CLASSIFICATION OF TOPOLOGICAL INSULATORS AND SUPERCONDUCTORS, RESPONSES AND QUANTUM ANOMALIES

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work done in collaboration with:

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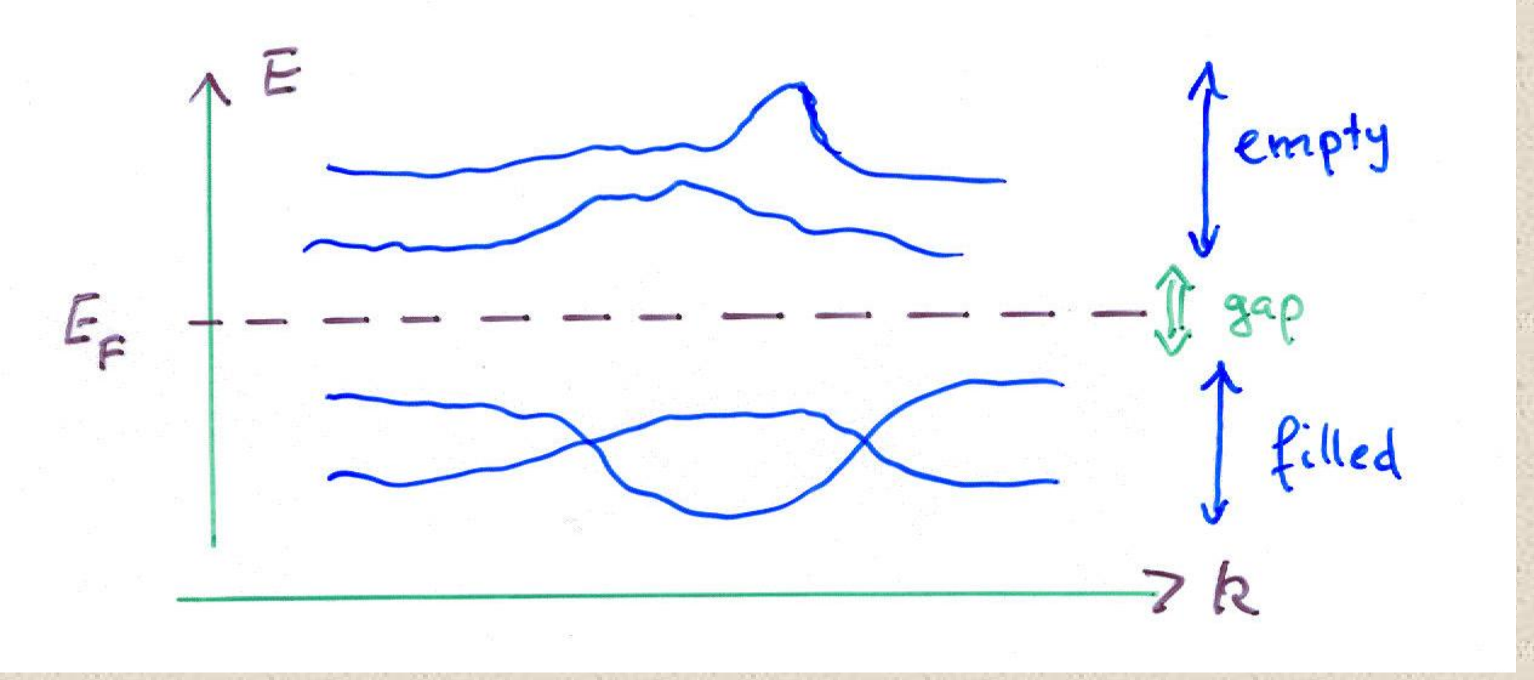
-PRB 78, 195125 (2008),
-PRL 102, 196804 (2009).
-Landau Memorial Conference, Landau Institute, AIP Conf. Proc. 1134, 10 (2009) [http://landau100.itp.ac.ru/Talk/ludwig.pdf],
-New Journal of Physics 12, 065010 (2010).

- Shinsei Ryu +Joel Moore (UC-Berkeley)

- arXiv: 1010.0936

TOPOLOGICAL INSULATOR: typical example

``Band Insulator''



•**PREVIOUSLY KNOWN:** ONE TOPOLOGICAL INSULATOR - spin-orbit ("Symmetry Class AII")

(Fu,Kane, Mele, Moore, Balents, Zhang , Qi, …)

•**CLASSIFICATION**: There are FIVE Topological Insulators (superconductors) in every dimension

THE ADDITIONAL ONES INCLUDE (in d=3):

* Helium 3B
* Superconductors with spin-orbit interactions

} "DIII"

* Singlet Superconductors

} "CI"

THREE METHODS OF CLASSIFICATION:

--: ANDERSON LOCALIZATION

[Ryu,Schnyder,Furusaki,Ludwig]

--: TOPOLOGY (K-THEORY)

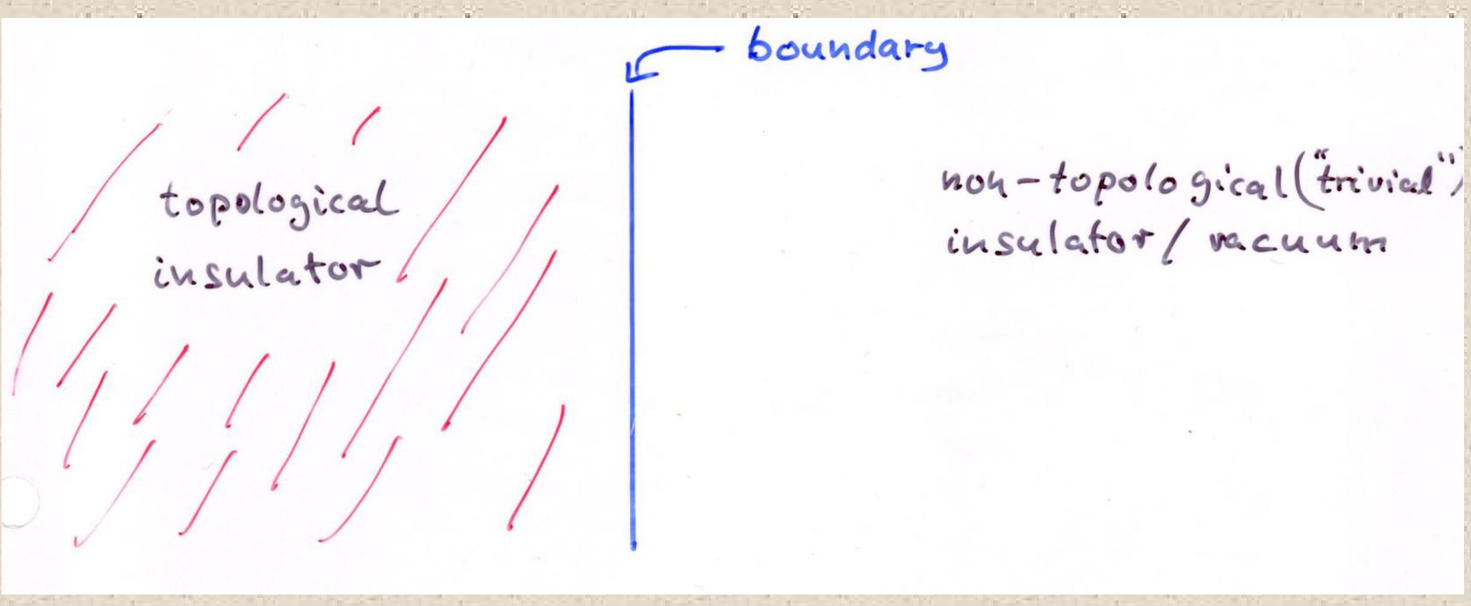
[Kitaev]

--: QUANTUM ANOMALIES (ALSO: INTERACTIONS)

[Ryu,Moore,Ludwig]

ONE METHOD OF CLASSIFICATION :

Classification of topological properties of bulk insulators (or: superconductors) by looking at their boundaries

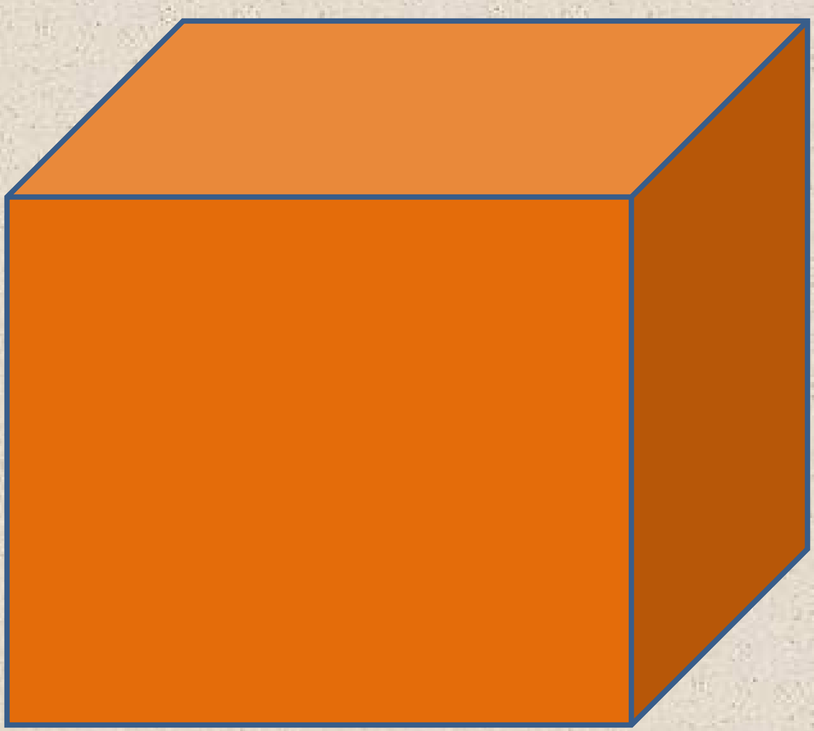


The defining characteristic of the topological properties of the bulk is the appearance of extended, gapless boundary degrees of freedom:

* topologically protected against any perturbation [respecting the `symmetries of the system' (more specific later)]
* including disorder

Topological Insulator (Superconductor):

- Bulk (electrical or thermal) insulator
- Boundary conducts (electricity or heat) similar to a metal



Because of the presence of the ***bulk gap***, bulk phases are ***robust against the addition of disorder***.

