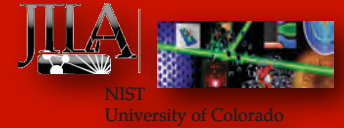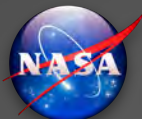# "A Hyperspherical Perspective of Few-body Physics"

Jose P. D'Incao

JILA, Dept. f Physics, University of Colorado at Boulder and NIST

National Science Foundation

CAL/JPL
Cold Atom Laboratory

How do we solve few-body problems?

For a system of few particles ...

$$\hat{H} = -\sum_i \frac{\nabla_i^2}{2m_i} + \sum_{i<j} V(r_{ij})$$

... angles + set of non-compact
coordinates $\quad r_{ij} \rightarrow [0, \infty]$

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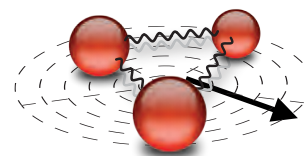coordinates $r_{ij} \rightarrow [0, \infty]$

... the hyperspherical way !!!

$$\hat{H} = -\frac{1}{2\mu}\frac{d^2}{dR^2} + \frac{\hat{\Lambda}^2(\Omega)}{2\mu R^2} + V(R, \Omega)$$

(Hyperradial Kinetic   (Hyperangular Kinetic
Energy)   Energy)

hyperradius R: overall size
(collective motion)

$$R \rightarrow [0, \infty]$$
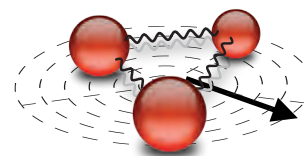
hyperangles $\{\Omega\}$: internal motion

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Democratic hyperangles:
(Smith-Whitten, Johnson, Kuppermann, Aquilanti)
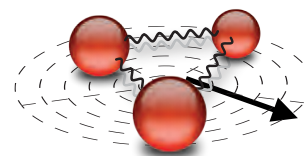
☑ Fragmentation thresholds
☑ Symmetrization is simpler

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... the hyperspherical way !!!

$$\hat{H} = -\frac{1}{2\mu}\frac{d^2}{dR^2} + \frac{\hat{\Lambda}^2(\Omega)}{2\mu R^2} + V(R, \Omega)$$

(Hyperradial Kinetic Energy) (Hyperangular Kinetic Energy)

hyperradius R: overall size (collective motion)

$$R \to [0, \infty]$$

hyperangles $\{\Omega\}$: internal motion

Anisotropic interactions
[**Dipolar:** Wang, D'Incao & Greene, PRL (2011);
Q2D: D'Incao & Esry, (in preparation)]

More bound states (chemistry)
[Wang, D'Incao & Greene, PRA (2011),
Wang, D'Incao, Esry & Greene, PRL (2012)]

Four-body physics (difficult!)
[D'Incao, Rittenhouse, Mehta & Greene, PRA (2009),
von Stecher, D'Incao & Greene, Nat. Phys. (2009)]

Democratic hyperangles:
(Smith-Whitten, Johnson, Kuppermann, Aquilanti)

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For a system of few particles ...

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... the hyperspherical way !!!

$$\hat{H} = -\frac{1}{2\mu}\frac{d^2}{dR^2} + \frac{\hat{\Lambda}^2(\Omega)}{2\mu R^2} + V(R,\Omega)$$

(Hyperradial Kinetic Energy) (Hyperangular Kinetic Energy)

Adiabatic representation:

$$\Psi(R,\Omega) = \sum_\nu F_\nu(R)\Phi_\nu(R;\Omega)$$

(... solve for fixed R)

(effective potential)

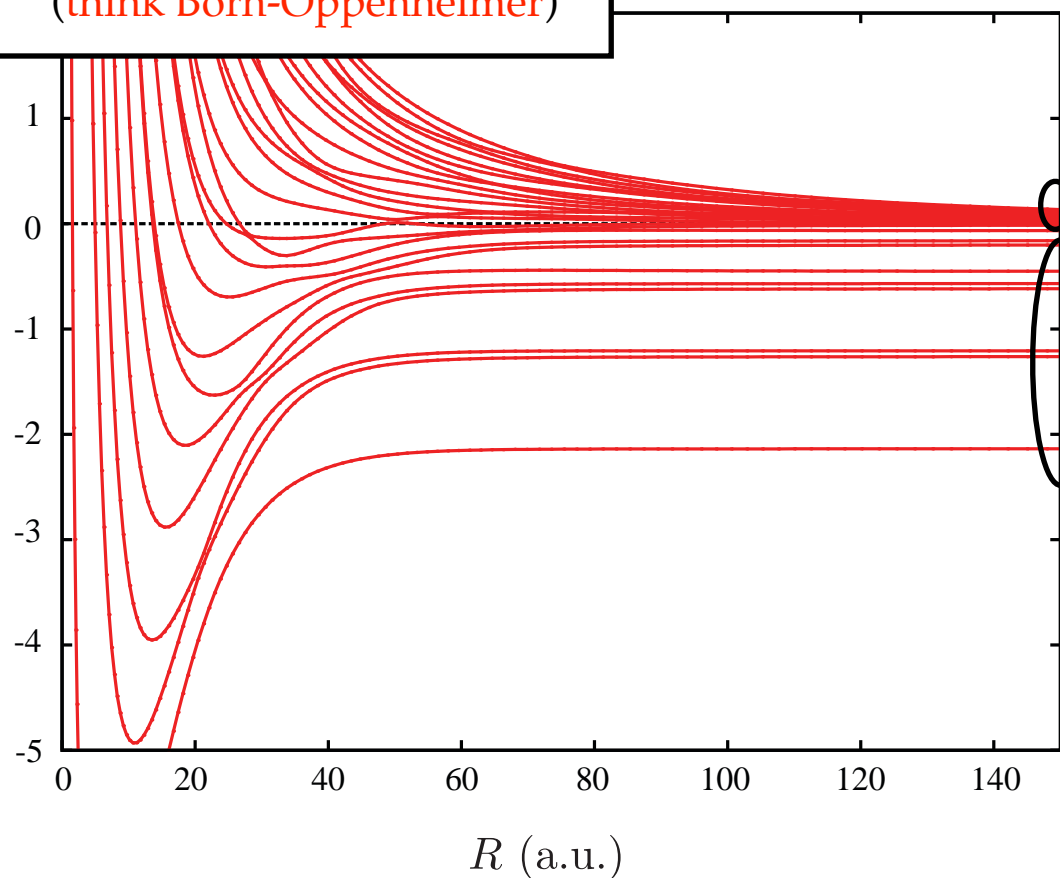$$\left[\frac{\hat{\Lambda}^2(\Omega)}{2\mu R^2} + V(R,\Omega)\right]\Phi_\nu(R;\Omega) = U_\nu(R)\Phi_\nu(R;\Omega)$$

For a system of few particles ...

$$\hat{H} = -\sum_i \frac{\nabla_i^2}{2m_i} + \sum_{i<j} V(r_{ij})$$

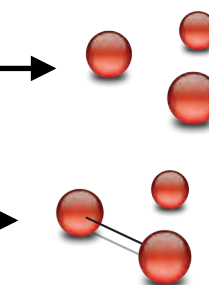**... the hyperspherical way !!!**

$$\hat{H} = -\frac{1}{2\mu}\frac{d^2}{dR^2} + \frac{\hat{\Lambda}^2(\Omega)}{2\mu R^2} + V(R,\Omega)$$

(Hyperradial Kinetic   (Hyperangular Kinetic
Energy)   Energy)

Hyperspherical Potentials
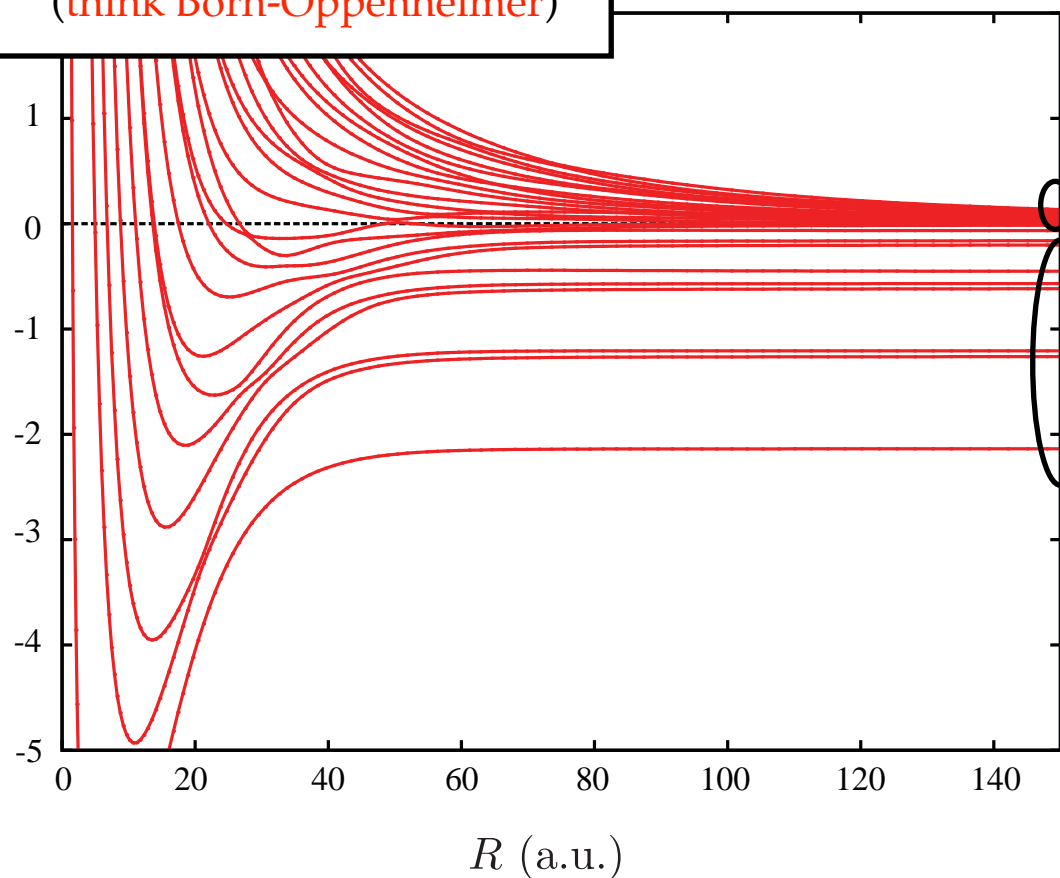(think Born-Oppenheimer)

(three-body continuum)

(bound channels)

R (a.u.)

