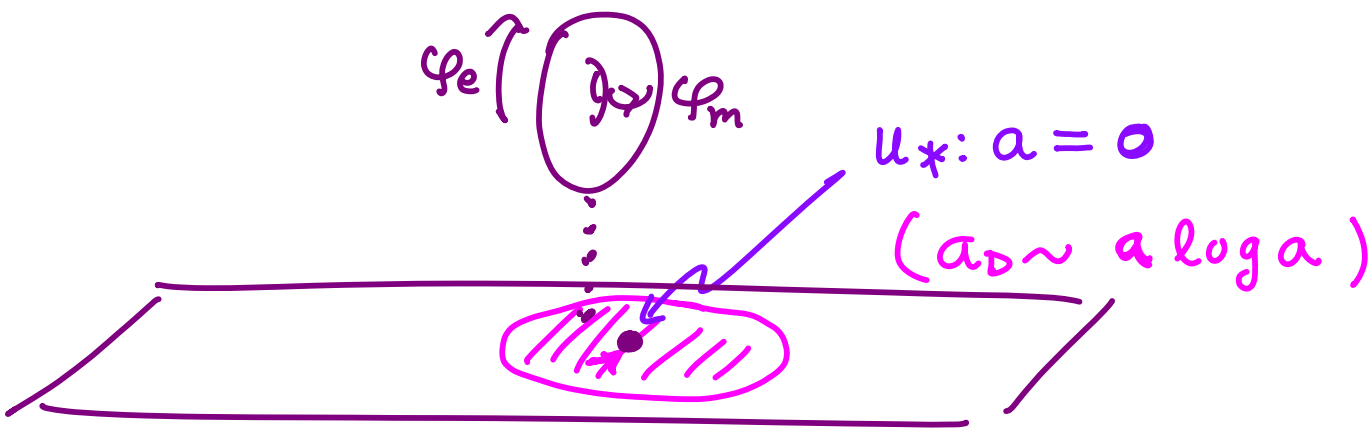
6. SINGLE-PARTICLE CORRECTIONS

NOW WE INCLUDE THE FIRST Q.C.

- FOR SIMPLICITY CONSIDER $r = 1$.

- CONSIDER A POINT $u_* \in \mathcal{B}$ WHERE A SINGLE HM HAS $M \to 0$

CHOOSE DUALITY FRAME SO IT HAS CHARGE $(q, 0)$, $q > 0$ & CENTRAL CHARGE $Z = qa$



<u>SCALING LIMIT:</u> $aR = \varepsilon \ll 1$, $R \to \infty$ $\Rightarrow$

DOMINANT CONTRIBUTION FROM A SINGLE 4D BPS H.M.

COMPUTATION OF THE METRIC

- INCLUDE Φ_{HM} IN 4D EFF. ACTION
- KK REDUCE

KK REDUCTION $\Longrightarrow$ TARGET SPACE METRIC IS A GIBBONS-HAWKING ANSATZ: [Seiberg Witten; Ooguri Vafa; Seiberg Shenker]

$$g = V^{-1}(\vec{x}) \left(\frac{d\varphi_m}{2\pi} + A \right)^2 + V(\vec{x})\, d\vec{x}^2$$

$$F = * dV \qquad \vec{x} \in \mathbb{R}^3$$

INTEGRATE OUT KK TOWER:

$$V(\vec{x}) = \frac{g^2 R}{4\pi} \sum_{n \in \mathbb{Z}} \frac{1}{\sqrt{g^2 R^2 |a|^2 + \left(g \frac{\varphi_e}{2\pi} + n \right)^2}}$$

$$a = x^1 + i x^2$$

$$\varphi_e = 2\pi R\, x^3 \qquad \text{PERIODIC}$$

$$V(\vec{x}) = V^{sf} + V^{inst}$$

$$V^{sf} = -\frac{g^2 R}{4\pi}\left(\log\frac{a}{\Lambda} + \log\frac{\bar{a}}{\bar{\Lambda}}\right)$$

$$V^{inst} = \frac{g^2 R}{2\pi}\sum_{n\neq 0} e^{ing\varphi_e} K_0\left(2\pi R|ng a|\right)$$

$\sim e^{-2\pi R|ng a|}$: INSTANTON CONTRIBUTION

HK STRUCTURE: $\alpha = 1,2,3$:

$$\omega^\alpha = dx^\alpha \wedge \left(\frac{d\varphi_m}{2\pi} + A\right) + \frac{1}{2}V\epsilon^{\alpha\beta\gamma}dx^\beta dx^\gamma$$

$$\boxed{\Rightarrow \text{COMPUTE } \varpi_\zeta = -\frac{i}{2\zeta}\omega_+ + \omega_3 - \frac{i}{2}\zeta\omega_-}$$

NOW, WHAT ARE THE HOLO. FUNCTIONS ON TWISTOR SPACE?

ALGEBRA OF HOLO FUNCTIONS $\{\chi_\gamma\}$ ON TWISTOR SPACE IS GENERATED

$$\chi_e := \chi_{(1,0)} = \exp\{ i\varphi_e + \cdots \}$$

$$\chi_m := \chi_{(0,1)} = \exp\{ i\varphi_m + \cdots \}$$

$$\chi_{(a,b)} = \chi_e^a \chi_m^b$$

DETERMINE χ_e AND χ_m FROM A DIFFERENTIAL EQUATION

$$\boxed{\varpi_5 = -\frac{1}{4\pi^2 R} \frac{d\chi_e}{\chi_e} \wedge \frac{d\chi_m}{\chi_m}}$$

WE FIND:

$$\chi_e = \chi_e^{sf} = \exp\left[\frac{\pi R}{\zeta} a + i\varphi_e + \pi R \zeta \bar{a} \right]$$