→ Thus, one is led to seek a classification of ground states of (in general) gapped random Hamiltonians.

→ There are only 10 classes of random Hamiltonians!

This underlies the well known classification of random matrices, and universality classes of Anderson localization transitions.

[Zirnbauer (1996), Altland+Zirnbauer(1997), Heinzner+Huckleberry+Zirnbauer (2004); Bernard+LeClair(2001)]

BRIEF REVIEW: CLASSIFICATION OF RANDOM FERMION HAMILTONIANS -- THE 10-FOLD WAY

• In classifying random Hamiltonians one must consider only the most generic symmetries,

time reversal $(\mathbf{T})$

and

charge-conjugation (particle-hole) $(\mathbf{C})$.

• There are only 10 possible behaviors of a Hamiltonian under $\mathbf{T}$ and $\mathbf{C}$. (10 “symmetry classes”)

H = second quantized hamiltonian =

$$= \psi_A^\dagger \ \mathcal{H}_{A,B} \ \psi_B$$

The basic idea is simple: $\mathcal{H} = 1st$ quantized Hamiltonian

$\mathbf{T}$ is antiunitary : $\mathbf{T} = U_T \cdot K$

$$\mathbf{T}: \qquad U_T \, \mathcal{H}^* \, U_T^\dagger \; = \; \mathcal{H}$$

$$\mathbf{T} = \begin{cases} 0, & \text{no time reversal invariance} \\ +1 & \text{time reversal invariance and } \mathbf{T}^2 = +1 \\ -1 & \text{time reversal invariance and } \mathbf{T}^2 = -1 \end{cases}$$

$\mathbf{C}$ is antiunitary : $\mathbf{C} = U_C \cdot K$

$$\mathbf{C}: \qquad U_C \, \mathcal{H}^* \, U_C^\dagger \; = \; -\,\mathcal{H}$$

$$\mathrm{C} = \begin{cases} 0, & \text{no particle -- hole symmetry} \\ +1 & \text{particle -- hole symmetry and } \mathbf{C}^2 = +1 \\ -1 & \text{particle -- hole symmetry and } \mathbf{C}^2 = -1 \end{cases}$$

* There are $3 \times 3 = 9$ choices for T $\times$ C

* For 8 of these choices the value of $\mathbf{S} := \mathbf{T}\cdot\mathbf{C}$ is uniquely fixed : these are all except for "A" and "AIII".

* For "A" and "AIII" :

T	C
0	0

free to choose $S = 0$ or $S = 1$, yielding "A" and "AIII"

- Total of 10 choices -

TABLE - "Ten Fold Way" [`CARTAN Classes']

Examples

Name (Cartan)	T	C	S= T C	Time evolution operator $\mathcal{U}(t) = \exp\{it\mathcal{H}\}$	Anderson Localization NLSM Manifold G/H [compact (fermionic) sector]	SU(2) spin con-served	Some Examples of Systems
A (unitary)	0	0	0	U(N)	U(2n)/U(n)xU(n)	yes/ no	IQHE Anderson
AI (orthogonal)	+1	0	0	U(N)/O(N)	Sp(4n) /Sp(2n)xSp(2n)	yes	Anderson
AII (symplectic)	-1	0	0	U(2N)/Sp(2N)	SO(2n)/SO(n)xSO(n)	no	*Quantum spin Hall Z2Top.Ins. Anderson(spinorbit)*
AIII (chiral unitary)	0	0	1	U(N+M)/U(N)xU(M)	U(n)	yes/ no	Random Flux Gade SC
BDI (chiral orth.)	+1	+1	1	SO(N+M)/SO(N)xSO(M)	U(2n)/Sp(2n)	yes/ no	Bipartite Hopping Gade
CII (chiral sympl.)	-1	-1	1	Sp(2N+2M) /Sp(2N)xSp(2M)	U(n)/O(n)	no	Bipartite Hopping Gade
D	0	+1	0	O(N)	O(2n)/U(2n)	no	(px+ipy)-wave 2D SC w/spin-orbit TQHE
C	0	-1	0	Sp(2N)	Sp(2n)/U(2n)	yes	Singlet SC +mag.field (d+id)-wave SQHE
DIII	-1	+1	1	O(2N)/U(2N)	O(n)	no	SC w/ spin-orbit He-3 B
CI	+1	-1	1	Sp(2N)/U(2N)	Sp(2n)	yes	Singlet SC

EXAMPLE: Symmetry class AI (time-reversal with $\mathbf{T}^2 = +\mathbf{1}$)

$$\begin{aligned} \mathcal{H} &= \frac{\mathcal{H}+\mathcal{H}^t}{2} + \frac{\mathcal{H}-\mathcal{H}^t}{2} \\ &= \mathcal{H}_s + \mathcal{H}_A \end{aligned}$$

Note: $e^{it\mathcal{H}_A} \in O(N)$

Thus: $\mathcal{H}_s \in Lie[U(N)] - Lie[O(N)]$

CLASSIFICATION FROM ANDERSON LOCALIZATION:

Non-linear Sigma Model (NLSM) at the $\bar{d}$-dimensional boundary of the d= ($\bar{d}$+1)-dimensional Topological Insulator:

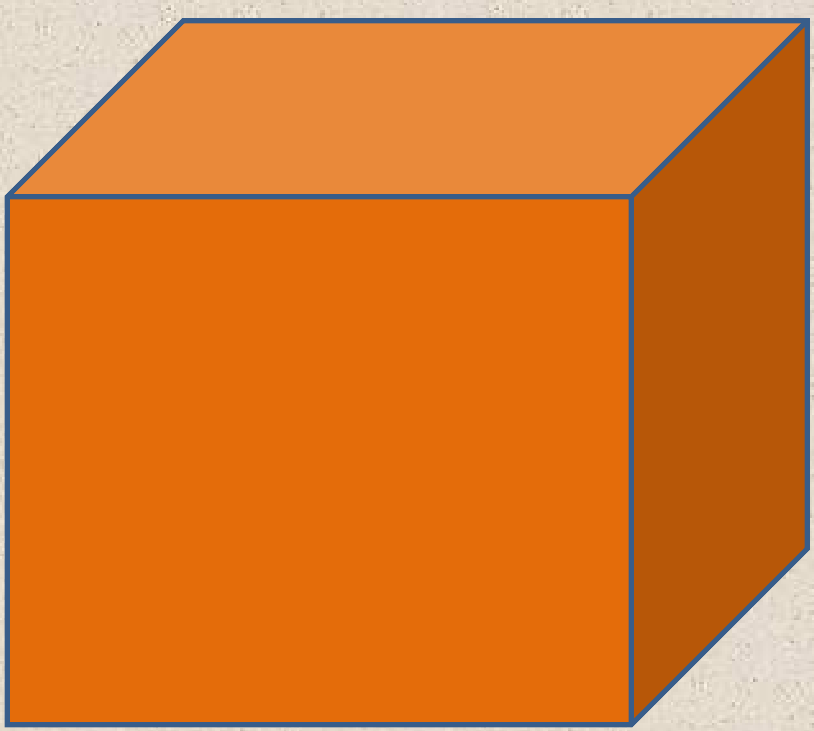
$$S \;=\; \frac{1}{g} \int d^{\bar{d}} r \;\; Tr\left(\partial_\mu \Phi(r) \partial_\mu \Phi(r)\right) \;\; + \;\; S_{top.}$$

$$\Phi(r) \;=\; \text{matrix field} \;=\; \text{element of symmetric space } G/H$$

[e.g. $G/H = U(2n)/[U(n) \times U(n)] =$ {"target space"}]

Topological Insulator (Superconductor):

- Bulk (electrical or thermal) insulator
- Boundary conducts (electricity or heat) similar to a metal



→ A Topological Insulator (Superconductor) exists in a given symmetry class in $d = (\bar{d}+1)$ dimensions

if and only if

a term of topological origin can be added to the NLSM at the $\bar{d}$-dimensional boundary, which prevents Anderson localization (top. term without tuneable parameter).