Bound and Scattering Properties

$$\left[-\frac{1}{2\mu}\frac{d^2}{dR^2} + U_\nu(R) - E \right] F_\nu(R) + \sum_{\nu'} W_{\nu\nu'}(R)F_{\nu'}(R) = 0$$

(Hyperradial Schrodinger Equation)

Hyperspherical Potentials
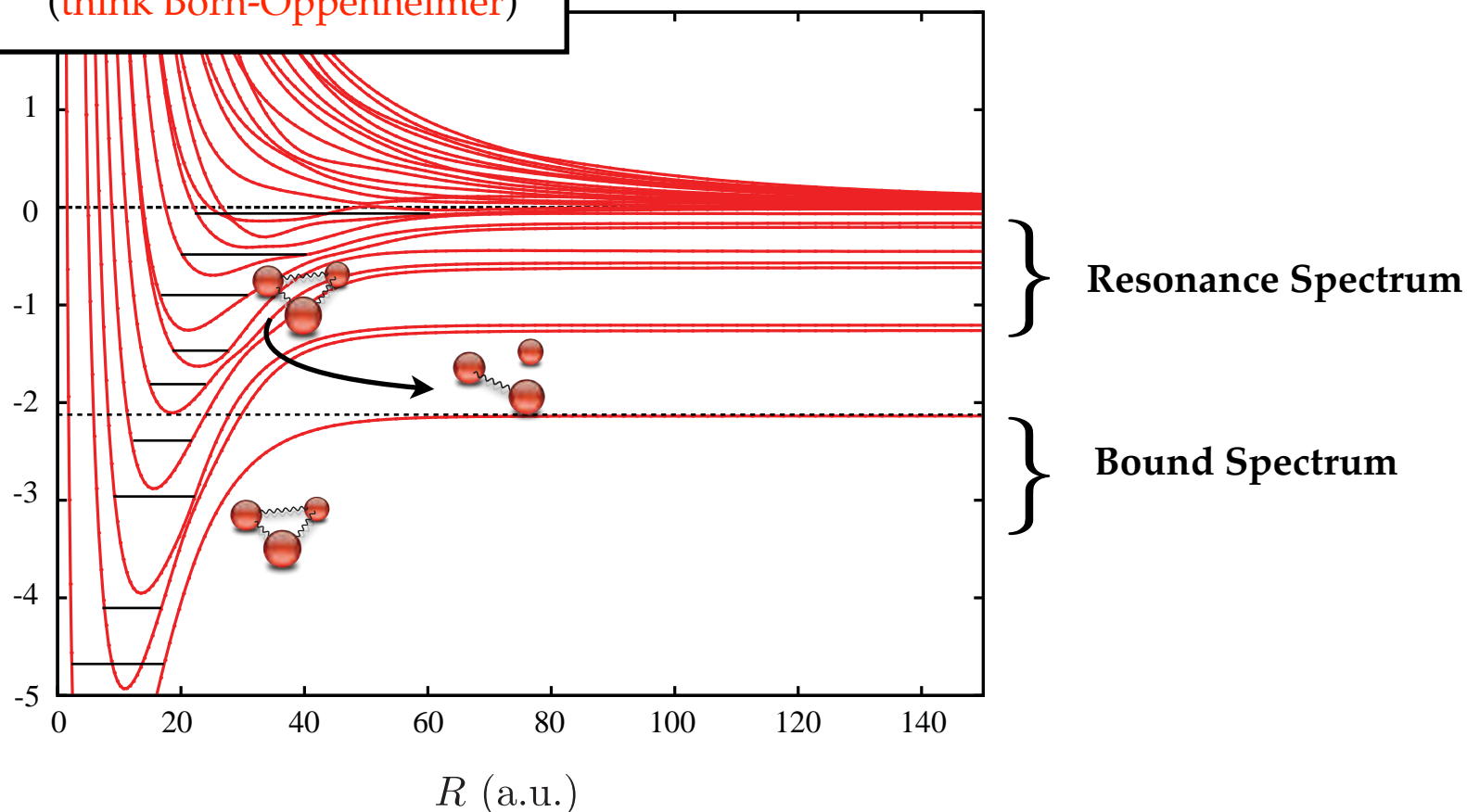(think Born-Oppenheimer)

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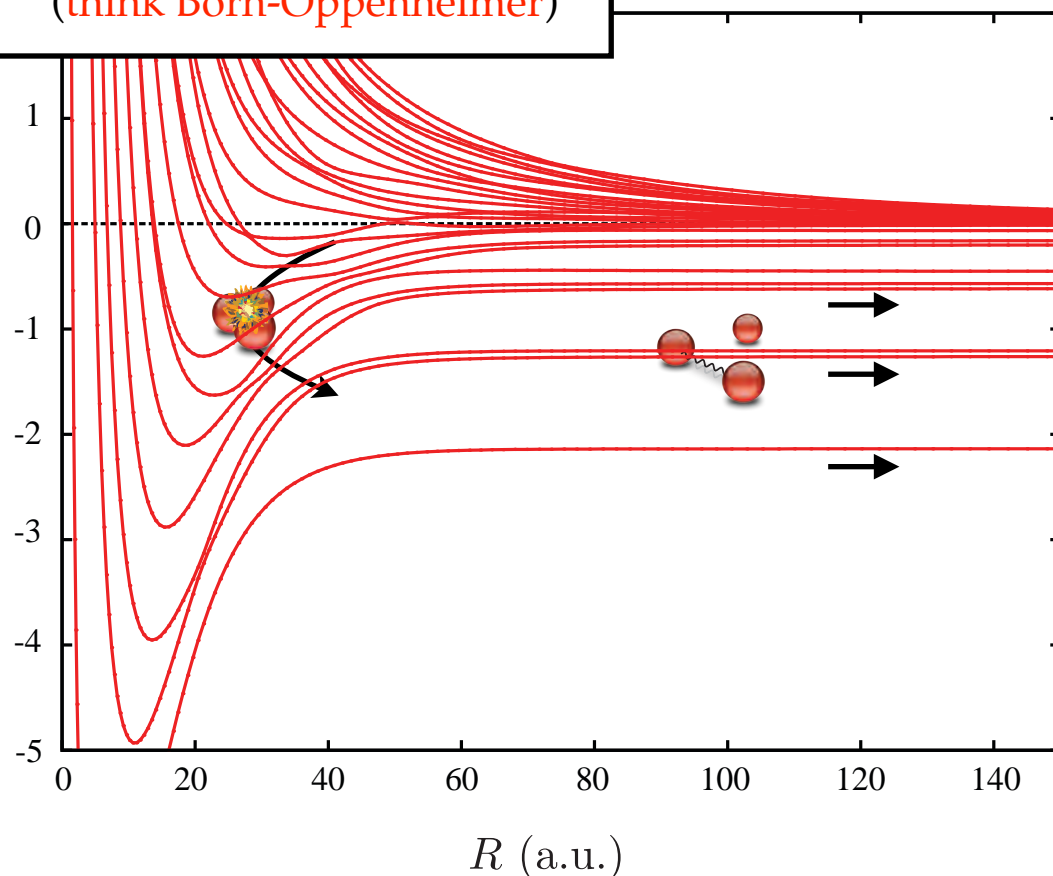Resonance Spectrum

Bound Spectrum

R (a.u.)

Bound and Scattering Properties

$$\left[-\frac{1}{2\mu}\frac{d^2}{dR^2} + U_\nu(R) - E \right] F_\nu(R) + \sum_{\nu'} W_{\nu\nu'}(R) F_{\nu'}(R) = 0$$

(Hyperradial Schrodinger Equation)

Hyperspherical Potentials
(think Born-Oppenheimer)

R (a.u.)

Ultracold Few-body Collisions

$W_{\nu\nu'}(R)$: non-adiabatic couplings
(drive inelastic transitions)

Typical length & energy scales:
– Van der Waals length: $r_0 \approx 100 a_0$
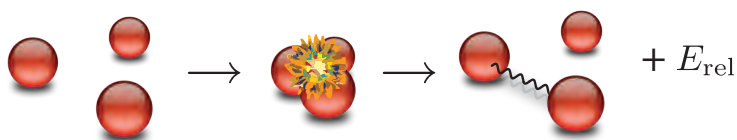– Temperature: $T \approx 100\text{nK}$

$\rightarrow$ Solve Schrödinger equation for $R \approx 10^6 a_0$
(0.05mm!!!)

Bound and Scattering Properties

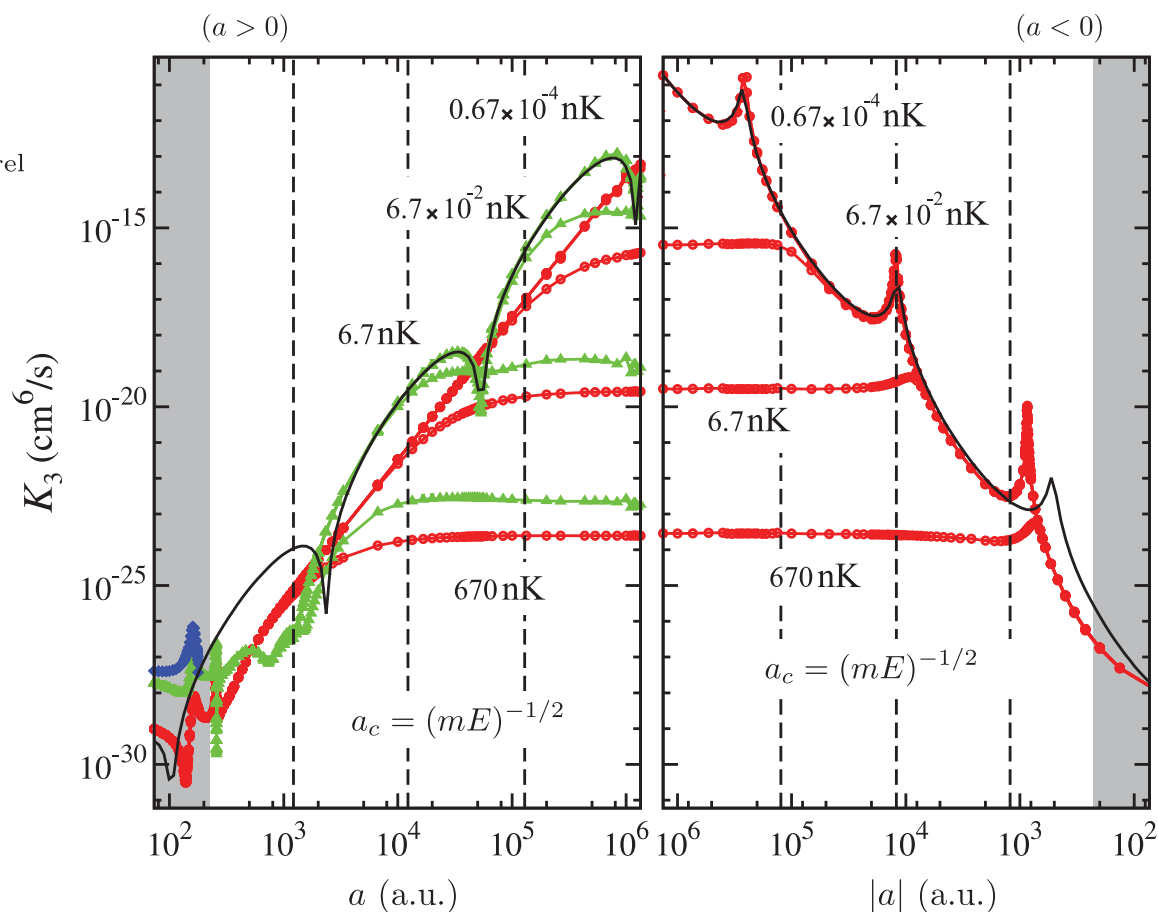$$\left[-\frac{1}{2\mu}\frac{d^2}{dR^2} + U_\nu(R) - E\right]F_\nu(R) + \sum_{\nu'}W_{\nu\nu'}(R)F_{\nu'}(R) = 0$$

