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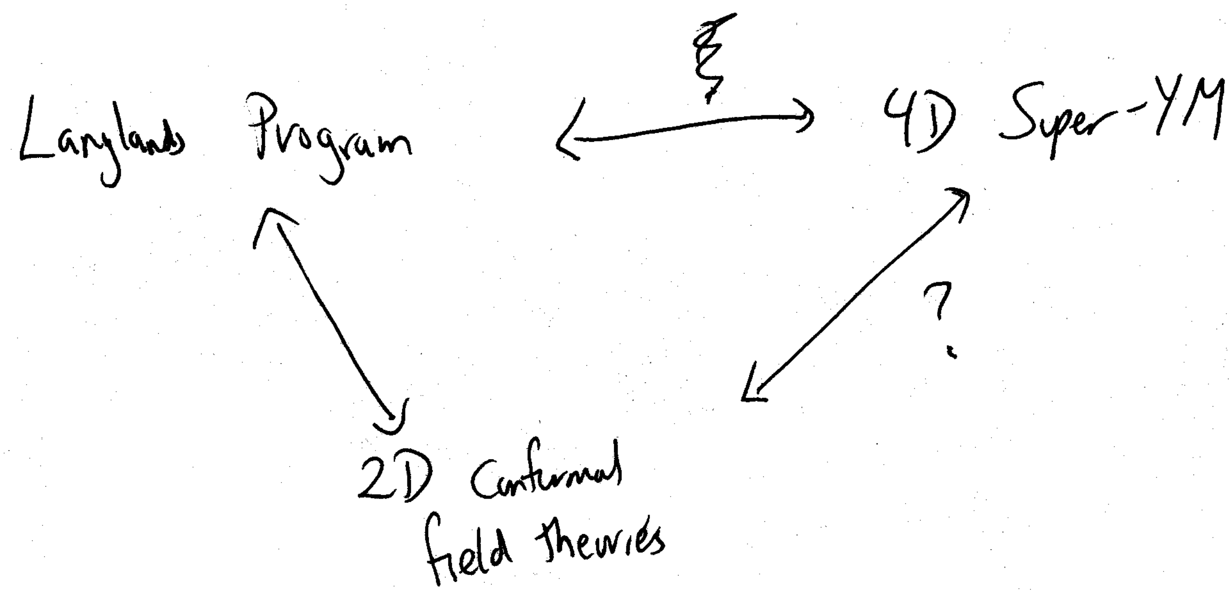
Overview of the links between Langlands program and 4D Super-Yang-Mills theory



Langlands program, or, categorical Langlands correspondence

Let C be a smooth ~~projective~~ curve$/\mathbb{C}$ (= compact Riemann surface)

Let G be a reductive Lie group$/\mathbb{C}$.

G_c = compact form of G.

Let Bun_G = moduli stack of G-bundles on C (= holo G-bundle

Loc_G = moduli stack of flat G-bundles on C

$\mathcal{E} = (E, \nabla)$

holo bundle $\rightarrow E$, (flat) connection $\rightarrow \nabla$

$\nabla = \partial_z + A(z)$ (the (1,0) part of a connection, whose (0,1) part is $\bar\partial$)

$G \longleftrightarrow {}^LG$ = Langlands dual group.

$$\begin{tikzcd} \boxed{\begin{array}{c}\mathcal{O}\text{-modules}\\ \text{on } Loc_{{}^LG}\end{array}} \arrow[loop left, "\substack{\text{Wilson}\\ \text{functors}\\ W_{x,V}}"] \arrow[r, leftrightarrow, "\substack{\text{categorical}\\ \text{Langlands}\\ \text{correspondence}}"] & \boxed{\begin{array}{c}\mathcal{D}\text{-modules on}\\ Bun_G\end{array}} \arrow[loop right, "\substack{\text{Hecke}\\ \text{functors}\\ H_{x,V}}"] \end{tikzcd}$$

((quasi)-coherent sheaves)

$\mathcal{D}$ = sheaf of diff. operators on Bun_G

- Derived categories
- Correspondence holds when G is abelian, say $G = GL_1$.
  The correspondence is an enhanced Fourier-Mukai transform.
  (Laumon-Rothstein)
- Wilson functors + Hecke functors should be intertwined by the correspondence labeled by $x \in C$, $V \in Rep({}^LG)$
- If $\mathcal{E} \in Loc_{{}^LG}$, can consider $\mathcal{O}_\mathcal{E}$ = skyscraper sheaf on $Loc_{{}^LG}$ supported at $[\mathcal{E}]$