→ This is possible, if and only if

the target space G/H of the NLSM allows for:

(a) : a $\mathbf{Z}_2$ top. term

$$\Longleftrightarrow \quad \pi_{\bar{d}}(\mathrm{G/H}) = \pi_{d-1}(\mathrm{G/H}) = \mathbf{Z}_2$$

or:

(b) : a Wess – Zumino – Witten term

$$\Longleftrightarrow \quad \pi_d(\mathrm{G/H}) = \pi_{\bar{d}+1}(\mathrm{G/H}) = \mathbf{Z}$$

Table of homotopy groups $\pi_{\bar{d}}(\mathrm{G/H})$:

$d = (\bar{d} + 1)$ dimensional top.insulator $\left(\bar{d} = \text{ dimension of boundary}\right)$

AZ	G/H	$\bar{d}=0$	$\bar{d}=1$	$\bar{d}=2$	$\bar{d}=3$	$\bar{d}=4$	$\bar{d}=5$	$\bar{d}=6$	$\bar{d}=7$
Complex case:									
A	$\mathrm{U}(N+M)/\mathrm{U}(N)\times\mathrm{U}(M)$	$\longleftarrow\mathbb{Z}$	$\mathbf{0}$	$\longleftarrow\mathbb{Z}$	$\mathbf{0}$	$\longleftarrow\mathbb{Z}$	$\mathbf{0}$	$\longleftarrow\mathbb{Z}$	$\mathbf{0}\longleftarrow$
AIII	$\mathrm{U}(N)$	$\mathbf{0}$	$\longleftarrow\mathbb{Z}$	$\mathbf{0}$	$\longleftarrow\mathbb{Z}$	$\mathbf{0}$	$\longleftarrow\mathbb{Z}$	$\mathbf{0}$	$\longleftarrow\mathbb{Z}$
Real case:									
AI	$\mathrm{Sp}(N+M)/\mathrm{Sp}(N)\times\mathrm{Sp}(M)$	$\longleftarrow\mathbb{Z}$	0	0	$\mathbf{0}$	$\longleftarrow\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbf{0}\longleftarrow$
BDI	$\mathrm{U}(2N)/\mathrm{Sp}(2N)$	$\mathbf{0}$	$\longleftarrow\mathbb{Z}$	0	0	$\mathbf{0}$	$\longleftarrow\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	$\mathrm{O}(2N)/\mathrm{U}(N)$	$\mathbb{Z}_2$	$\mathbf{0}$	$\longleftarrow\mathbb{Z}$	0	0	$\mathbf{0}$	$\longleftarrow\mathbb{Z}$	$\mathbb{Z}_2$
DIII	$\mathrm{O}(N)$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbf{0}$	$\longleftarrow\mathbb{Z}$	0	0	$\mathbf{0}$	$\longleftarrow\mathbb{Z}$
AII	$\mathrm{O}(N+M)/\mathrm{O}(N)\times\mathrm{O}(M)$	$\longleftarrow\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbf{0}$	$\longleftarrow\mathbb{Z}$	0	0	$\mathbf{0}\longleftarrow$
CII	$\mathrm{U}(N)/\mathrm{O}(N)$	$\mathbf{0}$	$\longleftarrow\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbf{0}$	$\longleftarrow\mathbb{Z}$	0	0
C	$\mathrm{Sp}(2N)/\mathrm{U}(N)$	0	$\mathbf{0}$	$\longleftarrow\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbf{0}$	$\longleftarrow\mathbb{Z}$	0
CI	$\mathrm{Sp}(2N)$	0	0	$\mathbf{0}$	$\longleftarrow\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbf{0}$	$\longleftarrow\mathbb{Z}$

(a): There is a $(\bar{d}+1)-$dimensional $\mathbf{Z}_2$ topological insulator, whenever $\pi_{\bar{d}}(\mathrm{G/H}) = \mathbf{Z}_2$

(b): There is a $(\bar{d}+1)-$dimensional $\mathbf{Z}$ topological insulator, whenever $\pi_{\bar{d}+1}(\mathrm{G/H}) = \mathbf{Z}$

This yields the following

TABLE OF TOPOLOGICAL INSULATORS (SUPERCONDUCTORS):

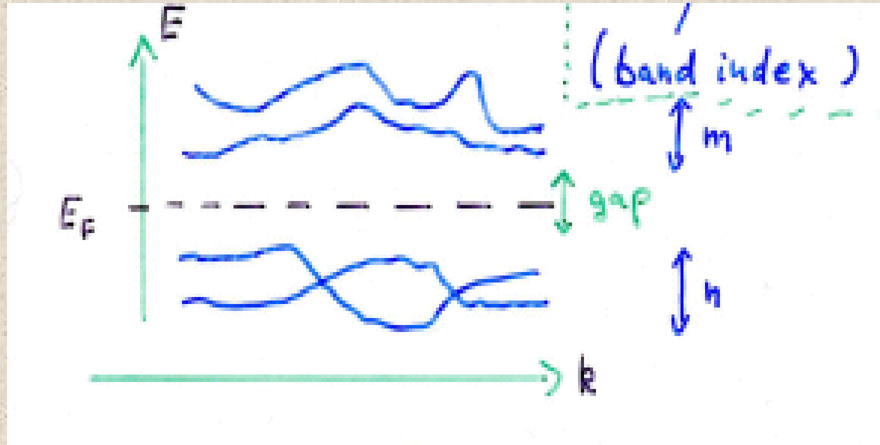
	d												
Cartan	0	1	2	3	4	5	6	7	8	9	10	11	…
Complex case:													
A	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	…
AIII	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	…
Real case:													
AI	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	…
BDI	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	…
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	…
DIII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	…
AII	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	…
CII	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	…
C	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	…
CI	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	…

(Ryu, Schnyder, Furusaki, Ludwig; Kitaev))

ORIGIN OF TOPOLOGY --- TRANSLATIONALLY INVARIANT CASE

Translational invariance →
Ground states of non-interacting Fermions (=band insulators) are filled Fermi seas in the Brillouin Zone:

k= momentum: $$\mathcal{H}(k)\ |u_a(k)\rangle = E_a(k)\ |u_a(k)\rangle$$



"SPECTRAL PROJECTOR":

$$\mathrm{P}(k) := \sum_{a}^{filled} |u_a(k)\rangle\ \langle u_a(k)|$$

Define: $$\mathrm{Q}(k) := 1 - 2\mathrm{P}(k)$$

$$\left(\mathrm{Q}^\dagger = \mathrm{Q}, \quad \mathrm{Q}^2 = 1, \quad tr\mathrm{Q} = m - n \right)$$

$\mathbf{Q}$ = Hamiltonian where $E_a(k)$ $\begin{cases} -1 & \text{filled} \\ +1 & \text{empty} \end{cases}$ ("flattened spectrum")

Consider the case of a Hamiltonian without any symmetry conditions (for simplicity): class A

class A: the Hamiltonian $\mathcal{H}$ is a general (m+n) x (m+n)-Hermitian matrix

-: set of eigenvectors = arbitrary unitary matrix $\in$ U(m+n)

-: gauge symmetry: U(m)$\times$ U(n) = relabeling filled and empty states

→: $Q(k) \in U(m+n)/[U(m)\times U(n)]$ = `Grassmannian'

$$Q: \quad BZ \rightarrow U(m+n)/[U(m)\times U(n)]$$
$$k \rightarrow Q(k)$$

The Quantum ground state is a map from the Brillouin Zone into the Grassmannian.

<u>Note</u>: an element of the "Grassmanian" can be written as

$$Q = U^\dagger \cdot \Lambda \cdot U$$

$$\Lambda = \begin{pmatrix} \mathbf{1}_m & 0 \\ 0 & -\mathbf{1}_n \end{pmatrix}, \qquad U \in U(m+n)$$

eigenvalues eigenvectors of Q

- <u>How many inequivalent (= not deformable into each other) ground states (= maps) are there ?</u>

This is answered by the **Homotopy Group**:

In d=2: $\pi_2 \left[U(m+n)/[U(m) \times U(n)] \right] = \mathbf{Z} =$

= counts the number of edge states of d=2 integer Quantum Hall states.

In d=3: $\pi_3 \left[U(m+n)/[U(m) \times U(n)] \right] = \mathbf{1}$

There are no topological insulators in d=3 dimensions in symmetry class A.

