Dynamics of entanglement in spin chains with diffusive transport

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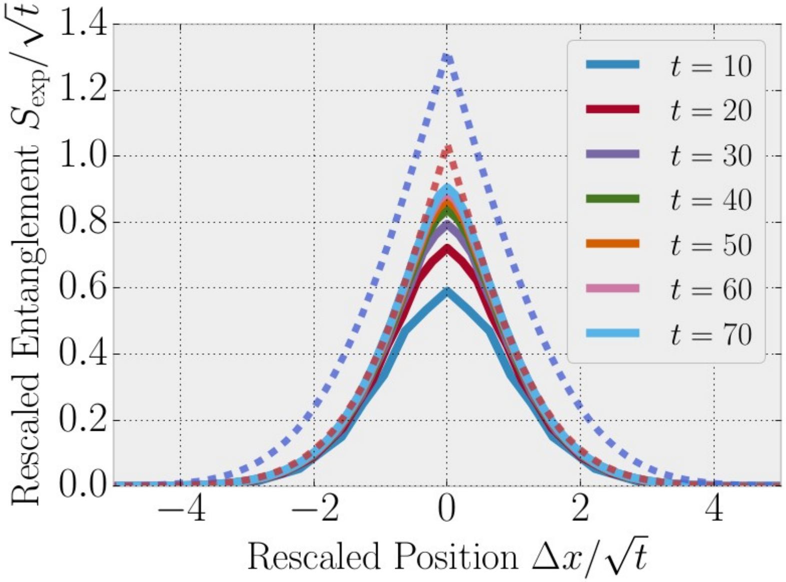
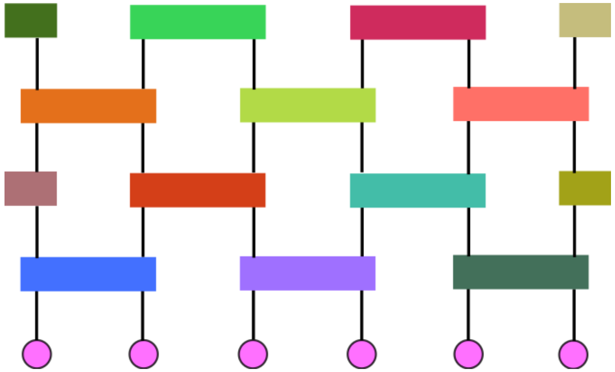
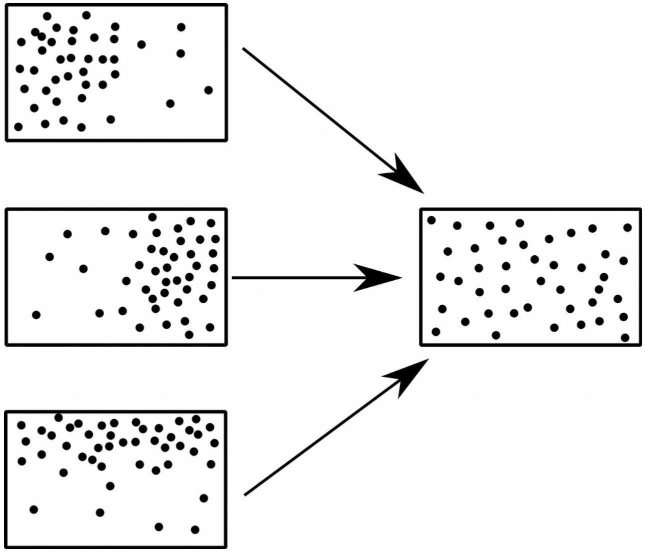
Collaboration with Curt von Keyserlingk and Frank Pollmann





KITP, Santa Barbara, 05.09.18

Thermalization: information of initial state is lost locally



$|\Psi(t)\rangle = \hat{U}(t)|\Psi_0\rangle$ remains a pure state

$\lim_{t\to\infty} \rho_A \approx \rho_{\text{Gibbs}}$ for subsystems L_A << L

"Scrambling" of information ⇨ Growth of entanglement

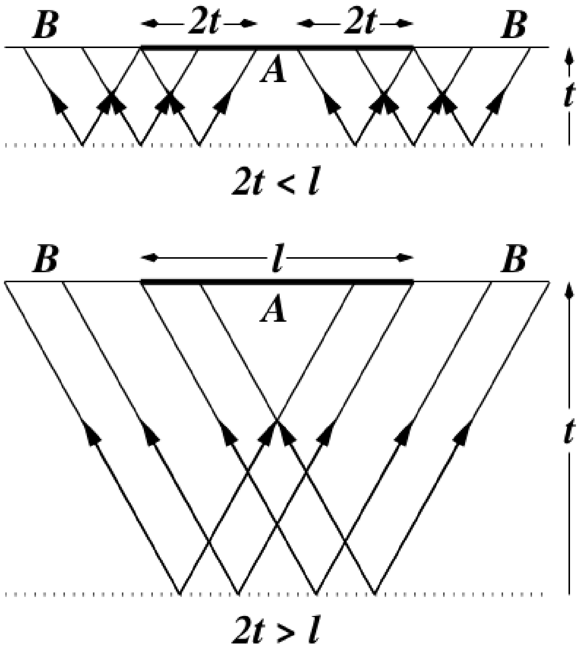
Spreading of local operators

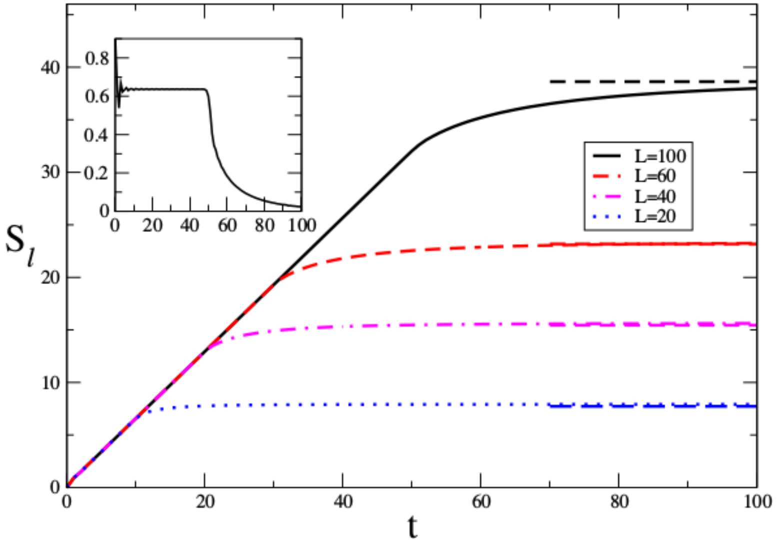
How does entanglement grow?

ρ_{Gibbs} has extensive von Neumann entropy

→ takes time to build up from short-range correlated initial state

Integrable systems / CFT (relax to generelized Gibbs) → quasiparticle picture





Calabrese, Cardy: JSTAT (2005)

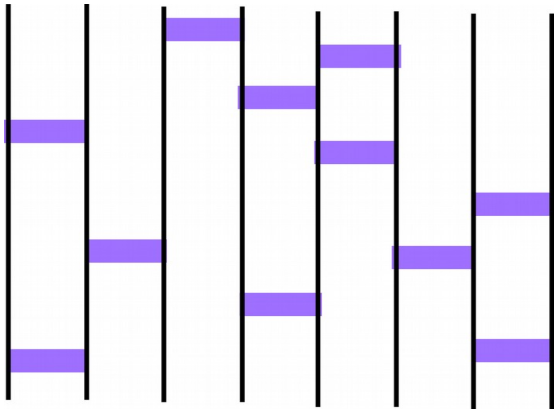
What about systems with no well-defined quasiparticles?

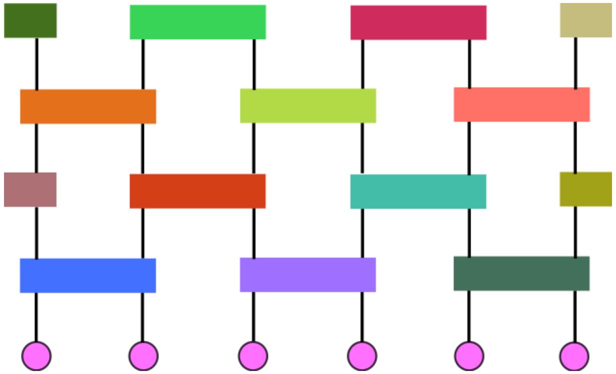
Linear entanglement growth is more generic – see: Kim, Huse: PRL (2013)

How to describe? Are there other universal features?

Minimal model: local random unitary circuits

Keep unitarity, locality (+ conservation laws), throw away all other structure





Entanglement growth (from product state): linear + KPZ fluctuations

Nahum Ruhman, Vijay, Haah: PRX (2017); Zhou, Nahum (arXiv 1804.09737)

Can be interpreted as 'energy' of a directed polymer / minimal cut

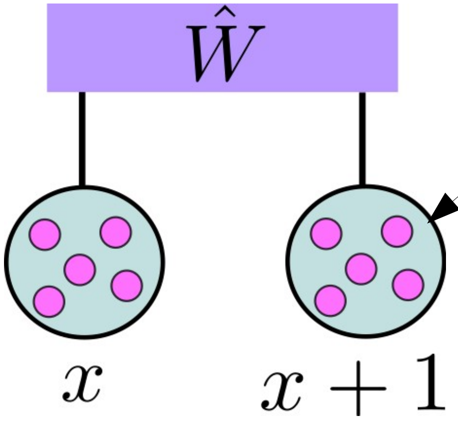
Jonay, Huse, Nahum (arXiv 1803.00089)

Related to operator spreading

Ho, Abanin: PRB (2017)
von Keyserlingk, TR, Pollmann, Sondhi: PRX (2018); Nahum, Vijay, Haah: PRX (2018)

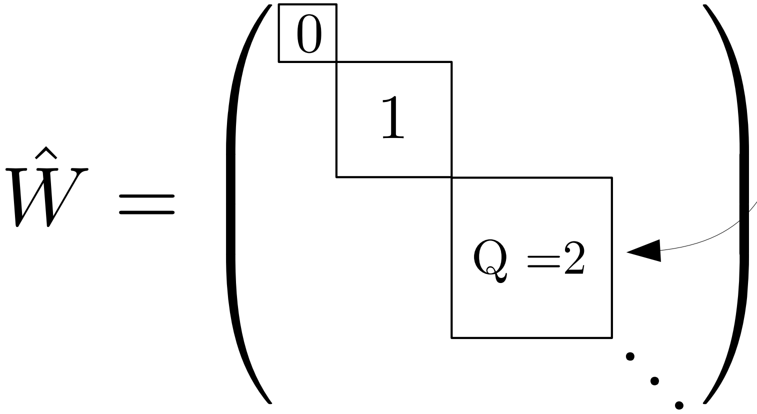
How is this affected by the presence of slow diffusive modes?