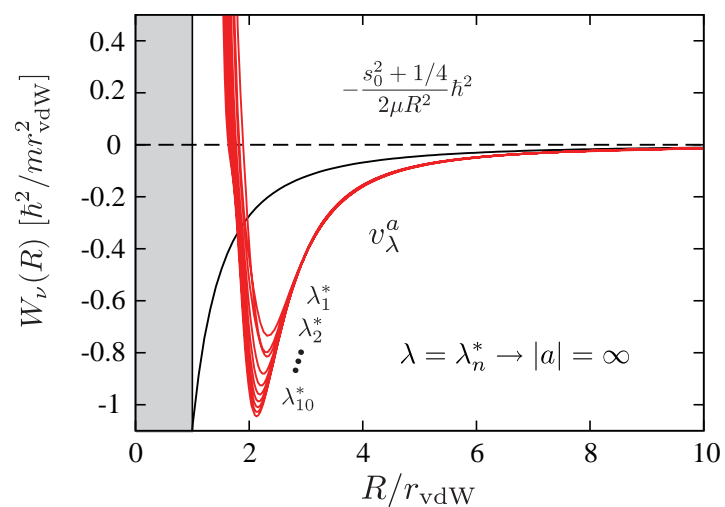(Hyperradial Schrodinger Equation)

Three-body Recombination

$+ E_{\text{rel}}$

Includes:
- finite energy effects
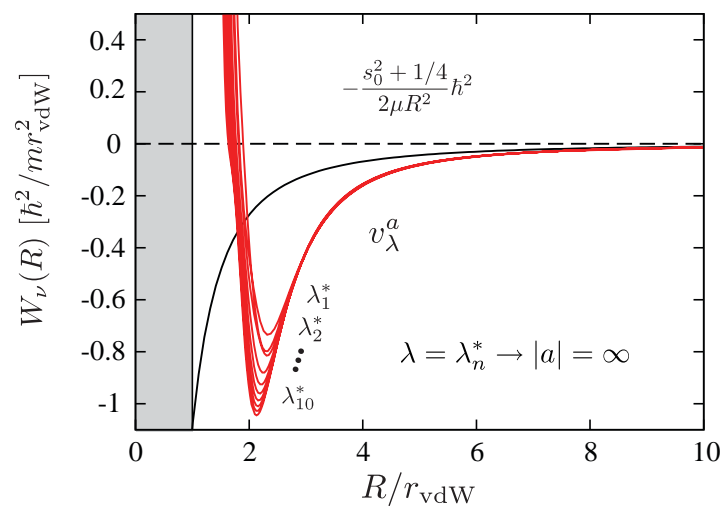- finite-range effects
- multiple bound states

Universality of the Three-body Parameter

[Wang, D'Incao, Esry & Greene (2012); Wang, Wang, D'Incao & Greene (2012); Naidon, Endo & Ueda (2014); Mestrom, Wang, Greene & D'Incao (2016)]

Universality of the Three-body Parameter
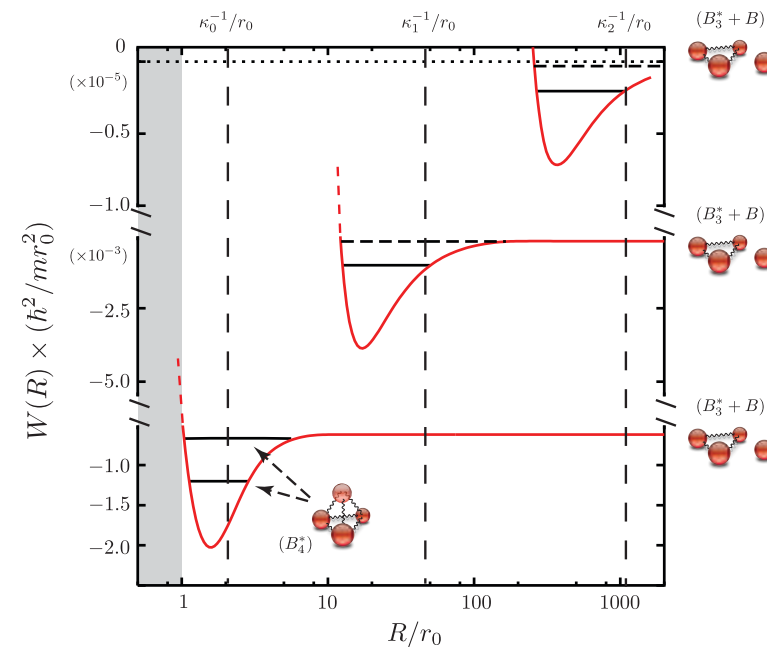

[Wang, D'Incao, Esry & Greene (2012);
Wang, Wang, D'Incao & Greene (2012);
Naidon, Endo & Ueda (2014); Mestrom,
Wang, Greene & D'Incao (2016)]

Four-bosons Universality

[von Stecher, D'Incao, Greene (2009); D'Incao,
von Stecher & Greene (2009)]

Universality of the Three-body Parameter

$$-\frac{s_0^2 + 1/4}{2\mu R^2}\hbar^2$$

v_λ^a

λ_1^*
λ_2^*
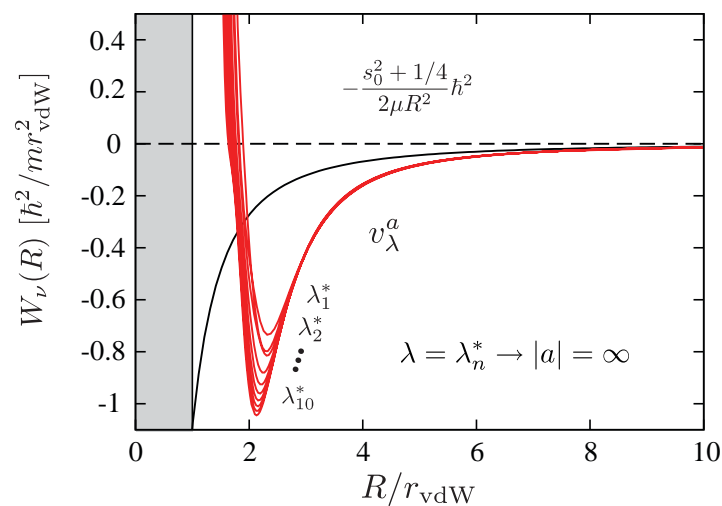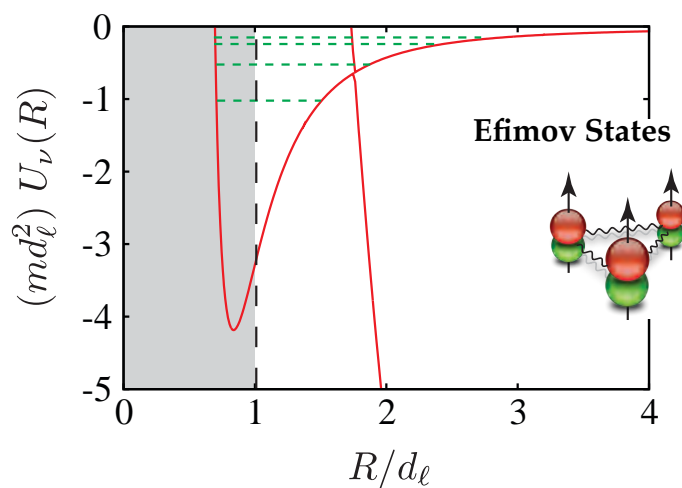
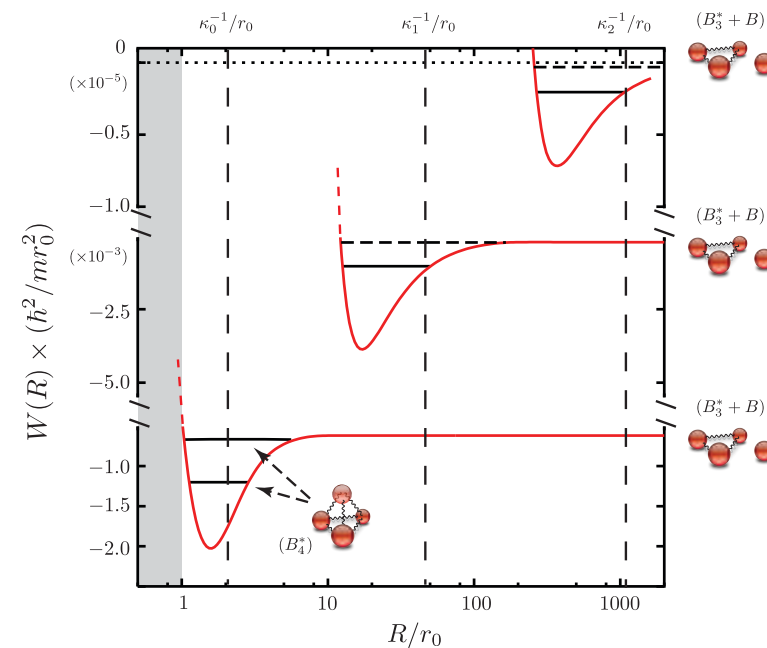$\lambda = \lambda_n^* \to |a| = \infty$

λ_{10}^*

[Wang, D'Incao, Esry & Greene (2012); Wang, Wang, D'Incao & Greene (2012); Naidon, Endo & Ueda (2014); Mestrom, Wang, Greene & D'Incao (2016)]
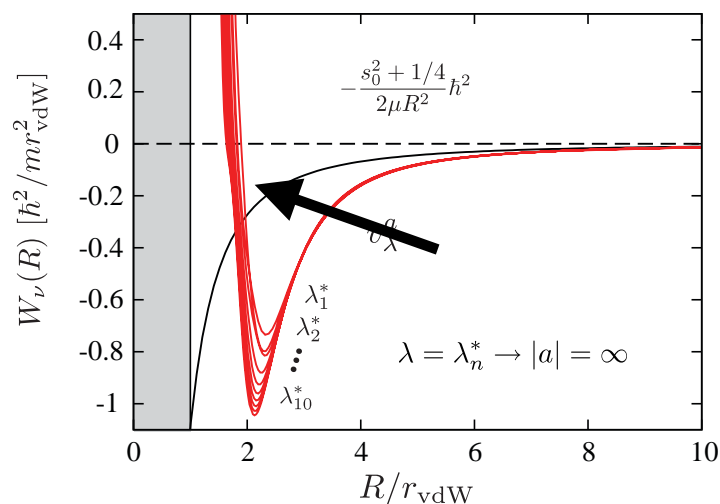
Few-body Dipolar Systems

Efimov States

[Wang, D'Incao & Greene, PRL (2011); Wang, D'Incao & Greene (2011)]

Four-bosons Universality

κ_0^{-1}/r_0 κ_1^{-1}/r_0 κ_2^{-1}/r_0 $(B_3^* + B)$

$(B_3^* + B)$

$(B_3^* + B)$

(B_4^*)

[von Stecher, D'Incao, Greene (2009); D'Incao, von Stecher & Greene (2009)]