GENERAL SYMMETRY CLASSES:

$$G_{m,n}(\mathbf{C}) := U(n+n)/[U(m) \times U(n)] =$$ "Grassmanian"

AZ class	Space of projectors in momentum space
A	$\{Q(k) \in G_{m,m+n}(\mathbb{C})\}$
AI	$\{Q(k) \in G_{m,m+n}(\mathbb{C}) \mid Q(k)^* = Q(-k)\}$
AII	$\{Q(k) \in G_{2m,2(m+n)}(\mathbb{C}) \mid (i\sigma_y)Q(k)^*(-i\sigma_y) = Q(-k)\}$
AIII	$\{q(k) \in \mathrm{U}(m)\}$
BDI	$\{q(k) \in \mathrm{U}(m) \mid q(k)^* = q(-k)\}$
CII	$\{q(k) \in \mathrm{U}(2m) \mid (i\sigma_y)q(k)^*(-i\sigma_y) = q(-k)\}$
D	$\{Q(k) \in G_{m,2m}(\mathbb{C}) \mid \tau_x Q(k)^* \tau_x = -Q(-k)\}$
C	$\{Q(k) \in G_{m,2m}(\mathbb{C}) \mid \tau_y Q(k)^* \tau_y = -Q(-k)\}$
DIII	$\{q(k) \in \mathrm{U}(2m) \mid q(k)^T = -q(-k)\}$
CI	$\{q(k) \in \mathrm{U}(m) \mid q(k)^T = q(-k)\}$

from: Schnyder, Ryu, Furusaki, Ludwig [PRB78, 19512 (2008) – Table III]

Comment:

In the presence of “chiral” symmetry (S=1), Q can brought in the form

$$Q = \begin{pmatrix} 0 & q \\ q^\dagger & 0 \end{pmatrix}, \qquad q = \text{ unitary}$$

• When momentum - (and position-) space is d=0 dimensional, or at points in momentum space where k =(- k) :

→ The spaces of the projectors are again the 10 Cartan symmetric spaces.

→ These are the "Classifying Spaces" of K-Theory.

THREE OCCURRANCES OF TEN CARTAN SPACES:

Cartan label	Time evolution operator $\exp\{it\mathcal{H}\}$	Fermionic replica NLσM target space	Classifying space
A	$U(N)\times U(N)/U(N)$	$U(2n)/U(n)\times U(n)$	$U(N+M)/U(N)\times U(M)=C_0$
AIII	$U(N+M)/U(N)\times U(M)$	$U(n)\times U(n)/U(n)$	$U(N)\times U(N)/U(N)=C_1$
AI	$U(N)/O(N)$	$Sp(2n)/Sp(n)\times Sp(n)$	$O(N+M)/O(N)\times O(M)=R_0$
BDI	$O(N+M)/O(N)\times O(M)$	$U(2n)/Sp(2n)$	$O(N)\times O(N)/O(N)=R_1$
D	$O(N)\times O(N)/O(N)$	$O(2n)/U(n)$	$O(2N)/U(N)=R_2$
DIII	$SO(2N)/U(N)$	$O(n)\times O(n)/O(n)$	$U(2N)/Sp(2N)=R_3$
AII	$U(2N)/Sp(2N)$	$O(2n)/O(n)\times O(n)$	$Sp(N+M)/Sp(N)\times Sp(M)=R_4$
CII	$Sp(N+M)/Sp(N)\times Sp(M)$	$U(n)/O(n)$	$Sp(N)\times Sp(N)/Sp(N)=R_5$
C	$Sp(2N)\times Sp(2N)/Sp(2N)$	$Sp(2n)/U(n)$	$Sp(2N)/U(N)=R_6$
CI	$Sp(2N)/U(N)$	$Sp(2n)\times Sp(2n)/Sp(2n)$	$U(N)/O(N)=R_7$

$$\pi(\bar{\mathbf{T}}^{d}; R_q) = \pi_0(R_{q-d}) \oplus \underbrace{\bigoplus_{s=0}^{d-1} \binom{d}{s} \ \pi_0(R_{q-s})}_{\text{weak top. insulators}}$$

where : $\pi(pt, R_q) := \pi_0(R_q)$

and similarly: $R_q \ \longrightarrow \ C_q$

[A. Kitaev (2009)]

This yields the following

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Complex case:													
A	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	…
AIII	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	…
Real case:													
AI	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	…
BDI	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	…
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	…
DIII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	…
AII	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	…
CII	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	…
C	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	…
CI	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	…

(Ryu, Schnyder, Furusaki, Ludwig; Kitaev))

R_{4-p} R_p

THREE OCCURRANCES OF TEN CARTAN SPACES:

Cartan label	Time evolution operator $\exp\{it\mathcal{H}\}$	Fermionic replica NLσM target space		Classifying space
A	$\mathrm{U}(N)\times\mathrm{U}(N)/\mathrm{U}(N)$	$\mathrm{U}(2n)/\mathrm{U}(n)\times\mathrm{U}(n)$	$=C_0$	$\mathrm{U}(N+M)/\mathrm{U}(N)\times\mathrm{U}(M)=C_0$
AIII	$\mathrm{U}(N+M)/\mathrm{U}(N)\times\mathrm{U}(M)$	$\mathrm{U}(n)\times\mathrm{U}(n)/\mathrm{U}(n)$	$=C_1$	$\mathrm{U}(N)\times\mathrm{U}(N)/\mathrm{U}(N)=C_1$
AI	$\mathrm{U}(N)/\mathrm{O}(N)$	$\mathrm{Sp}(2n)/\mathrm{Sp}(n)\times\mathrm{Sp}(n)$	$=R_4$	$\mathrm{O}(N+M)/\mathrm{O}(N)\times\mathrm{O}(M)=R_0$
BDI	$\mathrm{O}(N+M)/\mathrm{O}(N)\times\mathrm{O}(M)$	$\mathrm{U}(2n)/\mathrm{Sp}(2n)$	$=R_3$	$\mathrm{O}(N)\times\mathrm{O}(N)/\mathrm{O}(N)=R_1$
D	$\mathrm{O}(N)\times\mathrm{O}(N)/\mathrm{O}(N)$	$\mathrm{O}(2n)/\mathrm{U}(n)$	$=R_2$	$\mathrm{O}(2N)/\mathrm{U}(N)=R_2$
DIII	$\mathrm{SO}(2N)/\mathrm{U}(N)$	$\mathrm{O}(n)\times\mathrm{O}(n)/\mathrm{O}(n)$	$=R_1$	$\mathrm{U}(2N)/\mathrm{Sp}(2N)=R_3$
AII	$\mathrm{U}(2N)/\mathrm{Sp}(2N)$	$\mathrm{O}(2n)/\mathrm{O}(n)\times\mathrm{O}(n)$	$=R_0$	$\mathrm{Sp}(N+M)/\mathrm{Sp}(N)\times\mathrm{Sp}(M)=R_4$
CII	$\mathrm{Sp}(N+M)/\mathrm{Sp}(N)\times\mathrm{Sp}(M)$	$\mathrm{U}(n)/\mathrm{O}(n)$	$=R_7$	$\mathrm{Sp}(N)\times\mathrm{Sp}(N)/\mathrm{Sp}(N)=R_5$
C	$\mathrm{Sp}(2N)\times\mathrm{Sp}(2N)/\mathrm{Sp}(2N)$	$\mathrm{Sp}(2n)/\mathrm{U}(n)$	$=R_6$	$\mathrm{Sp}(2N)/\mathrm{U}(N)=R_6$
CI	$\mathrm{Sp}(2N)/\mathrm{U}(N)$	$\mathrm{Sp}(2n)\times\mathrm{Sp}(2n)/\mathrm{Sp}(2n)$	$=R_5$	$\mathrm{U}(N)/\mathrm{O}(N)=R_7$

