The quasi-linear evolution of gravitating systems:

Applications to ridge formation, disc thickening
Streams & stellar clusters around black holes.

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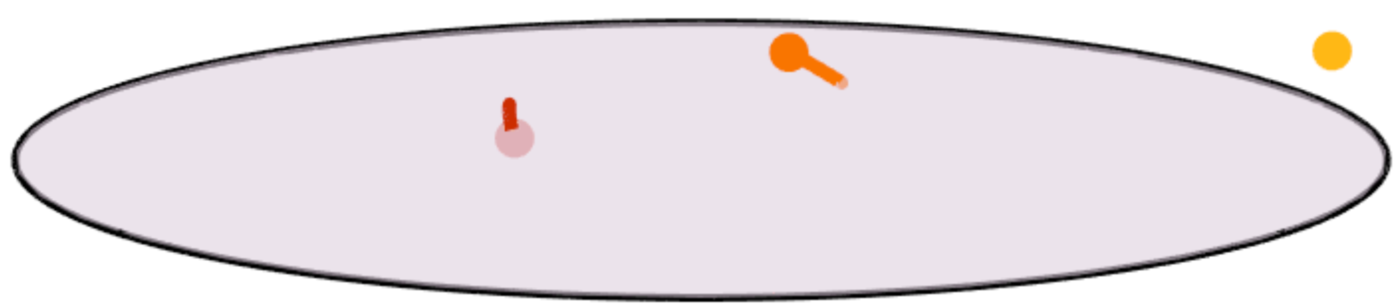


cosmicorigin.org Spine

in collaboration with J. Binney, P.H Chavanis S. Prunet & J. Magorrian

inspired by M. Weinberg's & J. Heyvaerts work

Bring Home Messages

What: Quasi linear theory = stellar version of dissipation-fluctuation theorem
= How do ***orbital structure*** of galaxies ***diffuse*** away from mean field locked trajectory.

How: time decoupling + matrix method (long range : non local + resonances)

Why:

✴Non-linear: qualify perturbation properties as well as equilibrium:
(address nature - nurture conundrum or probe DH).
✴Break the pb by scale (cf zoom simulation) **or** by component (BL - FP);
✴Statistical ≡ ensemble average of sims (cf cosmology, because NL)
✴Captures climate not weather;
✴Theory (can be parametrised, expressed in WKB limit, switch off gravity etc..).
✴ BL version provide correct description of Chandrasekhar friction

Cons:

✴Time decoupling, doesn't capture today's weather;
✴Assumes integrability (but...)
✴Technically not trivial to implement in its full glory (but ...)

For gaia : account for NLs+ resonances for streams + thick disc+ bars+cusp-core

Why Secular Dynamics?

What happens to stable self-gravitating galactic discs on a Hubble time?

How does a galaxy respond

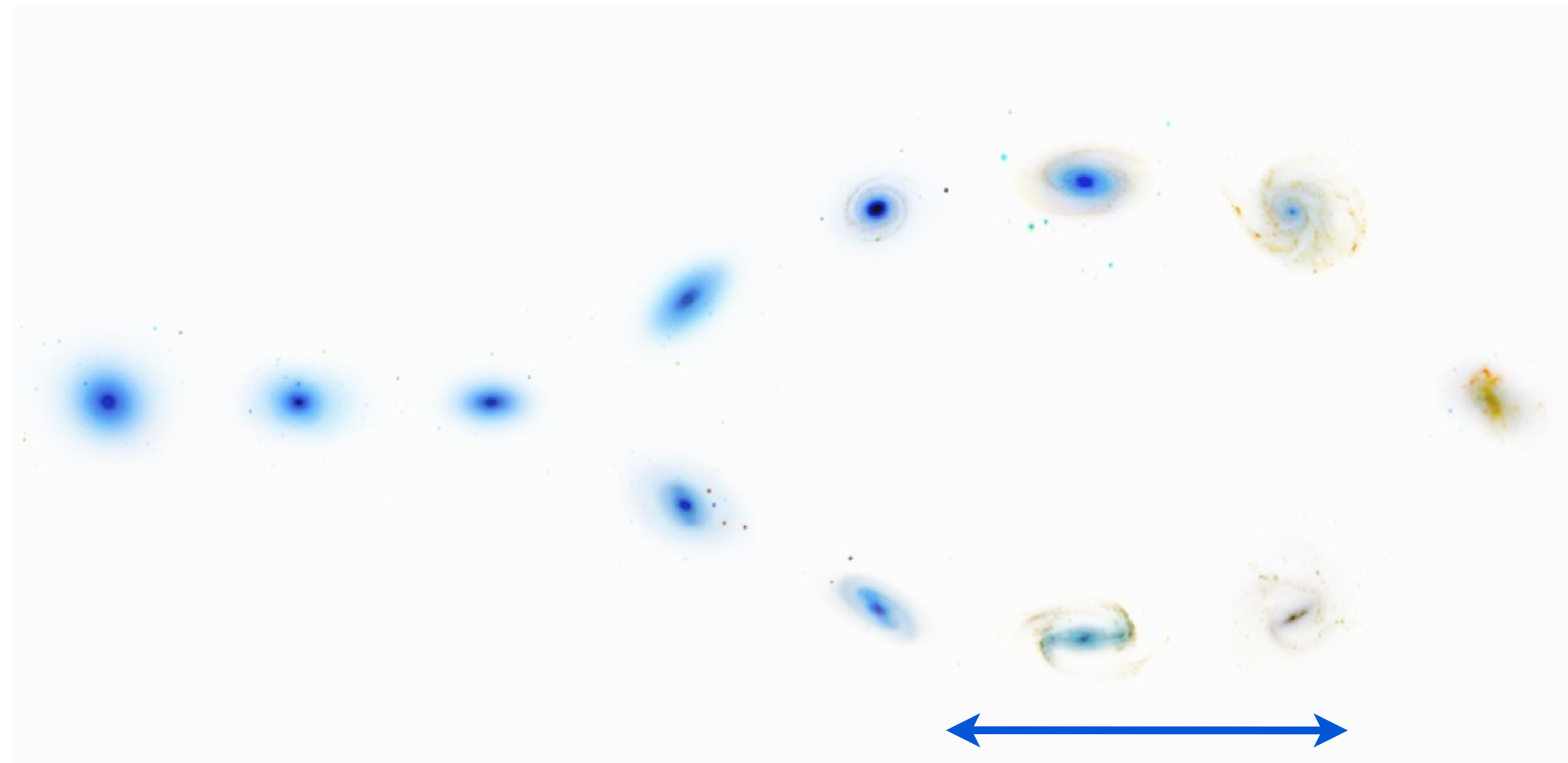
- *to its environment?* Nurture

  Dressed Fokker Planck diffusion

- *to its internal graininess?* Nature

  Balescu-Lenard diffusion

- *Which process matters most on cosmic timescales?*



Move along Hubble Fork

Of interest for galactic chemodynamics (GAIA), Galactic Centre, planetesimals, DM haloes...

Provide quasi-linear theories accounting for non-linear gravity for $t \gg t_{\rm dyn}$

- **Resonant** effects $\Longrightarrow$ **Secular evolution**

$\boxed{\text{What happens to } \textbf{orbital structures} \text{ on } \textbf{cosmic age}\text{?}}$

Fluctuations and dissipation

- Einstein (1905) and Perrin (1908): we know how ink diffuses in water.