Universality of the Three-body Parameter

$$-\frac{s_0^2 + 1/4}{2\mu R^2}\hbar^2$$

U_λ^a

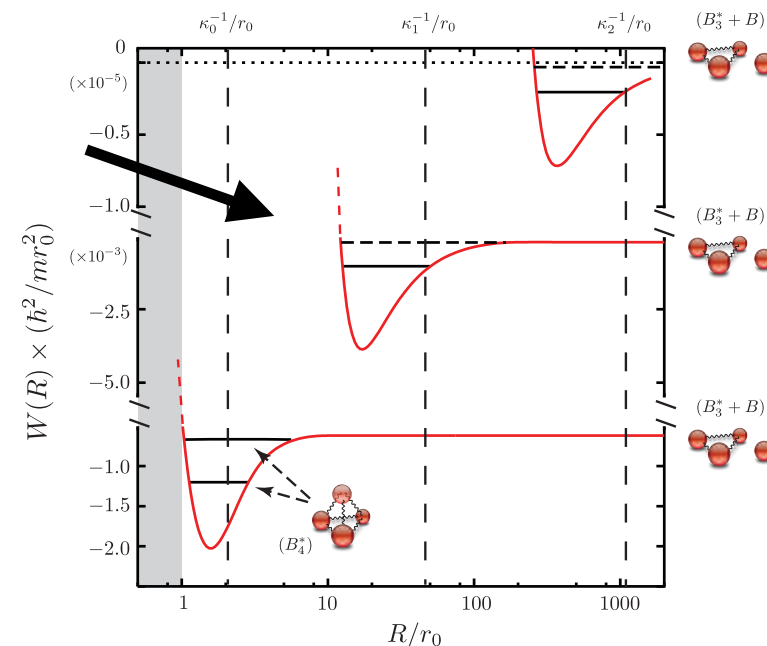λ_1^*
λ_2^*
$\vdots$
λ_{10}^*

$\lambda = \lambda_n^* \rightarrow |a| = \infty$

[Wang, D'Incao, Esry & Greene (2012); Wang, Wang, D'Incao & Greene (2012); Naidon, Endo & Ueda (2014); Mestrom, Wang, Greene & D'Incao (2016)]
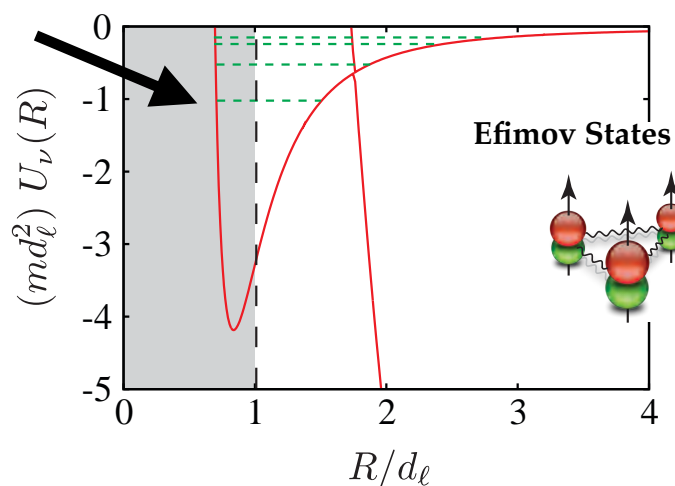
Repulsive barrier controlling the universal properties of the system.

Four-bosons Universality

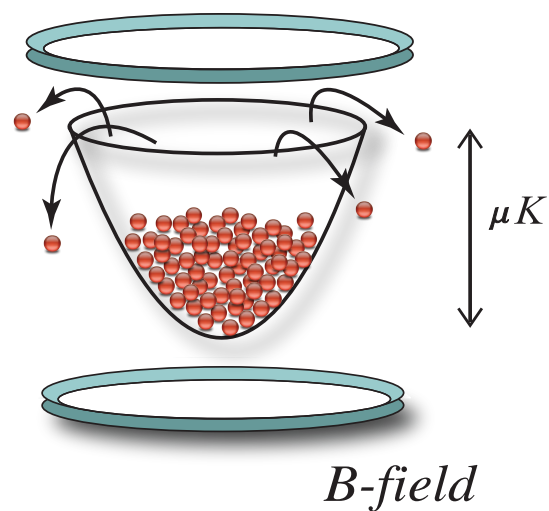[von Stecher, D'Incao, Greene (2009); D'Incao, von Stecher & Greene (2009)]

Few-body Dipolar Systems

Efimov States

[Wang, D'Incao & Greene, PRL (2011); Wang, D'Incao & Greene (2011)]

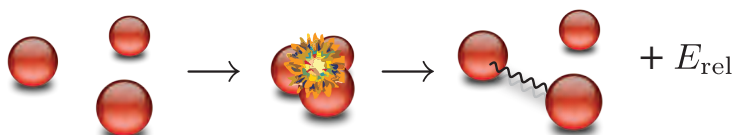# How does a simple picture emerge?

Ultracold Quantum Gases

μK

B-field

Inelastic Collisions $\longrightarrow$ **Stability and Lifetime of condensates !!!**

D'Incao & Esry PRL (2005)

Three-body Recombination

$+ E_{\rm rel}$

$$K_3 = n! \sum_{fi} 32\pi^2 \frac{(2J+1)}{\mu k^4} |T_{fi}|^2$$

Three-body Recombination

$+ E_{\rm rel}$
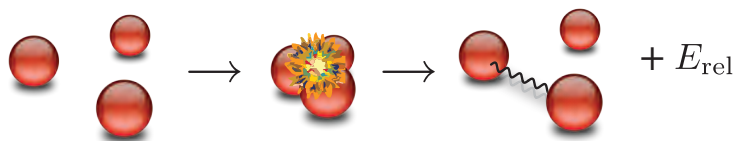
$$K_3 = n! \sum_{fi} 32\pi^2 \frac{(2J+1)}{\mu k^4} |T_{fi}|^2$$

Pathway analysis [D'Incao & Esry PRL (2005)]

$$|T_{fi}|^2 = \left| \sum_j |A_j^{fi}| e^{i\phi_j} \right|^2 = \sum_{j,k} |A_j^{fi}||A_k^{fi}| \cos(\phi_j - \phi_k)$$

Amplitudes and phases (WKB)

$$|A_j^{fi}| \to P_{x \to y}^{(\nu)} = \exp\left[-2 \left| \int_y^x \sqrt{2\mu \left(W_\nu(R) + \frac{1/4}{2\mu R^2} \right)} dR \right| \right]$$

$$\phi_j = \int \sqrt{-2\mu \left(W(R) + \frac{1/4}{2\mu R^2} \right)} dR - \frac{\pi}{2}$$

Three-body Recombination

$$+ E_{\rm rel}$$

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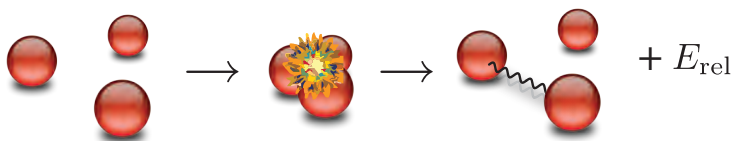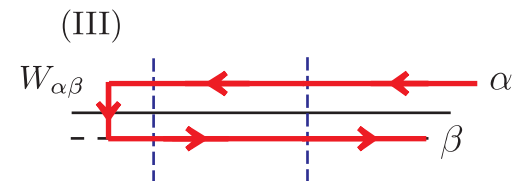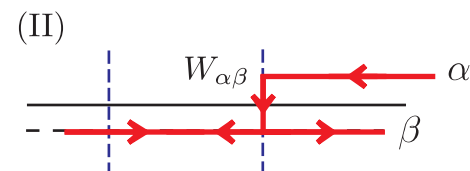
Pathways for Recombination (a>0)

(a) $a > 0$

α

$$\frac{s_\nu^2 - 1/4}{2\mu R^2}$$

$$\frac{\lambda(\lambda+4) + 15/4}{2\mu R^2}$$

R

$W_{\alpha\beta}$

$$E_{v0}^* + \frac{l(l+1)}{2\mu R^2}$$

a

$W_\nu(R)$

$W_{\alpha\beta}$
$W_{\beta\gamma}$ $W_{\alpha\gamma}$

β

$$-\frac{s_0^2 + 1/4}{2\mu R^2}$$

r_0

$$P_{\rm decay} = (1 - e^{-4\eta})$$

$$\gamma \equiv \{\gamma_1, \gamma_2, ...\}$$

$$E_{vl'} + \frac{l(l+1)}{2\mu R^2}$$

r_0 a

(I)

$E = 0$

$W_{\alpha\beta}$ α

R

a

β

r_0 a

(II)

$W_{\alpha\beta}$ α

β

(III)

$W_{\alpha\beta}$ α

β

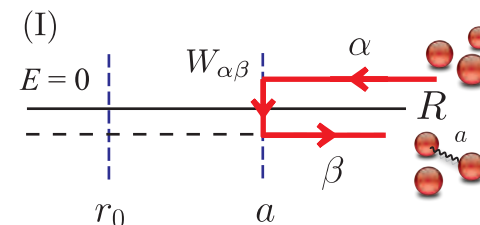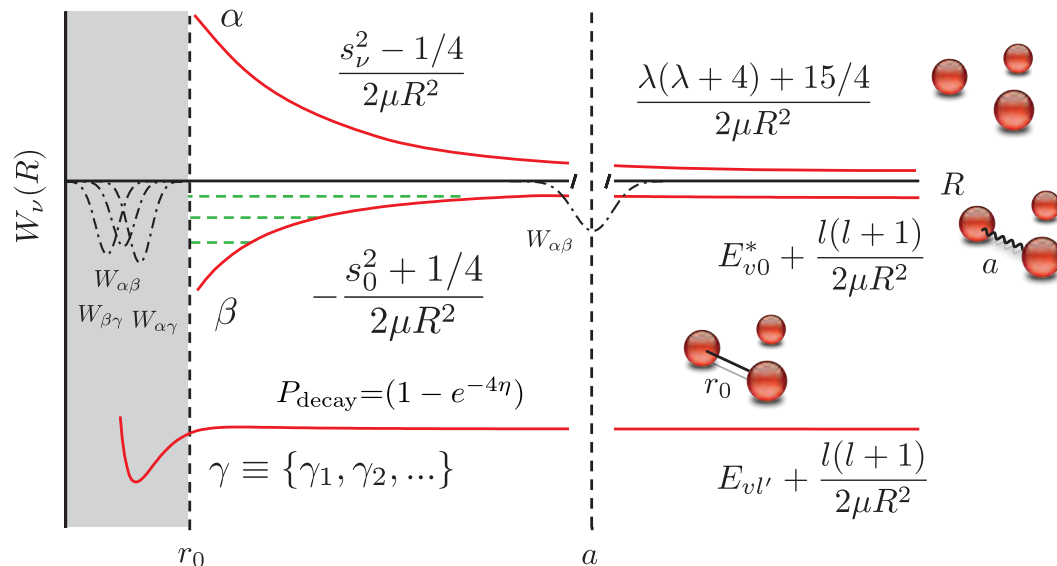## Three-body Recombination

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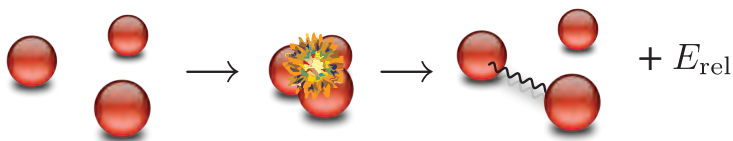
Pathways for Recombination (a>0)