BUT

$$\chi_m = \chi_m^{s.f.} \cdot \chi_m^{inst.}$$

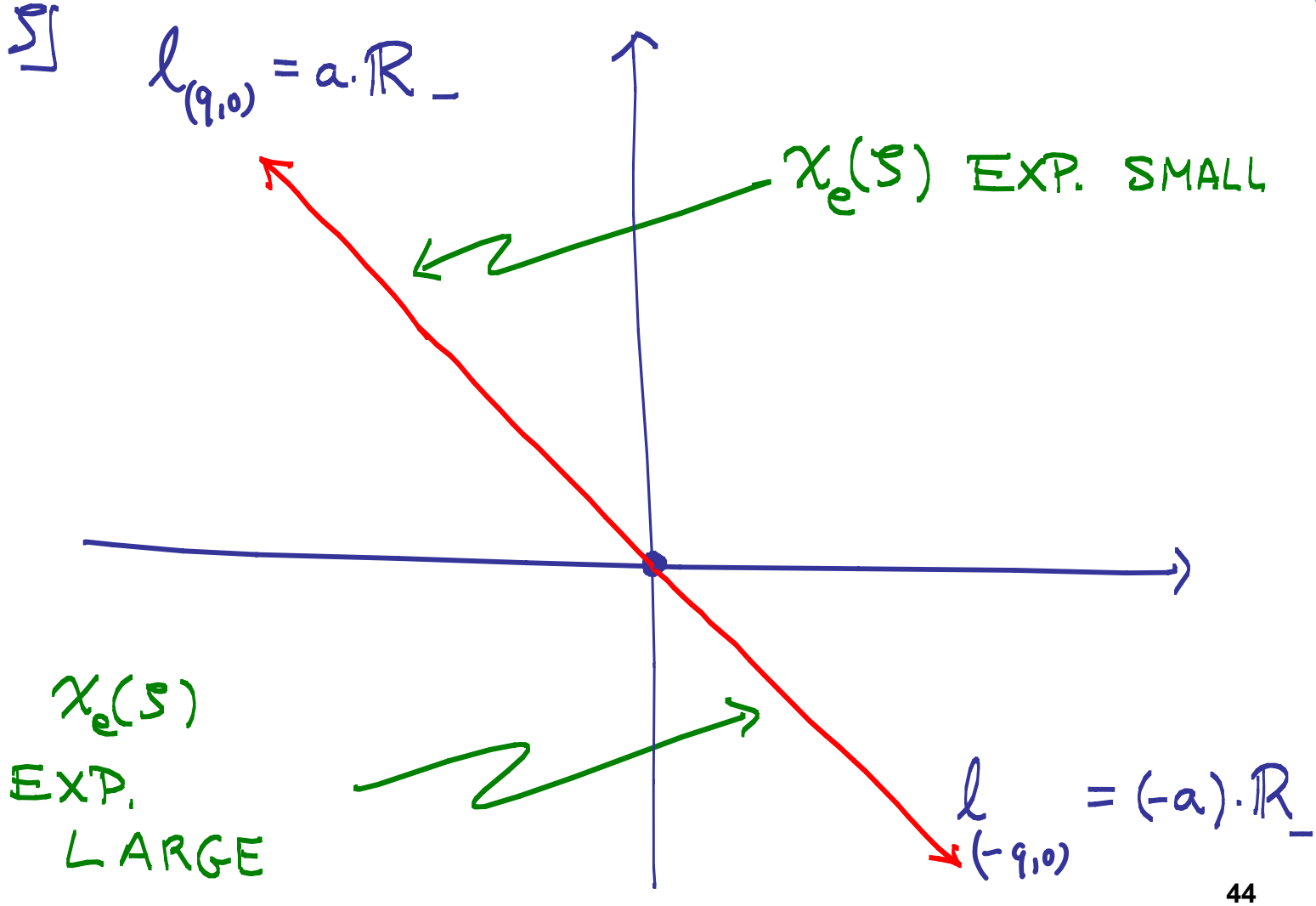
$$\chi_m^{sf} = \exp\left[\frac{\pi R}{\zeta} \cdot a_D + i\varphi_m + \pi R \zeta \bar{a}_D \right]$$

$$a_D = \frac{g^2}{2\pi i}\left(a \log \frac{a}{e\Lambda} \right)$$

χ_m^{inst} = INSTANTON CONTRIBUTION

$$\boxed{\begin{aligned}\mathcal{X}_m^{inst}(\zeta) = \exp\Bigg\{ & \frac{iq}{4\pi}\int_{\ell_{(q,0)}} \frac{d\zeta'}{\zeta'}\,\frac{\zeta'+\zeta}{\zeta'-\zeta}\log\left(1-\mathcal{X}_e(\zeta')^{q}\right) \\ & - \frac{iq}{4\pi}\int_{\ell_{(-q,0)}} \frac{d\zeta'}{\zeta'}\,\frac{\zeta'+\zeta}{\zeta'-\zeta}\log\left(1-\mathcal{X}_e(\zeta')^{-q}\right)\Bigg\}\end{aligned}}$$

ζ



EMERGENCE OF THE KS TRANSFORMATION

AS A FUNCTION OF ζ, $\mathcal{X}_m$ IS DISCONTINUOUS ACROSS THE BPS RAYS OF THE HYPERMULTIPLET OF CHARGE $(\pm q, 0)$

$$\ell_{\gamma,u} := \left\{ \zeta \;\middle|\; \frac{Z_\gamma(u)}{\zeta} \in \mathbb{R}_- \right\}$$

ACROSS THESE RAYS:

$$(\mathcal{X}_e, \mathcal{X}_m)^{cw} = \left(\mathcal{X}_e,\ \mathcal{X}_m (1 - \mathcal{X}_e^{\pm q})^{\mp q}\right)^{ccw}$$

$$= K_{(\pm q, 0)} (\mathcal{X}_e, \mathcal{X}_m)^{ccw}$$

$$= K_{(\pm q, 0)}^{\Omega(q,0)} (\mathcal{X}_e, \mathcal{X}_m)^{ccw}$$

6 KEY PROPERTIES OF $\mathcal{X}_\gamma$

1. $\mathcal{X}_\gamma$ ARE HOLOMORPHIC ON $\mathcal{Z}$

2. $\mathcal{X}_\gamma \cdot \mathcal{X}_{\gamma'} = \mathcal{X}_{\gamma+\gamma'}$

3. $\mathcal{X}_\gamma(\zeta) = \overline{\mathcal{X}_{-\gamma}(-1/\bar{\zeta})}$

4. $\mathcal{X}_\gamma \sim \mathcal{X}_\gamma^{s.f.}$ FOR $R \to \infty$

5. $$\left.\begin{array}{l} \lim\limits_{\zeta \to 0} \mathcal{X}_\gamma \exp\left(-\frac{\pi R}{\zeta} Z_\gamma(u)\right) \\ \lim\limits_{\zeta \to \infty} \mathcal{X}_\gamma \exp\left(-\pi R \zeta \overline{Z_\gamma(u)}\right) \end{array}\right\} \text{FINITE}$$

6. $\mathcal{X}_{\gamma'}(\zeta)$ TRANSFORMS BY $K_\gamma^{\Omega(\gamma;u)}$ AS ζ CROSSES THE BPS RAY $\ell_{\gamma,u}$.