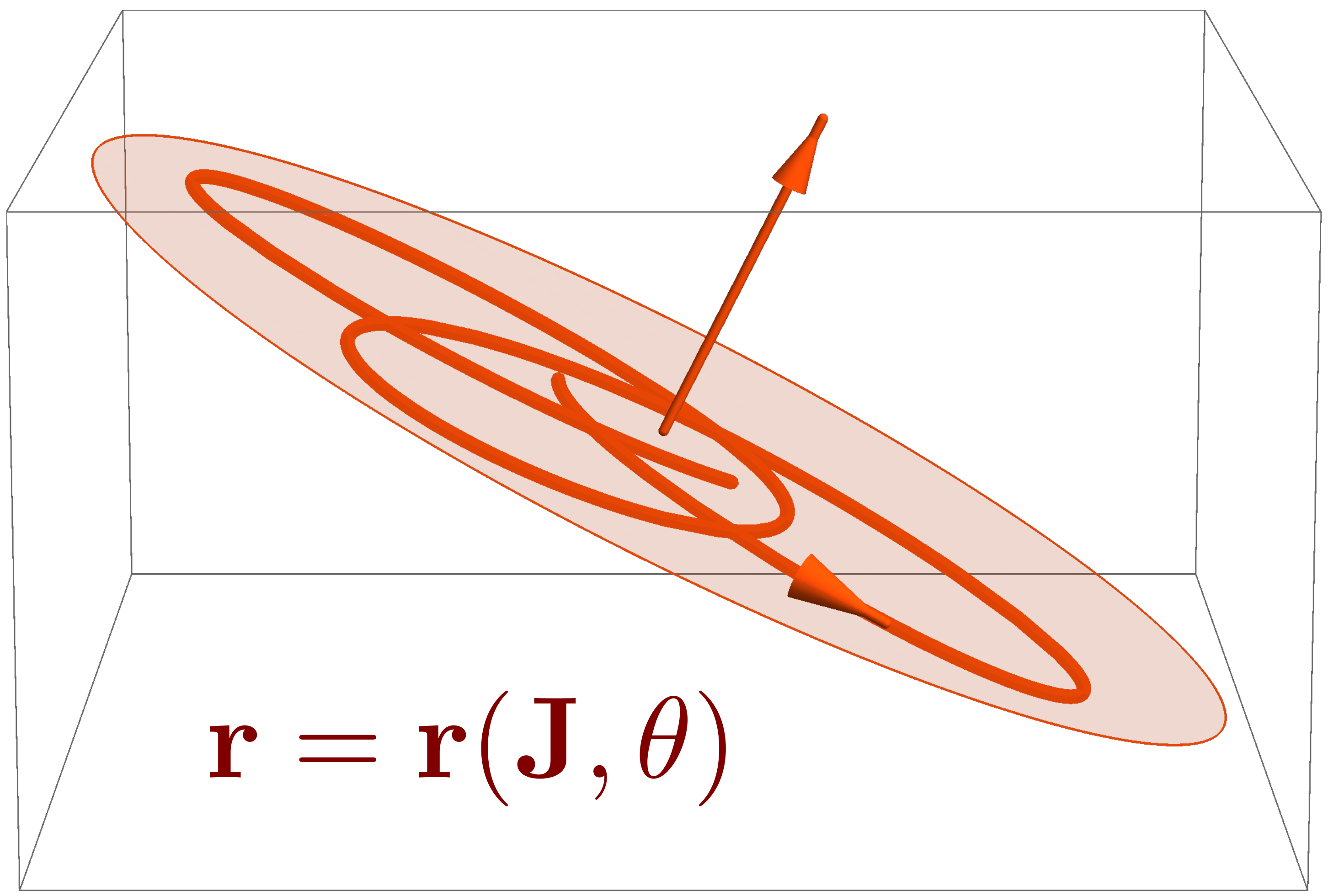
- **Fluctuation-Dissipation Theorem**

Diffusion rate	$\Longleftrightarrow$	Power spectrum **fluctuating forces**

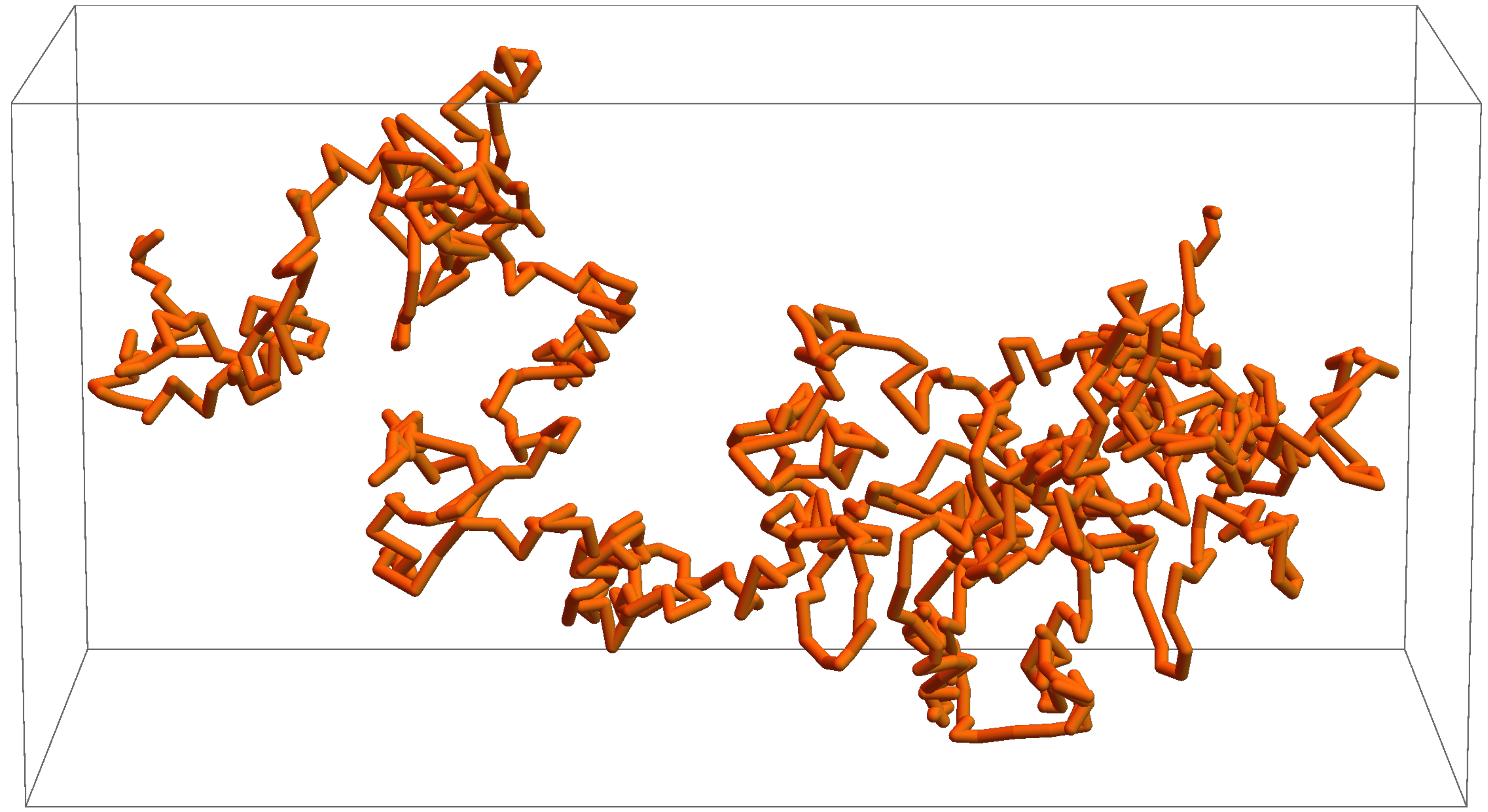
- Stars in cold galaxies undergo the same process
  $\Longrightarrow$ But, gravity is a **long-range interaction**.
  - To diffuse, stars need to **resonate**, otherwise follow the **mean field**.
  - Fluctuations are boosted by **collective effects**.

How do stars' orbits distort on cosmic times?

Along the unperturbed orbit



Potential fluctuate (through resonances)

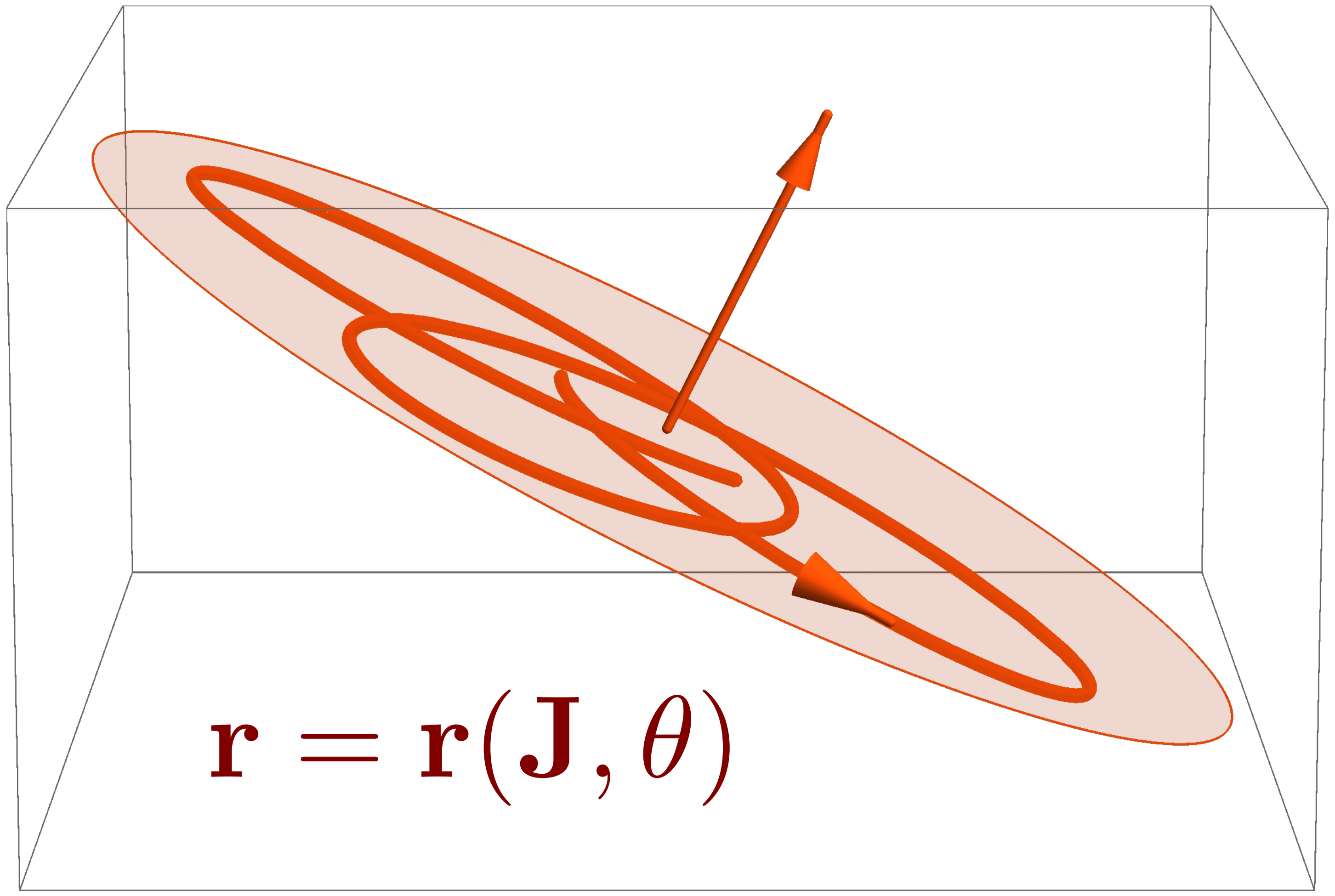


$$\delta\psi(\mathbf{r}, t) \rightarrow \sum_{\mathbf{m}} \delta\hat{\psi}_{\mathbf{m}}(\mathbf{J}, \omega) \exp\left(i\mathbf{m}\cdot\theta - \omega t\right)$$

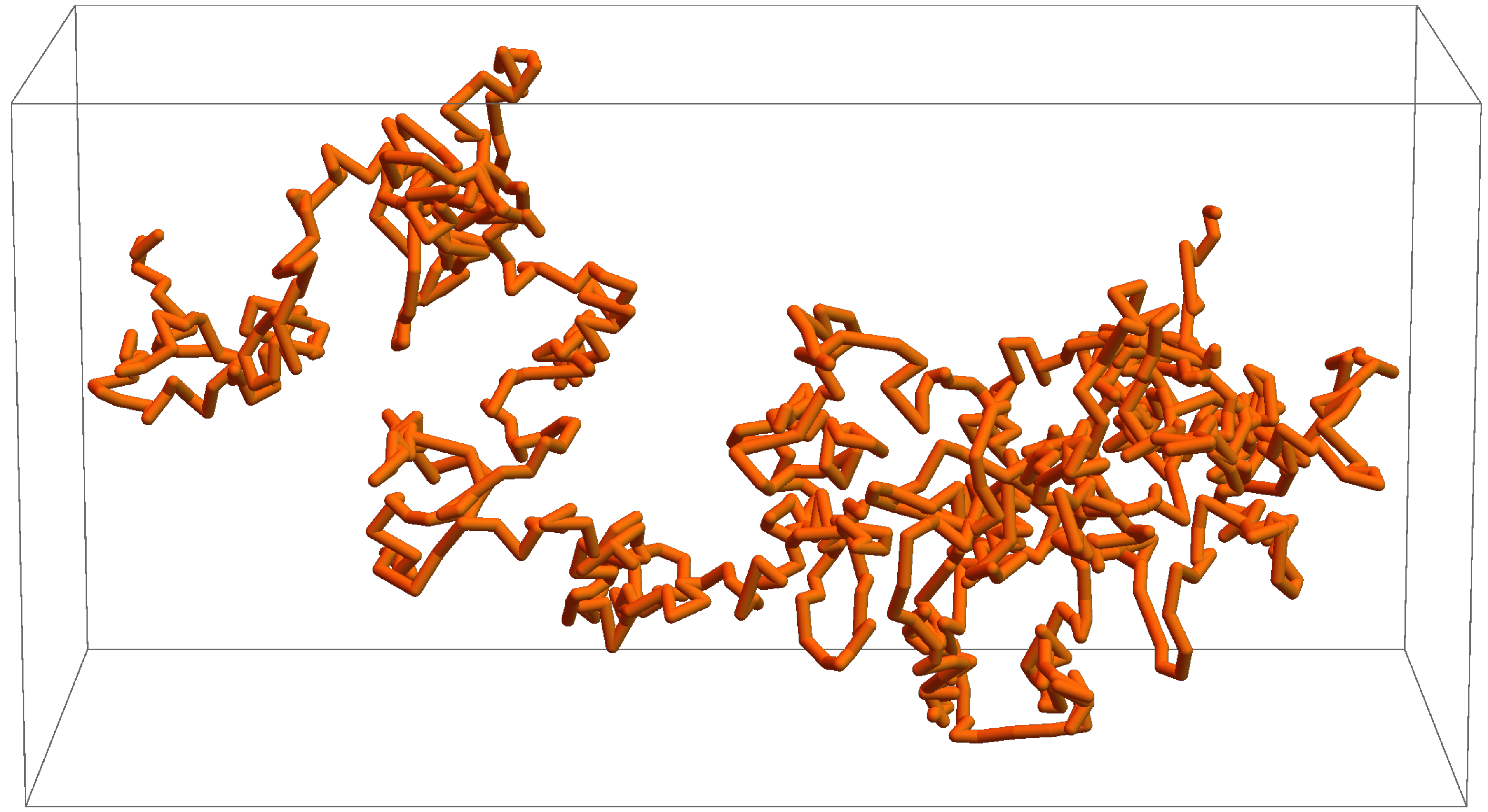
Fluctuating potential

Harmonic component

Along the unperturbed orbit



Potential fluctuate (through resonances)