Random circuit model with conserved U(1) charge



Each dot contains N quibits: $\bigotimes_{j=1}^{N} |\alpha_j = 0, 1\rangle$

Local charge density: $\hat{Q}_x = \frac{1}{N} \sum_{j=1}^{N} |1\rangle\langle 1|_j$



$d_Q \times d_Q$ Haar random

$$d_Q = \binom{2N}{Q}$$

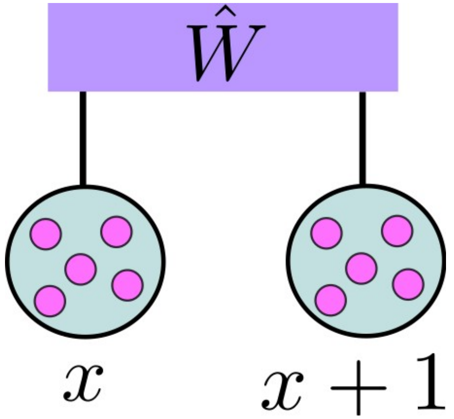
TR, von Keyserlingk, Pollmann: PRX (2018)

See also: Khemani, Vishwanath, Huse: PRX (2018)

$\hat{Q}_x$ obeys random walk / diffusion on average at all times

➡ Shows up both in transport for inhomogenous states
and in fluctuations for a global quench - see: Lux, Müller, Mitra, Rosch: PRA (2014)

Neglecting fluctuations leads to a simple growth rule for the entanglement in the large N limit



Assume that the total charge on x,x+1 is sharply peaked

In the limit N → ∞ this is true e.g. in local equilibrium

$$\rho_{x,x+1} \propto e^{-\mu(x,t)(\hat{Q}_x+\hat{Q}_{x+1})}$$

Evolution of half-chain entropy across a cut at position x:

$$\boxed{e^{-S(x,t)} \to e^{-s_{\mu(x,t)}}(e^{-S(x-1,t)} + e^{-S(x+1,t)})}$$

$$s_\mu \sim \log d_{\bar{Q}}$$

⇨ $S(x,t) \to \min[S(x-1,t), S(x+1,t)] + s_{\mu(x,t)}$

Space-time dependent surface growth / directed polymer problem

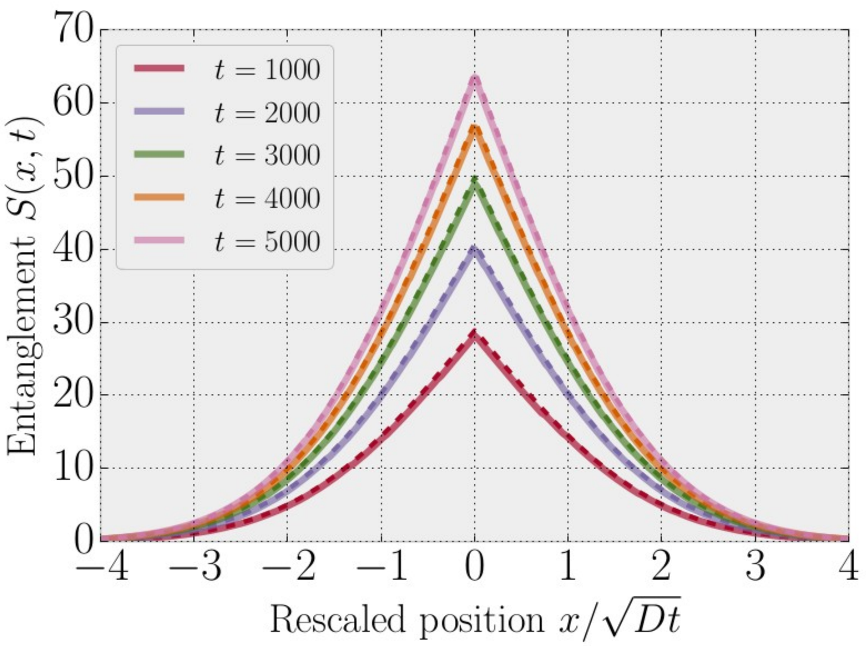
For constant μ this coincides with the model of Nahum el. al. (PRX, 2017)

For domain wall, diffusion implies t$^{1/2}$ entanglement growth

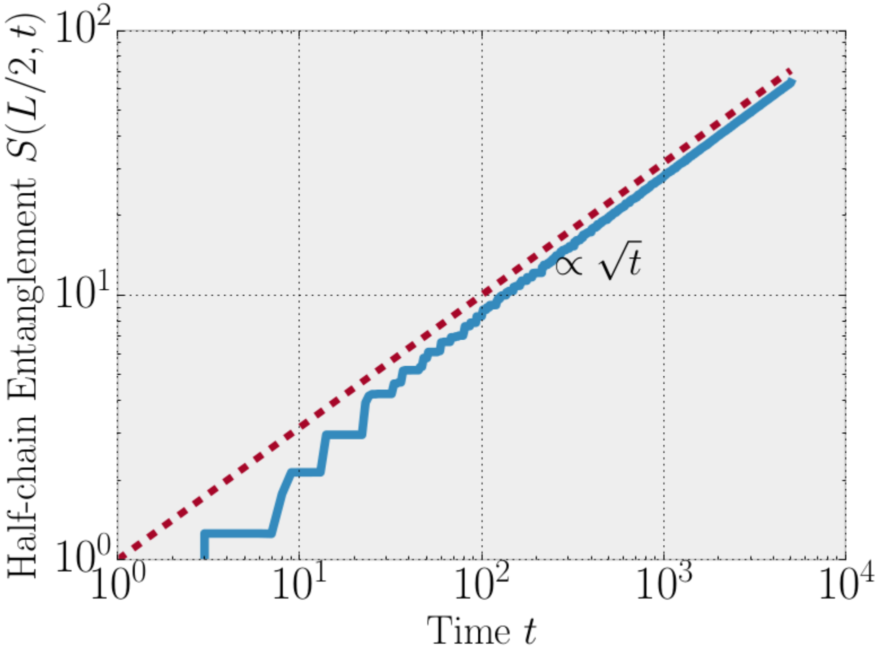
Subadditivity + local eq. → $|S(x+1,t) - S(x,t)| \leq s_{\mu(x,t)}$

Domain wall: $Q(x,0) = \begin{cases} 0 \text{ if } x \leq 0 \\ 1 \text{ if } x > 0 \end{cases} \rightarrow Q(x,t) = \dfrac{1 + \mathrm{erf}(x/\sqrt{Dt})}{2}$

$$\Rightarrow \quad \boxed{S(x,t) \leq \int_0^x \mathrm{d}x\, s_{\mu(x,t)} = \sqrt{Dt}\, f\left(\frac{x}{\sqrt{Dt}}\right)}$$



N = ∞



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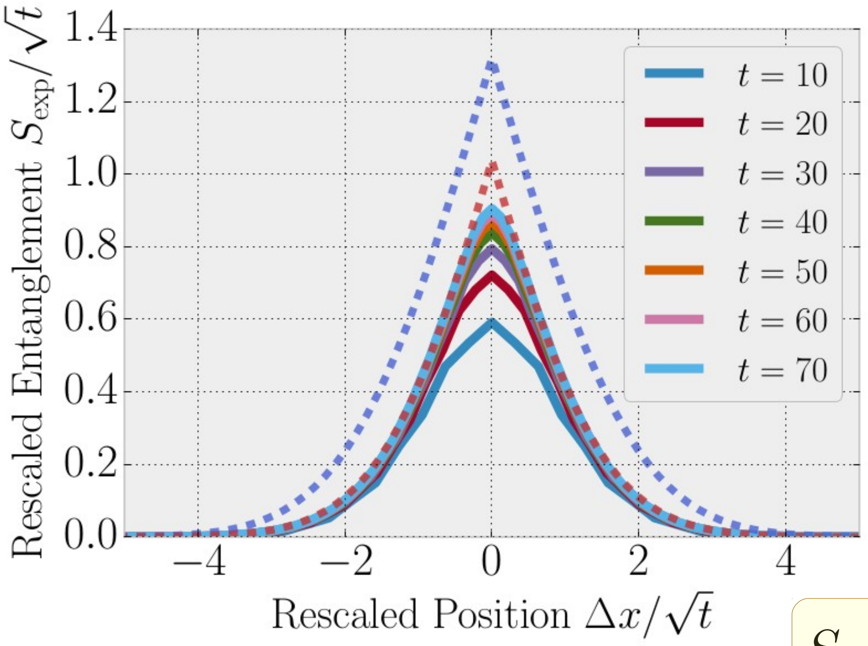
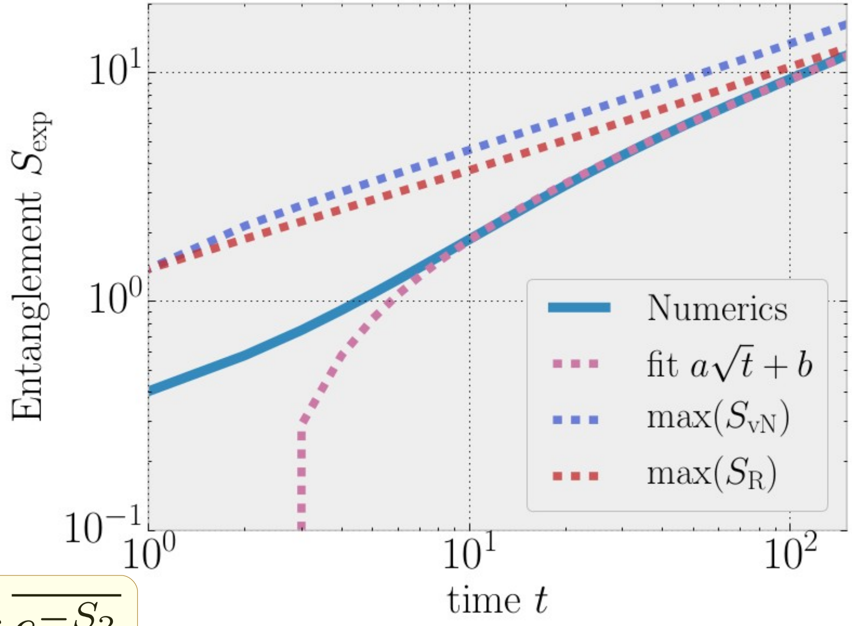