$\updownarrow$

$\mathcal{F}_\mathcal{E}$ = a $\mathcal{D}$-module on Bun_G

$\mathcal{O}_{\mathcal{E}}$ is an eigensheaf of the Wilson functors $W_{x,V}$

i.e.,
$$W_{x,V}(\mathcal{O}_{\mathcal{E}}) \simeq \underline{V} \otimes \mathcal{O}_{\mathcal{E}}$$

$\Rightarrow$
$$H_{x,V}(\mathcal{F}_{\mathcal{E}}) \simeq \underline{V} \otimes \mathcal{F}_{\mathcal{E}}$$
, which is a non-trivial property: $\mathcal{F}_{\mathcal{E}}$ is a <u>Hecke eigensheaf</u>.

Now vary x: get $\mathcal{E} \times_{{}^L G} V$, a flat vector bundle

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How is this related to the original Langlands correspondence?

$$C/\mathbb{C} \longmapsto C/\mathbb{F}_q \qquad F = \mathbb{F}_q(C)$$

$$\begin{array}{ccc} \mathcal{E} = (E, \nabla) & & \\ \pi_1(C,p) \to {}^L G & & \sigma : Gal(\bar{F}/F) \to {}^L G \\ \updownarrow & & \updownarrow \\ \mathcal{F}_{\mathcal{E}} & & \pi_\sigma \text{ - automorphic representation of } G(\mathbb{A}_F) \end{array}$$

Kapustin–Witten

4D $\mathcal{N}=4$ supersymmetric gauge theory

G_c = compact Lie group (gauge group)

$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}$, complexified coupling.

$$\left({}^L G_c, -\frac{1}{\tau}\right) \xleftrightarrow{\text{S-duality}} (G_c, \tau)$$

Note: exchanges g and g^{-1}

Well-understood in the abelian case.

GL-twist ("Geometric Langlands twist")

$Q = u Q_\ell + v Q_r$, $Q^2 = 0$

$$t = \frac{v}{u}$$

Under S-duality, $t \to {}^L t = \frac{\tau}{|\tau|}$

Special values of parameters:

$$\left({}^L G_c, \begin{array}{c} {}^L\tau = \infty \\ {}^L t = i \end{array}\right) \longleftrightarrow (G_c, \tau = 0, t = 1)$$

$$\mathcal{M}_H({}^LG) \xleftrightarrow{\text{SYZ mirror symmetry}} \mathcal{M}_H(G)$$

B-model, complex str. J

A-model, ω_K-symplectic structure

G-Higgs field

(E,ϕ), $E = G$-bundle on C

$\phi \in \Gamma(C, \mathfrak{g}_E \otimes K_C)$

$\mathfrak{g}_E = E \times_G \mathfrak{g}$

$\cong \mathcal{Y}({}^LG)$ - moduli space of flat ${}^LG$-bundles

$\cong T^* Bun_G$

$${}^LF_b \xleftrightarrow{\text{dual tori}} F_b$$

$b \in B =$ Hitchin base

$$\boxed{\text{B-branes on } \mathcal{Y}({}^LG)} \xleftrightarrow[\text{(HMS)}]{\text{homological mirror symmetry}} \boxed{\text{A-branes on } \mathcal{M}_H(G)}$$

A-branes on $\mathcal{M}_H(G)$ $\ni$ Lagrangian in $\mathcal{M}_{H,F}$

(L, ∇) with a flat unitary vector bundle on L

(+ generalization: co-isotropic branes)

$$\begin{tikzcd} \boxed{\mathcal{D}\text{-modules on } \mathrm{Bun}_G} \arrow[d, leftrightarrow, "\text{microlocalization}"] \\ \boxed{A\text{-branes on } \mathcal{M}_H(G)} \end{tikzcd}$$

What happens to $\mathcal{O}_\mathcal{E}$, $\mathcal{F}_\mathcal{E}$?

$\mathcal{F}_\mathcal{E}$: notoriously complicated

What is the A-brane $A_\mathcal{E}$ corresponding to the skyscraper sheaf $\mathcal{O}_\mathcal{E}$?