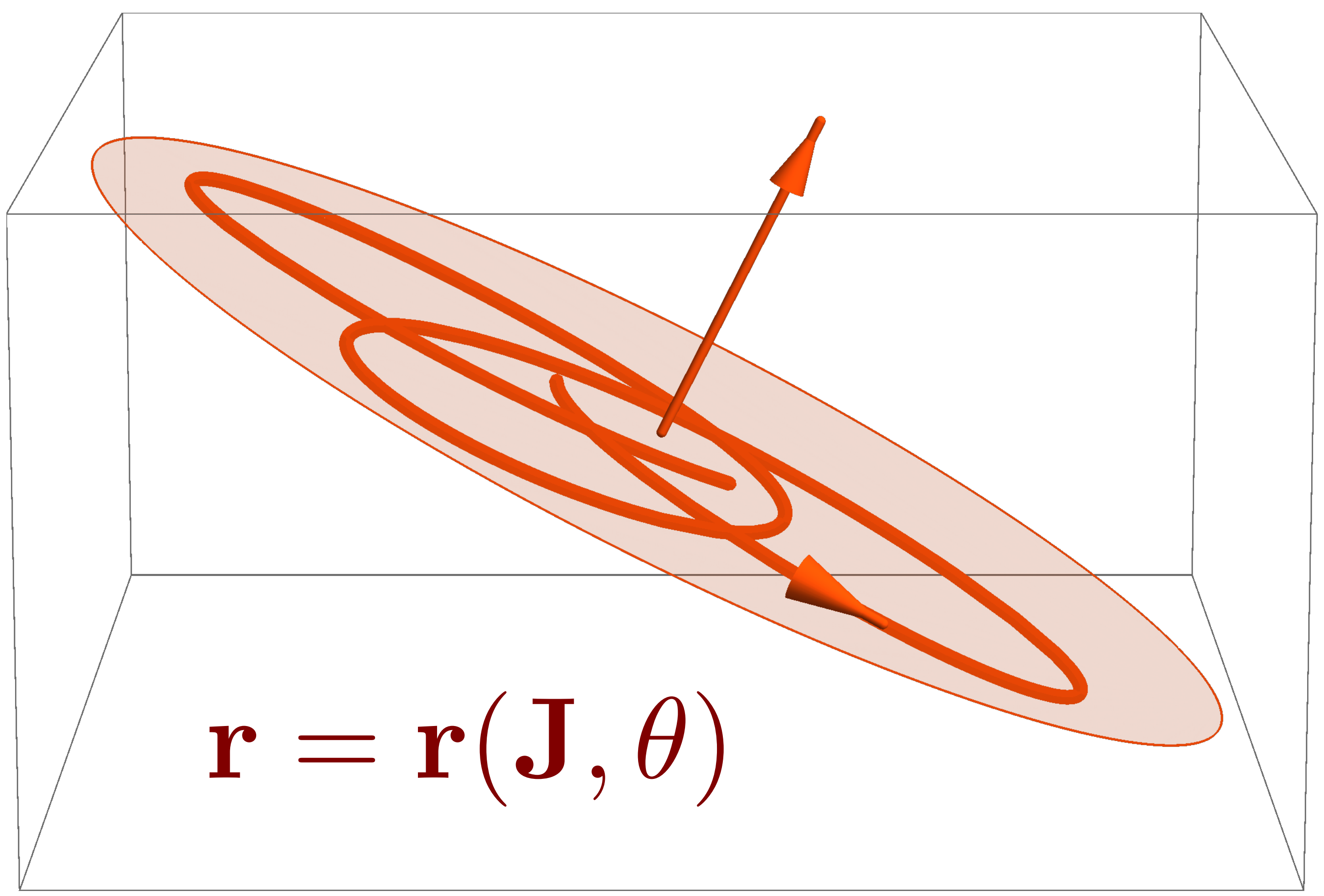
$$\delta\psi(\mathbf{r}, t) \to \sum_{\mathbf{m}} \delta\hat{\psi}_{\mathbf{m}}(\mathbf{J}, \omega) \exp\left(i\mathbf{m}\cdot\theta - \omega t\right)$$

Fluctuating potential

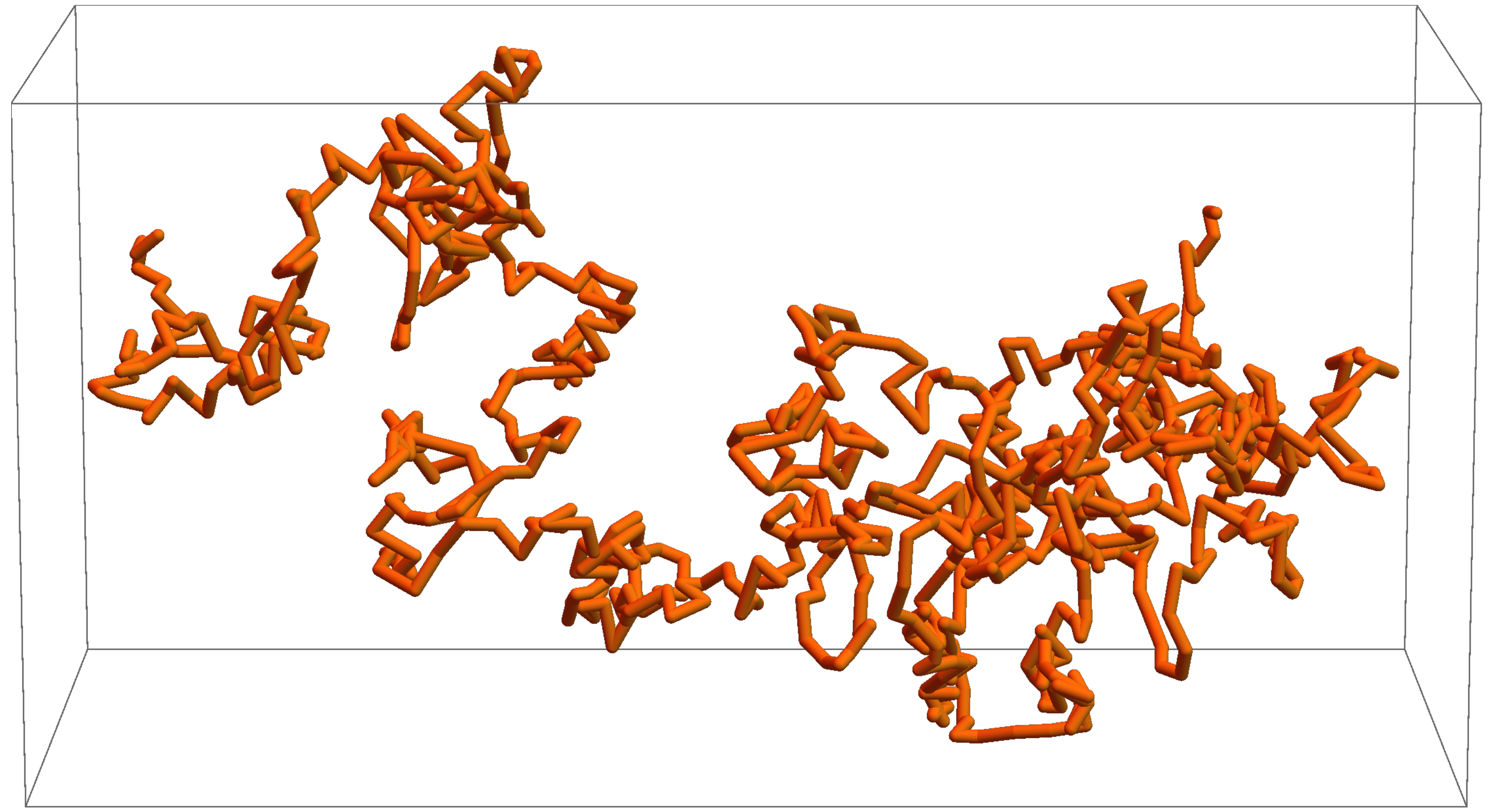
Harmonic component

$$\to \langle|\delta\hat{\psi}_{\mathbf{m}}(\mathbf{J}, \omega)|^2\rangle$$

Along the unperturbed orbit



Potential fluctuate (through resonances)



$$\delta\psi(\mathbf{r}, t) \to \sum_{\mathbf{m}} \delta\hat{\psi}_{\mathbf{m}}(\mathbf{J}, \omega) \exp\left(i\mathbf{m}\cdot\theta - \omega t\right)$$

Fluctuating potential

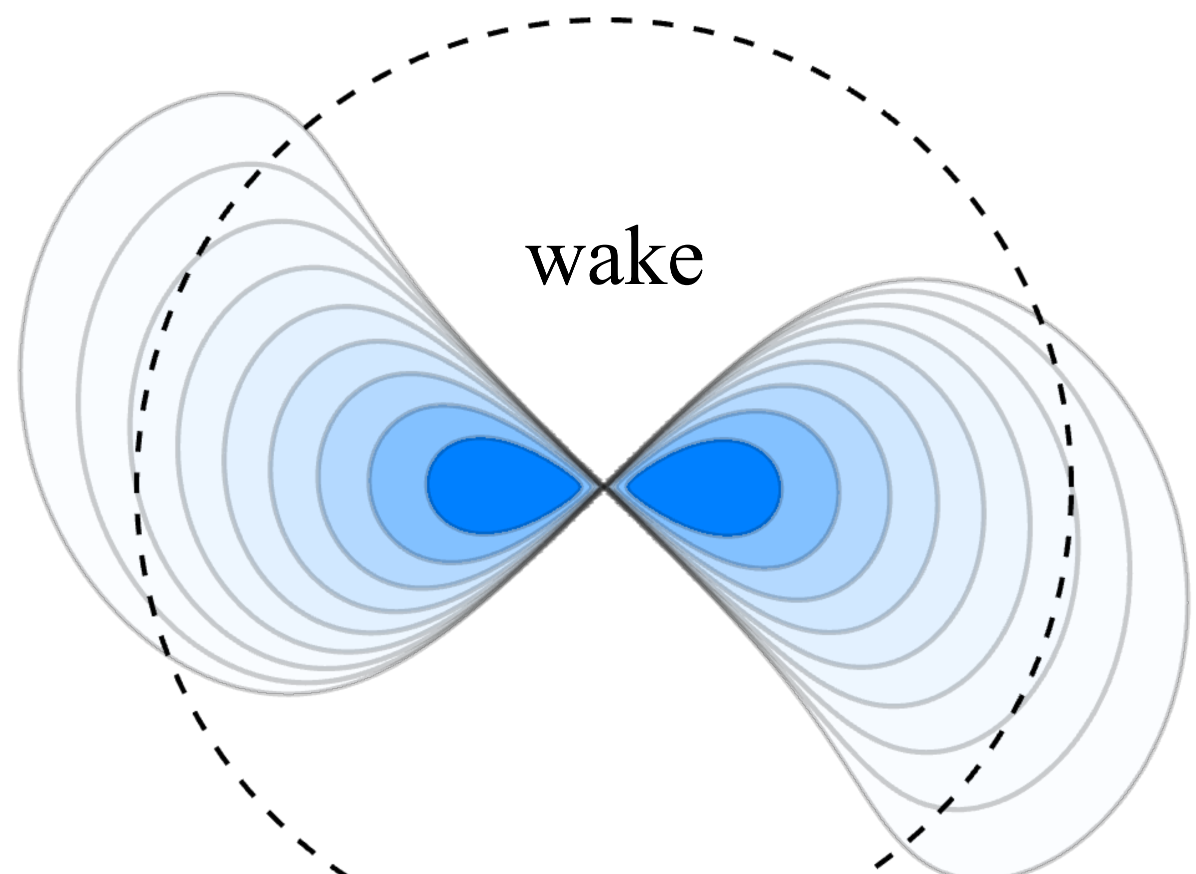
Harmonic component

$$\to \langle|\delta\hat{\psi}_{\mathbf{m}}(\mathbf{J}, \omega)|^2\rangle$$

$$\to \langle|\delta\hat{\psi}_{\mathbf{m}}^{\mathrm{dressed}}(\mathbf{J}, \mathbf{m}\cdot\Omega)|^2\rangle$$

dressed by wake

@ resonance



Heuristic derivation

$$\boxed{\begin{array}{l} \dfrac{\partial F}{\partial t} + [H, F] = 0 \quad \text{with} \quad H = \dfrac{v}{2} + \psi \\ F = f(\mathbf{I}, t) + \delta f(\mathbf{I}, \theta, t) \quad \text{with} \quad \dfrac{\partial \delta f}{\partial t} \gg \dfrac{\partial f}{\partial t} \end{array}}$$