N = 1



$S_{\mathrm{exp}} \equiv -\log \overline{e^{-S_2}}$

Domain wall quickly approaches local equilibrium

$$S_{\text{exp}} \equiv -\log \overline{e^{-S_2}}$$

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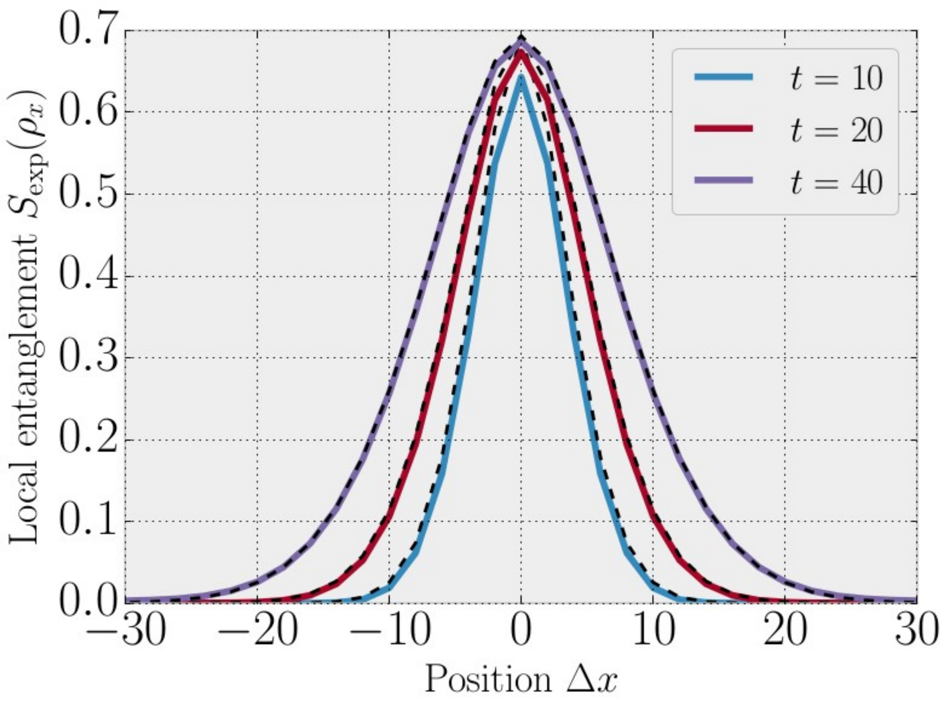
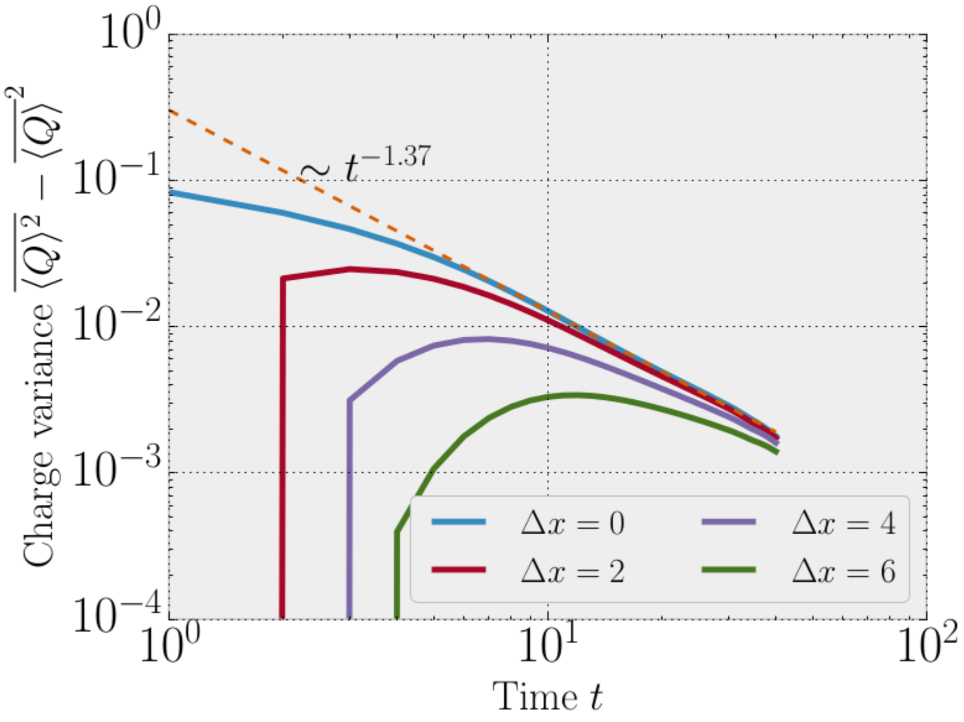
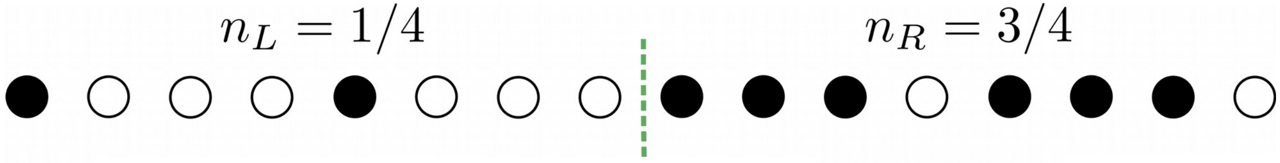
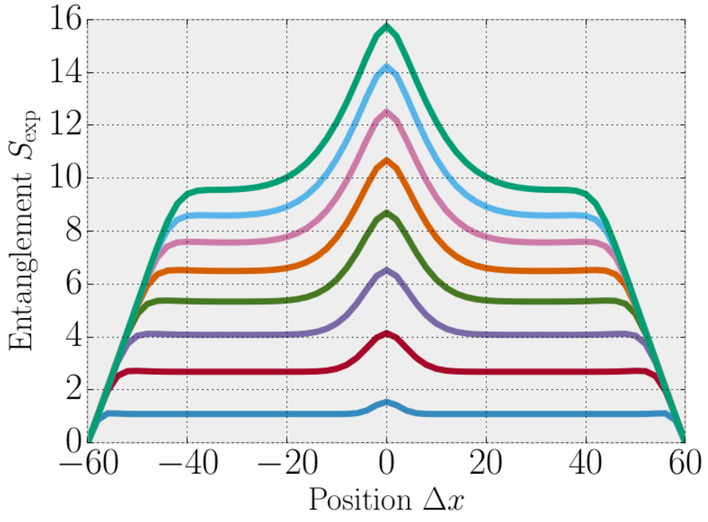
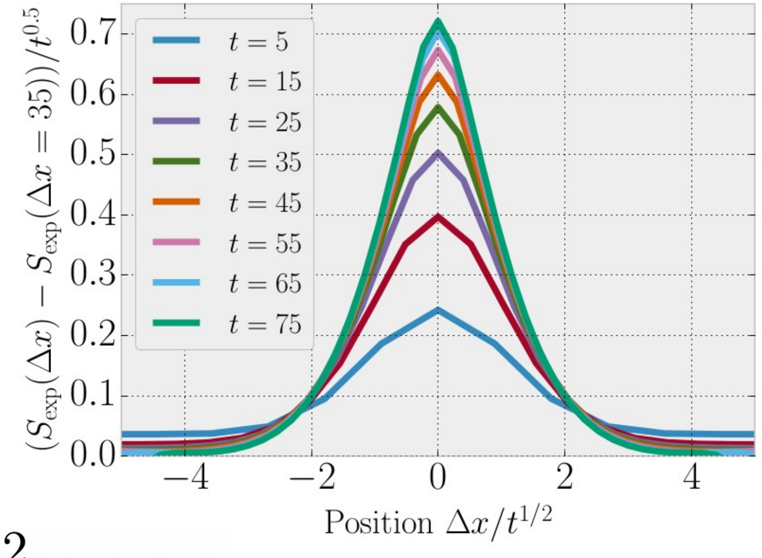