CLASSIFICATION OF TOPOL. INSUL. (SUPERCOND.) FROM QUANTUM ANOMALIES

⇨ WELL DEFINED IN PRESENCE OF INTERACTIONS

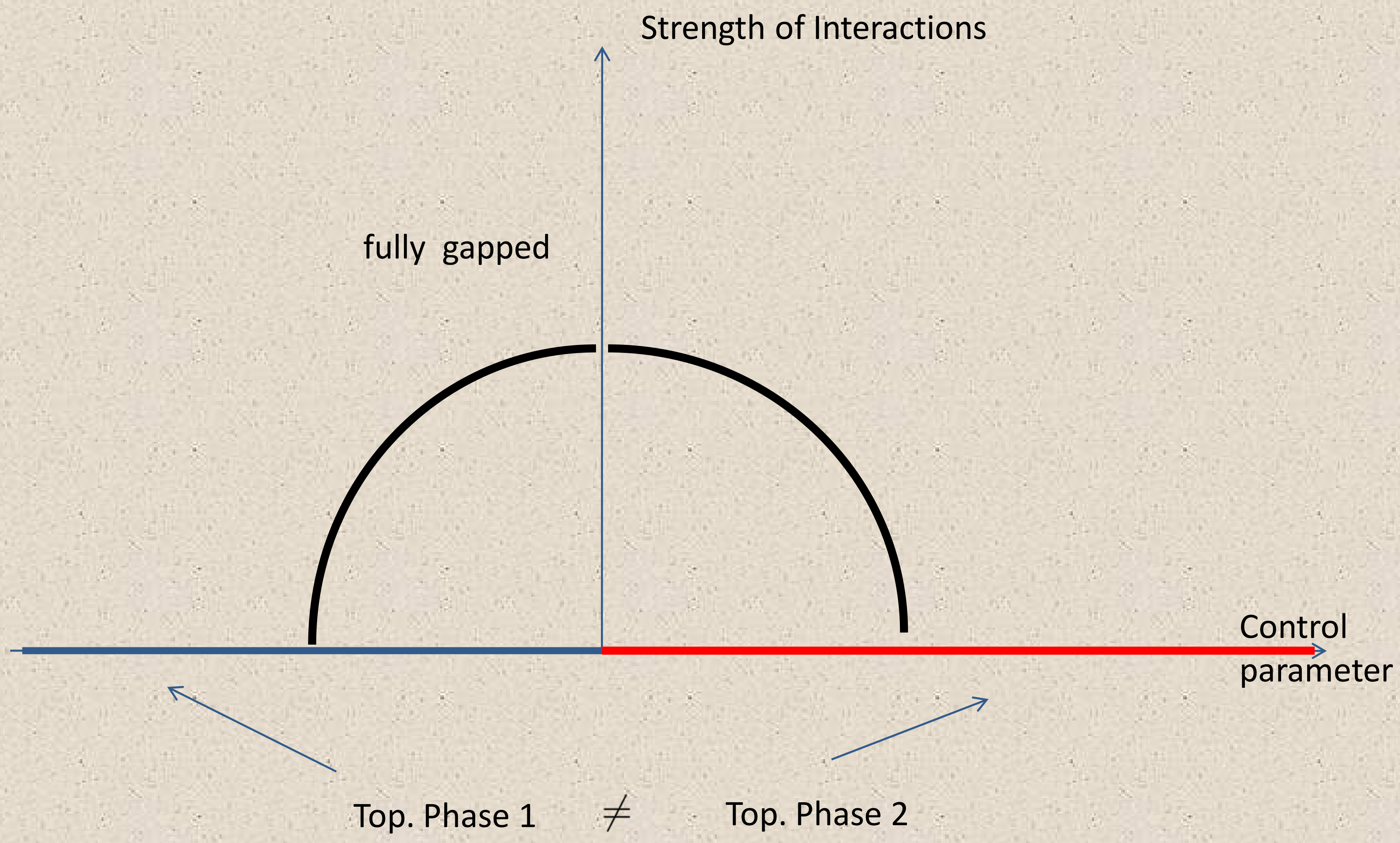
Anomaly: Loss of a symmetry of a Quantum (Field) Theory due to quantum effects

HERE: - Every Topological Insulator (Superconductor) Phase, in any dimension, has a **massive Dirac Hamiltonian representative** in the same topological class [Ryu, Schnyder, Furusaki, Ludwig, NJPhys 12 (2010)].

- Since only interested in **topological features**, we are free to consider the **Dirac Hamiltonian representative**

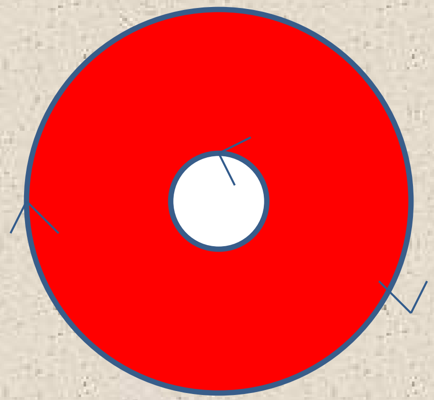
- Couple the Dirac Hamiltonian representative to a suitable space-time dependent background fields [gauge, gravitational (="thermal")].

- Integrate out the massive Dirac Fermions (in space-time). Obtain **effective space-time action** for the **background fields**.


Strength of Interactions
fully gapped
Control parameter
Top. Phase 1 ≠ Top. Phase 2

TWO OCCURRANCES OF AN ANOMALY: **TYPE (i) and TYPE (ii)**

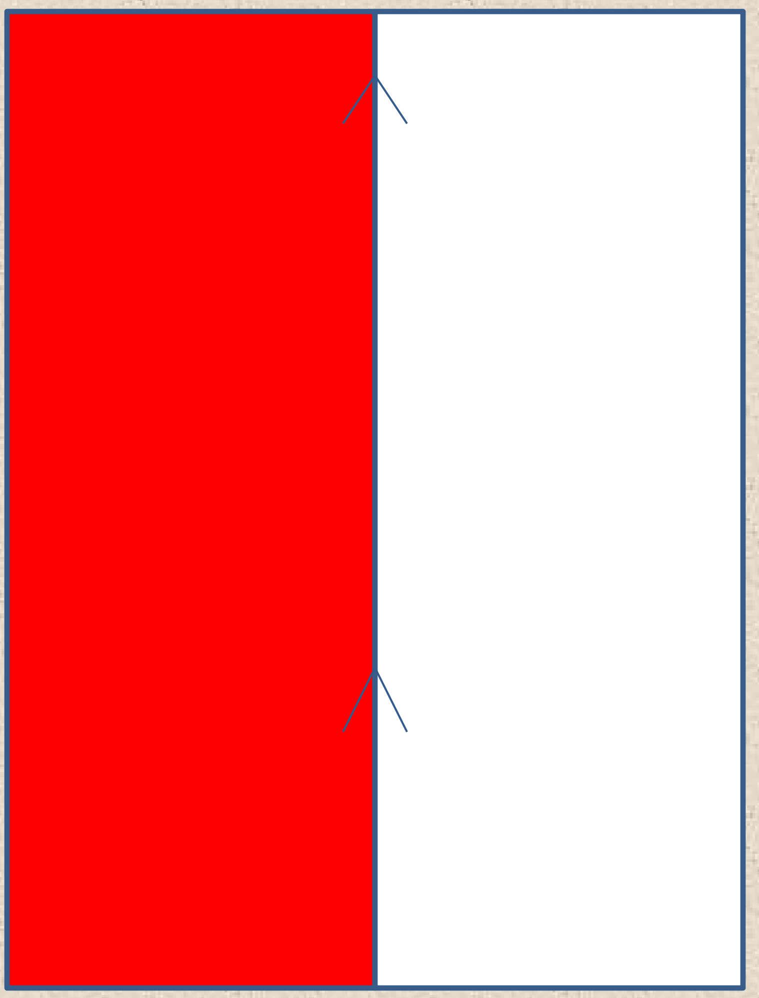
Example for **TYPE (i)**: Integer Quantum Hall Effect in d=2



*Charge conservation at the boundary (=edge) is spoiled by Quantum Mechanics: `**Charge can leak into the bulk'.**

* A characteristic of the Quantum Hall Effect

* This indicates that the boundary doesn't exist in isolation, but is the **boundary of a topological insulator in one dimension higher.**



Example for **TYPE (ii)** : Chiral Anomaly

Applies for systems with Chiral Symmetry: $$S\mathcal{H}S^\dagger = -\mathcal{H}$$

Then, in some basis: $$\mathcal{H} = \begin{pmatrix} 0 & h \\ h^\dagger & 0 \end{pmatrix}, \text{ and } \quad S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Massive Dirac Fermion Representative:

Action: $$\mathcal{S} = \int d^D x \quad \mathcal{L}$$ (D = (d+1) Eulidean Space-Time dim's)

$$\mathcal{L} = \bar{\psi}\,(\nabla_\mu\ \gamma_\mu + m)\,\psi = \bar{\psi}'\,(\nabla_\mu\ \gamma_\mu + m'(\alpha))\,\psi'$$

where $\psi = e^{i\alpha\gamma_5/2}\ \psi'$

and : $m'(\alpha) = me^{i\alpha\gamma_5} = m[\cos(\alpha) + i\gamma_5 \sin(\alpha)]$

$$m'(\alpha = 0) = m, \qquad m'(\alpha = \pi) = -m$$

(topologically trivial) (topologically non-trivial)

- **Euclidean Effective Action (for space-time dependent "background fields"):**

$$e^{-W_{eff}} := \int \mathcal{D}[\bar{\psi}, \psi] \; e^{-S[\bar{\psi},\psi]}$$

