Non-Equilibrium Casimir Force between two moving plates

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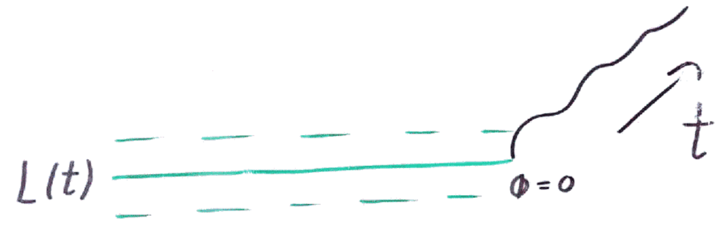
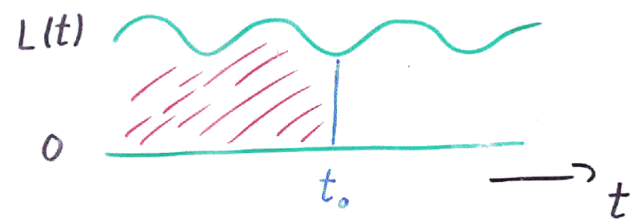
KITP Santa Barbara 10/29/08



$$\gamma \partial_t \phi = \nabla^2 \phi + \eta$$

$$\langle \eta(\vec{r},t)\,\eta(\vec{r}',t')\rangle = 2\gamma k_B T\, \delta(\vec{r}-\vec{r}')\,\delta(t-t')$$



quasi-static: $F(t_0) = \dfrac{\zeta(3)}{8\pi} \dfrac{k_B T}{L(t_0)^3}$ "PFT"

non-equilibrium: $F(\gamma, t_0)$ probes history $t' \le t_0$

relevant for $\Omega = \gamma \omega_0 L^2 \gtrsim 1$

typical values:

$\gamma \approx 10^{11} \frac{s}{m^2}$ liquid crystal

$\omega_0 \approx 10^3 s^{-1}$ resonance frequency in MEMS

$L \approx 100$ nm

$\Rightarrow \Omega \approx 10^0 = O(1)$

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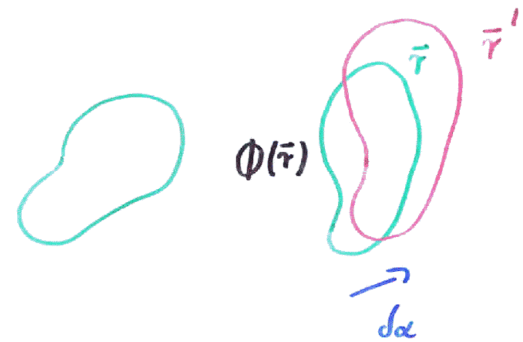
Stress Tensor



$$H\{\phi\} = \int d^3r \, \frac{1}{2} (\nabla \phi)^2$$

coordinate transformation:

$$\vec{r}\,' = \vec{r} + \vec{a}(\vec{r})$$

$$\Rightarrow \delta H = H' - H = \int d^3r \, \frac{\partial a_\ell}{\partial x_k} T_{\ell k}(\vec{r})$$

$$T_{\ell k} = \partial_\ell \phi \, \partial_k \phi - \frac{1}{2} \delta_{\ell k} (\nabla \phi)^2$$

$$\delta E = \langle \delta H \rangle \, , \quad F = - \frac{\delta E}{\delta \alpha}$$

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Langevin Dynamics:

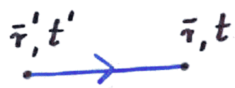
$$H\{\phi\} = \frac{1}{2}\int d^3r \left[\bar{\nabla}\phi(\bar{r})\right]^2$$

$$\gamma \frac{\partial}{\partial t}\phi(\bar{r},t) = \bar{\nabla}^2\phi(\bar{r},t) + \eta(\bar{r},t)$$

$$\langle \eta(\bar{r},t)\eta(\bar{r}',t')\rangle = 2\gamma k_B T\delta(\bar{r}-\bar{r}')\delta(t-t')$$