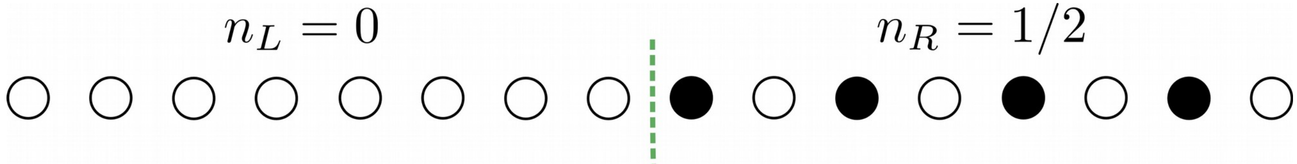


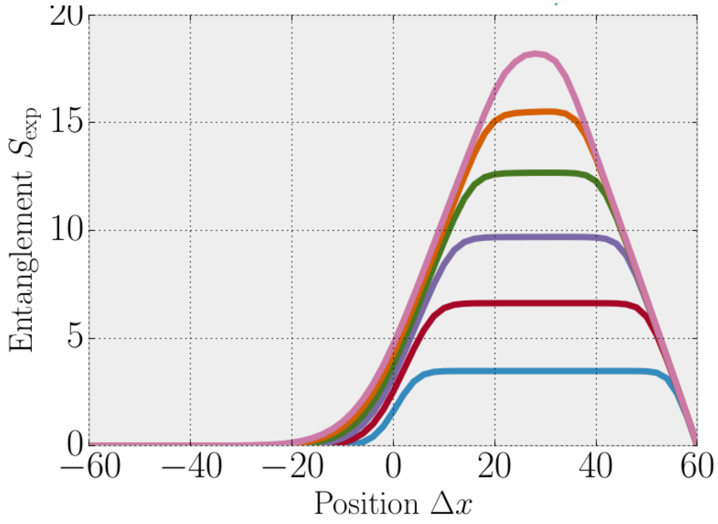
Similar scaling appears for more general domain walls

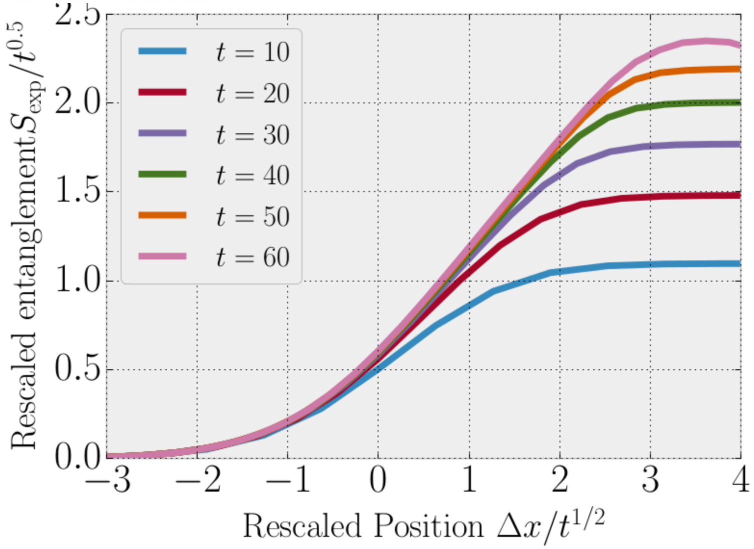






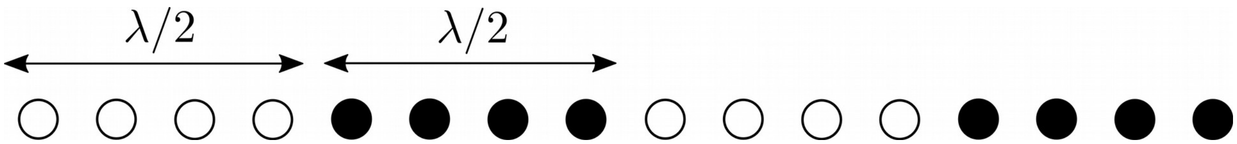


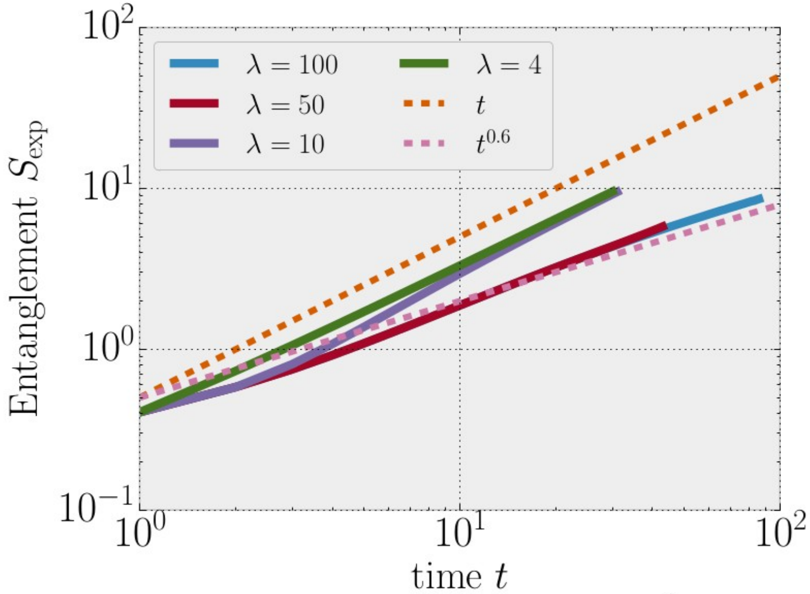


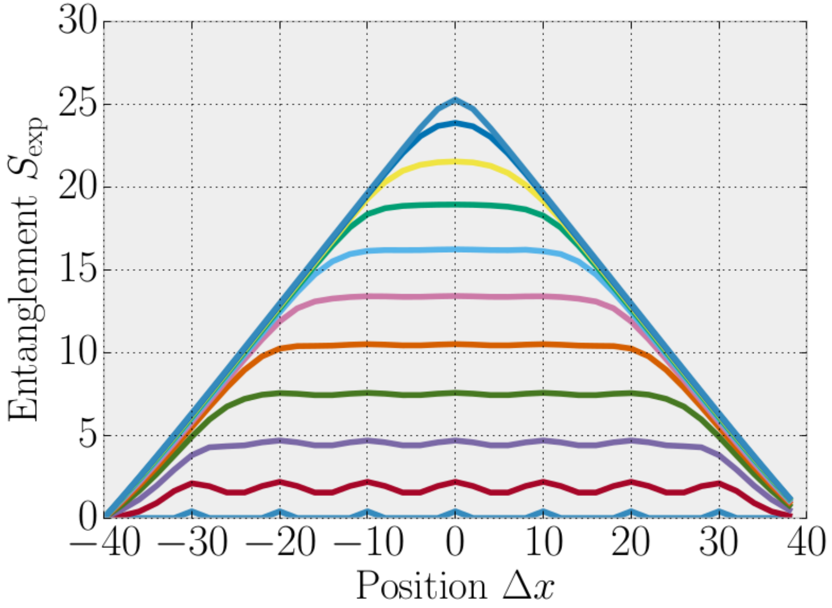


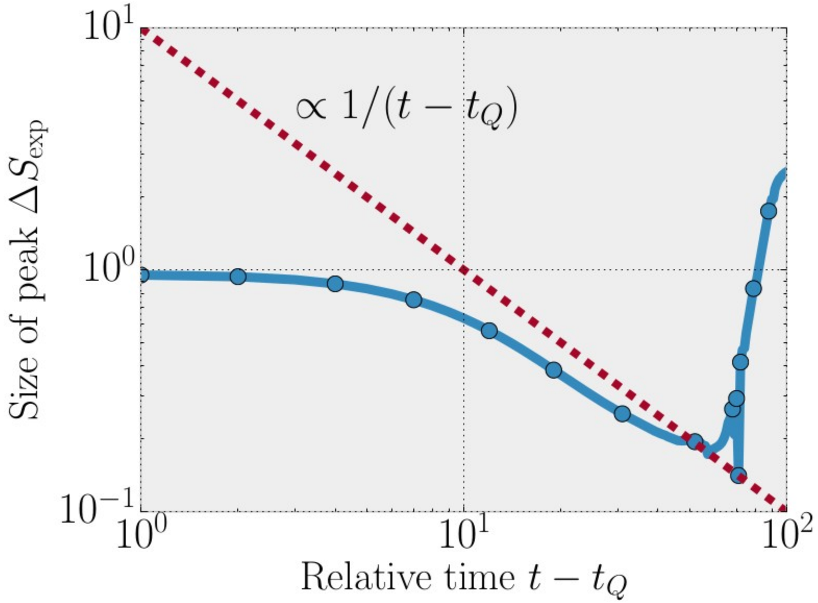
‘Bumps’ smooth out at long times

Charge-density wave:









Slow growth of entanglement for domain wall is present in a deterministic, periodically driven spin chain