We have $\mathcal{E} \in \mathcal{Y}({}^L G)$, assume it is generic.

It has an image $b \in B$, that has a fiber $F_b \subset \mathcal{M}_H(G)$.

$$A_\mathcal{E} = (F_b, \nabla_\mathcal{E})$$

$\nabla_\mathcal{E}$: connection on F_b determined by the point $\mathcal{E} \in {}^L F_b$.

$M_4 = \Sigma \times C$, compactify on C

Get an effective theory on Σ

There is a $N=(2,2)$ (twisted) sigma model on Σ with the target $\mathcal{M}_H(G,C) = \mathcal{M}_H(G)$, the Hitchin moduli space of Higgs G-bundles.

Connection to 2D CFT

$\mathcal{E} \mapsto \mathcal{F}_{\mathcal{E}}$ using methods of conformal field theory

$\mathcal{F}_{\mathcal{E}}$: $\mathcal{D}$-module on $\underbrace{Bun_G}_{\text{moduli of } G\text{-bundles on } C}$ (e.g. vector bundle with a flat connection)

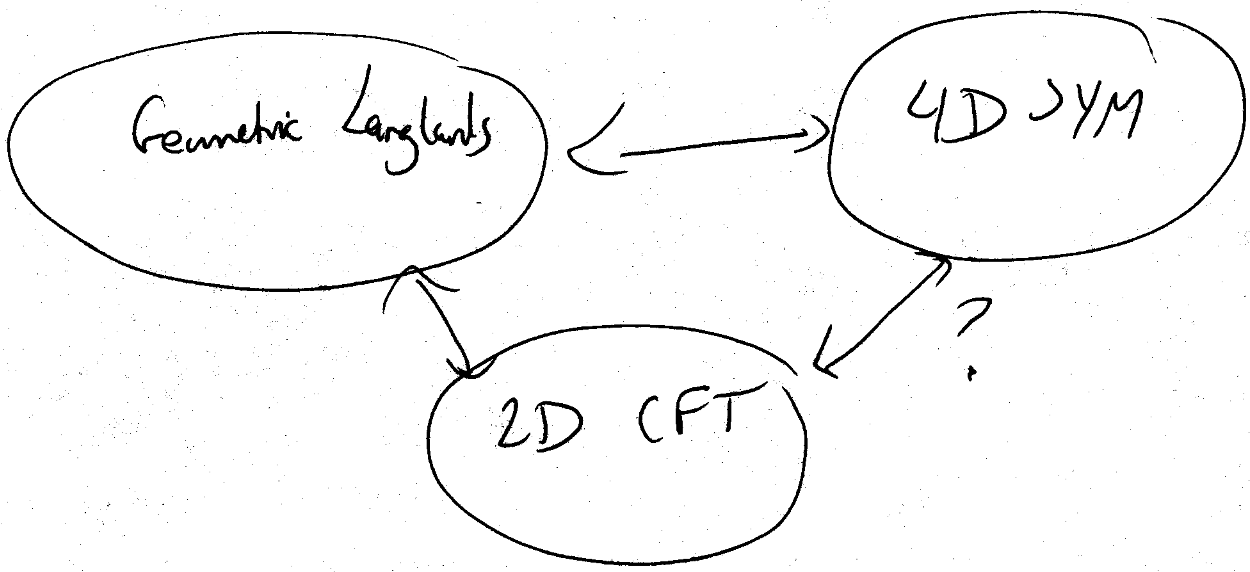
In CFT: spaces of conformal blocks combine into a flat vector bundle on $\mathcal{M}_{g,n}$. Virasoro symmetry $T(z)$

$\hat{\mathfrak{g}}$ - affine Kac-Moody algebra uniformizes Bun_G.

In $\mathfrak{g}$-CFT, RCFT (e.g. WZW model) - flat v.b. on Bun_G

Theory with $\hat{\mathfrak{g}}$-symmetry $k=-h^\vee \mapsto$ $\mathcal{D}$-modules on Bun_G.

Beilinson-Drinfeld: get $\mathcal{F}_{\mathcal{E}}$ (for a special class - the $\mathcal{E}$-opers)



Idea: deform the parameters

"quantum geometric Langlands"