Propagator:

$$\phi(\bar{r},t) = \int d^3r' \int_{-\infty}^{t} dt'\, \eta(\bar{r}',t')G(\bar{r}',t';\bar{r},t)$$



$$\left(\gamma\frac{\partial}{\partial t} - \bar{\nabla}^2\right)G(\bar{r}',t';\bar{r},t) = \delta(\bar{r}-\bar{r}')\delta(t-t')$$

Causality: $G(\bar{r}',t';\bar{r},t) = 0$ for $t' > t$

Correlation function:

$$\langle \phi(\bar{r}',t')\phi(\bar{r},t)\rangle = 2\gamma k_B T \int d^3\rho \int_{-\infty}^{\infty} d\tau\; G(\bar{r}',t';\bar{\rho},\tau)G(\bar{\rho},\tau;\bar{r},t)$$

Unbounded bulk

$$G_b(\bar{r}',t';\bar{r},t)\Theta(t-t') = \int \frac{d^3q}{(2\pi)^3} e^{i\bar{q}\cdot(\bar{r}-\bar{r}')} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega(t-t')}\, g_b(q,\omega)$$

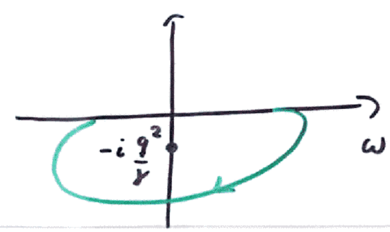
$$g_b(q,\omega) = \frac{1}{q^2 - i\gamma\omega}$$

Causality:

$$g_b(q,t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\frac{e^{-i\omega(t-t')}}{q^2 - i\gamma\omega} = -\frac{1}{\gamma}\frac{1}{2\pi i}\int_C d\omega \frac{e^{-i\omega(t-t')}}{\omega + i\frac{q^2}{\gamma}}$$

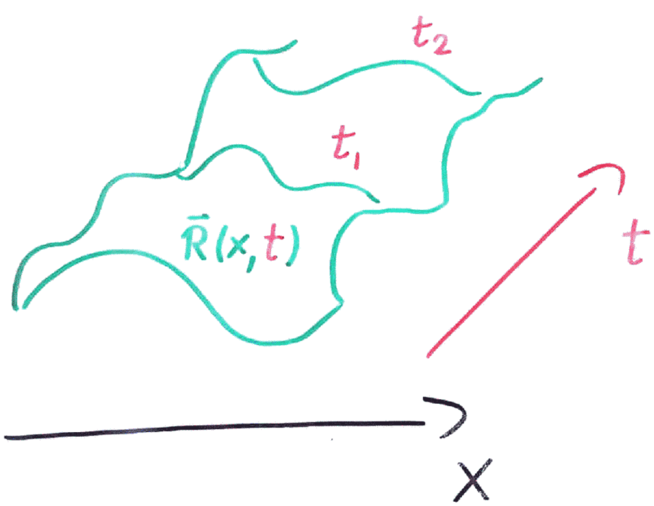
$$= \begin{cases} \frac{1}{\gamma} e^{-\frac{q^2}{\gamma}(t-t')} &, \; t-t' > 0 \\ 0 &, \; t-t' < 0 \end{cases}$$

✓

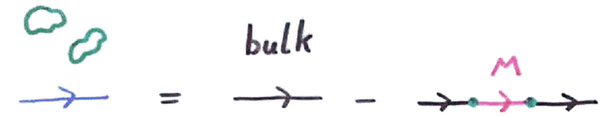


Moving Surface

$\vec{R}(\bar{x}, t) \qquad \bar{x} \in \mathbb{R}^2$



$(\bar{x}, t) \in \mathbb{R}^2 \times \mathbb{R}$ base plane



$$G(\vec{r}\,',t';\vec{r},t) = G_b(\vec{r}\,',t';\vec{r},t)$$

$$- \left(\int d^2x \int_{-\infty}^{\infty} d\tau\right)\left(\int d^2y \int_{-\infty}^{\infty} d\sigma\right)$$

$$\times\, G_b\left[\vec{r}\,',t';\vec{R}(\bar{x},\tau),\tau\right] M(\bar{x},\tau;\bar{y},\sigma)\, G_b\left[\vec{R}(\bar{y},\sigma),\sigma;\vec{r},t\right]$$