$$U_{\mathrm{F}} = e^{-i\tau H_4} e^{-i\tau H_3} e^{-i\tau H_2} e^{-i\tau H_1}$$

$$T = 4\tau = 1$$

$$H_1 = J_z^{(1)} \sum_r \hat{Z}_r \hat{Z}_{r+1}$$

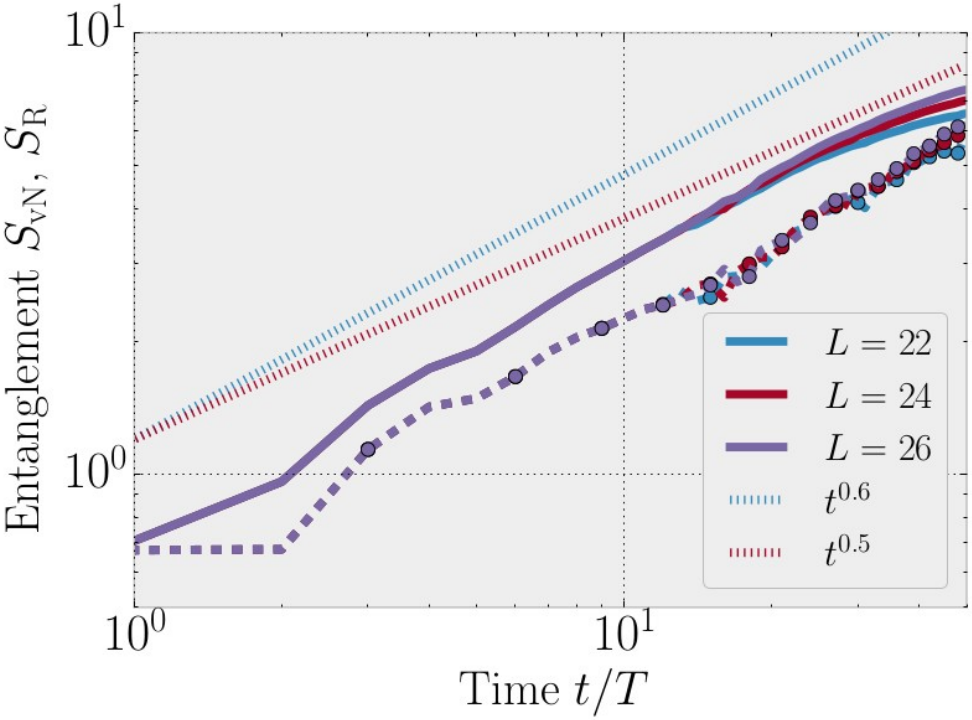
$$J_z^{(1)} = (\sqrt{3}+5)/6$$

$$H_3 = J_z^{(2)} \sum_r \hat{Z}_r \hat{Z}_{r+2}$$

$$J_z^{(2)} = \sqrt{5}/2$$

$$J_{xy} = (2\sqrt{3}+3)/7$$

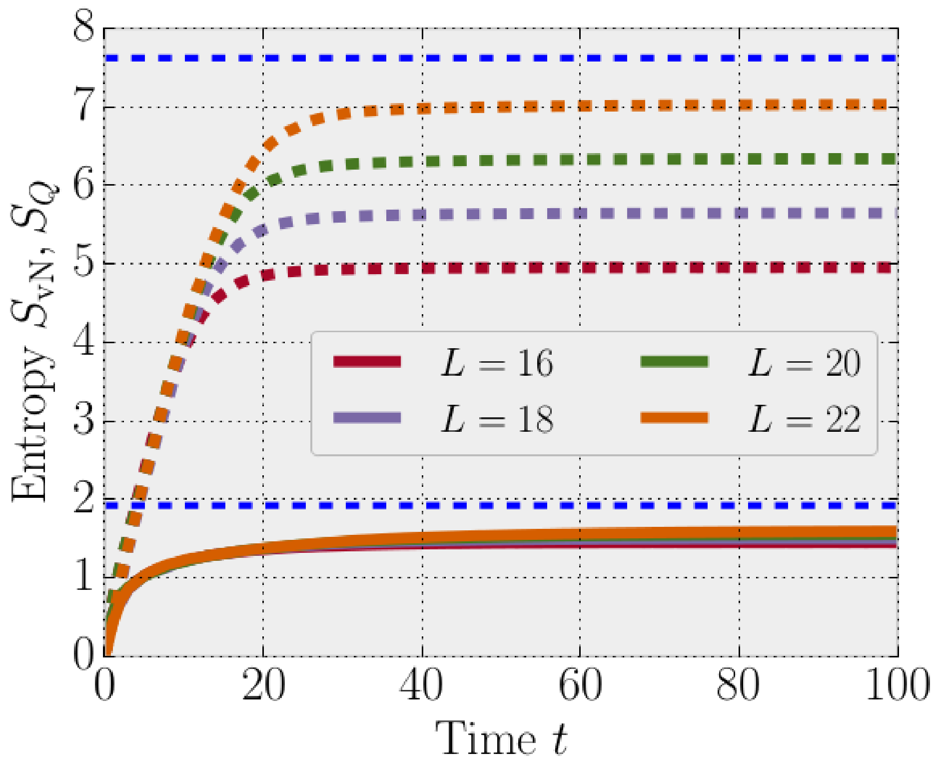
$$H_2 = H_4 = J_{xy} \sum_r \left(\hat{X}_r \hat{X}_{r+1} + \hat{Y}_r \hat{Y}_{r+1} \right),$$

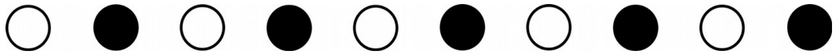


Diffusion can slow down entanglement growth even for homogenous initial states, due to charge fluctuations

Subsystem density matrix is block-diagonal in charge:

$$\rho = \sum_Q p_Q \rho^{(Q)} \rightarrow S_{\mathrm{vN}}[\rho] = -\sum_Q p_Q \log(p_Q) - \sum_Q p_Q \mathrm{tr}(\rho^{(Q)} \log \rho^{(Q)})$$





Infinite temperature state: p_Q is binomial → $S_Q = \log\left(\sqrt{L_A}\right) + \mathrm{const.}$

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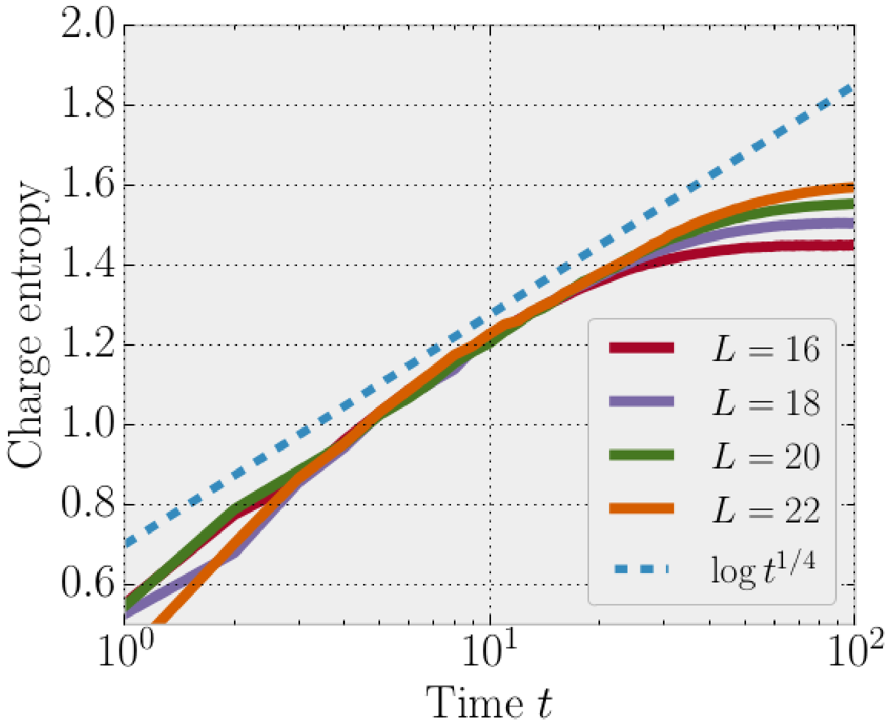

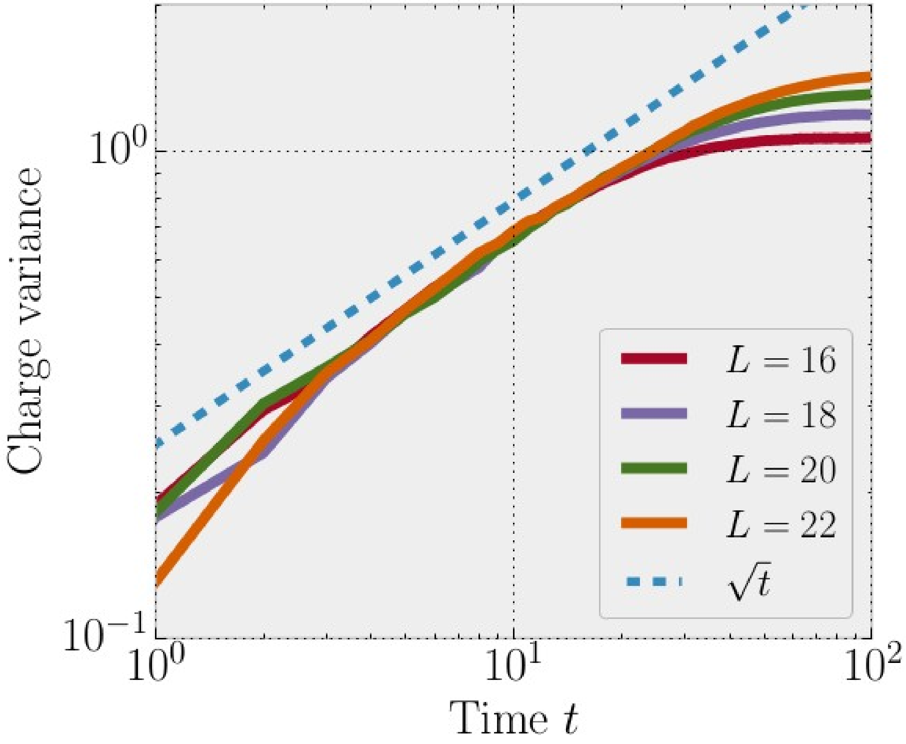

What about states with large initial fluctuations?

Contrived example: put a cat state on left / right halves of the chain

$$|\psi_0\rangle = \left[\frac{|\uparrow\ldots\uparrow\rangle + |\downarrow\ldots\downarrow\rangle}{\sqrt{2}}\right]_L \bigotimes \left[\frac{|\uparrow\ldots\uparrow\rangle + |\downarrow\ldots\downarrow\rangle}{\sqrt{2}}\right]_R$$

How do the different entanglement entropies grow in the middle?