Easy to derive

$$\frac{\partial f}{\partial t} = -\langle [\delta f, \delta \Phi] \rangle$$

where $[,]$ a Poisson bracket and $\langle . \rangle$ is ensemble average

f evolves because fluctuations in f and Φ correlated

- δf depends on $\delta\phi$ through eqns of motion
- $\delta\Phi$ depends on δf through Poisson eqn

Heuristic derivation

$$\frac{\partial F}{\partial t} + [H, F] = 0 \quad \text{with} \quad H = \frac{v}{2} + \psi$$

$$F = f(\mathbf{I}, t) + \delta f(\mathbf{I}, \theta, t) \quad \text{with} \quad \frac{\partial \delta f}{\partial t} \gg \frac{\partial f}{\partial t}$$

Diffusion Tensor

$$\frac{\partial f(\mathbf{J}, t)}{\partial t} = \frac{\partial}{\partial \mathbf{J}} \cdot \left(\mathbf{D} \cdot \frac{\partial f}{\partial \mathbf{J}} \right) \quad \text{where} \quad \mathbf{D} = \sum_{\mathbf{m}} D_{\mathbf{m}} \, \mathbf{m} \otimes \mathbf{m}$$

Dressed fluctuations

Nurture

At **resonance**

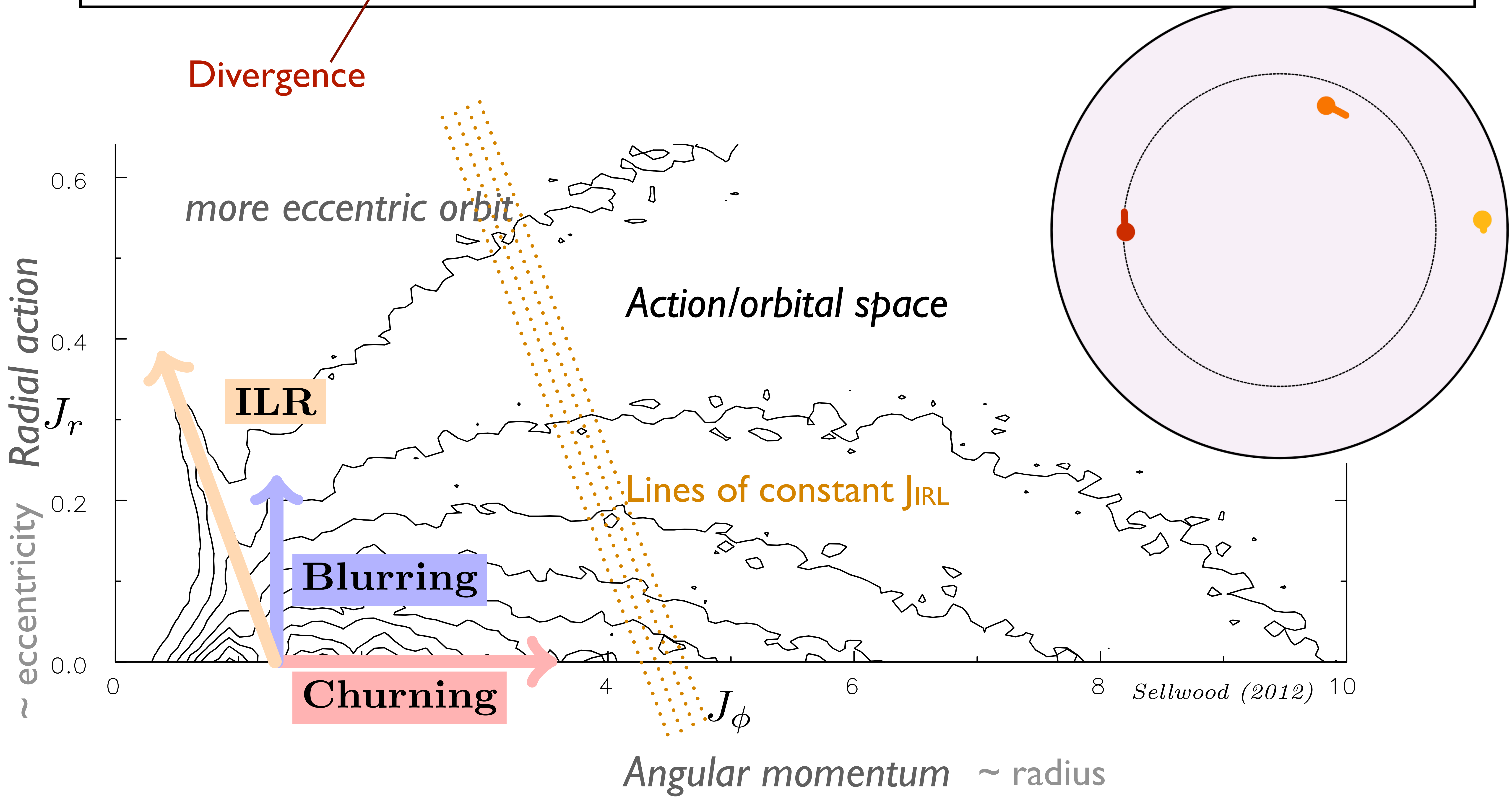
$$D_{\boldsymbol{m}}(\boldsymbol{J}) = \left\langle \left| \psi_{\boldsymbol{m}}^{\mathrm{tot}}(\omega) \right|^2 \right\rangle (\omega = \boldsymbol{m} \cdot \boldsymbol{\Omega}) = \frac{\left\langle \boxed{\left| \psi_{\boldsymbol{m}}^{\mathrm{ext}}(\omega) \right|}^2 \right\rangle}{\left| \varepsilon_{\boldsymbol{m}}(\boldsymbol{J}, \omega) \right|^2} (\omega = \boldsymbol{m} \cdot \boldsymbol{\Omega})$$

Nature

Diffusion is anisotropic

$$\boxed{\frac{\partial f(\mathbf{J},t)}{\partial t} = \frac{\partial}{\partial \mathbf{J}} \cdot \left(\mathbf{D} \cdot \frac{\partial f}{\partial \mathbf{J}} \right) \quad \text{where} \quad \mathbf{D} = \sum_{\mathbf{m}} D_{\mathbf{m}}\, \mathbf{m} \otimes \mathbf{m}}$$

Divergence



Diffusion is anisotropic

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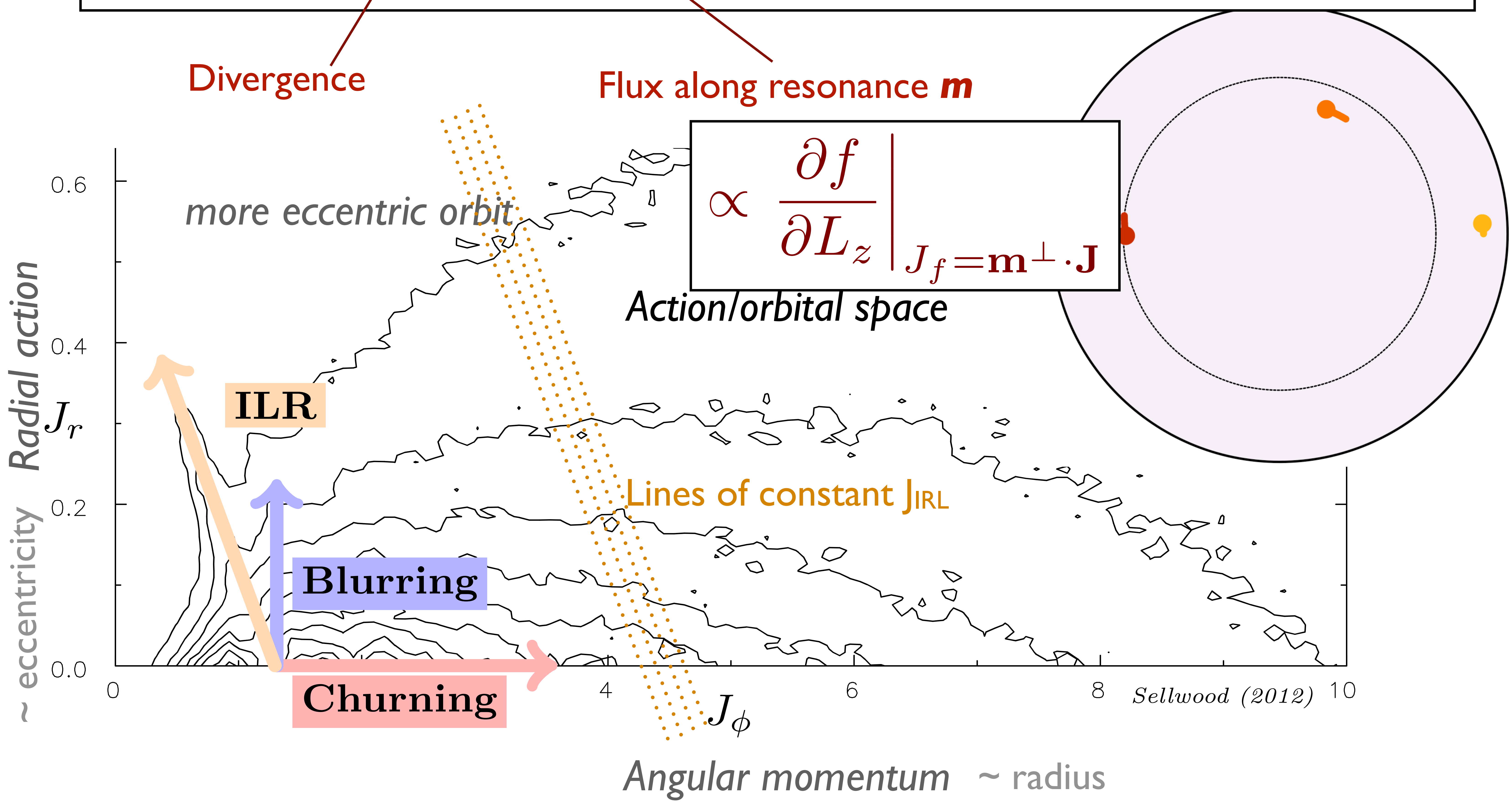
Divergence

Flux along resonance ***m***

$$\propto \left. \frac{\partial f}{\partial L_z} \right|_{J_f = \mathbf{m}^{\perp} \cdot \mathbf{J}}$$