$M(\bar{x},\tau;\bar{y},\sigma)$ inverse to $G_b\left[\vec{R}(\bar{x},\tau),\tau;R(\bar{y},\sigma),\sigma\right]$:

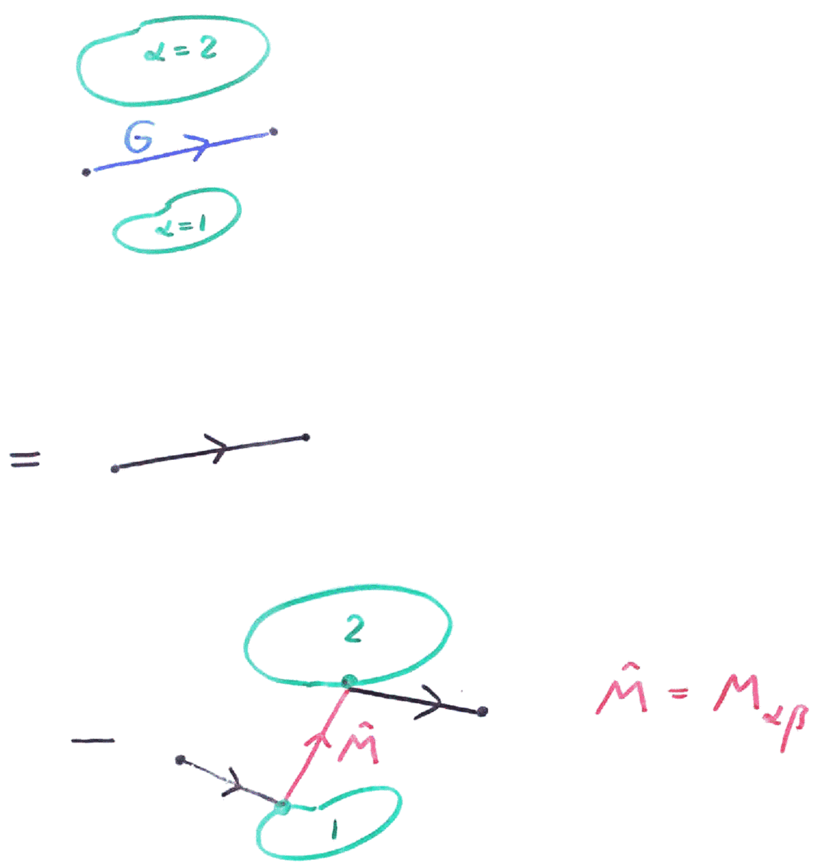
$$\left(\int d^2y \int_{-\infty}^{\infty} d\sigma\right) M(\bar{x},\tau;\bar{y},\sigma)\, G_b\left[R(\bar{y},\sigma),\sigma;R(\bar{u},t),t\right] = \delta(\bar{x}-\bar{y})\,\delta(\tau-t)$$

Check: $\vec{r}$ on surface at time $t \Rightarrow \exists \bar{u} \in \square^{2} : \vec{r} = \vec{R}(\bar{u},t)$

$$G\left[\vec{r}\,',t';\vec{R}(\bar{u},t),t\right] = G_b\left[\vec{r}\,',t';\vec{R}(\bar{u},t),t\right]$$

$$- \left(\int d^2x \int_{-\infty}^{\infty} d\tau\right) G_b\left[\vec{r}\,',t';\vec{R}(\bar{x},\tau),\tau\right] \delta(\bar{x}-\bar{u})\,\delta(\tau-t) = 0$$

Derived using path-integral method with constraints:
A. Hanke and M. Kardar, Phys. Rev. E **65**, 046121 (2002)