7. MULTI-PARTICLE CONTRIBUTIONS

TO TAKE INTO ACCOUNT ALL BPS PARTICLES WE CANNOT USE A LOW ENERGY EFFECTIVE LAG., BECAUSE THE PARTICLES WILL BE MUTUALLY NONLOCAL.

PROPOSAL: PROPERTIES 1–6

HOLD FOR THE <u>EXACT</u> FUNCTIONS $\mathcal{X}_\gamma$, USING ALL THE BPS RAYS ℓ_γ WITH DISCONTINUITY $K_\gamma^{\Omega(\gamma;u)}$

THIS WILL DETERMINE THEM UNIQUELY

NOW: FINDING $\mathcal{X}_\gamma$ SATISFYING PROPERTIES $1 \to 6$ IS <u>EQUIVALENT</u> TO SOLVING A RIEMANN-HILBERT PROBLEM:

<u>RH</u>: FIND A PIECEWISE HOLOMOR. FUNCTION WITH PRESCRIBED SINGULARITIES AND ASYMPTOTICS.

SUMMARIZING THE $\mathcal{X}_\gamma$ BY A SINGLE MAP $\mathcal{X}$ (RECALL $\mathcal{X}_\gamma = \mathcal{X}^*(X_\gamma)$)

$\Rightarrow$ A RIEMANN-HILBERT PROBLEM IN THE ζ-PLANE FOR THE MAP

$$\mathcal{X}(\cdot, \zeta) : \mathcal{M}^\zeta \longrightarrow \mathcal{T} = \Gamma^* \otimes_{\mathbb{Z}} \mathbb{C}^*$$

PIECEWISE HOLOMORPHIC IN ζ

RIEMANN-HILBERT PROBLEM:

1.) $\mathcal{X}(\zeta)$ IS DISCONTINUOUS ACROSS BPS RAYS ℓ_γ:

$$\mathcal{X}^{cw} = S_\gamma(\mathcal{X}^{ccw})$$

[RECALL: $S_\gamma = \prod_{\ell_{\gamma'}=\ell_\gamma} K_{\gamma'}^{\Omega(\gamma';u)}$]

2.) $\mathcal{X}(\zeta)$ HAS ASYMPOTICS FOR $\zeta \to 0, \infty$ GIVEN BY $\mathcal{X}^{sf}(\zeta)$, UP TO $O(1)$ CORRECTIONS

$$Y := (\mathcal{X}^{sf})^{-1}\mathcal{X} : \mathcal{M} \to \mathcal{M}$$

i.e.

$$Y_0 = \lim_{\zeta\to 0} Y(\zeta) \;\&\; Y_\infty = \lim_{\zeta\to\infty} Y(\zeta)$$

EXIST

SOLUTION:

$$\mathcal{X}_\gamma(\zeta) = \mathcal{X}_\gamma^{sf}(\zeta) \cdot$$

$$\exp\left\{ -\frac{1}{4\pi i} \sum_{\gamma' \in \Gamma} \Omega(\gamma'; u) \langle \gamma, \gamma' \rangle \cdot \right.$$

$$\left. \cdot \int_{\ell_{\gamma'}} \frac{d\zeta'}{\zeta'} \frac{\zeta' + \zeta}{\zeta' - \zeta} \log\left[1 - \mathcal{X}_{\gamma'}(\zeta')\right] \right\}$$

ITERATING THIS EQUATION

(AS A SUM OVER TREES...)

GIVES THE FULL INSTANTON EXPANSION!

$\Rightarrow$ EXPLICIT CONSTRUCTION OF TWISTOR COORD'S

NOW WE <u>CONSTRUCT</u> THE METRIC FROM:

$$\omega_\zeta = \frac{1}{4\pi^2 R} \chi^*(\cdot,\zeta)(\omega^T)$$

- CONTINUITY IN ζ ?

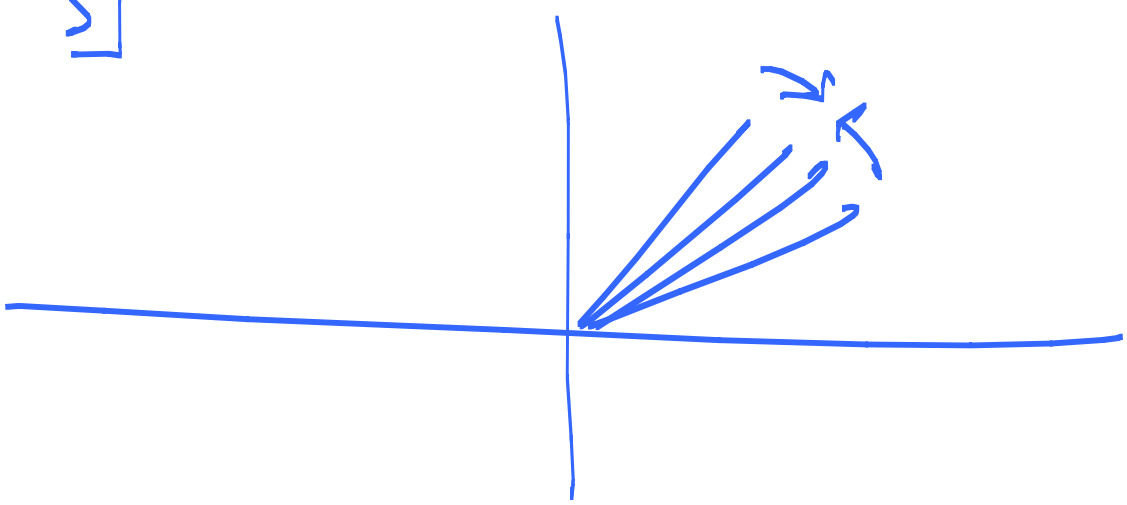
O.K. BECAUSE DISCONTINUITIES OF $\chi(\cdot,\zeta)$ ARE <u>SYMPLECTOMORPHISMS</u>

- BUT WHAT ABOUT CONTINUITY AS A FUNCTION OF u ??