Action/orbital space



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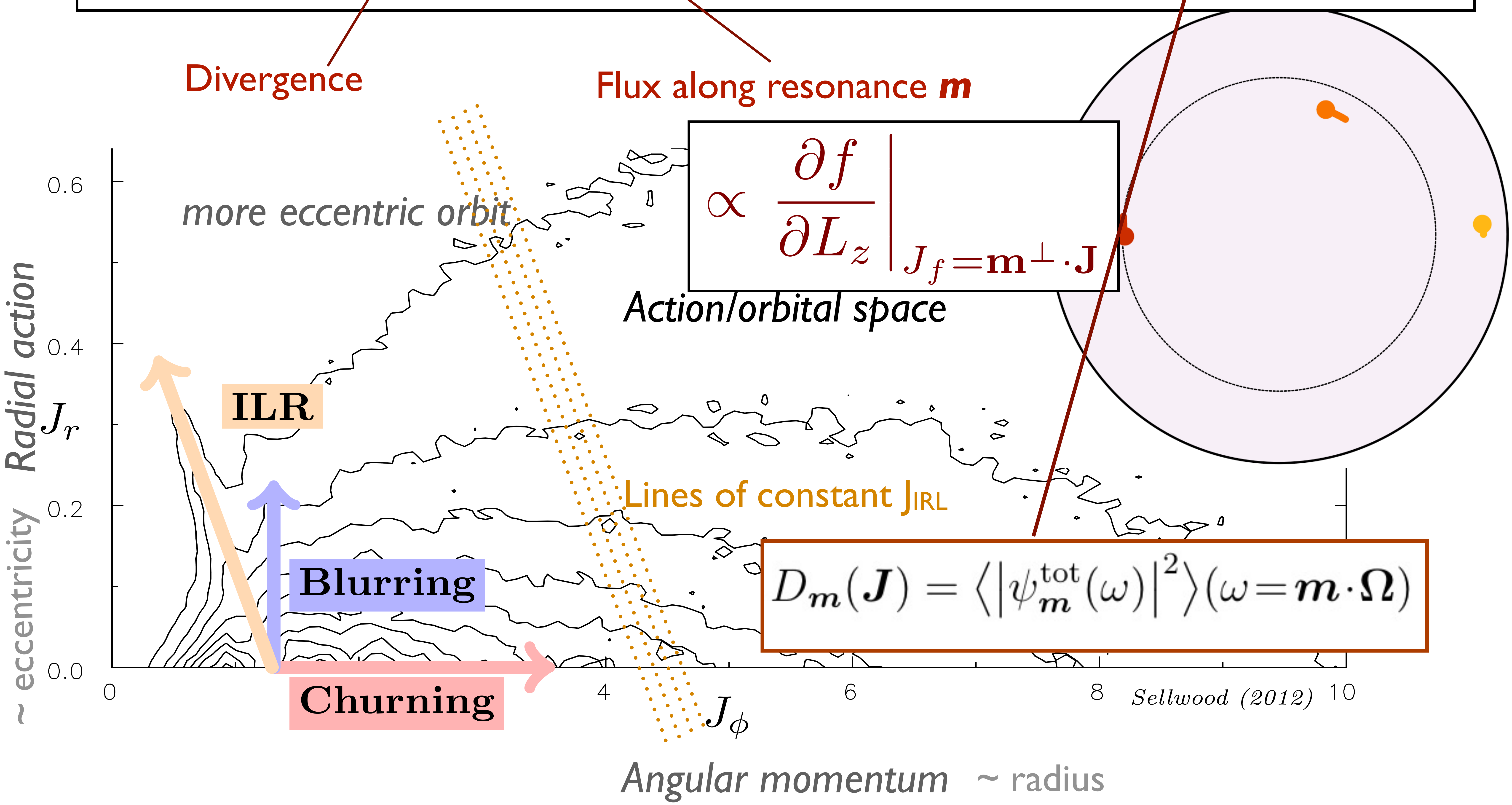
$$\boxed{\frac{\partial f(\mathbf{J},t)}{\partial t} = \frac{\partial}{\partial \mathbf{J}} \cdot \left(\mathbf{D} \cdot \frac{\partial f}{\partial \mathbf{J}}\right) \quad \text{where} \quad \mathbf{D} = \sum_{\mathbf{m}} D_{\mathbf{m}}\, \mathbf{m} \otimes \mathbf{m}}$$

Divergence

Flux along resonance ***m***

$$\boxed{\propto \left.\frac{\partial f}{\partial L_z}\right|_{J_f=\mathbf{m}^\perp\cdot\mathbf{J}}}$$

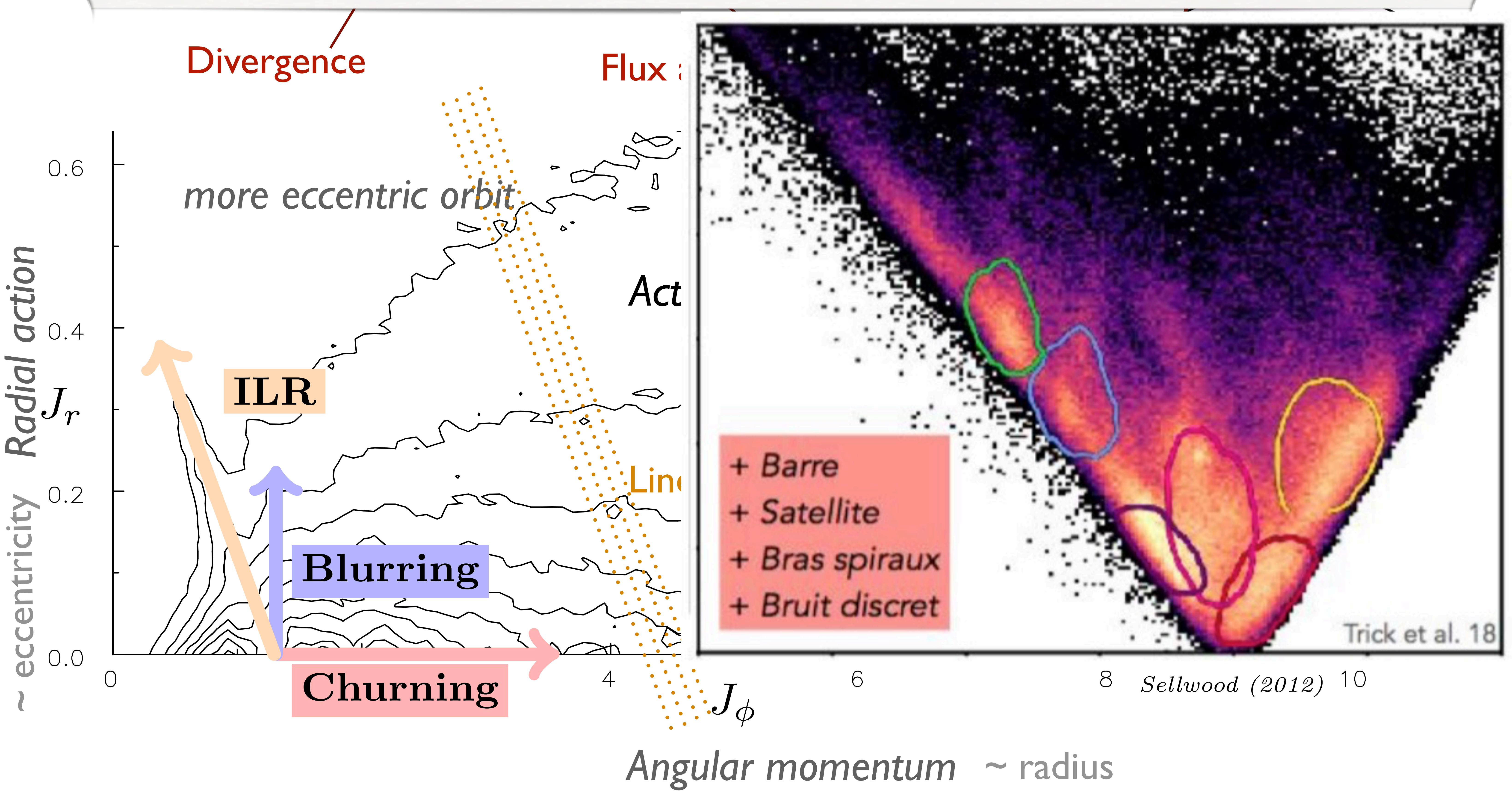


$$\boxed{D_{\boldsymbol{m}}(\boldsymbol{J}) = \left\langle \left|\psi^{\rm tot}_{\boldsymbol{m}}(\omega)\right|^2 \right\rangle (\omega = \boldsymbol{m}\cdot\boldsymbol{\Omega})}$$

If perturbation is low frequency low order resonances dominate

Diffusion is anisotropic

Anything (e.g. *Sgr*) making the potential fluctuate at low frequency will create ridges in phase space as found e.g. by Gaia.



If perturbation is low frequency low order resonances dominate

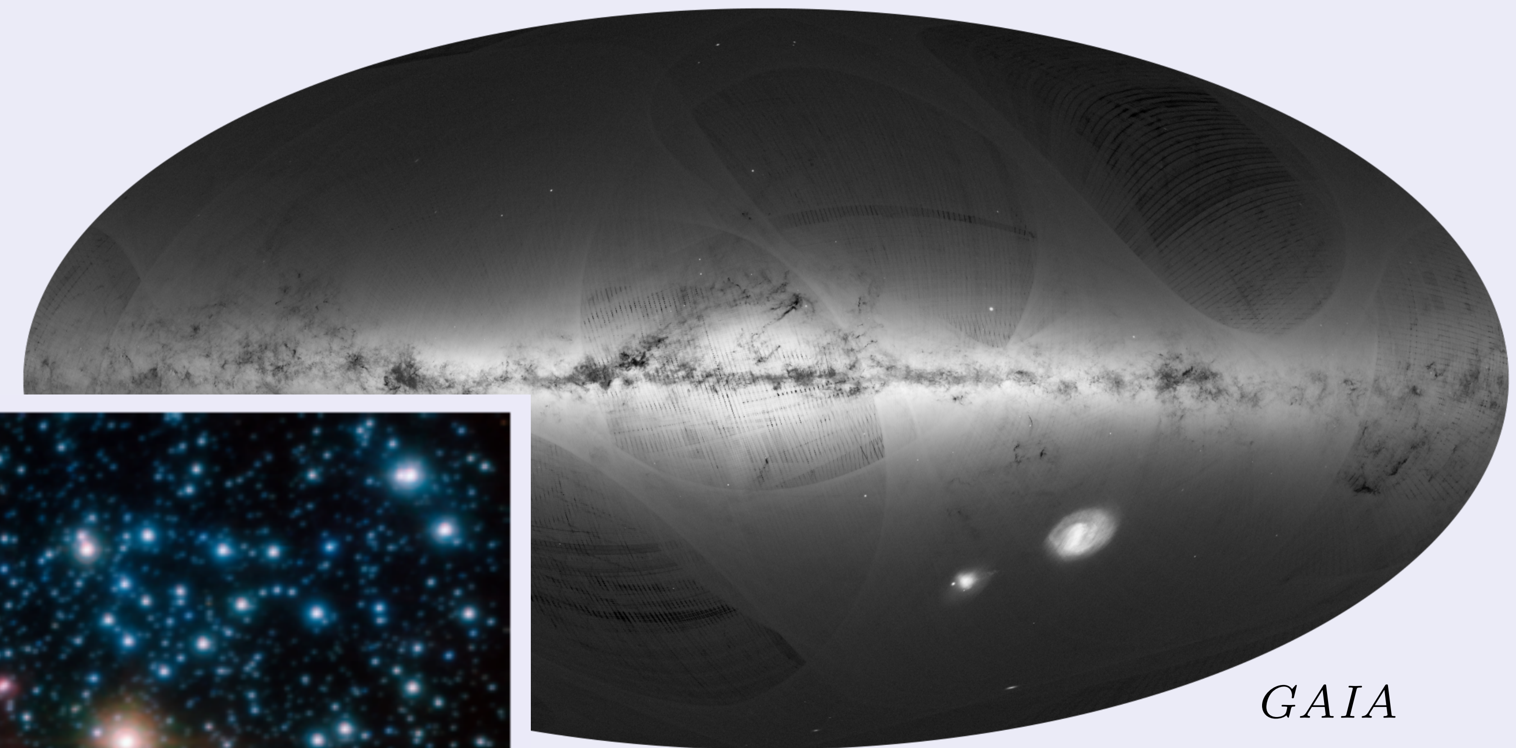
PROSPECTS

Powerful framework applicable to a whole range of **nested scales**

- **Galactic discs**
  - Radial migration, chemistry and galactic archeology (**GAIA**)
  - Disc thickening
- **Galactic centres**
  - BH feeding/spin up
  - Last parcsec pb
- **Galactic haloes**
  - Cusp-core and feedback
  - Impact of cosmic environment