Amplitudes and phases (WKB)

$$|A_{\text{I}}^{\beta\alpha}|^2 = P_{r_c \to a}^{(\alpha)} P_{a \to \infty}^{(\beta)},$$

$$|A_{\text{II}}^{\beta\alpha}|^2 = P_{r_c \to a}^{(\alpha)} P_{a \to r_0}^{(\beta)} P_{r_0 \to a}^{(\beta)} P_{a \to \infty}^{(\beta)},$$
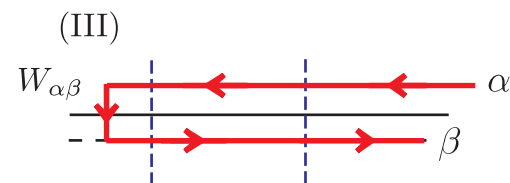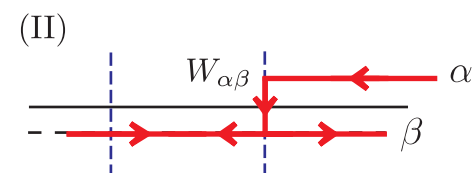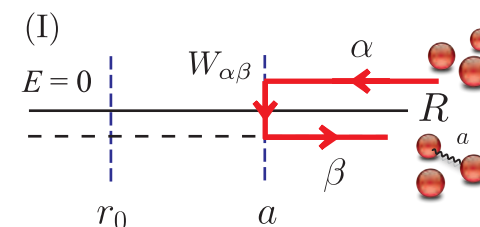
$$|A_{\text{III}}^{\beta\alpha}|^2 = P_{r_c \to a}^{(\alpha)} P_{a \to r_0}^{(\alpha)} P_{r_0 \to a}^{(\beta)} P_{a \to \infty}^{(\beta)}$$

$$[r_c = (\lambda + 2)/k: \text{ classical turning point}]$$

Three-body Recombination

 $+ E_{\text{rel}}$

$$K_3 = n! \sum_{fi} 32\pi^2 \frac{(2J+1)}{\mu k^4} |T_{fi}|^2$$
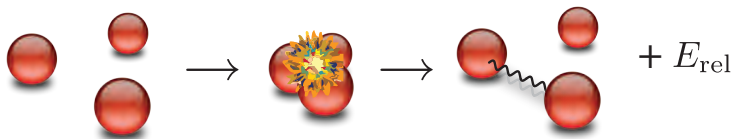
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Pathways for Recombination (a>0)

Amplitudes and phases (WKB)

$$|A_{\text{I}}^{\beta\alpha}|^2 = A^2 (ka)^{2\lambda+4},$$
$$|A_{\text{II}}^{\beta\alpha}|^2 = A^2 e^{-4\eta} (ka)^{2\lambda+4},$$
$$|A_{\text{III}}^{\beta\alpha}|^2 = B^2 e^{-4\eta} \left(\frac{r_0}{a}\right)^{2s_1} (ka)^{2\lambda+4},$$

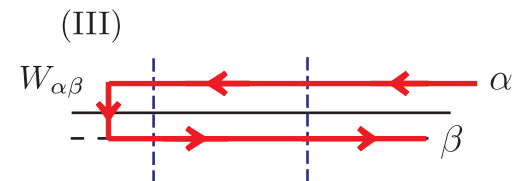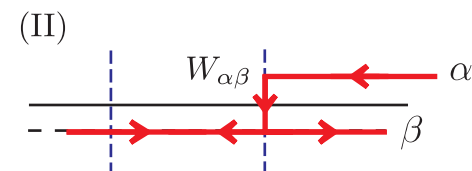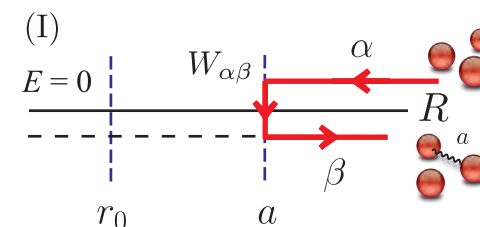$[\text{Phases: } \phi_{\text{I}} - \phi_{\text{II}} = -2[s_0 \ln(a/r_0) + \Phi],$
$\phi_{\text{I}} - \phi_{\text{III}} = -[s_0 \ln(a/r_0) + \Phi],$
$\phi_{\text{II}} - \phi_{\text{III}} = s_0 \ln(a/r_0) + \Phi]$

(I)

(II)

(III)

Three-body Recombination

$+ E_{\rm rel}$
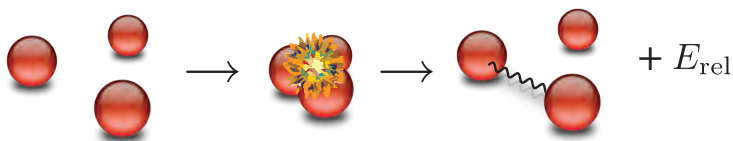
$$K_3 = n! \sum_{fi} 32\pi^2 \frac{(2J+1)}{\mu k^4} |T_{fi}|^2$$

$$K_3^w = \frac{\hbar}{\mu} \left[4A_w^2 e^{-2\eta} \left(\sin^2 \left[s_0 \ln(a/a_+) \right] + \sinh^2 \eta \right) \right.$$
$$+ 2A_w B_\eta \frac{\sin \left[s_0 \ln(a/a_+) \right]}{e^{2\eta}(1 + e^{-2\eta})^{-1}} \left(\frac{r_0}{a} \right)^{s_1}$$
$$\left. + B_\eta^2 e^{-4\eta} \left(\frac{r_0}{a} \right)^{2s_1} \right] k^{2\lambda} a^{2\lambda+4},$$

$[A_w^2 \approx 67.1177\sqrt{3}/4]$

[Nielsen & Macek PRL (1999); Esry, Greene & Burke PRL (1999); Bedaque, Braaten & Hammer PRL (2000); Petrov (2005); Macek, Ovchinnikov & Gasaneo PRA (2006); Gogolin, Mora & Egger PRL (2008); Helfrich, Hammer & Petrov PRA (2010)]

Pathway analysis   [D'Incao & Esry PRL (2005)]

$$|T_{fi}|^2 = \left| \sum_j |A_j^{fi}| e^{i\phi_j} \right|^2 = \sum_{j,k} |A_j^{fi}||A_k^{fi}| \cos(\phi_j - \phi_k)$$

Recombination (a>0)

(a) $BBB(J^\pi = 0^+, a > 0)$

$\left(\frac{r_0}{a} \right)^{s_1} \simeq \left(\frac{r_0}{a} \right)^{4.47}$

K_3^w

K_3^d

e^{π/s_0}

$\eta=0.01$
$\eta=0.1$
$\eta=0.5$
$\eta=1$

$K_3^w, K_3^d \ (A_w^2 \hbar a^4/\mu)$

a/r_0

Three-body Recombination

$$+ E_{\text{rel}}$$

$$K_3 = n! \sum_{fi} 32\pi^2 \frac{(2J+1)}{\mu k^4} |T_{fi}|^2$$

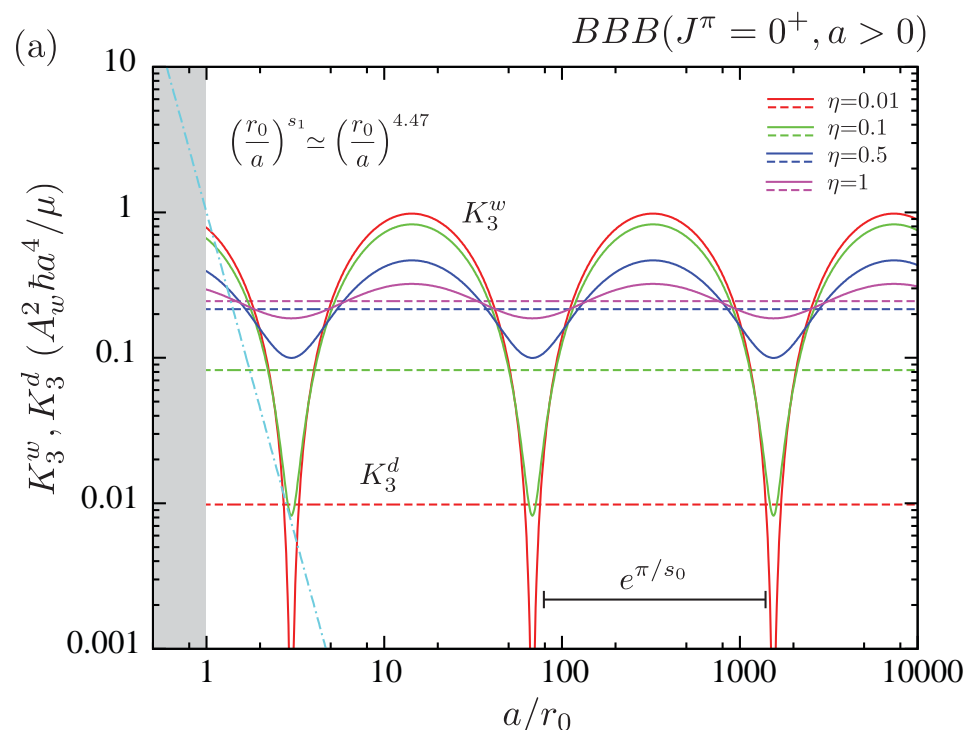**Pathway analysis**   [D'Incao & Esry PRL (2005)]

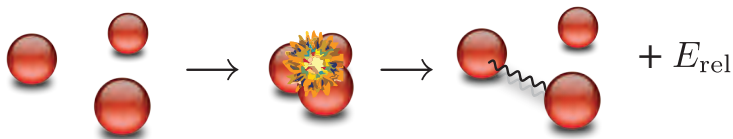$$|T_{fi}|^2 = \left| \sum_j |A_j^{fi}| e^{i\phi_j} \right|^2 = \sum_{j,k} |A_j^{fi}||A_k^{fi}| \cos(\phi_j - \phi_k)$$

Pathways for Recombination (a<0)

(b) $a < 0$

$$\frac{s_\nu^2 - 1/4}{2\mu R^2}$$

$$\frac{\lambda(\lambda+4)+15/4}{2\mu R^2}$$

$$-\frac{s_0^2 + 1/4}{2\mu R^2}$$

$$P_{\text{decay}} = (1 - e^{-4\eta})$$

$$\gamma \equiv \{\gamma_1, \gamma_2, ...\}$$

$$E_{vl'} + \frac{l(l+1)}{2\mu R^2}$$

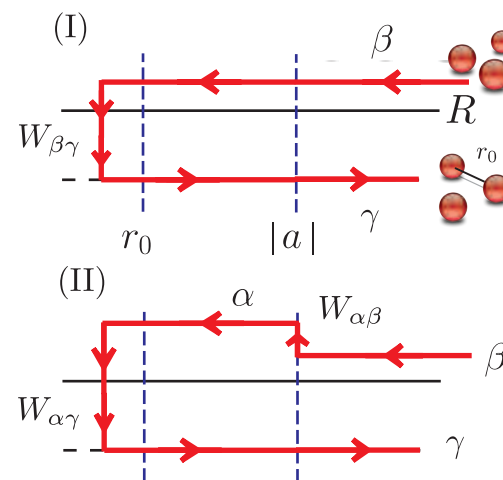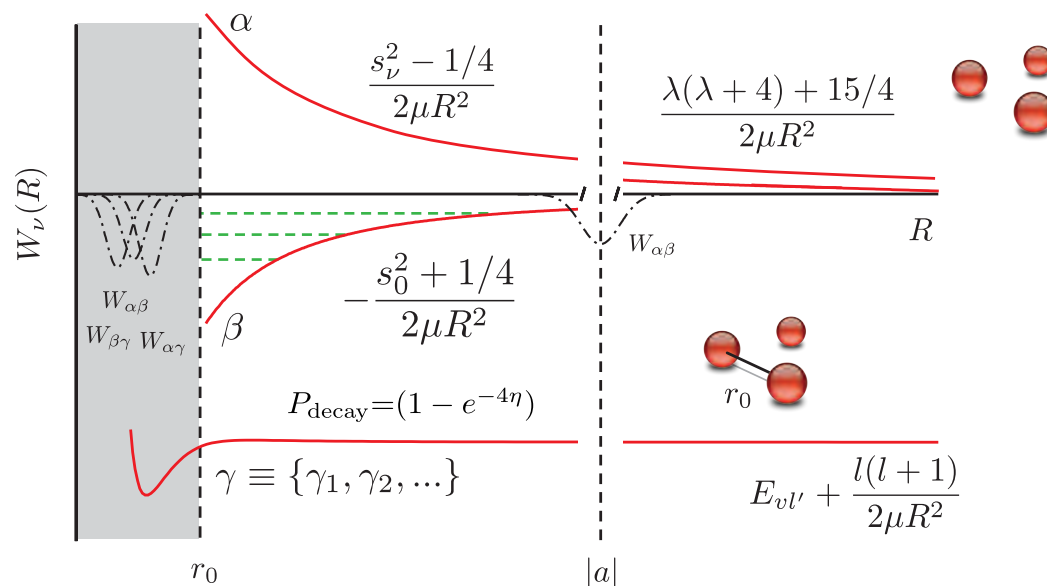## Three-body Recombination