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Contrived example: put a cat state on left / right halves of the chain

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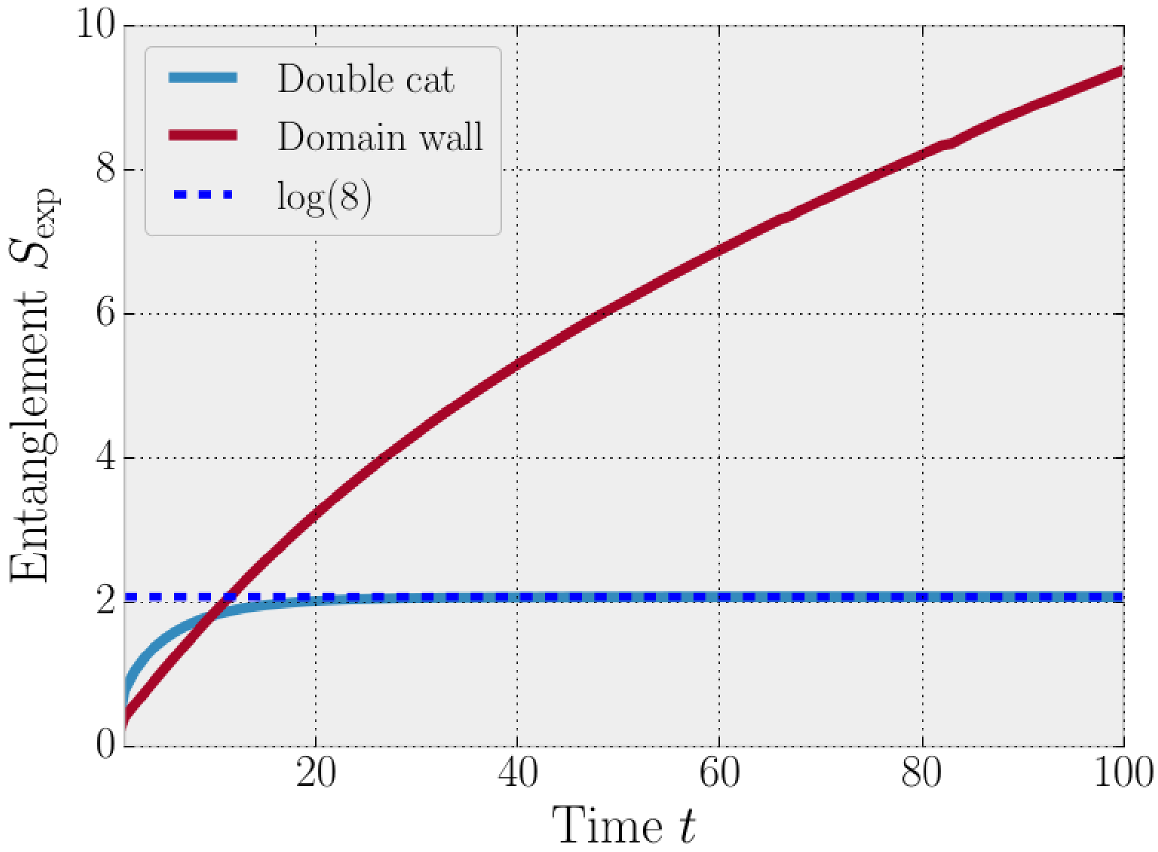

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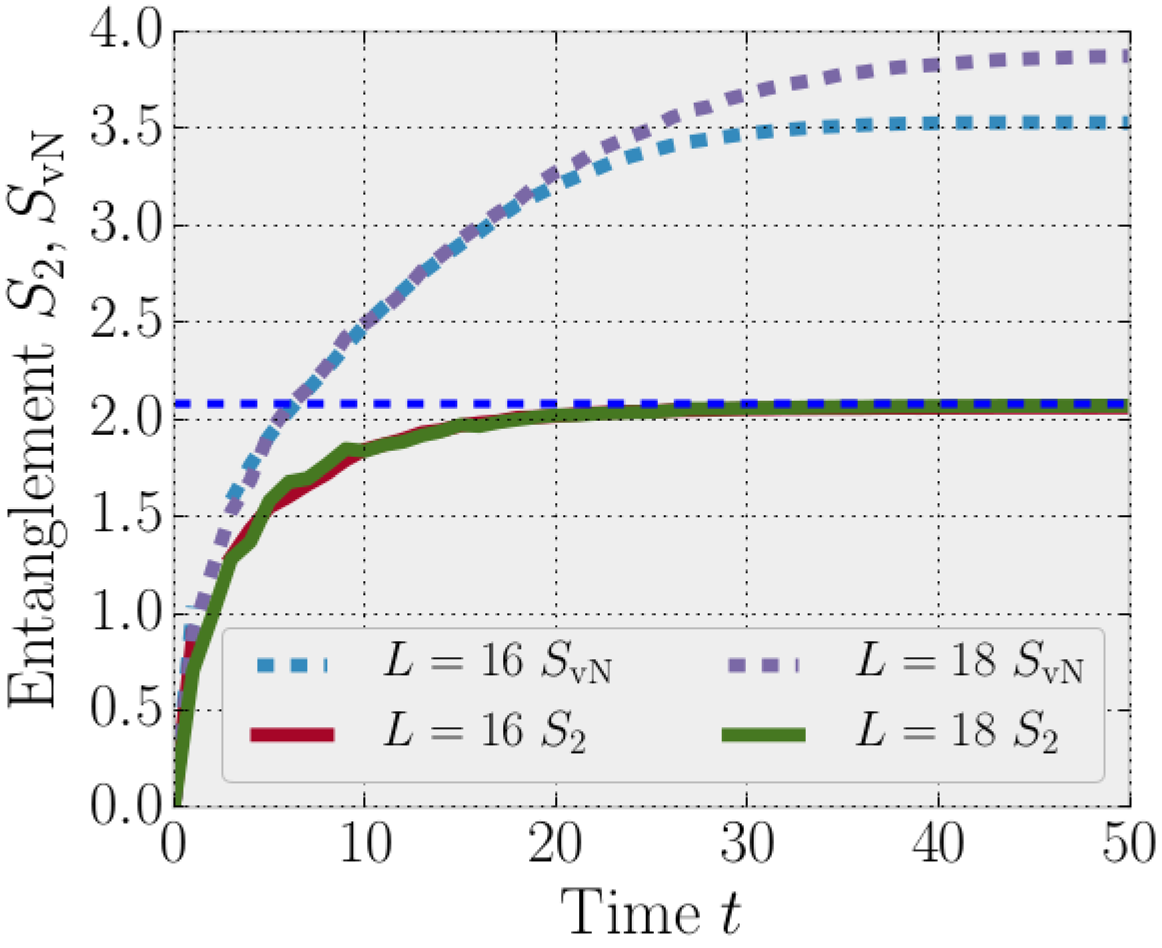
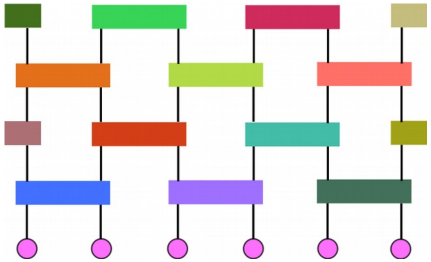
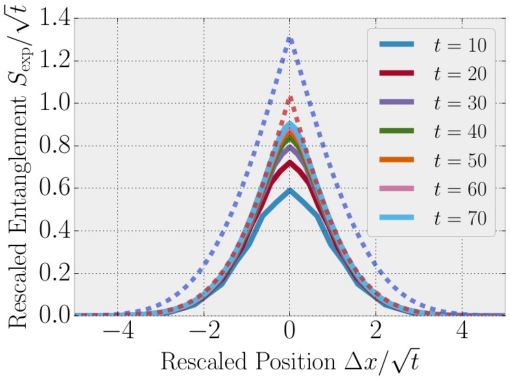


Conclusions and outlook

- Random circuits are a useful toy model for chaotic dynamics



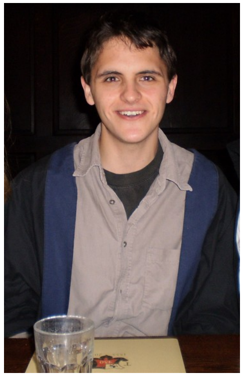
- ‘Coarse-grained’ limit yields exact EOM for the entanglement



- Space-time dependent surface growth model / minimal cut

- Different time regimes: 1) Local equilibration
  2) Charge transport → bumpy entanglement profile
  3) Bumps smooth out

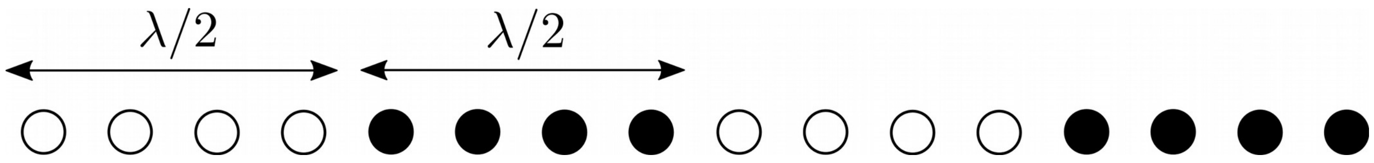
- Charge-fluctuations can also lead to slower growth