- consider case where α not constant: $\alpha(x)$

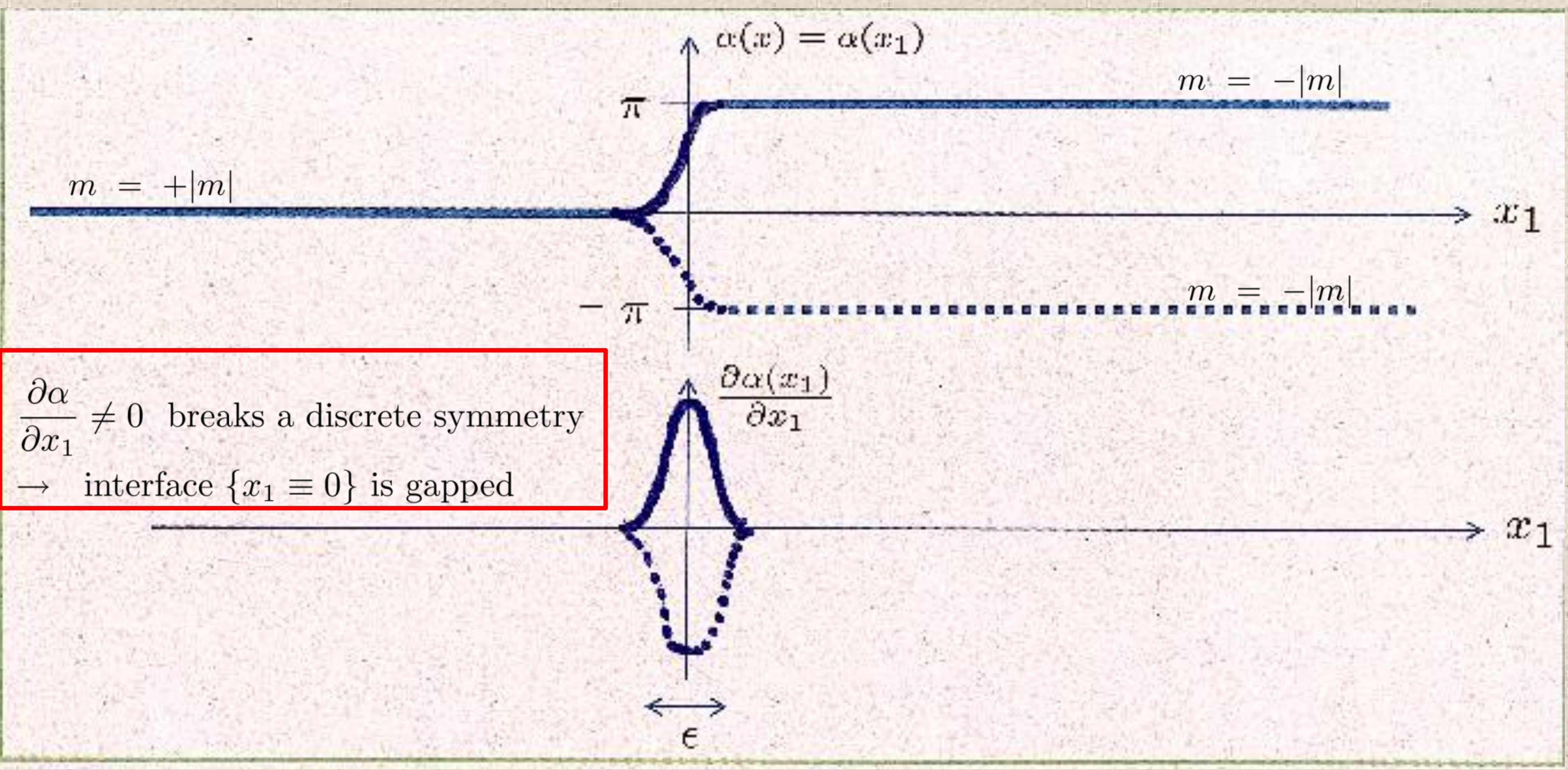
$$i \, \mathrm{Im} \; W_{eff} = i \, \mathrm{Im} \int d^D x \; \frac{1}{2}\alpha(x) \boxed{D_\mu J_5^\mu(x)}$$

$$\boxed{D_\mu J_5^\mu(x) = 2i \; m \; \bar{\psi}\gamma_5\psi + 2i \; \mathcal{A}_D(x)}$$

real → imaginary → (Chiral Anomaly)

In general: $$\Omega_D(x) \ := \ \mathcal{A}_D(x)\ d^D x \ = \ d\Omega^{(0)}_{D-1}$$

= is a total derivative (= exact differential form)



⇨ <u>Integral over the interface:</u> **general version of Chern-Simons term**

$$i \ \mathrm{Im}\ W_{eff} \ = i \int_{x_1 \equiv 0} d^{D-1}x \ \ \Omega^{(0)}_{D-1}(x)$$

Example: Symmetry Class AIII/ U(1) gauge theory in D=4 space-time dimensions

$$\mathcal{A}_{D=4}(x) = \frac{(-1)}{16\pi} \frac{1}{2\pi} \epsilon^{\mu\nu\kappa\lambda} F_{\mu\nu} F_{\kappa\lambda}$$

$$= \frac{(-1)}{16\pi} \frac{1}{2\pi} 4\partial_\sigma \left(\epsilon^{\mu\nu\rho\sigma} A_\mu \partial_\nu A_\rho\right)$$

(This idea was first used by Qi,Hughes, Zhang for E+M responses in symmetry class AII.)

$$i \, \mathrm{Im} W_{eff} =$$

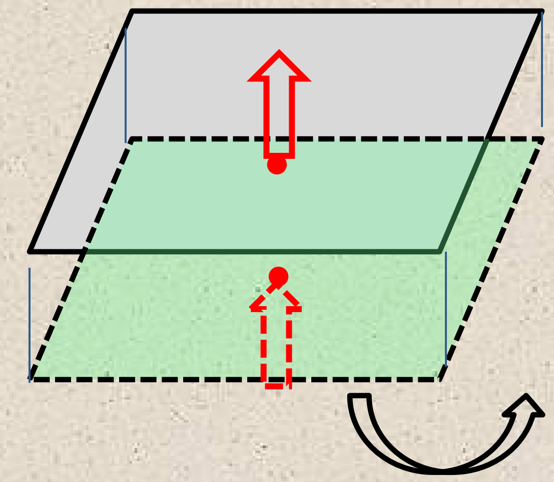
$$= i \frac{(-1)}{16\pi} \frac{1}{2} \int d^4x \, \frac{\alpha(x_1)}{\pi} \epsilon^{\mu\nu\kappa\lambda} F_{\mu\nu} F_{\kappa\lambda} =$$

$$= i \frac{(-1)}{16\pi} \frac{4}{2} \left[\int dx_1 \frac{\partial}{\partial x_1} \left(\frac{\alpha(x_1)}{\pi}\right)\right] \int d^3x \, \epsilon^{1\mu\nu\rho} A_\mu \partial_\nu A_\rho$$

$$\underset{\epsilon \downarrow 0^+}{\rightarrow} i \frac{(-1)}{4\pi} \sigma_{xy} \int d^3x \, \epsilon^{1\mu\nu\rho} A_\mu \partial_\nu A_\rho, \quad \sigma_{xy} = \begin{cases} +1/2 \\ -1/2 \end{cases}$$

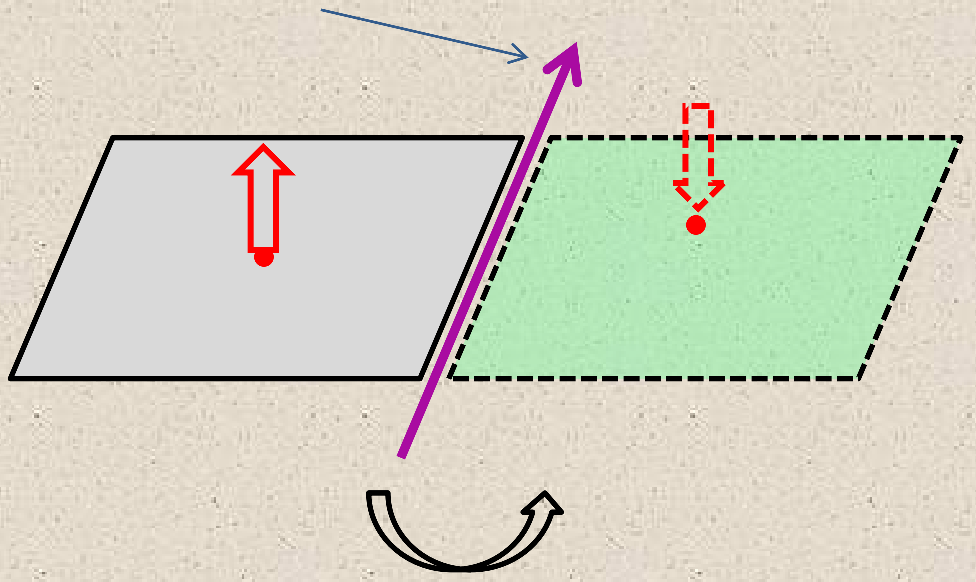
(U(1) Chern-Simons term)

Slab of Top. Insulator:



<u>edge state</u>

conductance : $G = 2 \times \sigma_{xy}$ (units of e^2/h)



(Chiral) Anomaly - General Properties:

(D space-time dimensions)

$$D_\mu J_5^\mu(x) \;=\; 2im\bar{\psi}\gamma_{D+1}\psi \;+\; 2i\mathcal{A}_D(x)$$

- Closed form expressions = Generating Functions:

$$\mathcal{F} := \frac{1}{2}F_{\mu\nu}\,dx^\mu \wedge dx^\nu; \qquad ch(\mathcal{F}) := r + (\frac{i}{2\pi})tr\mathcal{F} + \frac{1}{2!}(\frac{i}{2\pi})^2 tr\mathcal{F}^2 + \ldots$$ (Chern-Character)

$$\mathcal{R}_\mu{}^\nu \;:=\; \frac{1}{2}\,R_{\alpha\beta\mu}{}^\nu\;dx^\alpha \wedge dx^\beta;$$

$$\hat{A}(\mathcal{R}) \;:=\; 1 + \frac{1}{(4\pi)^2}\,\frac{1}{12}tr\mathcal{R}^2 + \frac{1}{(4\pi)^2}\left[\frac{1}{288}\left(tr\mathcal{R}^2\right)^2 + \frac{1}{360}tr\mathcal{R}^4\right] + \ldots$$ (Dirac Genus)