 $+ E_{\rm rel}$

$$K_3 = n! \sum_{fi} 32\pi^2 \frac{(2J+1)}{\mu k^4} |T_{fi}|^2$$

Pathway analysis [D'Incao & Esry PRL (2005)]

$$|T_{fi}|^2 = \left| \sum_j |A_j^{fi}| e^{i\phi_j} \right|^2 = \sum_{j,k} |A_j^{fi}||A_k^{fi}| \cos(\phi_j - \phi_k)$$
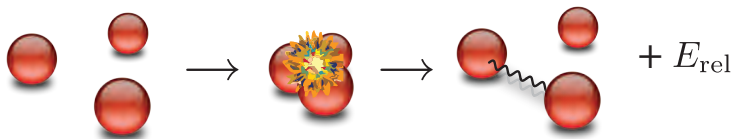
Pathways for Recombination (a<0)

Amplitudes and phases (WKB)

$$|A_{\rm I,j}^{\gamma\beta}|^2 = A^2 (1 - e^{-4\eta})(e^{-4\eta})^{j-1}(k|a|)^{2\lambda+4},$$

$$|A_{\rm II}^{\gamma\beta}|^2 = B^2 (1 - e^{-4\eta}) \left(\frac{r_0}{|a|} \right)^{2s_1} (k|a|)^{2\lambda+4},$$
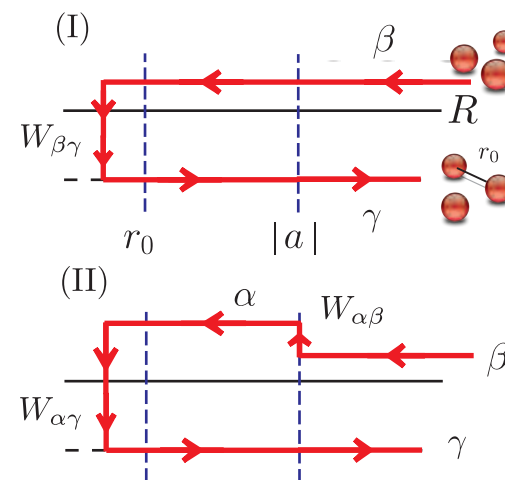
$$(j = 1, 2, ..., \infty)$$

[Phases: $\phi_{\rm I,j} - \phi_{\rm II} = (2j-1)[s_0 \ln(|a|/r_0) + \Phi]$

$\phi_{\rm I,j} - \phi_{\rm I,k} = 2(j-k)[s_0 \ln(|a|/r_0) + \Phi]]$

Three-body Recombination

$$+ E_{\text{rel}}$$

$$K_3 = n! \sum_{fi} 32\pi^2 \frac{(2J+1)}{\mu k^4} |T_{fi}|^2$$

$$K_3^d = \frac{\hbar}{\mu} \left[\frac{A_d^2}{2} \frac{\sinh 2\eta}{\sin^2 \left[s_0 \ln(|a/a_-|) \right] + \sinh^2 \eta} \right. $$

$$+ \frac{A_d B_\eta}{(1 - e^{-2\eta})^{-1}} \frac{\sinh 2\eta \cos \left[s_0 \ln(|a/a_-|) \right]}{\sin^2 \left[s_0 \ln(|a/a_-|) \right] + \sinh^2 \eta} \left(\frac{r_0}{|a|} \right)^{s_1}$$

$$\left. + B_\eta^2 \left(\frac{r_0}{|a|} \right)^{2s_1} \right] k^{2\lambda} |a|^{2\lambda+4}$$

$$[A_d^2 \approx 4590(2\sqrt{3})]$$

[Braaten & Hammer PRA (2004);
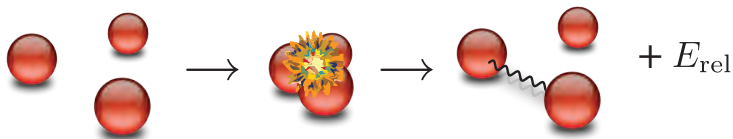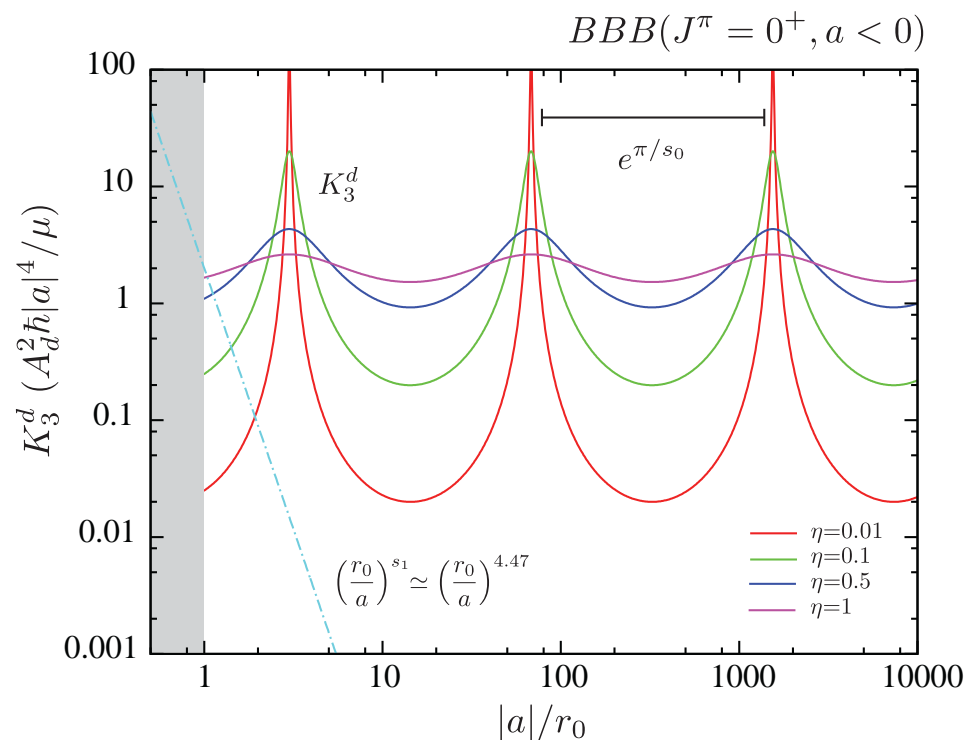D'Incao & Esry PRL (2005)]

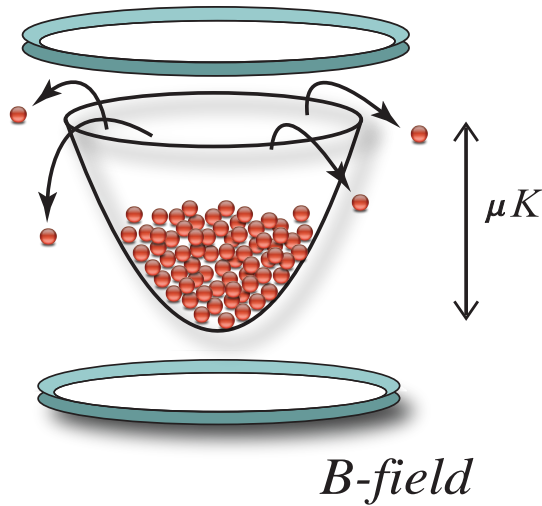**Pathway analysis**   [D'Incao & Esry PRL (2005)]

$$|T_{fi}|^2 = \left| \sum_j |A_j^{fi}| e^{i\phi_j} \right|^2 = \sum_{j,k} |A_j^{fi}||A_k^{fi}| \cos(\phi_j - \phi_k)$$

Recombination (a<0)

$$BBB(J^\pi = 0^+, a < 0)$$

K_3^d

e^{π/s_0}

$$\left(\frac{r_0}{a} \right)^{s_1} \simeq \left(\frac{r_0}{a} \right)^{4.47}$$

K_3^d $(A_d^2 \hbar |a|^4/\mu)$

$|a|/r_0$

η=0.01
η=0.1
η=0.5
η=1

Ultracold Quantum Gases

μK

B-field

Elastic Collisions $\longrightarrow$ **Correlations and Interactions !!!**

D'Incao & Esry PRL (2005)

Three-body Elastic Collisions
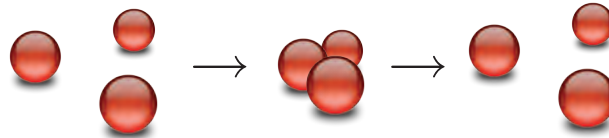
For $R \gg |a|$:

$$W_\nu(R) = \frac{\lambda(\lambda + 4) + 15/4}{2\mu R^2} = \frac{l_{\text{eff}}(l_{\text{eff}} + 1)}{2\mu R^2}$$
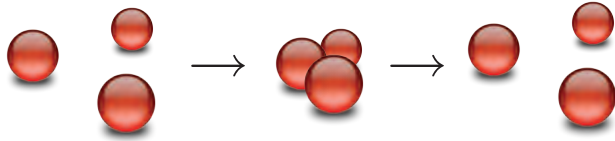
where $l_{\text{eff}} = \lambda + 3/2$

Three-body scattering length (L^4)

$$A_{3b}^{(\lambda)} = -\lim_{k \to 0} \text{Re} \left[\frac{\tan \delta_{3b}^{(\lambda)}(k)}{k^{2\lambda + 4}} \right]$$

[Amado & Rubin, PRL (1970); Efimov, SJNP (1970); {Braaten, Hammer, & Mehen, PRL (2002); Braaten & Hammer, PRep (2006); Shina Tan (DAMOP2010)]

Three-body Elastic Collisions

Three-body scattering length (L^4)

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Pathway analysis [D'Incao & Esry PRL (2005)]

$$\text{Re}[T] = -\text{Re}\left[\sum_j |A_j| e^{i\phi_j}\right] = -\sum_j |A_j| \cos \phi_j$$
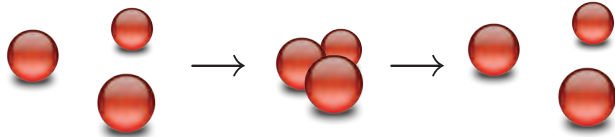
$$= \text{Re}\left[\tan \delta_{3b}\right]$$

[Amado & Rubin, PRL (1970); Efimov, SJNP (1970);
{Braaten, Hammer, & Mehen, PRL (2002);
Braaten & Hammer, PRep (2006);
Shina Tan (DAMOP2010)]