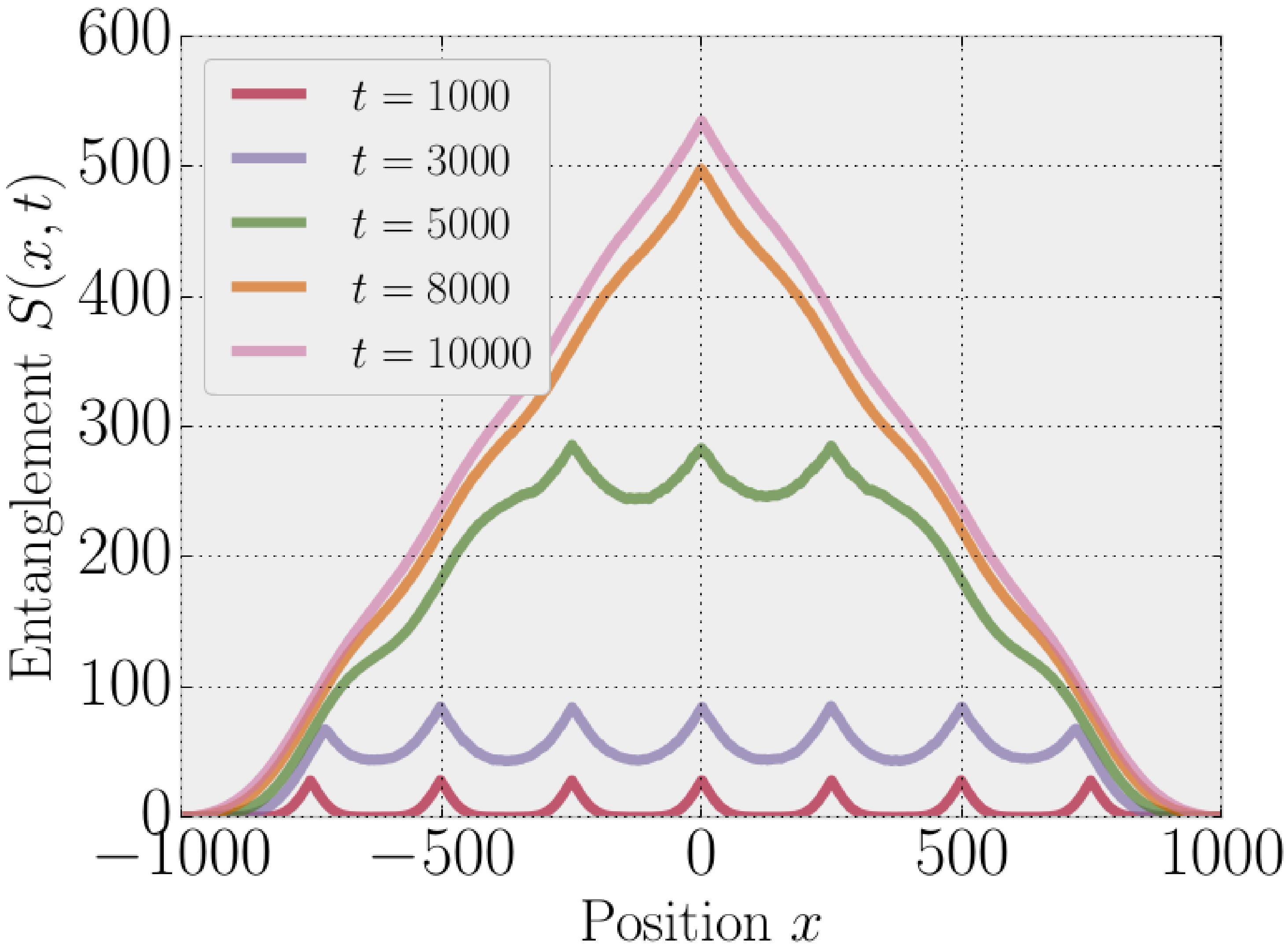


C. V. Keyserlingk

F. Pollmann

At longer times the entanglement profile smooths out



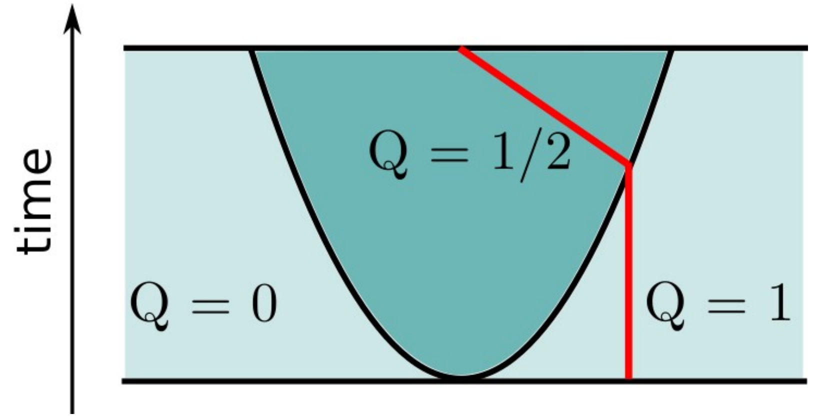


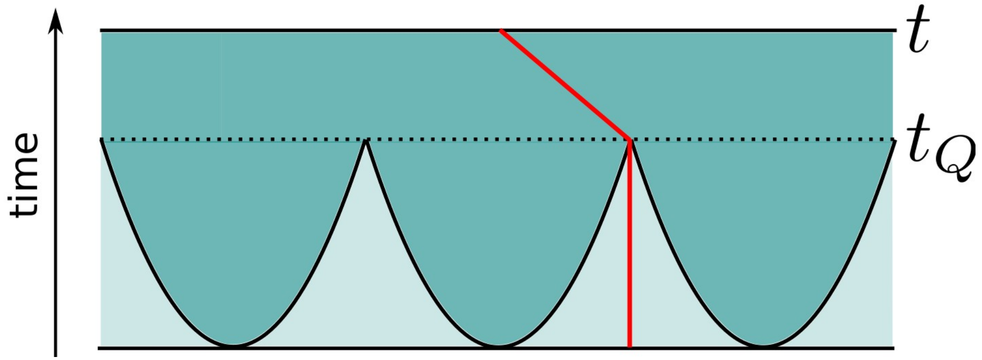
Main features are captured by ‘minimal cut’ picture

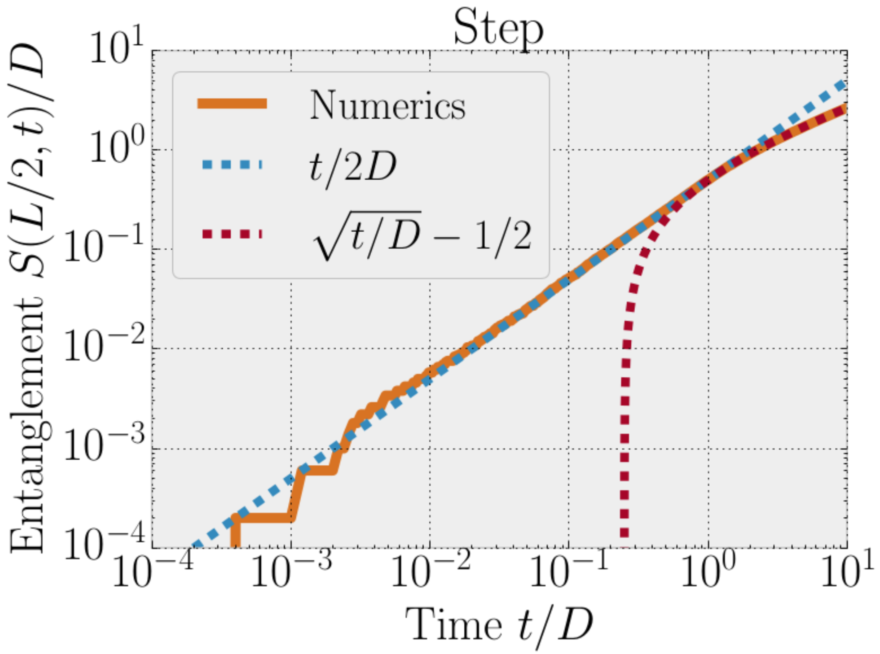
‘Energy’ of a cut:

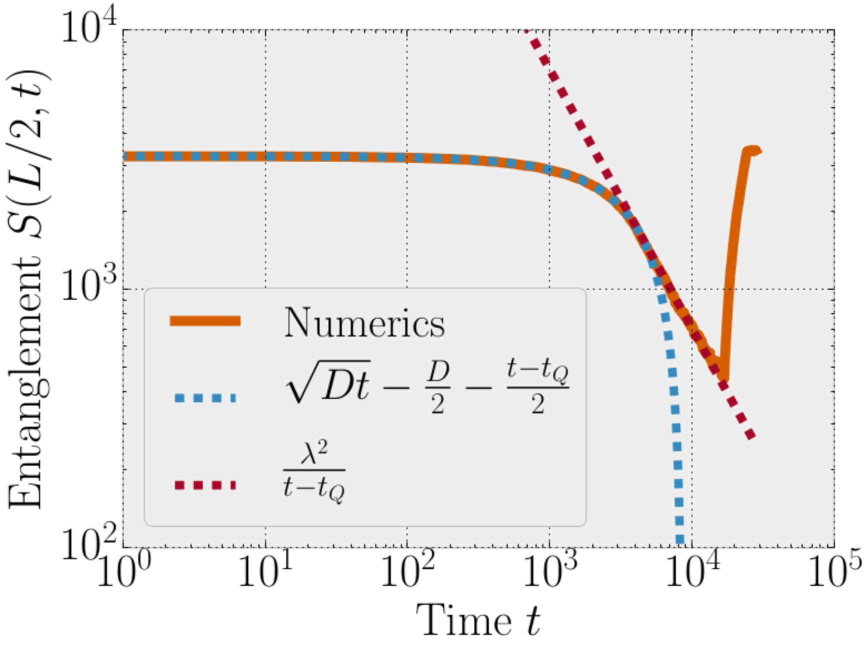
$$E[v(t')] = \int_0^t \mathrm{d}t'\, s(x,t')\,\frac{1+v(t')^2}{2}$$

Entanglement = minimum of energy over cuts







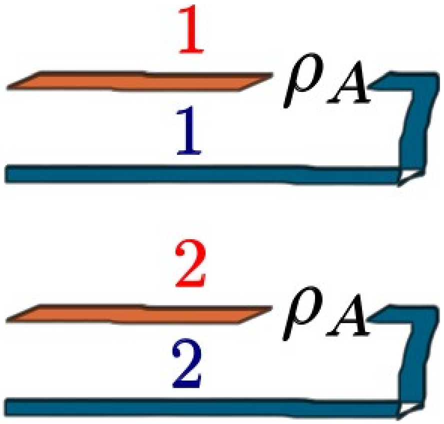


Calculation of average purity can be mapped to a classical partition function

$$e^{-S_2} = \mathrm{tr}(\rho_A^2) = \mathrm{tr}(\mathcal{S}_A \cdot [\rho_A \otimes \rho_A])$$