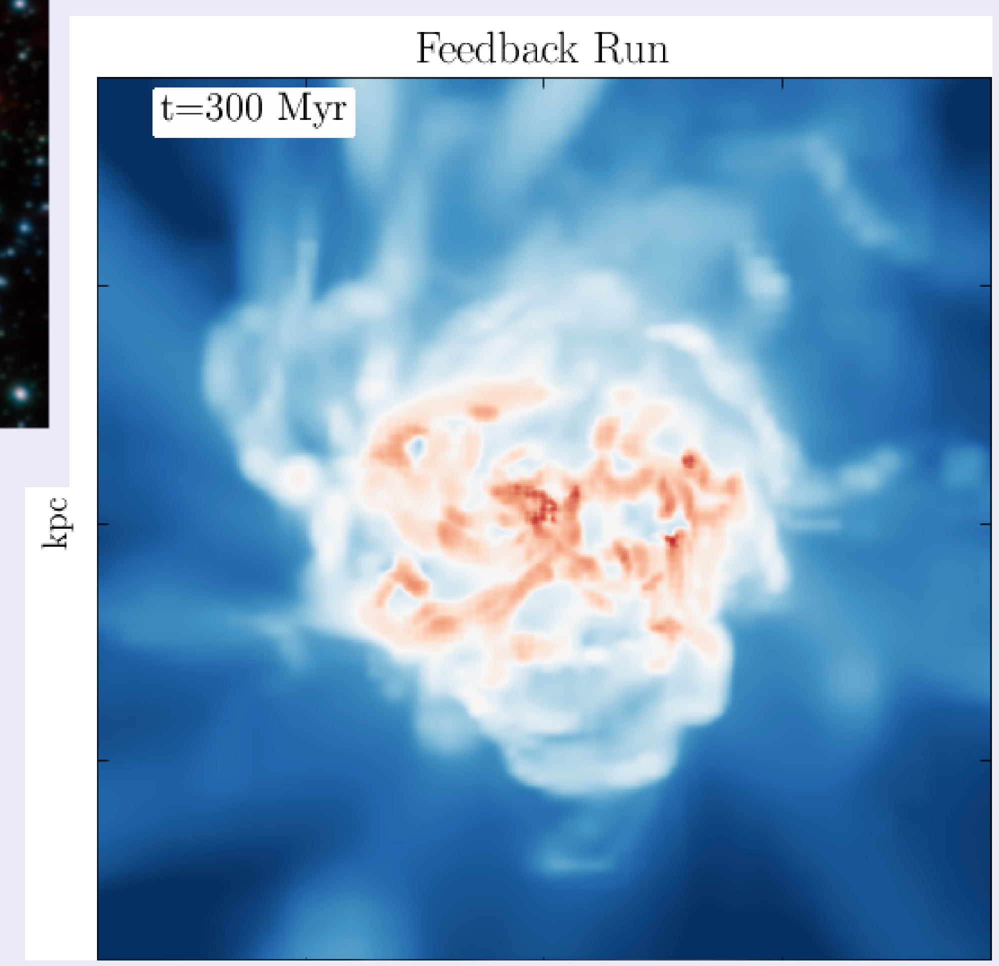
Galactic Streams

- Probe properties of DHs
- Impact of SgrA on MW



GAIA

NACO - VLT

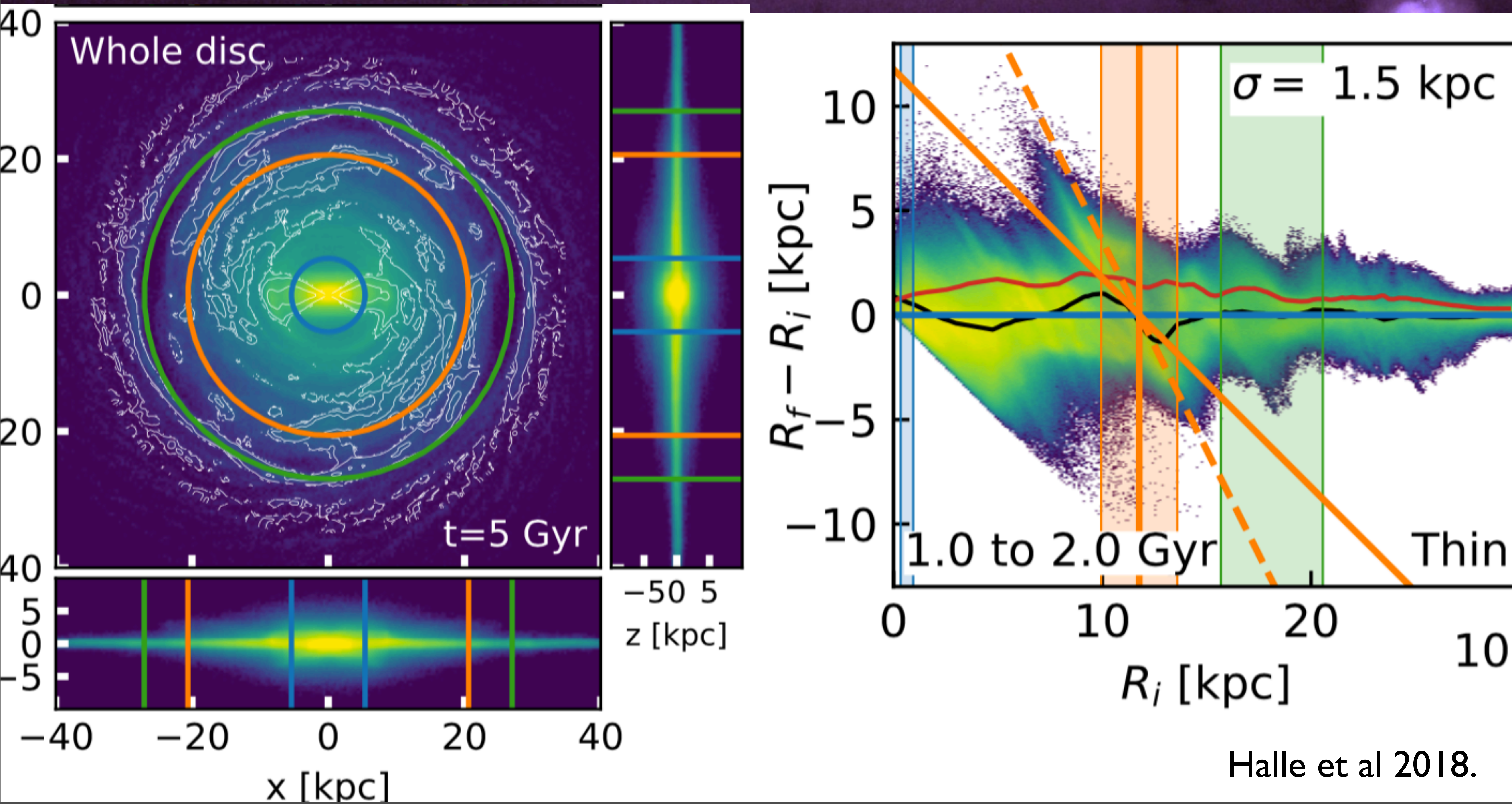


Rebekka Bieri - IAP

Radial migration: churning and blurring and thickening

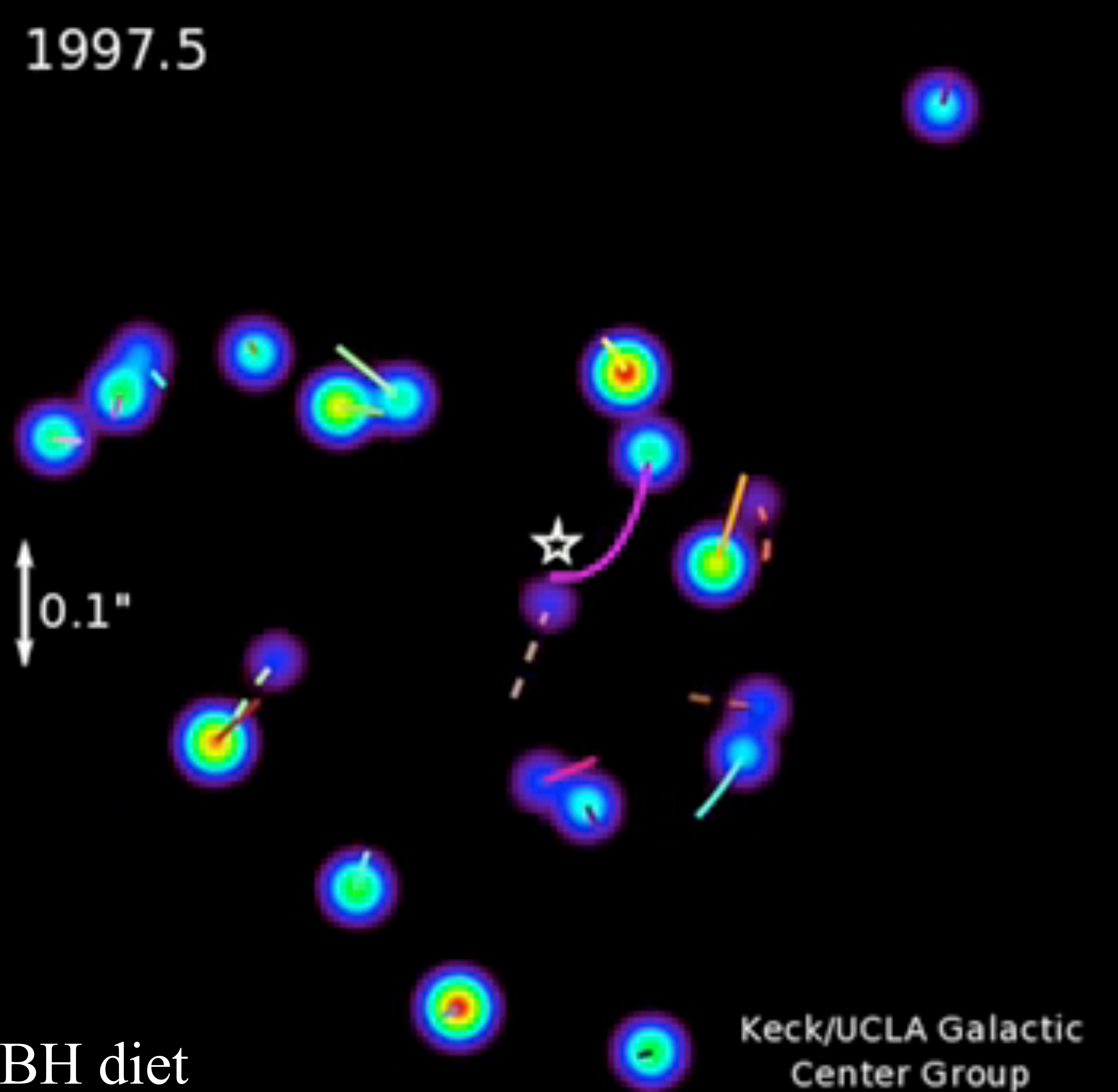
traced by stellar time capsules

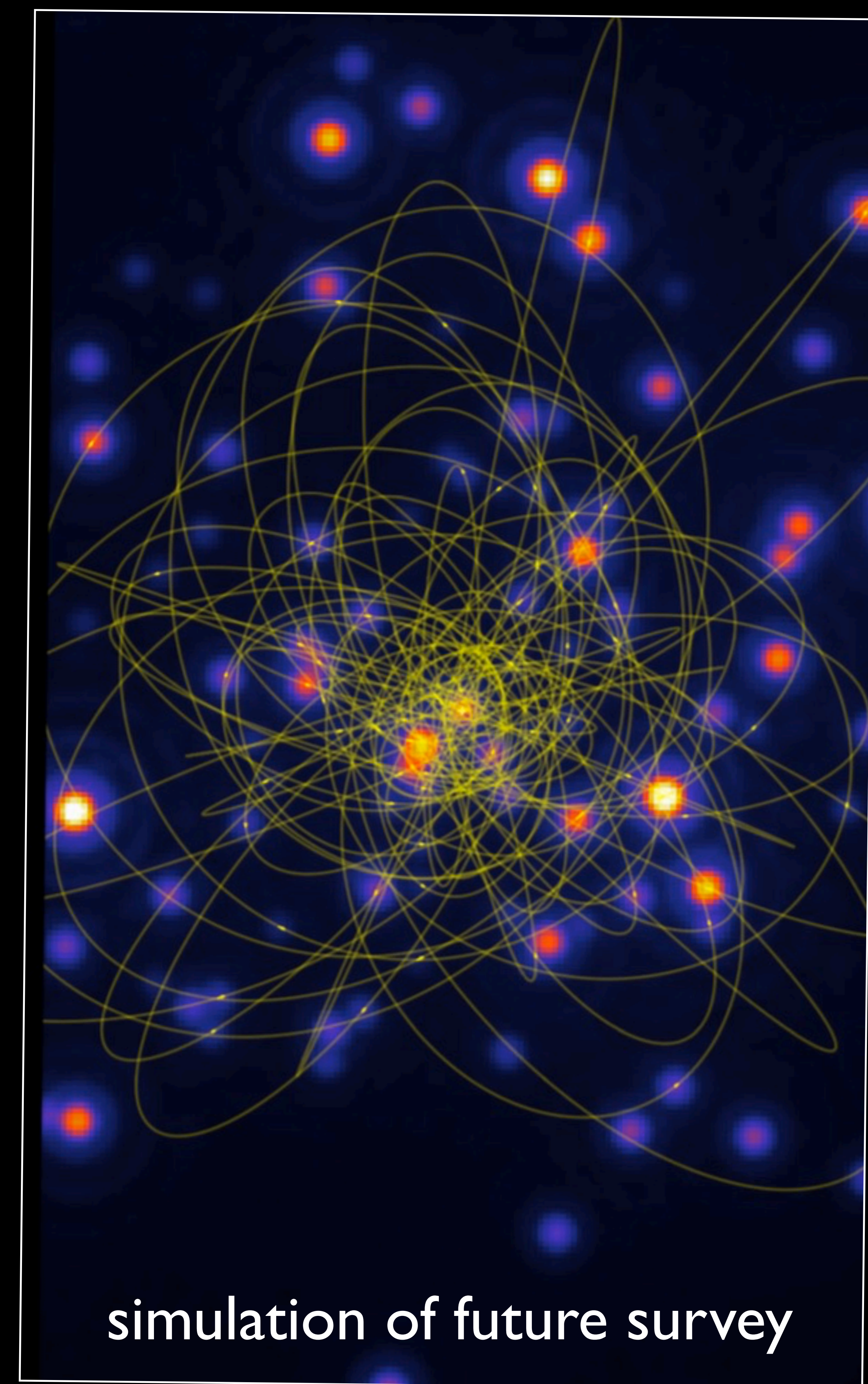
Time reverse cosmic evolution of MW?



Halle et al 2018.

Galactic Center stellar cluster





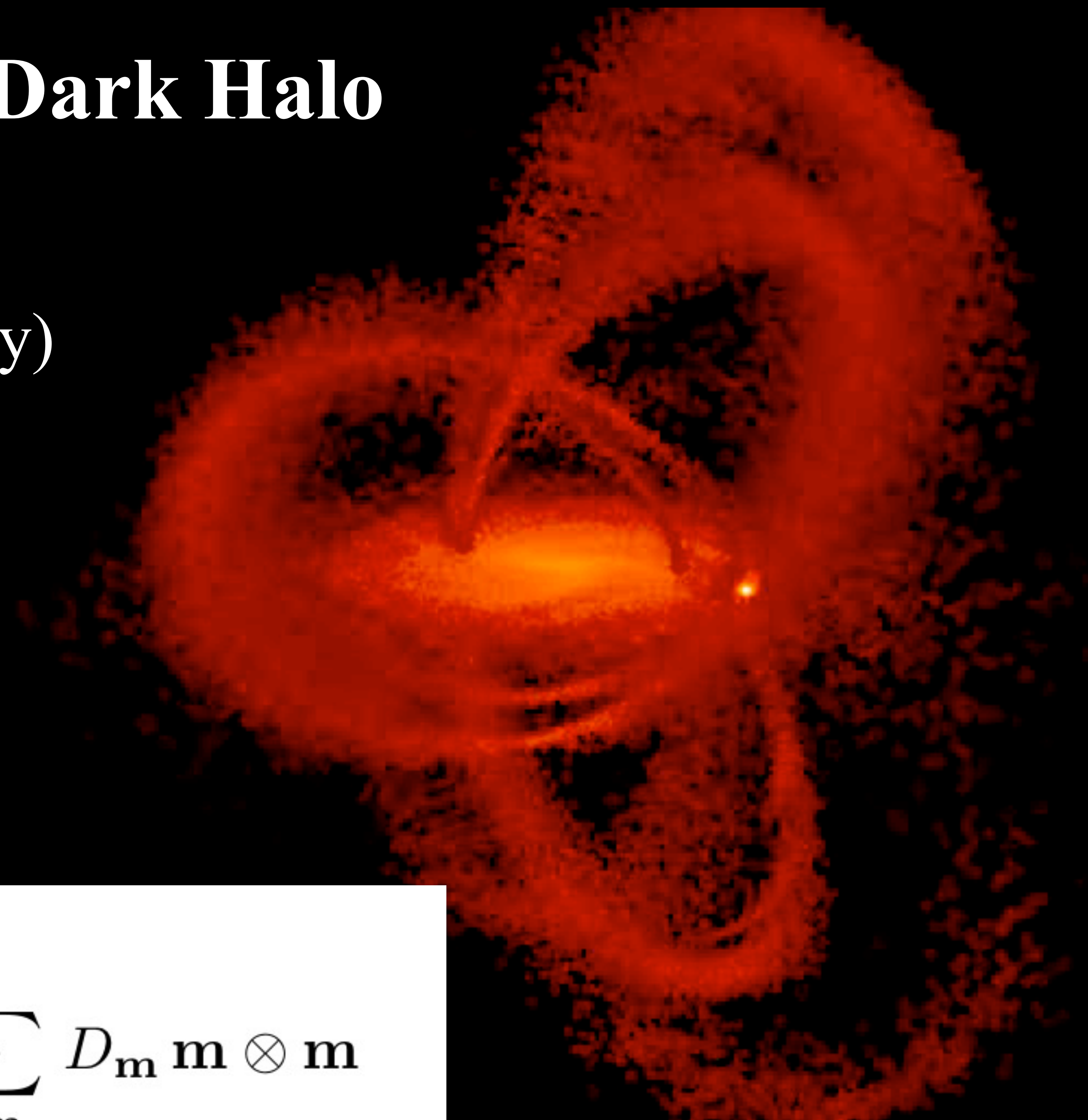
- BH diet
- BH spin up
- Cluster dark component

Diffusion of Streams to probe Dark Halo

- DH clumpiness (through diffusion)
- DH flattening (through induced stochasticity)

$$\frac{\partial \mathbf{I}}{\partial t} \propto \sqrt{\mathbf{D}}\,\xi$$

Stochastic Langevin "Ito" Process



Diffusion Tensor

$$\frac{\partial f(\mathbf{J},t)}{\partial t} = \frac{\partial}{\partial \mathbf{J}} \cdot \left(\mathbf{D} \cdot \frac{\partial f}{\partial \mathbf{J}}\right) \quad \text{where} \quad \mathbf{D} = \sum_{\mathbf{m}} D_{\mathbf{m}}\, \mathbf{m} \otimes \mathbf{m}$$