α = 2
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α = 1
=
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2
M̂
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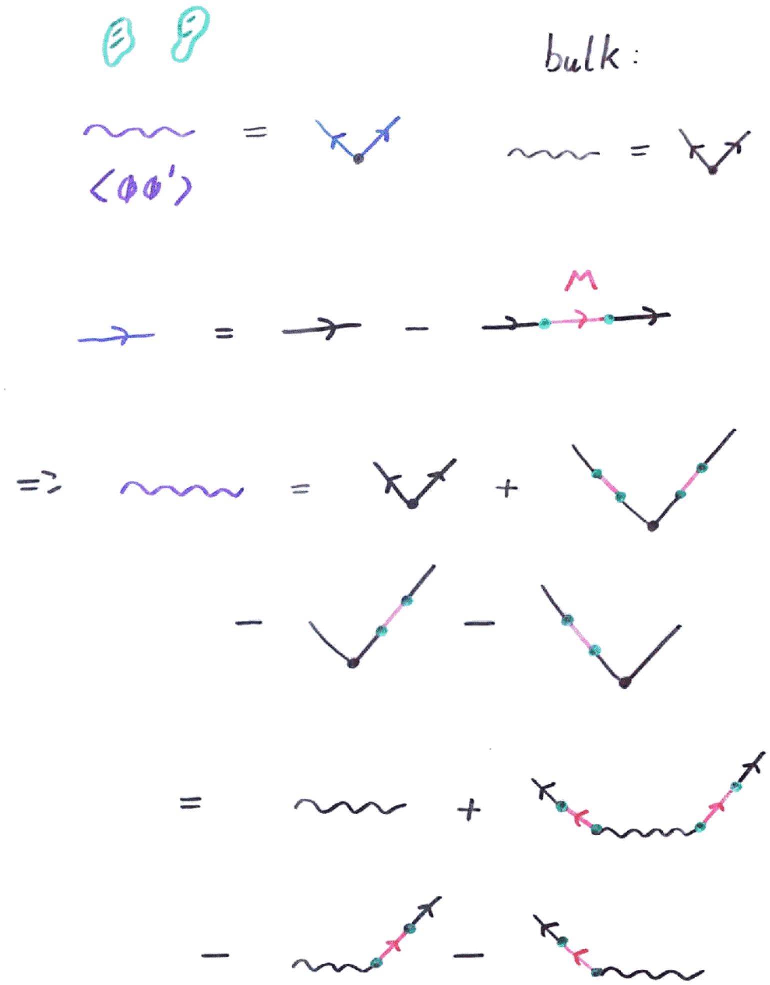
M̂ = M_αβ


bulk:
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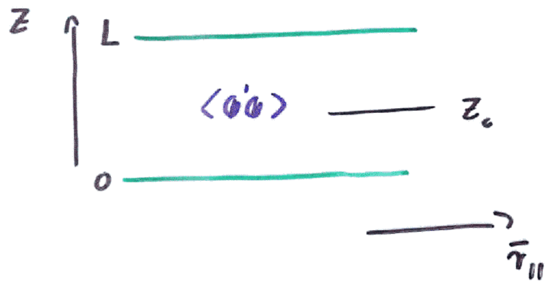
Equilibrium Casimir force between two plates



Equilibrium force per unit area on plate at $z = 0$:

$$F_0 = \langle T_{zz}(z_0,t) \rangle \quad ,$$

independent of t and $z_0 \in [0, L]$ ($\Rightarrow$check of calculation!)

Choosing $z_0 \to 0$:

$$F_0 = \lim_{z_0 \to 0} \langle T_{zz}(z_0,t) \rangle = \frac{1}{2} \frac{\partial}{\partial z'} \frac{\partial}{\partial z} \langle \phi(\vec{r}_{\|}, z', t) \phi(\vec{r}_{\|}, z, t) \rangle \Big|_{z \to 0,\, z' \to 0}$$

pz-representation:

$$G_b(\vec{r}\,', t'; \vec{r}, t)\,\Theta(t - t') = \int \frac{d^2 p}{(2\pi)^2} e^{ip\cdot(\vec{r}_{\|} - \vec{r}_{\|}\,')} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega(t-t')}\, g(z', z; \omega, p)$$