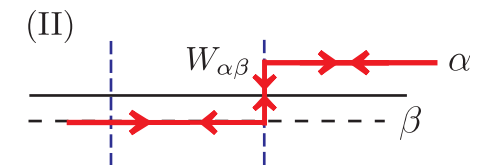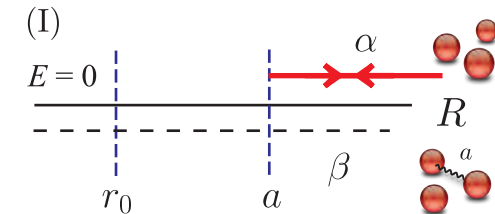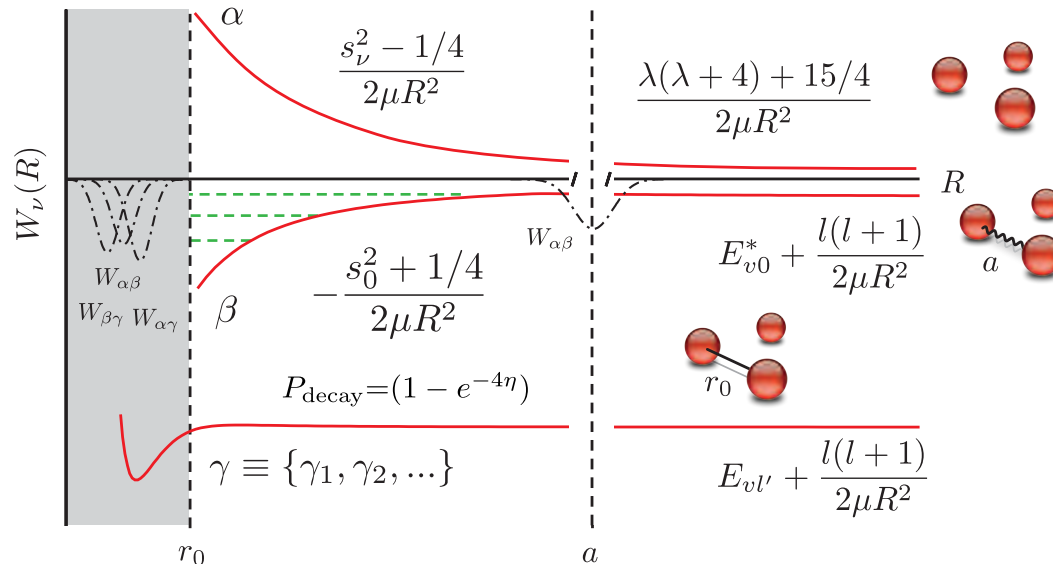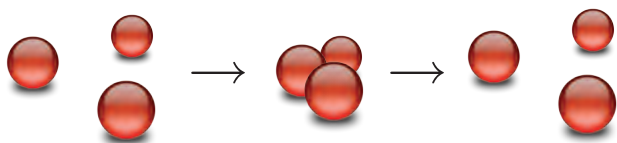## Three-body Elastic Collisions

Three-body scattering length (L^4)

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$$= \mathrm{Re}\left[\tan \delta_{3b}\right]$$

Pathways for Elastic (a>0)

(a) $a > 0$

α

$$\frac{s_\nu^2 - 1/4}{2\mu R^2}$$

$$\frac{\lambda(\lambda+4) + 15/4}{2\mu R^2}$$

$W_\nu(R)$

$W_{\alpha\beta}$

$W_{\beta\gamma}\; W_{\alpha\gamma}$

β

$$-\frac{s_0^2 + 1/4}{2\mu R^2}$$

$$E_{v0}^* + \frac{l(l+1)}{2\mu R^2}$$

$$P_{\mathrm{decay}} = (1 - e^{-4\eta})$$

$$\gamma \equiv \{\gamma_1, \gamma_2, ...\}$$

$$E_{vl'} + \frac{l(l+1)}{2\mu R^2}$$

r_0

a

r_0

(I)

$E = 0$

α

R

β

a

r_0

a

(II)

$W_{\alpha\beta}$

α

β

Three-body Elastic Collisions

Three-body scattering length (L^4)

$$A_{3b}^{(\lambda)} = -\lim_{k \to 0} \mathrm{Re}\left[\frac{\tan \delta_{3b}^{(\lambda)}(k)}{k^{2\lambda+4}}\right]$$

Amplitudes and phases (WKB)

$$|A_{\mathrm{I}}^{\alpha\alpha}|^2 = A^4 (ka)^{4\lambda+8},$$
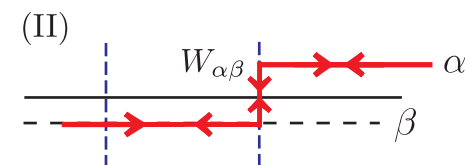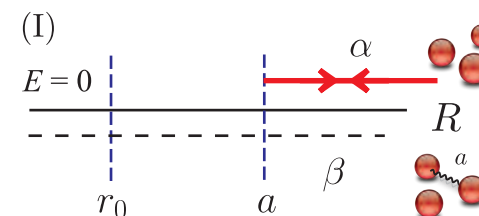$$|A_{\mathrm{II}}^{\alpha\alpha}|^2 = B^4 e^{-4\eta}(ka)^{4\lambda+8},$$
$$[\text{Phases: } \phi_{\mathrm{I}} = 0,$$
$$\phi_{\mathrm{II}} = 2[s_0 \ln(a/r_0) + \Phi] - \pi/2]$$

Pathway analysis [D'Incao & Esry PRL (2005)]

$$\mathrm{Re}[T] = -\mathrm{Re}\left[\sum_j |A_j| e^{i\phi_j}\right] = -\sum_j |A_j| \cos \phi_j$$
$$= \mathrm{Re}\left[\tan \delta_{3b}\right]$$

Pathways for Elastic (a>0)

Three-body Elastic Collisions
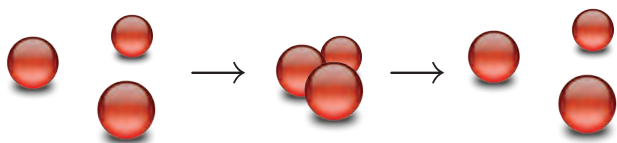
Three-body scattering length (L^4)

$$A_{3b}^{(\lambda)} = -\lim_{k \to 0} \mathrm{Re}\left[\frac{\tan \delta_{3b}^{(\lambda)}(k)}{k^{2\lambda+4}}\right]$$

$$A_{3b}^{(\lambda)} = \left[\left(A^2 - B^2 e^{-2\eta}\right) + 2B^2 e^{-2\eta} \sin^2[s_0 \ln(a/a_+) - \pi/4]\right] a^{2\lambda+4}$$
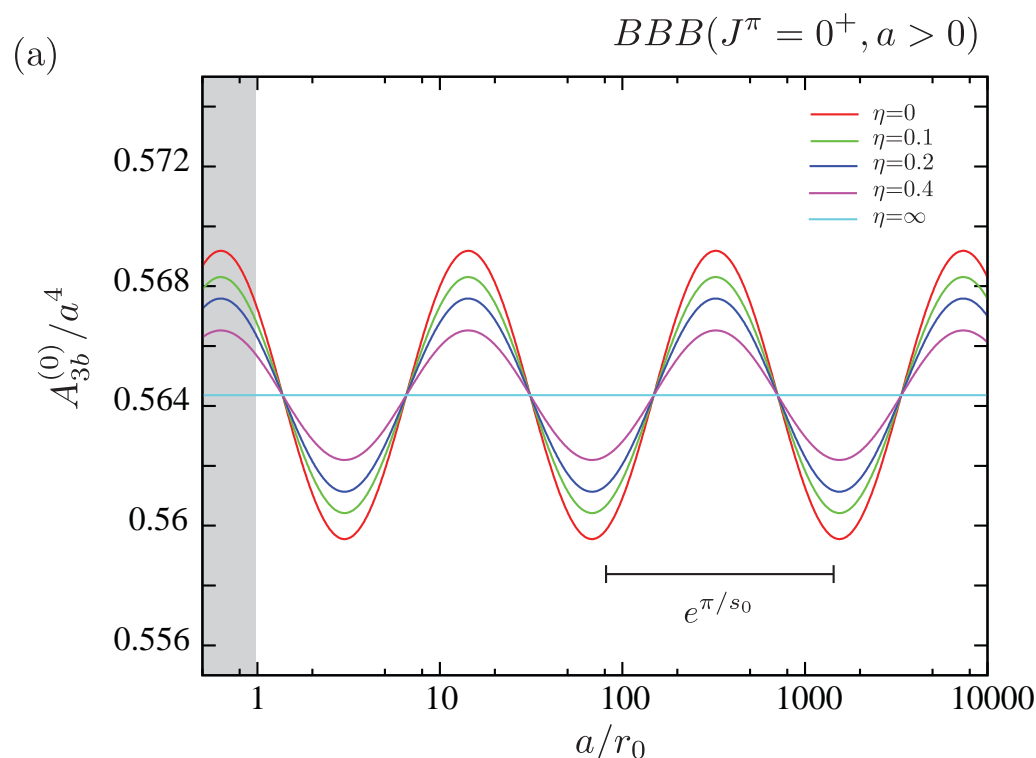
$[(A^2 - B^2) \approx 1.22(4\pi - 3\sqrt{3})/(2\sqrt{3}\pi),$

$2B^2 \approx 0.021(4\pi - 3\sqrt{3})/(2\sqrt{3}\pi)]$

[Braaten, Hammer, & Mehen, PRL (2002);
Braaten & Hammer, PRep (2006);
Shina Tan (DAMOP2010)]

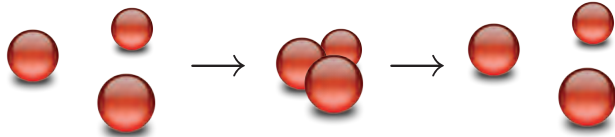**Pathway analysis** [D'Incao & Esry PRL (2005)]

$$\mathrm{Re}[T] = -\mathrm{Re}\left[\sum_j |A_j| e^{i\phi_j}\right] = -\sum_j |A_j| \cos \phi_j$$

$$= \mathrm{Re}\,[\tan \delta_{3b}]$$

Three-body Elastic (a>0)

$BBB(J^\pi = 0^+, a > 0)$

(a)