ROLE OF THE KS WCF:

- AS u CROSSES A WALL OF MS BPS RAYS PILE UP

$\boxed{5}$



$u \to u^+$: DISCONTINUITY IN RH PROBLEM ALONG $\ell_{\gamma_1} = \ell_{\gamma_2}$

$$= \overrightarrow{\prod_{\substack{n \geq 0 \\ m \geq 0}}} K_{n\gamma_1 + m\gamma_2}^{\Omega(n\gamma_1 + m\gamma_2 ; u^+)}$$

$u \to u^-$: DISCONTINUITY IS:

$$= \overleftarrow{\prod} K_{n\gamma_1 + m\gamma_2}^{\Omega(n\gamma_1 + m\gamma_2 ; u^-)}$$

THUS, THE RH PROBLEM REMAINS <u>UNCHANGED</u> AS u CROSSES THE WALL IF THE $\Omega(\gamma;u)$ OBEY THE KSWCF.

THUS: THE KS FORMULA GUARANTEES THE CONTINUITY OF THE HK. METRIC ACROSS WALLS OF M.S. !

BUT... WHY IS OUR PROPOSAL THE RIGHT ONE ?

WHY IS THE METRIC THE RIGHT ONE FOR THE PHYSICAL PROBLEM?

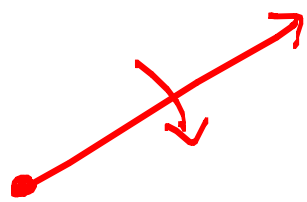
8. PHYSICAL PROOF OF THE KS FORMULA

RH IS EQUIVALENT TO A DIFF. EQ.:

$$A_\zeta = \mathcal{X}^{-1} \zeta \partial_\zeta \mathcal{X}$$

IS CONTINUOUS IN ζ-PLANE:

ACROSS ℓ_γ



$$\mathcal{X}^{-1} \zeta \partial_\zeta \mathcal{X} \to (S\mathcal{X})^{-1} \zeta \partial_\zeta (S\mathcal{X})$$
$$= \mathcal{X}^{-1} \zeta \partial_\zeta \mathcal{X}$$

$\Rightarrow A_\zeta$ IS HOLOMORPHIC FOR $\zeta \in \mathbb{C}^*$

$$\Longrightarrow \quad \zeta \partial_\zeta \chi = \chi \overleftarrow{\mathcal{A}}_\zeta$$

STRUCTURE GROUP: SYMPL($\mathbb{T}$)

ASYMPTOTICS $\Longrightarrow$

$$\mathcal{A}_\zeta = \zeta^{-1} \mathcal{A}_\zeta^{(-1)} + \mathcal{A}_\zeta^{(0)} + \zeta \mathcal{A}_\zeta^{(+1)}$$

(NOTE: $\mathcal{A}_\zeta^{(-1)}$ CONJUGATE TO $i\pi Z_{\gamma_i} \overleftarrow{\frac{\partial}{\partial \theta_i}}$)

SINCE S_γ IS INDPT. OF R, u, Λ ....

SAME ARGUMENT $\Rightarrow$ χ SATISFIES A SET OF DIFFERENTIAL EQUATIONS:

$$\frac{\partial}{\partial u}\chi = \chi \mathcal{A}_u$$

$$\frac{\partial}{\partial \bar{u}}\chi = \chi \bar{\mathcal{A}}_u$$

$$\Lambda \frac{\partial}{\partial \Lambda}\chi = \chi \mathcal{A}_\Lambda$$

$$\bar{\Lambda} \frac{\partial}{\partial \bar{\Lambda}}\chi = \chi \mathcal{A}_{\bar{\Lambda}}$$

$$R \frac{\partial}{\partial R}\chi = \chi \mathcal{A}_R$$

$$\zeta \frac{\partial}{\partial \zeta}\chi = \chi \mathcal{A}_\zeta$$

$$\mathcal{A}_i = \zeta^{-1}\mathcal{A}_i^{(-1)} + \mathcal{A}_i^{(0)} + \zeta\,\mathcal{A}_i^{(+1)}$$

SUMMARY OF THE LOGIC:

PROPERTIES 1–6 $\Longleftrightarrow$ R-H PROBLEM

R-H PROBLEM $\Longleftrightarrow$ SYSTEM OF DIFF. EQS. WITH PRESCRIBED "MONODROMY" (STOKES DATA).

(KNOWING $\mathcal{A}_i$ $\Longleftrightarrow$ KNOWING THE "MONODROMY")

KEY POINT: THE DIFF. EQS. ALL FOLLOW FROM THE PHYSICS OF THE 4D GAUGE THEORY!!

THE $\mathcal{X}_\gamma$ HAVE NICE PHYSICAL INTERPRETATIONS:

$$\begin{tikzcd}
\boxed{D=6 \quad \mathcal{N}=1 \text{ SYM}} \arrow[d, "T^2"] \arrow[dd, bend left=60, "T^3"] \\
\boxed{D=4 \quad \mathcal{N}=2 \text{ SYM}} \arrow[d, "S^1"] \\
\boxed{D=3, \mathcal{N}=4 \ \sigma\text{-MODEL}}
\end{tikzcd}$$

- 6D LIGHTLIKE LINE OPERATORS
- 4D 't HOOFT-WILSON-MALDACENA LOOP OPERATORS ANNIHILATED BY Q_ζ
- 3D CHIRAL RING OPERATORS ANN. BY Q_ζ.

- $$v^M = R\left(0,0,0,1,\frac{\zeta+\zeta^{-1}}{\sqrt{2}},\frac{i(\zeta-\zeta^{-1})}{\sqrt{2}}\right)$$

$$\oint dx^3\, v^M A_M = \oint dx^3 \left(A_3 + \zeta R \Phi + \zeta^{-1} R \bar{\Phi}\right)$$

- FOR SPIN(1,5) $(4 \otimes 4)_a \cong 6_v$

$V^M V_M = 0 \Rightarrow V^{rs}$ HAS KERNEL η_s

$$\Longrightarrow \quad [\eta_s Q_s, \; V^{rr'} A_{rr'}]$$

$$= \quad V^{rr'}(\eta_r \lambda_{s'} - \eta_{r'} \lambda_s) = 0.$$

- CALL THE SUSY Q_ζ FOR ABOVE NULL VECTOR

* $Q_\zeta = \bar{Q} + \zeta K_3$ IN 4D DONALDSON - WITTEN TWIST

* $Q_\zeta = Q_1 + \zeta Q_2$ IN 3D TFT FOR ROZANSKY_WITTEN TWIST

$$\chi_e^I = \exp \oint \left(A_3^I + R\zeta a^I + R\zeta^{-1} \bar{a}^I \right) dx^3$$

$$\chi_{m,I} = \exp \oint \left((A_3^D)_I + R\zeta\, a^D_I + R\zeta^{-1}\, \overline{a^D_I} \right) dx^3$$

ARE Q_ζ - CLOSED AND GENERATE THE CHIRAL RING OF THE 3D TFT.