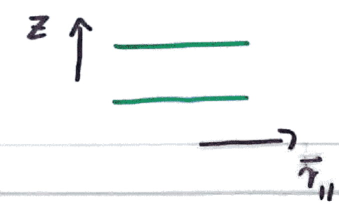
$$g(z', z; \omega, p) = \frac{1}{2Q} e^{-Q|z-z'|} , \quad Q = \sqrt{p^2 - i\gamma\omega}$$

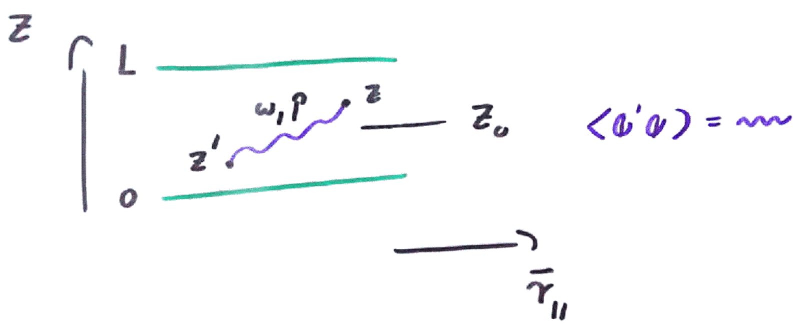
$$g^*(z', z; \omega, p) = g(z', z; -\omega, p) = \frac{1}{2Q^*} e^{-Q^*|z-z'|}$$

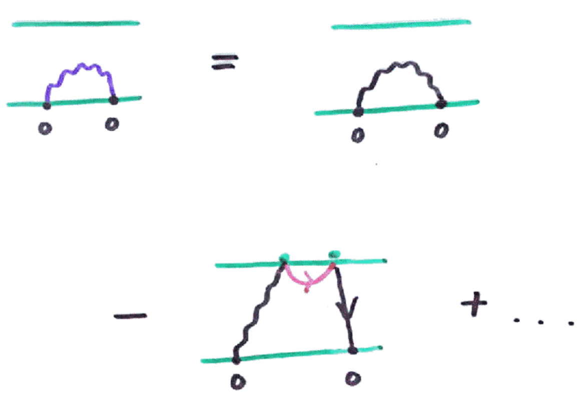
$$\langle \phi(\vec{r}\,', t') \phi(\vec{r}, t) \rangle = \frac{\gamma k_B T}{2} \int \frac{d^2 p}{(2\pi)^2} e^{ip\cdot(\vec{r}_{\|} - \vec{r}_{\|}\,')} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega(t-t')}\, C(z', z; \omega, p)$$

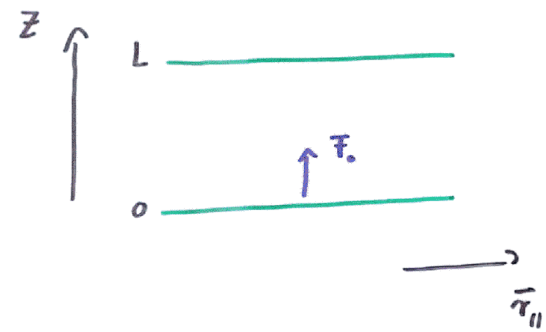
$$C_b(z', z; \omega, p) = 4 \int_{-\infty}^{\infty} d\zeta\, g^*(z', \zeta; \omega, p)\, g(\zeta, z; \omega, p)$$

$$= \frac{1}{i\gamma\omega} \frac{1}{QQ^*} \left[Q^* e^{-Q|z-z'|} - Q\, e^{-Q^*|z-z'|} \right]$$









$$F_0 = \frac{k_B T}{2} \int \frac{d^2 p}{(2\pi)^2} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{1}{i\omega} \left[Q^* \coth(Q^* L) - Q \coth(QL) \right]$$