Swaps copies on subsystem A

2 copies



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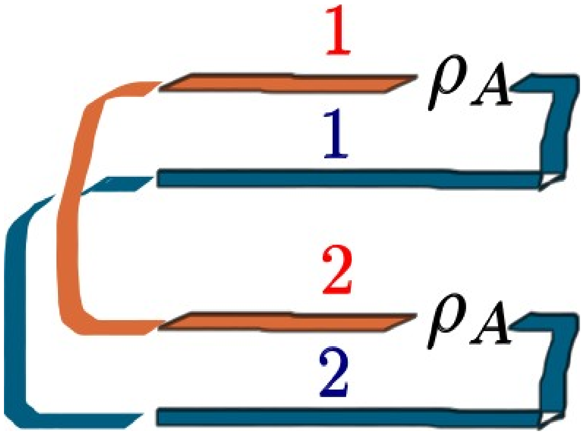

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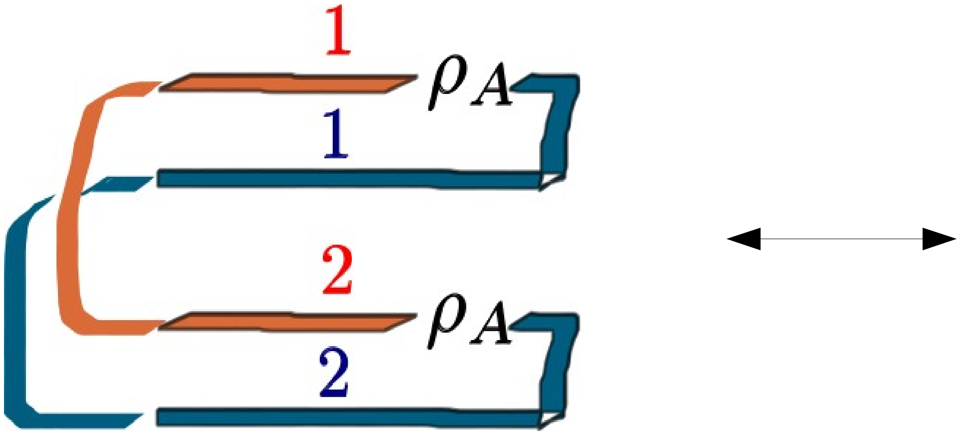
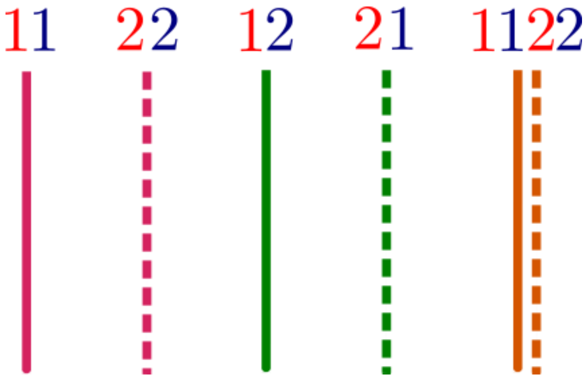
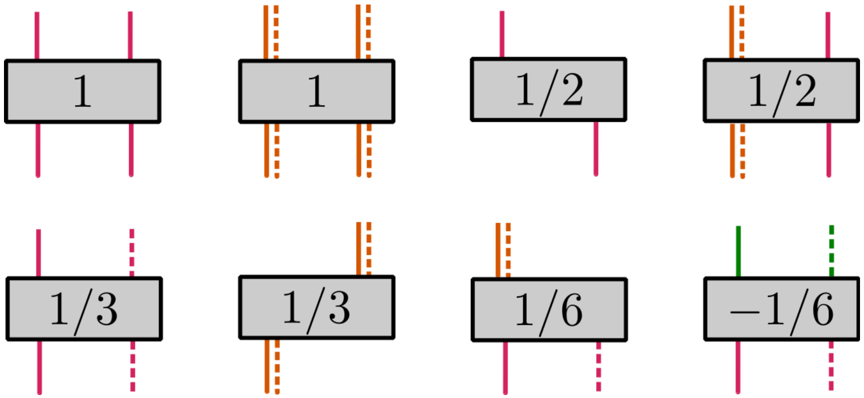


Six local states:



+ empty

Interactions terms arise at each gate:



+ boundary conditions from $\rho(0), \mathcal{S}_A$

Spatial spreading is ballistic at the edges

$$U_{\mathrm{F}} = e^{-i\tau H_4} e^{-i\tau H_3} e^{-i\tau H_2} e^{-i\tau H_1}$$

$$H_1 = J_z^{(1)} \sum_r \hat{Z}_r \hat{Z}_{r+1}$$

$$H_3 = J_z^{(2)} \sum_r \hat{Z}_r \hat{Z}_{r+2}$$

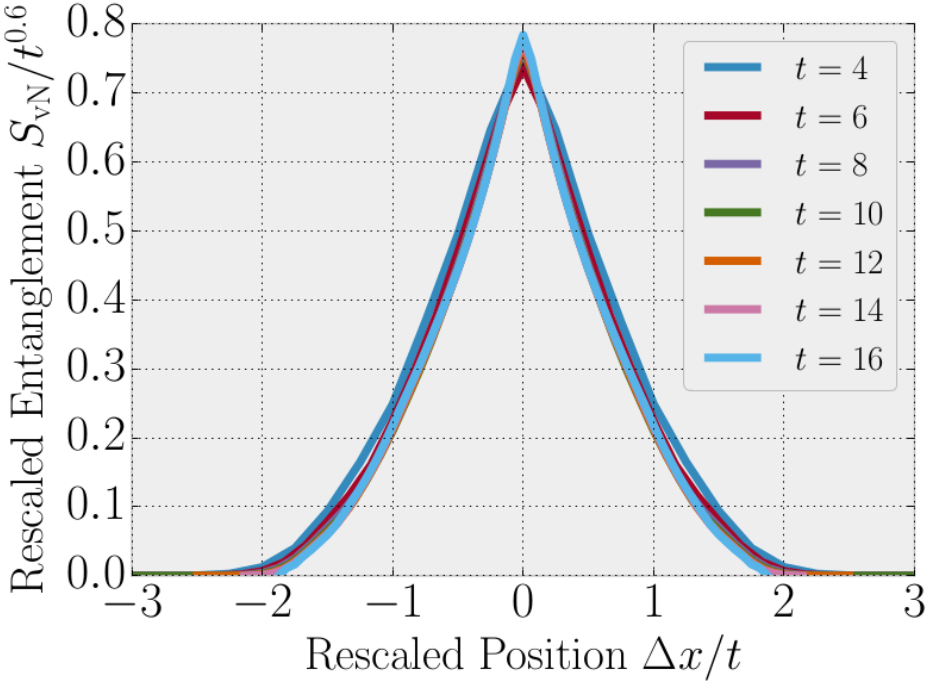
$$H_2 = H_4 = J_{xy} \sum_r \left(\hat{X}_r \hat{X}_{r+1} + \hat{Y}_r \hat{Y}_{r+1} \right),$$

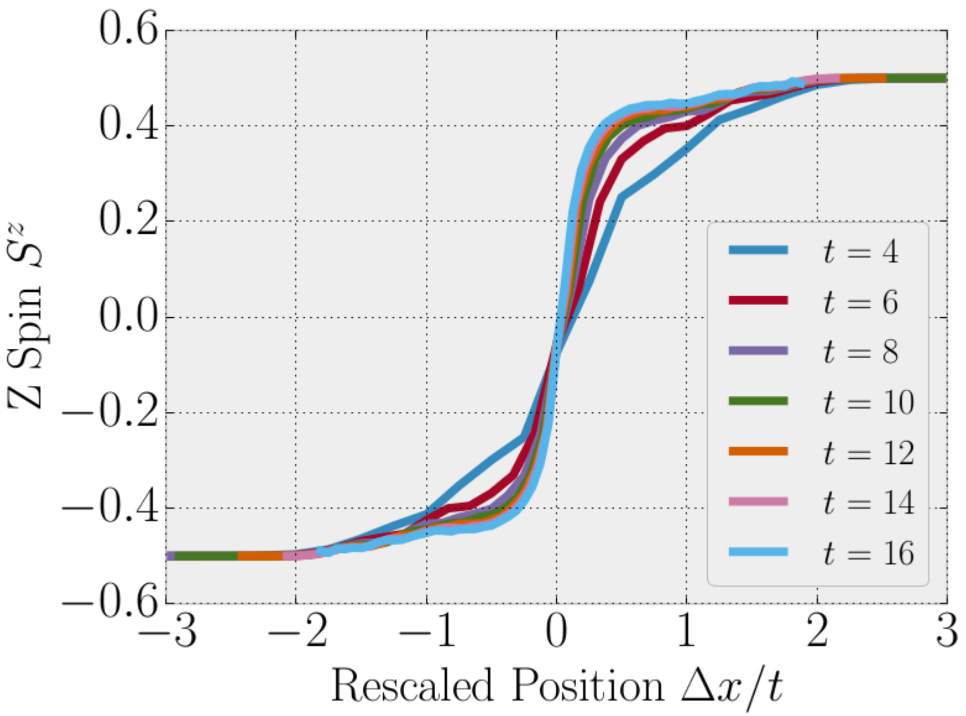
$$T = 4\tau = 1$$

$$J_z^{(1)} = (\sqrt{3} + 5)/6$$

$$J_z^{(2)} = \sqrt{5}/2$$

$$J_{xy} = (2\sqrt{3} + 3)/7$$





Diffusive spreading becomes clearer at finite filling

$$U_{\mathrm{F}} = e^{-i\tau H_4} e^{-i\tau H_3} e^{-i\tau H_2} e^{-i\tau H_1}$$

$$T = 4\tau = 1$$

$$H_1 = J_z^{(1)} \sum_r \hat{Z}_r \hat{Z}_{r+1}$$

$$J_z^{(1)} = (\sqrt{3} + 5)/6$$

$$H_3 = J_z^{(2)} \sum_r \hat{Z}_r \hat{Z}_{r+2}$$

$$J_z^{(2)} = \sqrt{5}/2$$

$$J_{xy} = (2\sqrt{3} + 3)/7$$

$$H_2 = H_4 = J_{xy} \sum_r \left(\hat{X}_r \hat{X}_{r+1} + \hat{Y}_r \hat{Y}_{r+1} \right),$$