- ANOMALOUS $U(1)_R$ WARD IDENTITIES

+ ANOMALOUS SCALING WARD IDENT'S

$\Longrightarrow$ R, ζ DIFFERENTIAL EQUATION.

$$\left.\begin{aligned} \frac{\partial}{\partial u}\chi &= \chi \mathcal{A}_u \\ \frac{\partial}{\partial \bar{u}}\chi &= \chi \mathcal{A}_{\bar{u}} \end{aligned}\right\}$$

HOLOMORPHY ON $\mathcal{M}^5$

ALSO HOLOMORPHY...

$$\left.\begin{aligned} \Lambda \frac{\partial}{\partial \Lambda}\chi &= \chi \mathcal{A}_\Lambda \\ \bar{\Lambda} \frac{\partial}{\partial \bar{\Lambda}}\chi &= \chi \mathcal{A}_{\bar{\Lambda}} \end{aligned}\right\}$$

VIEW Λ AS BACKGROUND VEV OF A VM.

$$\left.\begin{aligned} R \frac{\partial}{\partial R}\chi &= \chi \mathcal{A}_R \\ \zeta \frac{\partial}{\partial \zeta}\chi &= \chi \mathcal{A}_\zeta \end{aligned}\right\}$$

ANOMALOUS SCALE AND R-SYMMETRY

TO COMPLETE THE STORY WE MUST DERIVE THE CONNECTION $\mathcal{A}_i$, OR EQUIVALENTLY ITS "MONODROMY":

STOKES PHENOMENON

THE ζ- DIFF. EQ. HAS AN IRREGULAR SINGULAR POINT AT $\zeta = 0, \infty$; SOLUTIONS EXHIBIT STOKES PHENOM.

$\mathcal{A}_\zeta^{(-1)}$ IS CONJUGATE TO $Z \Rightarrow$

- STOKES RAYS = BPS RAYS ℓ_γ

DENOTE STOKES FACTORS BY $\mathcal{S}_\gamma$

REMAINING EQUATIONS:

ISOMONODROMIC DEFORMATION

$\Rightarrow$ STOKES FACTORS $\mathcal{S}_\gamma$ ARE INDP'T OF $R, u, \Lambda, \ldots$

$\Rightarrow$ CHECK AT LARGE R IN 1-INSTANTON APPROXIMATION:

$$\mathcal{S}_\gamma = S_\gamma^{K.S.}$$

THIS COMPLETES THE PROOF OF THE KS WCF FOR FIELD THEORY, BUT IT LEAVES SOME IMPORTANT OPEN PROBLEMS:

- RELATION TO WORK OF BRIDGELAND + TOLEDANO LAREDO.

- RELATION TO q-DEFORMED "MOTIVIC" FORMULAE OF K&S

- RELATION TO INTEGRABLE SYSTEMS.

9. NEW DIRECTIONS

- THERE ARE STRONG CONNECTIONS WITH THE tt^* EQUATIONS OF CECOTTI & VAFA.

$\mathcal{B}$ = FAMILY OF MASSIVE $d=2$ $\mathcal{N}=(2,2)$ Q.F.T.

$V \longrightarrow \mathcal{B}$ BUNDLE OF (c,c) OPERATORS ($\Longleftrightarrow$ R GROUNDSTATES)

ANALOGY $V = \mathcal{C}^\infty(\mathcal{M}_u)$

CECOTTI & VAFA DEFINE tt^* CONNECTION

$$\nabla_i \psi = \left(\frac{\partial}{\partial t^i} + \zeta^{-1} C_i + \bar{g}^{-1} \partial_i g \right) \psi = 0$$

$$\bar{\nabla}_i \psi = \left(\frac{\partial}{\partial \bar{t}^i} + \zeta \bar{C}_i \right) \psi = 0$$

FLATNESS $\Rightarrow$ tt^* EQUATIONS

ANALOGY: HOLOMORPHY OF $\mathcal{X}$ ON $\mathcal{M}^\zeta$

MOREOVER, ANOMALOUS SCALE + $U(1)_R$ SYMMETRY $\Longrightarrow$

$$\zeta \frac{\partial}{\partial \zeta} \psi = \left(R\zeta C + Q - R\zeta^{-1} \bar{C} \right) \psi$$

$$R \frac{\partial}{\partial R} \psi = \left(R\zeta C + Q + R\zeta^{-1} \bar{C} \right) \psi$$

BPS WALL-CROSSING EXPRESSED THROUGH STOKES FACTORS

STOKES FACTORS $S_{ij} = \mathbb{1} + e_{ij} \quad i \neq j$

BPS CHARGES LABELED BY PAIRS $i \neq j$

EXAMPLE $\mathrm{Rk}(V) = 3$. CROSSING A WALL FOR BPS STATE (13):

$$S_{12}^{\Omega_{12}} S_{13}^{\Omega_{13}^+} S_{23}^{\Omega_{23}} = S_{23}^{\Omega_{23}} S_{13}^{\Omega_{13}^-} S_{12}^{\Omega_{12}}$$

$$\Longleftrightarrow \qquad \Omega_{13}^+ = \Omega_{13}^- - \Omega_{12}\Omega_{23}$$

NOTE $Q_{ij} = \mathrm{Tr}_{\mathcal{H}_{ij}}\left(F(-1)^F e^{-R\cdot H}\right)$

4D ANALOG ?

- ANALOG FOR SUPERGRAVITY:

$\mathcal{M} \longrightarrow$ Q.K. MANIFOLD FIBERED BY HEISENBERG GROUPS

$\mathcal{T} \longrightarrow$ CANONICAL H.G. EXTENSION OF $T^* \otimes \mathbb{C}^*$ GIVEN BY Q.R.

HITCHIN THM. $\longrightarrow$ LE BRUN THM.

$\varpi_\zeta \longrightarrow$ CONTACT STRUCTURE

BUT.... CONVERGENCE ???