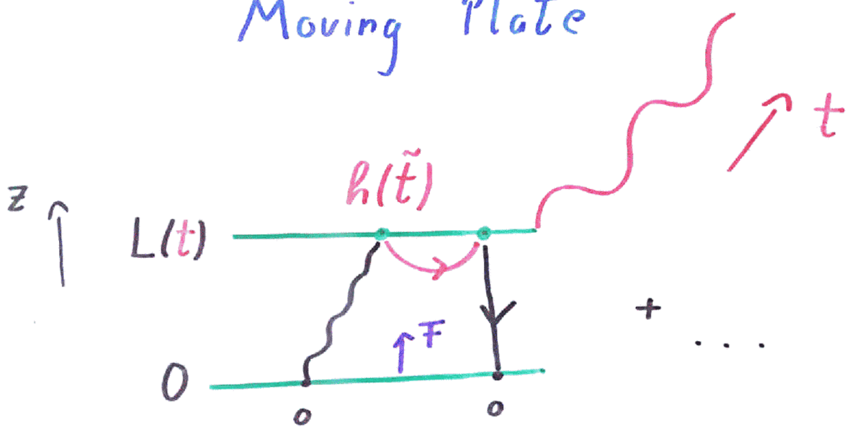
$$= \frac{k_B T}{2} \int \frac{d^2 p}{(2\pi)^2} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{Q^* - Q}{i\omega}$$ divergent part (2 half spaces)

$$+ \frac{k_B T}{2} \int \frac{d^2 p}{(2\pi)^2} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{1}{i\omega} \left\{ Q^* \left[\coth(Q^* L) - 1 \right] - Q \left[\coth(QL) - 1 \right] \right\}$$

Contour integration in finite part:

$$\Rightarrow F_0(L) = -\frac{\zeta(3)}{8\pi} \frac{k_B T}{L^3}$$ ✓

Moving Plate



$$F(t) = \frac{1}{2} \partial_{z'} \partial_z \langle \phi(\vec{r}, t)\, \phi(\vec{r}', t) \rangle |_0$$

$$L(t) = L_0 + h(t)$$

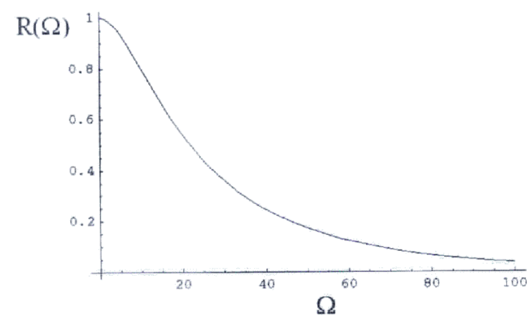
$$h(t) = a \cos(\omega_0 t)$$

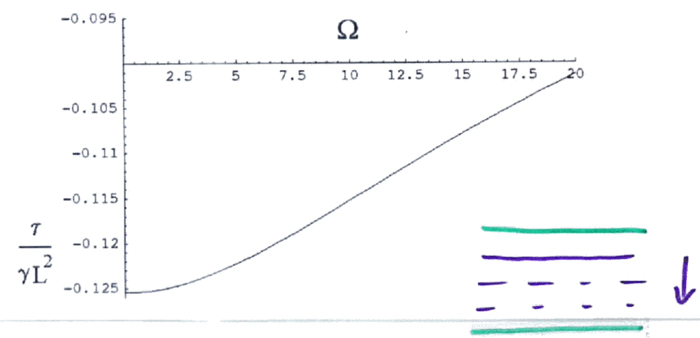
$\int_\omega$ picks out $\omega_0, -\omega_0$

$$L(t) = L_0 + a\cos(\omega_0 t)$$

$$F(t) = F_0(L_0)\left[1 - \frac{3a}{L_0} f(\omega_0 t, \Omega)\right], \quad \Omega = \omega_0 \gamma L^2$$

$$f(\omega_0 t, \Omega) = R(\Omega)\cos[\omega_0 (t+\tau)]$$





outlook

1. Casimir Force as force field with its own dynamics ?

→ field equations ?

2. True origin of EM Casimir Force ?

funded by AFOSR, KITP