- where $\Omega_D \;:=\; \mathcal{A}_D(x)\;\; d^D x$

and $\Omega_D \;:=\; ch(\mathcal{F})_{|_D}; \qquad := \; \hat{A}(\mathcal{R})_{|_D}; \qquad := \; \left(ch(\mathcal{F})\;\hat{A}(\mathcal{R})\right)_{|_D}$

↕ ↕ ↕

[**U(1)-;** **gravitational-;** **mixed-** **anomaly**]

Descent Relations:

$$\Omega_D = d\Omega^{(0)}_{D-1}, \quad \delta_v \Omega^{(0)}_{D-1} = d\Omega^{(1)}_{D-2}$$

(infinitesimal) gauge transformation

These imply:

- **Boundary** Chern-Simons term" ["**TYPE (ii)"**]:

$$\int_{M_D} \mathcal{A}_D \, d^d x = \int_{M_D} \Omega_D = \int_{M_D} d\Omega^{(0)}_{D-1} = \int_{\partial M_D} \Omega^{(0)}_{D-1}$$

D-dimensional space-time manifold with boundary

- **Bulk** Chern-Simons term ["**TYPE (i)**"]

$$\delta_v \int_{M_{D-1}} \Omega^{(0)}_{D-1} = \int_{M_{D-1}} d\Omega^{(1)}_{D-2} = \int_{\partial M_{D-1}} \Omega^{(1)}_{D-2}$$

Another Example: Symmetry Class DIII in D=4 space-time dimensions

* d=3: a.) superconductors with spin-orbit interactions (Zahid Hasan expt's); b.) He3-B.

* No conservation laws besides energy/momentum ->
-> can only couple to *gravitational* background . (Physically : heat transport <-> later.)

$$\Omega_D = d\Omega^{(0)}_{D-1} \quad \left\{ \begin{array}{l} \mathcal{A}_{D=4}(x) = \frac{1}{2}\frac{1}{4^2\times 12\times 2^2\times\pi^2}\,\epsilon^{cdef}\;R^a{}_{bcd}R^b{}_{aef} = \\ = \frac{1}{2}\frac{1}{4^2\times 12\times 2^2\times\pi^2}\,4\,\epsilon^{\mu\nu\rho\lambda}\partial_\mu\left[tr\left(\omega_\nu\partial_\rho\omega_\lambda+\frac{2}{3}\omega_\nu\omega_\rho\omega_\lambda\right)\right] \end{array}\right\}$$

("spin connection")

$$i\,\mathrm{Im}W_{eff} = i\int d^4x\,\sqrt{g}\,\alpha(x_1)\,\mathcal{A}_4(x)\rightarrow$$

$$\epsilon\downarrow 0^+ \qquad \rightarrow -i\,\frac{1}{4\pi}\,\frac{\sigma^{(T)}_{xy}}{T}\,\frac{c}{24}\int d^3x\,\epsilon^{1\nu\rho\lambda}\,tr\left(\omega_\nu\partial_\rho\omega_\lambda+\frac{2}{3}\omega_\nu\omega_\rho\omega_\lambda\right)$$

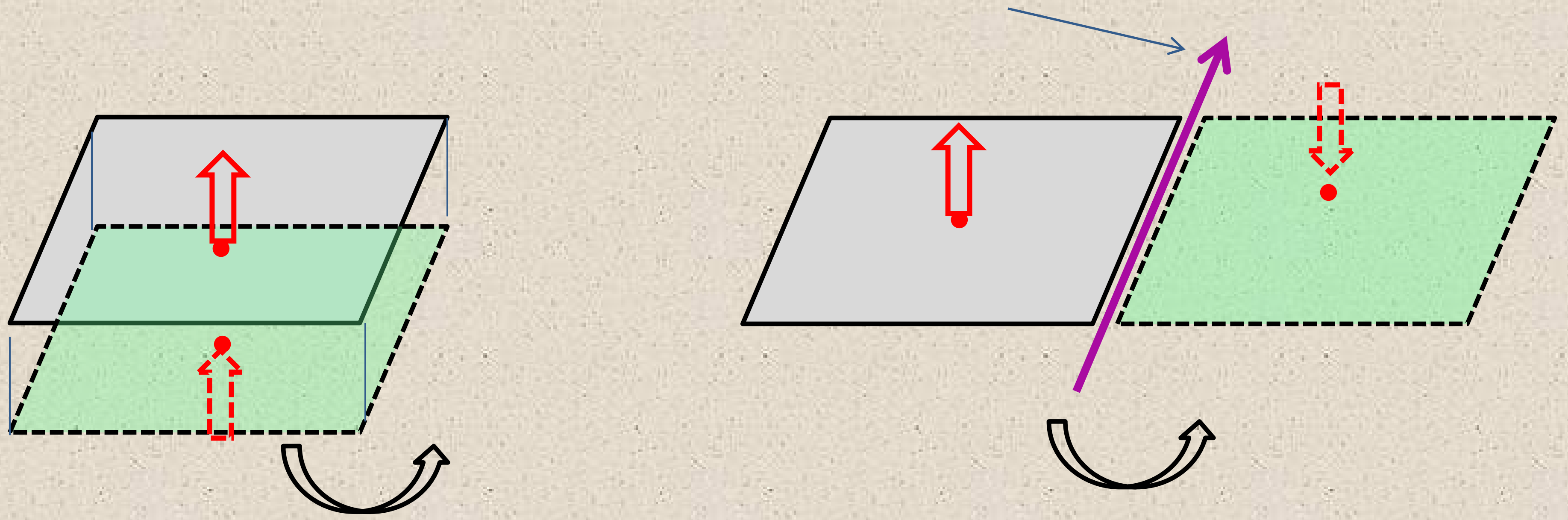
(Gravitational Chern-Simons term)

(see also: Qi,Zhang, arXiv Nov. 2010)

where $c = 1/2$, and $\frac{1}{T}\sigma^{(T)}_{xy} = \left\{\begin{array}{l} +1/2 \\ -1/2 \end{array}\right.$ ——— ...... $\left(\begin{array}{c}\text{units of} \\ \frac{1}{6}\,\frac{(\pi k_B)^2}{h}\end{array}\right)$

edge state

$$\text{thermal}-\text{conductance}: \quad \frac{1}{T}G^T = 2\times\frac{1}{T}\sigma_{xy}^T \quad (\text{units of } \frac{1}{6}\frac{(\pi k_B)^2}{h})$$



PHYSICAL MEANING:

A Temperature gradient in the x-direction

produces

a heat-current in the y-direction:

$$j_y^T = \sigma_{xy}^T \; (-\partial_x T)$$

Have shown for **significant part of Ten-Fold-Table that** the corresponding Topolog. Insulators (Superconducts.) **arise from an “anomaly”** . (Conjecture: true for most.)

-- **Those Top. Ins. well defined even in presence of interactions** --

Cartan\\d	0	1	2	3	4	5	6	7	8	9	10	11	$\cdots$
A	$\mathbf{Z}$	0	$\mathbf{Z}$	0	$\mathbf{Z}$	0	$\mathbf{Z}$	0	$\mathbf{Z}$	0	$\mathbf{Z}$	0	$\cdots$
AIII	0	$\mathbf{Z}$	0	$\mathbf{Z}$	0	$\mathbf{Z}$	0	$\mathbf{Z}$	0	$\mathbf{Z}$	0	$\mathbf{Z}$	$\cdots$
AI	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\cdots$
BDI	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	$\cdots$
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	$\cdots$
DIII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\cdots$
AII	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\cdots$
CII	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\cdots$
C	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\cdots$
CI	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	$\cdots$

TABLE II: Topological insulators (superconductors) with an integer ($\mathbb{Z}$) classification, (A): in the complex symmetry classes, predicted from the **chiral** $U(1)$ [$SU(2)$] **anomaly** (**bold face**), and (B): in the real symmetry classes, predicted from the gravitational anomaly (red), the chiral anomaly in the presence of background gravity (blue), and the ("mixed") chiral anomaly in the presence of both background gravity and $U(1)$ gauge field (green).