- $\Omega(\lambda\gamma) \sim e^{\text{const.}\ \lambda^2}$
- $\displaystyle\int_{\ell_{\lambda\gamma}} [d\zeta'] \log(1 - \chi^{sf}_{\lambda\gamma}) \sim e^{-\text{const.}\ \lambda}$

!?

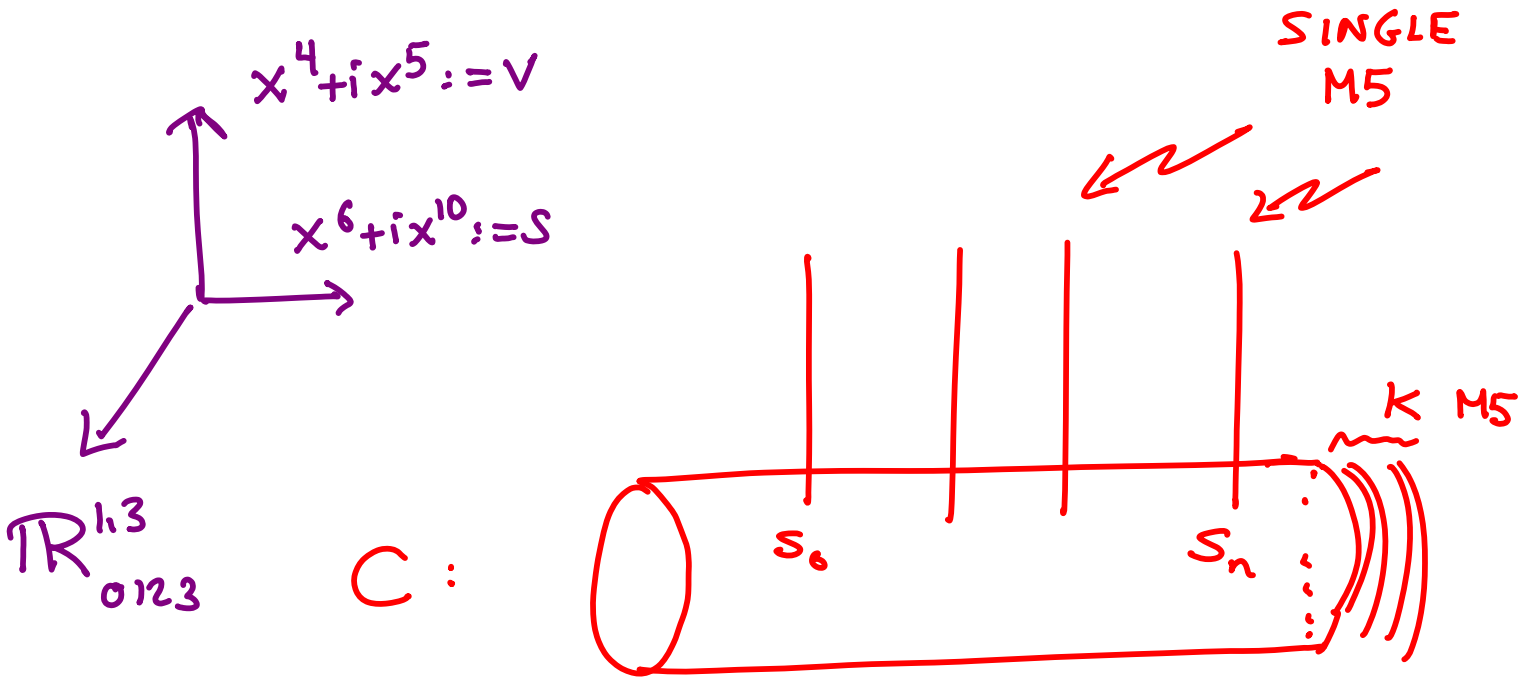
- RELATION TO HITCHIN SYSTEMS

FOR A LARGE CLASS OF $d=4, \mathcal{N}=2$ THEORIES THE MODULI SPACE $(\mathcal{M}, g)$ IS ALSO THE MODULI SPACE OF A HITCHIN SYSTEM (WITH SINGULARITIES) ON $\mathbb{P}^1$.

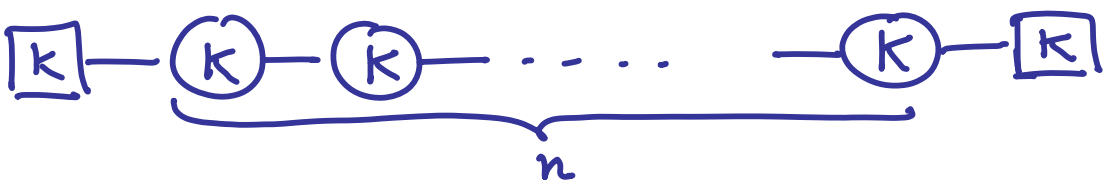
THIS GENERALIZES A RESULT OF CHERKIS + KAPUSTIN AND FOLLOWS FROM THE CONSTRUCTION OF $d=4, \mathcal{N}=2$ THEORIES USING M5-BRANES.