Three-body Elastic Collisions

Three-body scattering length (L^4)
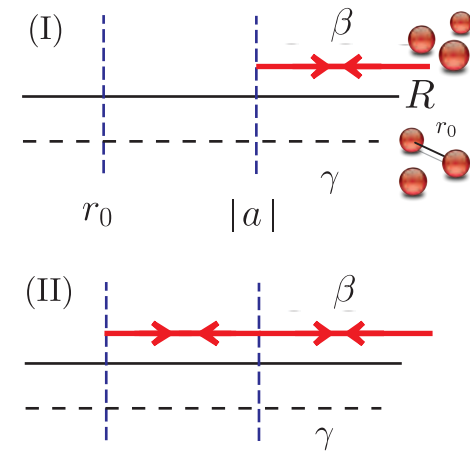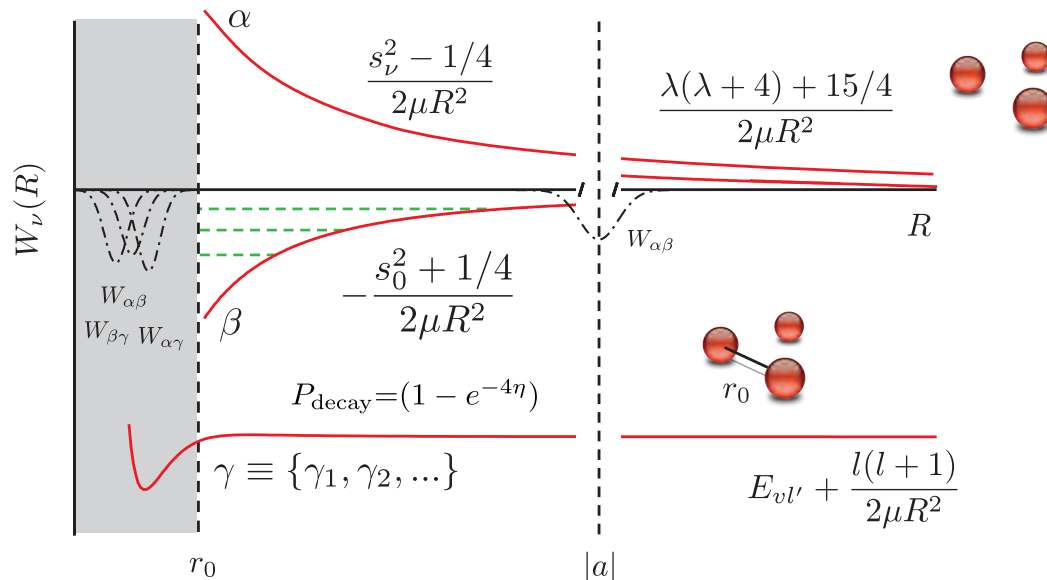
$$A_{3b}^{(\lambda)} = -\lim_{k \to 0} \text{Re}\left[\frac{\tan \delta_{3b}^{(\lambda)}(k)}{k^{2\lambda+4}}\right]$$

Pathway analysis [D'Incao & Esry PRL (2005)]

$$\text{Re}[T] = -\text{Re}\left[\sum_j |A_j| e^{i\phi_j}\right] = -\sum_j |A_j| \cos \phi_j$$

$$= \text{Re}\left[\tan \delta_{3b}\right]$$

Pathways for Elastic (a<0)

(b) $a < 0$

$\dfrac{s_\nu^2 - 1/4}{2\mu R^2}$

$\dfrac{\lambda(\lambda+4)+15/4}{2\mu R^2}$

$-\dfrac{s_0^2 + 1/4}{2\mu R^2}$

$P_{\text{decay}} = (1 - e^{-4\eta})$

$\gamma \equiv \{\gamma_1, \gamma_2, ...\}$

$E_{vl'} + \dfrac{l(l+1)}{2\mu R^2}$

Three-body Elastic Collisions

Three-body scattering length (L^4)

$$A_{3b}^{(\lambda)} = -\lim_{k \to 0} \mathrm{Re}\left[\frac{\tan \delta_{3b}^{(\lambda)}(k)}{k^{2\lambda+4}}\right]$$

Amplitudes and phases (WKB)

$$|A_{\mathrm{I}}^{\beta\beta}|^2 = A^4(k|a|)^{4\lambda+8},$$

$$|A_{\mathrm{II},j}^{\beta\beta}|^2 = B^4(e^{-4\eta})^j(k|a|)^{4\lambda+8},$$

$$(j = 1, 2, ..., \infty)$$

$$[\text{Phases: } \phi_{\mathrm{I}} = 0,$$

$$\phi_{\mathrm{II},j} = (2j)[s_0 \ln(|a|/r_0) + \Phi] - \pi/2]$$

Pathway analysis [D'Incao & Esry PRL (2005)]

$$\mathrm{Re}[T] = -\mathrm{Re}\left[\sum_j |A_j|e^{i\phi_j}\right] = -\sum_j |A_j|\cos\phi_j$$
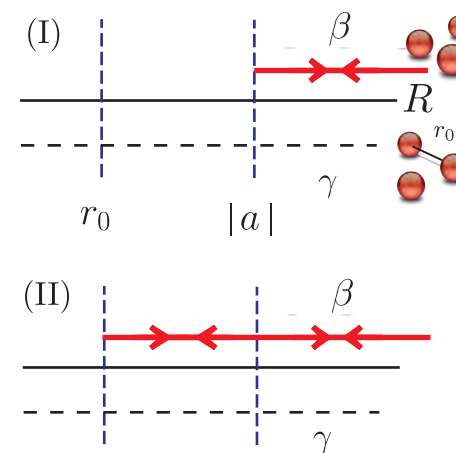
$$= \mathrm{Re}\left[\tan \delta_{3b}\right]$$

Pathways for Elastic (a<0)

Three-body Elastic Collisions
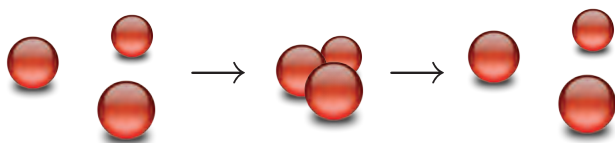
Three-body scattering length (L^4)

$$A_{3b}^{(\lambda)} = -\lim_{k \to 0} \mathrm{Re}\left[\frac{\tan \delta_{3b}^{(\lambda)}(k)}{k^{2\lambda+4}}\right]$$

$$A_{3b}^{(\lambda)} = \left[A^2 + \frac{B^2}{4}\frac{\sin[2s_0\ln(|a/a_-|)]}{\sin^2[s_0\ln(|a/a_-|)] + \sinh^2\eta}\right]|a|^{2\lambda+4}$$

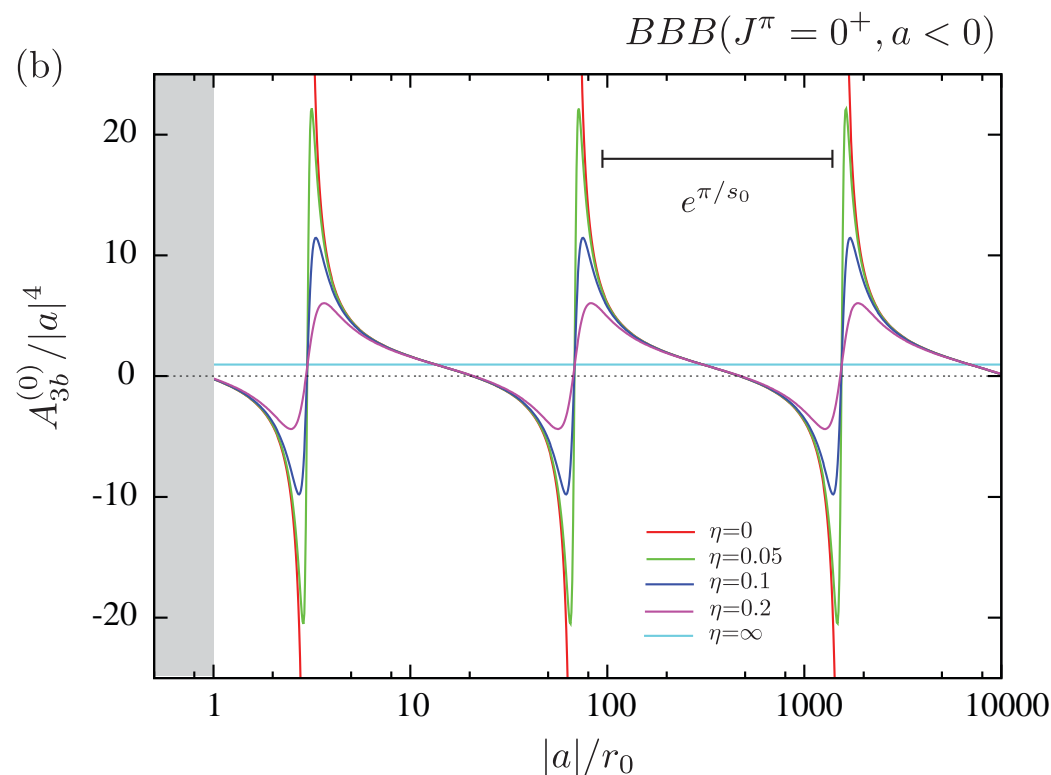$[A^2 \approx 1.23(4\pi - 3\sqrt{3})/(2\sqrt{3}\pi),$
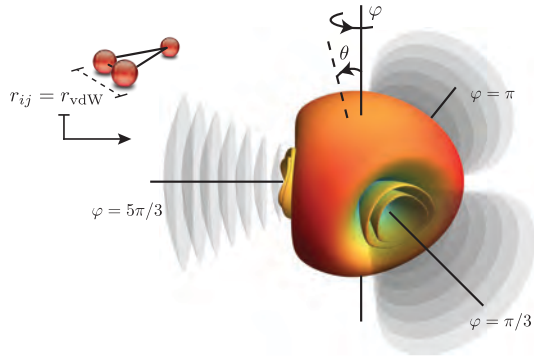$B^2/4 \approx 3.16(4\pi - 3\sqrt{3})/(2\sqrt{3}\pi)]$

[Braaten, Hammer, & Mehen, PRL (2002);
Braaten & Hammer, PRep (2006);
Shina Tan (DAMOP2010)]

Pathway analysis [D'Incao & Esry PRL (2005)]

$$\mathrm{Re}[T] = -\mathrm{Re}\left[\sum_j |A_j|e^{i\phi_j}\right] = -\sum_j |A_j|\cos\phi_j$$

$$= \mathrm{Re}\left[\tan\delta_{3b}\right]$$

Three-body Elastic (a<0)

$$BBB(J^\pi = 0^+, a < 0)$$

$r_{ij} = r_{vdW}$

φ

θ
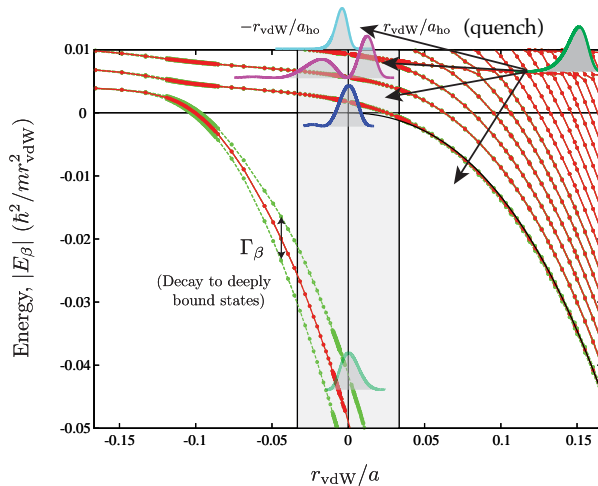
$\varphi = \pi$

$\varphi = 5\pi/3$

$\varphi = \pi/3$

• **Efimov physics plays an important role in determining the stability and lifetime of atomic gases as well as the nature of few-body interactions.**

• **Experimental advances inspired new theoretical insights on unexpected universal properties of atomic few-body systems which opened up new avenues to more quantitative predictions.**

ADVERTISEMENTS

$-r_{vdW}/a_{ho}$ r_{vdW}/a_{ho} (quench)

Γ_β

(Decay to deeply bound states)

Energy, $|E_\beta|$ $(\hbar^2/m r_{vdW}^2)$

r_{vdW}/a

• **Interesting universal few-body physics in 85Rb quenched BEC experiment at JILA (Eric Cornell). Formation of Efimov trimers, coherent association effects, etc.**

• **Few-body physics in a microgravity environment [Eric Cornell (JILA) & Peter Engles (WSU)]. Extremely low temperatures (T~10pK) and atomic densities (n~10^8/cm^3). Explore excited Efimov resonances.**

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