Classification:

Symmetry Classes A and AIII:

TYPE (ii): Chiral Anomaly in U(1) gauge background

$$\mathcal{A}_D(x) \neq 0 \quad \text{when} \quad D = 2k \;\; (k \in \mathbf{N})$$

⇨ Top. Insulators in **class AIII** exsist in
D= 2+ 2k space-time dimensions
d= 1 +2k spatial dimensions

(Chern-Simons form)

TYPE (i) : U(1) Chern-Simons term

Use Descent Relation : $$\Omega_{2k} \; = \; \mathcal{A}_{2k} d^D x \; = \; d\Omega^{(0)}_{2k-1}$$

Exists U(1)-Chern Simons term in D = 2k -1 space-time dimensions

⇨ Top. Insulators in **class A** exsist in
D= 1+ 2k space-time dimensions
d= 0 +2k spatial dimensions

RECALL:

(Chiral) Anomaly - General Properties:

(D space-time dimensions)

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$$\mathcal{R}_\mu{}^\nu \;:=\; \frac{1}{2}\, R_{\alpha\beta\mu}{}^\nu\; dx^\alpha \wedge dx^\beta;$$

$$\hat{A}(\mathcal{R}) \;:=\; 1 + \frac{1}{(4\pi)^2}\,\frac{1}{12}tr\mathcal{R}^2 + \frac{1}{(4\pi)^2}\left[\frac{1}{288}\left(tr\mathcal{R}^2\right)^2 + \frac{1}{360}tr\mathcal{R}^4\right] + \ldots$$ (Dirac Genus)

- where $\Omega_D \;:=\; \mathcal{A}_D(x)\;\; d^D x$

and $\Omega_D := ch(\mathcal{F})_{|_D}; \quad := \hat{A}(\mathcal{R})_{|_D}; \quad := \left(ch(\mathcal{F})\;\hat{A}(\mathcal{R})\right)_{|_D}$

↕ ↕ ↕

[**U(1)** ; **gravitational**; **mixed**]

Have shown for **significant part of Ten-Fold-Table that** the corresponding Topolog. Insulators (Superconducts.) **arise from an "anomaly"** . (Conjecture: true for most.)

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AIII	0	$\mathbf{Z}$	0	$\mathbf{Z}$	0	$\mathbf{Z}$	0	$\mathbf{Z}$	0	$\mathbf{Z}$	0	$\mathbf{Z}$	⋯
AI	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	⋯
BDI	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	⋯
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	⋯
DIII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	⋯
AII	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	⋯
CII	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	⋯
C	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	⋯
CI	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	⋯

TABLE II: Topological insulators (superconductors) with an integer ($\mathbb{Z}$) classification, (A): in the complex symmetry classes, predicted from the **chiral** $U(1)$ $[SU(2)]$ **anomaly (bold face)**, and (B): in the real symmetry classes, predicted from the gravitational anomaly (red), the chiral anomaly in the presence of background gravity (blue), and the ("mixed") chiral anomaly in the presence of both background gravity and $U(1)$ gauge field (green).

Symmetry Classes D, DIII, C, CI:

(No natural U(1) symmetry -> **gravitational anomaly**)

TYPE (ii): Chiral Anomaly in gravitational background

$$\mathcal{A}_D(x) \neq 0 \quad \text{when} \quad D = 4k \ (k \in \mathbf{N})$$

(equivalently: D=4+8k, D=8+8k)

$\Rightarrow$

Top. Insulators in **class DIII** exsist in
D= 4+ 8k space-time dimensions
d= 3 +8k spatial dimensions

and

Top. Insulators in **class CI** exsist in
D= 8+ 8k space-time dimensions
d= 7 +8k spatial dimensions

TYPE (i) : Gravitational Chern-Simons term

(Chern-Simons form)

Use Descent Relation : $\Omega_{4k} = \mathcal{A}_{4k} d^D x = d\Omega^{(0)}_{4k-1}$

Exists gravitational Chern-Simons term in D = 4k -1 space-time dimensions
(i.e. in D=3+8k, D= 7+8k).

$\Rightarrow$

Top. Insulators in **class D** exsist in
D= 3+ 8k space-time dimensions
d= 2 +8k spatial dimensions

and

Top. Insulators in **class C** exsist in
D= 7+ 8k space-time dimensions
d= 6 +8k spatial dimensions

RECALL:

(Chiral) Anomaly - General Properties:

(D space-time dimensions)

$$D_\mu J_5^\mu(x) = 2im\bar{\psi}\gamma_{D+1}\psi + 2i\mathcal{A}_D(x)$$

- Closed form expressions = Generating Functions:

$$\mathcal{F} := \frac{1}{2}F_{\mu\nu}\, dx^\mu \wedge dx^\nu; \quad ch(\mathcal{F}) := r + (\frac{i}{2\pi})tr\mathcal{F} + \frac{1}{2!}(\frac{i}{2\pi})^2 tr\mathcal{F}^2 + \ldots$$

(Chern-Character)

$$\mathcal{R}_\mu{}^\nu := \frac{1}{2} R_{\alpha\beta\mu}{}^\nu \, dx^\alpha \wedge dx^\beta;$$

$$\hat{A}(\mathcal{R}) := 1 + \frac{1}{(4\pi)^2}\frac{1}{12}tr\mathcal{R}^2 + \frac{1}{(4\pi)^2}\left[\frac{1}{288}\left(tr\mathcal{R}^2\right)^2 + \frac{1}{360}tr\mathcal{R}^4\right] + \ldots$$

(Dirac Genus)

- where $\Omega_D := \mathcal{A}_D(x) \; d^D x$

and $\Omega_D := ch(\mathcal{F})_{|_D}$; $:= \hat{A}(\mathcal{R})_{|_D}$; $:= \left(ch(\mathcal{F}) \; \hat{A}(\mathcal{R})\right)_{|_D}$

↕ ↕ ↕

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Cartan\\d	0	1	2	3	4	5	6	7	8	9	10	11	⋯
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AI	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	⋯
BDI	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	⋯
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	⋯
DIII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	⋯
AII	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	⋯
CII	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	⋯
C	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	⋯
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Symmetry Classes AI, AII, CII, BDI:

(Have natural U(1) symmetry -> **mixed anomaly**)

TYPE (ii): Chiral Anomaly in 'mixed' background

$$\mathcal{A}_D(x) \neq 0 \quad \text{when} \quad D = 4k+2 \ (k \in \mathbf{N})$$

(equivalently: D=2+8k, D=6+8k)

$\Rightarrow$

Top. Insulators in **class BDI** exsist in
D= 2+ 8k space-time dimensions
d= 1 +8k spatial dimensions

and

Top. Insulators in **class CII** exsist in
D= 6+ 8k space-time dimensions
d= 5 +8k spatial dimensions

TYPE (i) : 'Mixed' Chern-Simons term

(Chern-Simons form)

Use Descent Relation : $\Omega_{4k+2} = \mathcal{A}_{4k+2} d^D x = d\Omega^{(0)}_{4k+1}$

Exists 'mixed' Chern-Simons term in D = 4k +1 space-time dimensions
(i.e. in D=1+8k, D= 5+8k).

$\Rightarrow$

Top. Insulators in **class AI** exsist in
D= 1+ 8k space-time dimensions
d= 0 +8k spatial dimensions

and

Top. Insulators in **class AII** exsist in
D= 5+ 8k space-time dimensions
d= 4 +8k spatial dimensions

RECALL:

(Chiral) Anomaly - General Properties:

(D space-time dimensions)

$$D_\mu J_5^\mu(x) = 2im\bar{\psi}\gamma_{D+1}\psi + 2i\mathcal{A}_D(x)$$

- Closed form expressions = Generating Functions:

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$$\mathcal{R}_\mu{}^\nu := \frac{1}{2} R_{\alpha\beta\mu}{}^\nu\, dx^\alpha \wedge dx^\beta;$$

$$\hat{A}(\mathcal{R}) := 1 + \frac{1}{(4\pi)^2}\frac{1}{12} tr\mathcal{R}^2 + \frac{1}{(4\pi)^2}\left[\frac{1}{288}\left(tr\mathcal{R}^2\right)^2 + \frac{1}{360} tr\mathcal{R}^4\right] + \ldots$$ (Dirac Genus)

- where $\Omega_D := \mathcal{A}_D(x)\ d^D x$

and $\Omega_D := ch(\mathcal{F})_{|_D}$; $:= \hat{A}(\mathcal{R})_{|_D}$; $:= \left(ch(\mathcal{F})\ \hat{A}(\mathcal{R})\right)_{|_D}$

[**U(1)** ; **gravitational**; **mixed**]

Have shown for **significant part of Ten-Fold-Table that** the corresponding Topolog. Insulators (Superconducts.) **arise from an "anomaly"** . (Conjecture: true for most.)

-- **Those Top. Ins. well defined even in presence of interactions** --

Cartan$\backslash d$	0	1	2	3	4	5	6	7	8	9	10	11	$\cdots$
A	$\mathbf{Z}$	0	$\mathbf{Z}$	0	$\mathbf{Z}$	0	$\mathbf{Z}$	0	$\mathbf{Z}$	0	$\mathbf{Z}$	0	$\cdots$
AIII	0	$\mathbf{Z}$	0	$\mathbf{Z}$	0	$\mathbf{Z}$	0	$\mathbf{Z}$	0	$\mathbf{Z}$	0	$\mathbf{Z}$	$\cdots$
AI	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\cdots$
BDI	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	$\cdots$
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	$\cdots$
DIII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\cdots$
AII	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\cdots$
CII	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\cdots$
C	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\cdots$
CI	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	$\cdots$

TABLE II: Topological insulators (superconductors) with an integer ($\mathbb{Z}$) classification, (A): in the complex symmetry classes, predicted from the **chiral** $U(1)$ $[SU(2)]$ **anomaly (bold face)**, and (B): in the real symmetry classes, predicted from the gravitational anomaly (red), the chiral anomaly in the presence of background gravity (blue), and the ("mixed") chiral anomaly in the presence of both background gravity and $U(1)$ gauge field (green).