- KLEMM, LERCHE, MAYR, VAFA, WARNER '96
- WITTEN '97

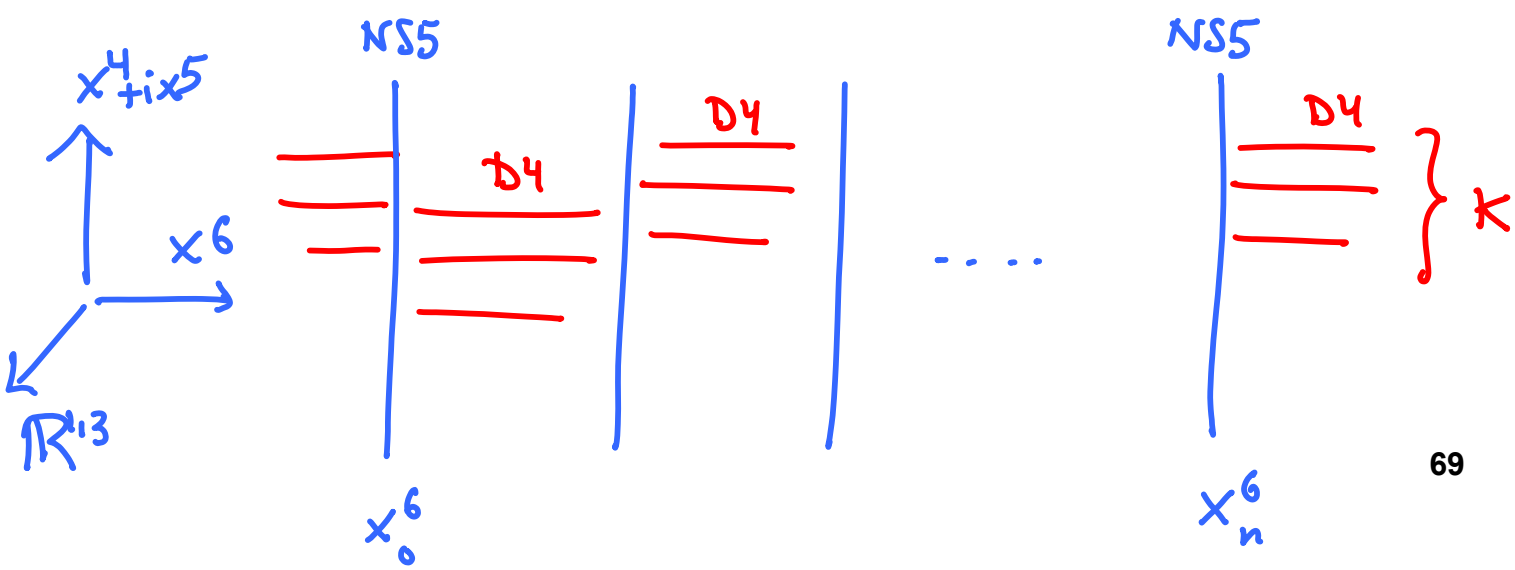
M-THEORY / $\underset{0123}{\mathbb{R}^{1,3}} \times \underset{6,10}{\mathbb{C}} \times \underset{4,5}{\mathbb{C}} \times \underset{7,8,9}{\mathbb{R}^3}$



4D IR : DESCRIBES LINEAR QUIVER



RECALL: PROJECTION TO IIA



ON THE COULOMB BRANCH THE M5 WRAP $\Sigma \subset T^*C$,

Σ = SW CURVE OF 4D THEORY

$C \cong \mathbb{C}^*$ $\qquad t = e^{-(x^6 + i x^{10})} \in \mathbb{C}^*$

$v = x^4 + i x^5 \in$ fiber

$$F(t,v) = \sum_{\alpha=0}^{n+1} t^{n+1-\alpha} q_\alpha(v) = 0.$$

$$q_\alpha(v) = c_\alpha \left(v^k - \mu^{(\alpha)} v^{k-1} - u_2^{(\alpha)} v^{k-2} - \cdots - u_k^{(\alpha)} \right)$$

gauge couplings

bifund. masses

adjoint vevs.

$$\lambda = v\, ds = -v \frac{dt}{t}$$

DECOUPLE GRAVITY: M5 $\Longrightarrow$ (2,0) THEORY

$$\begin{tikzcd}
\text{6D } (2,0)\ A_{K-1} \,/\, \mathbb{R}^{1,2} \times S^1_R \times C \ + \text{DEFECTS} \arrow[d, "\ell_C \ll \ell_{S^1}, \ell_{\mathbb{R}^{1,2}}"'] \arrow[dr, "\ell_{S^1} \ll \ell_C, \ell_{\mathbb{R}^{1,2}}"] & \\
\text{4D } \mathcal{N}=2 \text{ GAUGE THEORY} \,/\, \mathbb{R}^{1,2} \times S^1_R \arrow[dr, "\ell_{S^1} \ll \ell_{\mathbb{R}^{1,2}}"'] & \dfrac{\text{5D } U(K) \text{ SYM}}{\mathbb{R}^{1,2} \times C} \arrow[d, "\ell_C \ll \ell_{\mathbb{R}^{1,2}}"] \\
& \sigma\text{- MODEL: } \mathbb{R}^{1,2} \longrightarrow \mathcal{M}
\end{tikzcd}$$

ON THE OTHER HAND -- REDUCTION

OF THE 5D $U(K)$ (2,0) THEORY ON C $\Rightarrow$

HITCHIN EQS.
$$\begin{cases} F + R^2[\varphi, \bar{\varphi}] = 0 \\ \bar{\partial}_{\bar{A}} \varphi = 0 \qquad \partial_A \bar{\varphi} = 0 \end{cases}$$

$\varphi = (\Phi_4 + i\Phi_5)$: (1,0) FORM ON C

+ SINGULAR BC'S AT S_α

$\Rightarrow$ $\mathcal{M}$ = MODULI SPACE OF HITCHIN SYSTEM.

$v\,ds$ = Eigenvalues of φ $\Rightarrow$

$$\Sigma = \{(t,v) : \det(v + t\varphi_t) = 0\}$$

= Spectral curve

$$F(t,v) = \sum_{\alpha=0}^{n+1} t^{n+1-\alpha} q_\alpha(v)$$

$$= p_0(t) v^k + p_1(t) v^{k-1} + \cdots + p_k(t)$$

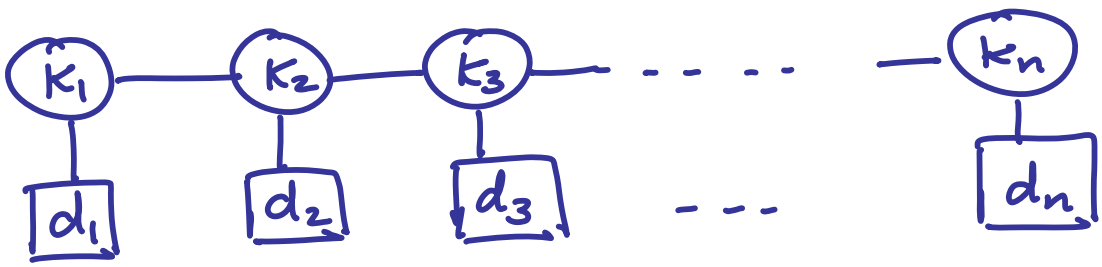
$$\Longrightarrow \underbrace{v^k + R_1(t) v^{k-1} + \cdots + R_k(t) = 0}$$

$$= \det(v - t\varphi_t)$$

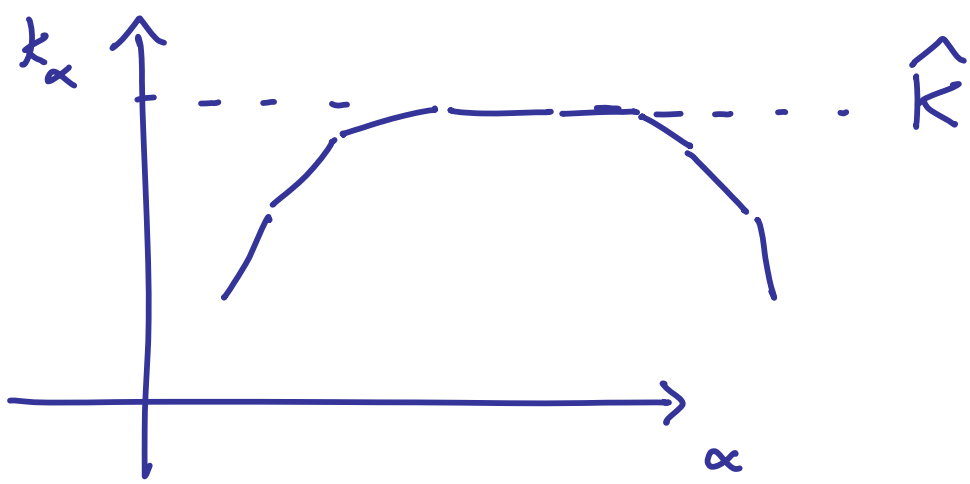
Behavior of roots $v(t)$

$\Longrightarrow$ Boundary Conditions on φ_t

BY PLACING M5-BRANES IN MULTI-CENTER TAUB-NUT THE WITTEN CONSTRUCTION GENERALIZES TO ASYMP. FREE LINEAR QUIVERS:



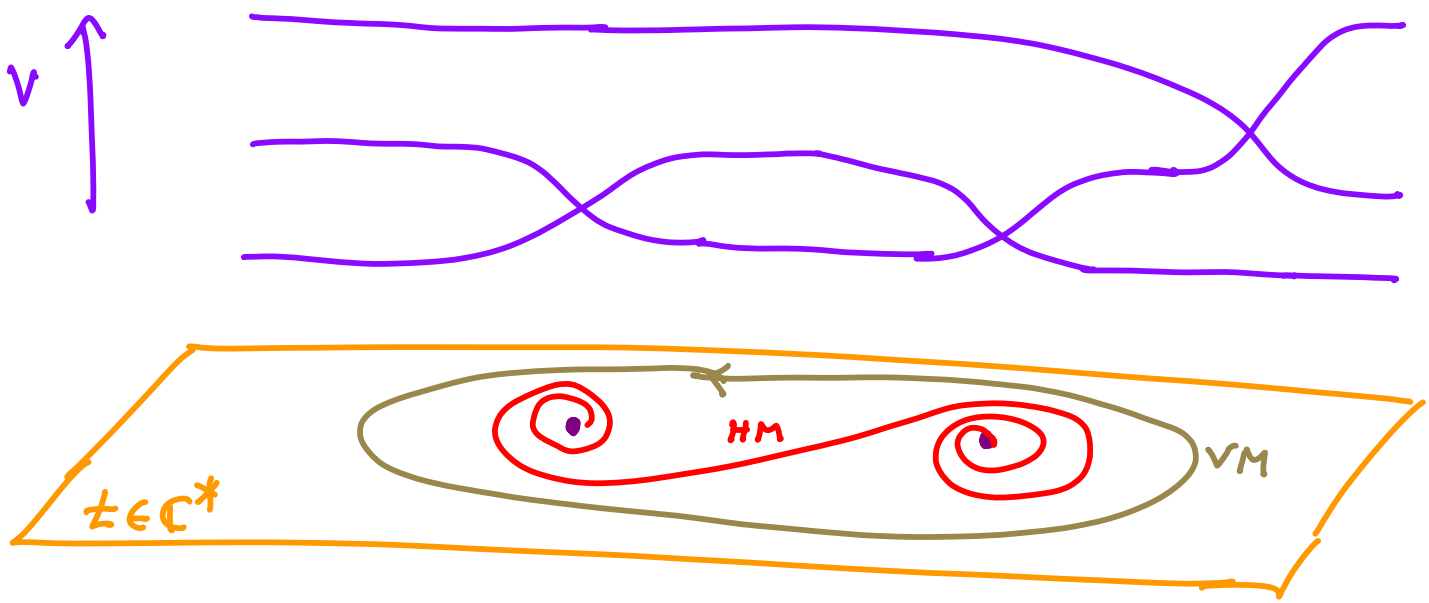
$$-2k_\alpha + k_{\alpha-1} + k_{\alpha+1} + d_\alpha \leq 0$$



$\Longrightarrow$ $U(\hat{k})$ HITCHIN SYSTEM

BPS STATES FROM OPEN M2-BRANES

Σ IS A BRANCHED COVER OF THE t-PLANE



- HM: PROJECT TO CURVES JOINING B.P.'S
- VM: PROJECT TO CLOSED CURVES,

$$\langle \lambda, \frac{\partial}{\partial t} \rangle = \zeta$$

[LERCHE et.al.; MIKHAILOV]

- IN GENERAL, WE GET STRING WEBS.

FOR THEORIES WITH $\hat{K}=2$:

$$\boxed{2} - (2) - (2) - \cdots\cdots - (2) - \boxed{2}$$

WE HAVE CONSTRUCTED THE $\mathcal{X}_\gamma$ AND VERIFIED THE PROPERTIES 1–6 ABOVE

THE KS TRANSFORMATIONS AND $\zeta \to 0, \infty$ ASYMPTOTICS EMERGE VERY NATURALLY...

CONJECTURALLY A GENERALIZED CONSTRUCTION APPLIES TO $\hat{K} > 2$.

SEE ANDY NEITZKE'S TALK.

10. TAKE-HOME SUMMARY

1. WE CONSTRUCT THE HK METRIC FOR CIRCLE-COMPACTIFICATION OF $\mathcal{N}=2, D=4$ FIELD THEORIES.

2. QUANTUM CORRECTIONS TO THE DIMENSIONAL REDUCTION METRIC COME FROM BPS STATES.

3. CONTINUITY OF THE HYPERKÄHLER METRIC FOLLOWS FROM THE KS WCF.

4. KS TMNS APPEAR AS DISCONTINUITIES OF HOLOMORPHIC FUNCTIONS $\mathcal{X}_\gamma(\zeta)$ THAT HAVE THE INTERPRETATION OF LINE OPER'S.

6. $\mathcal{M}$ IS ALSO THE MODULI SPACE OF SOLUTIONS TO A HITCHIN SYSTEM.