Broadening of stream with phase = measure of D_m $(\mathbf{J})$

Dressed fluctuations

Nurture

At **resonance**

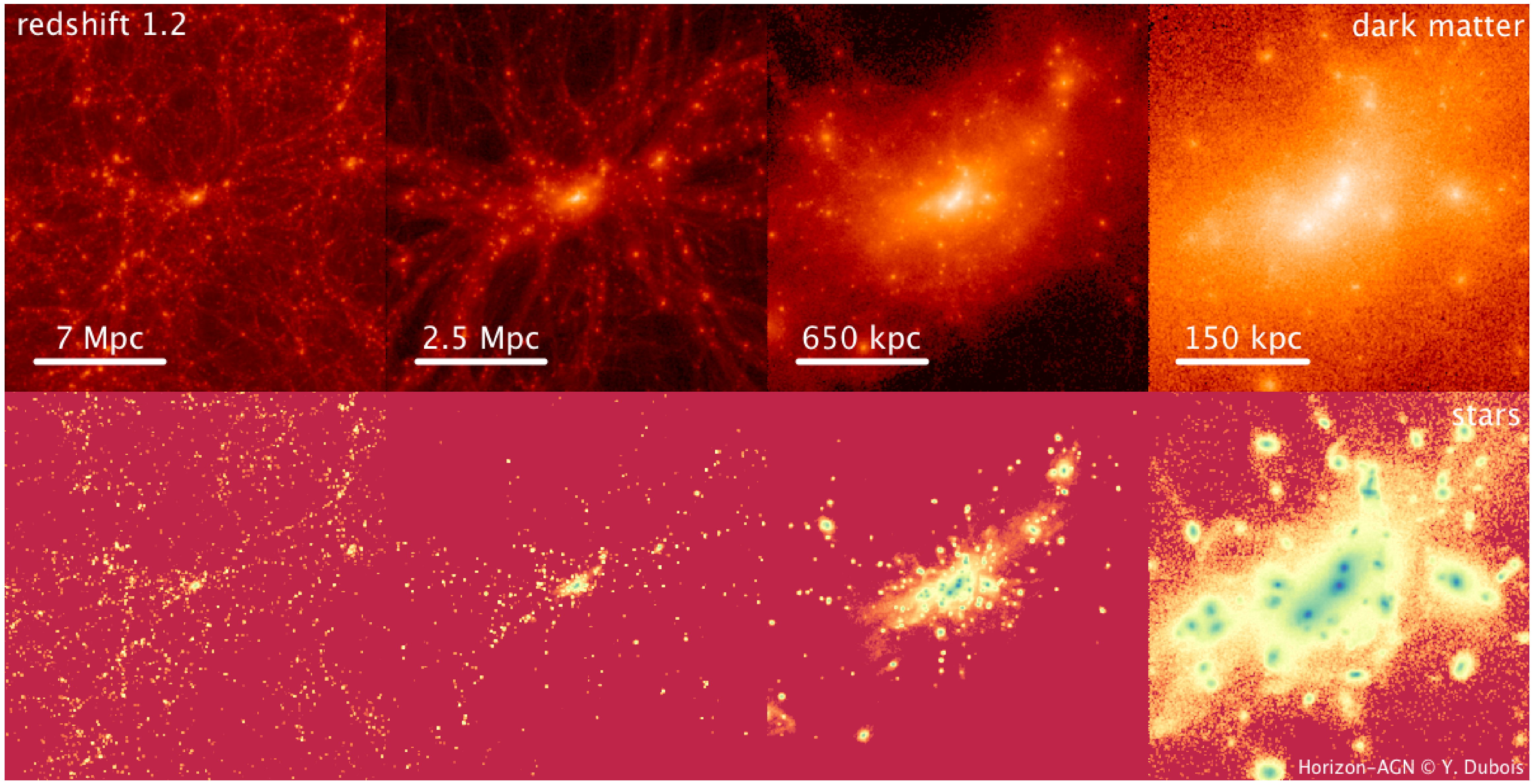
$$D_{\boldsymbol{m}}(\boldsymbol{J}) = \left\langle \left| \psi^{\mathrm{tot}}_{\boldsymbol{m}}(\omega) \right|^2 \right\rangle (\omega = \boldsymbol{m}\cdot\boldsymbol{\Omega}) = \frac{\left\langle \left| \psi^{\mathrm{ext}}_{\boldsymbol{m}}(\omega) \right|^2 \right\rangle}{\left| \varepsilon_{\boldsymbol{m}}(\boldsymbol{J},\omega) \right|^2} (\omega = \boldsymbol{m}\cdot\boldsymbol{\Omega})$$

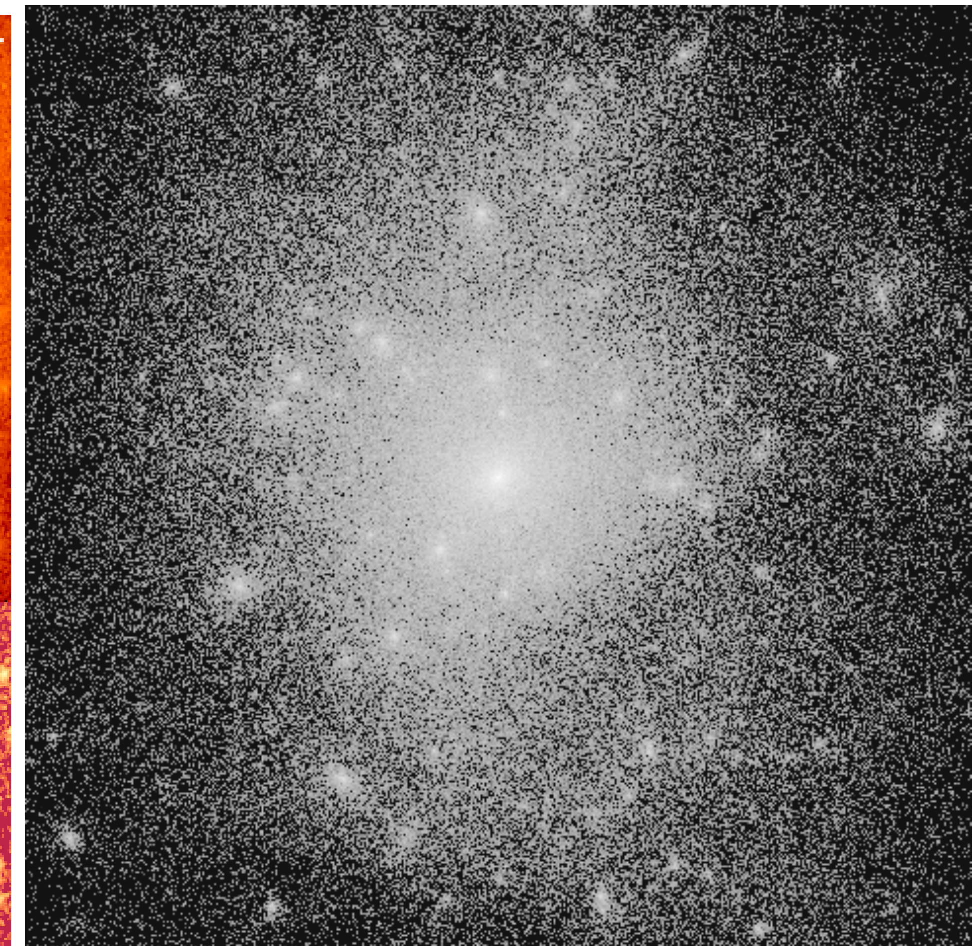
Nature

for streams, diffusion at fixed orbital parameters

Galaxies are perturbed

- **ΛCDM** paradigm $\Longrightarrow$ Live **cosmic environment**



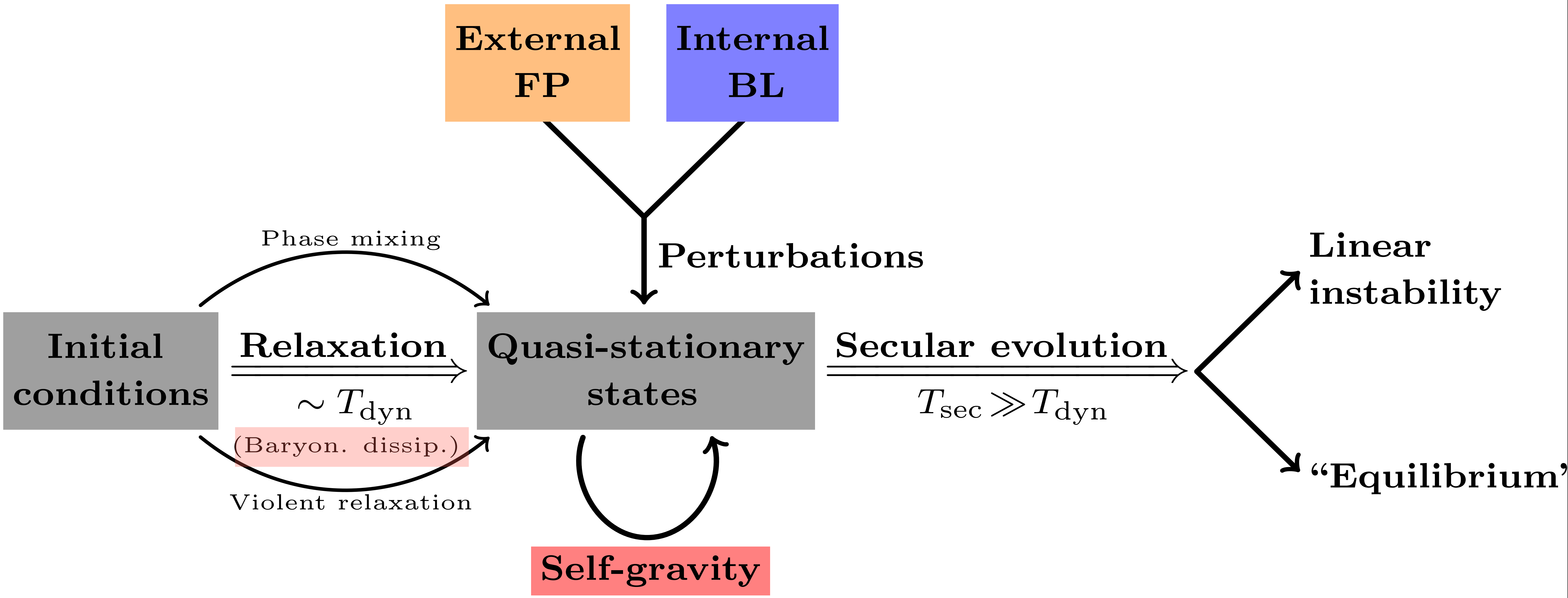


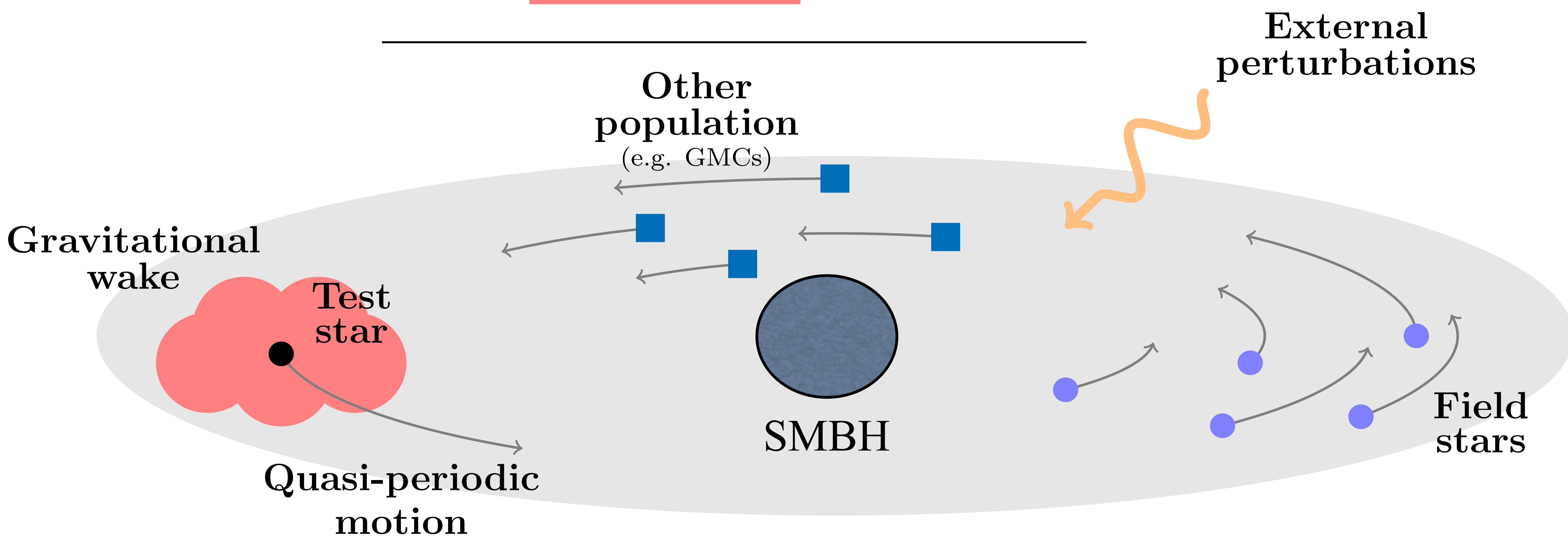
500 kpc

- Recent theoretical works to describe the effects of **fluctuations**:
  - **External perturbations** $\Longrightarrow$ **Dressed Fokker-Planck**
    (e.g., large scale structures, satellites)
  - **Internal perturbations** $\Longrightarrow$ **Balescu-Lenard**
    (e.g., graininess, GMCs)

Nature vs. **Nurture**?
Self-induced vs. **Externally**-induced?

The fate of self-gravitating systems

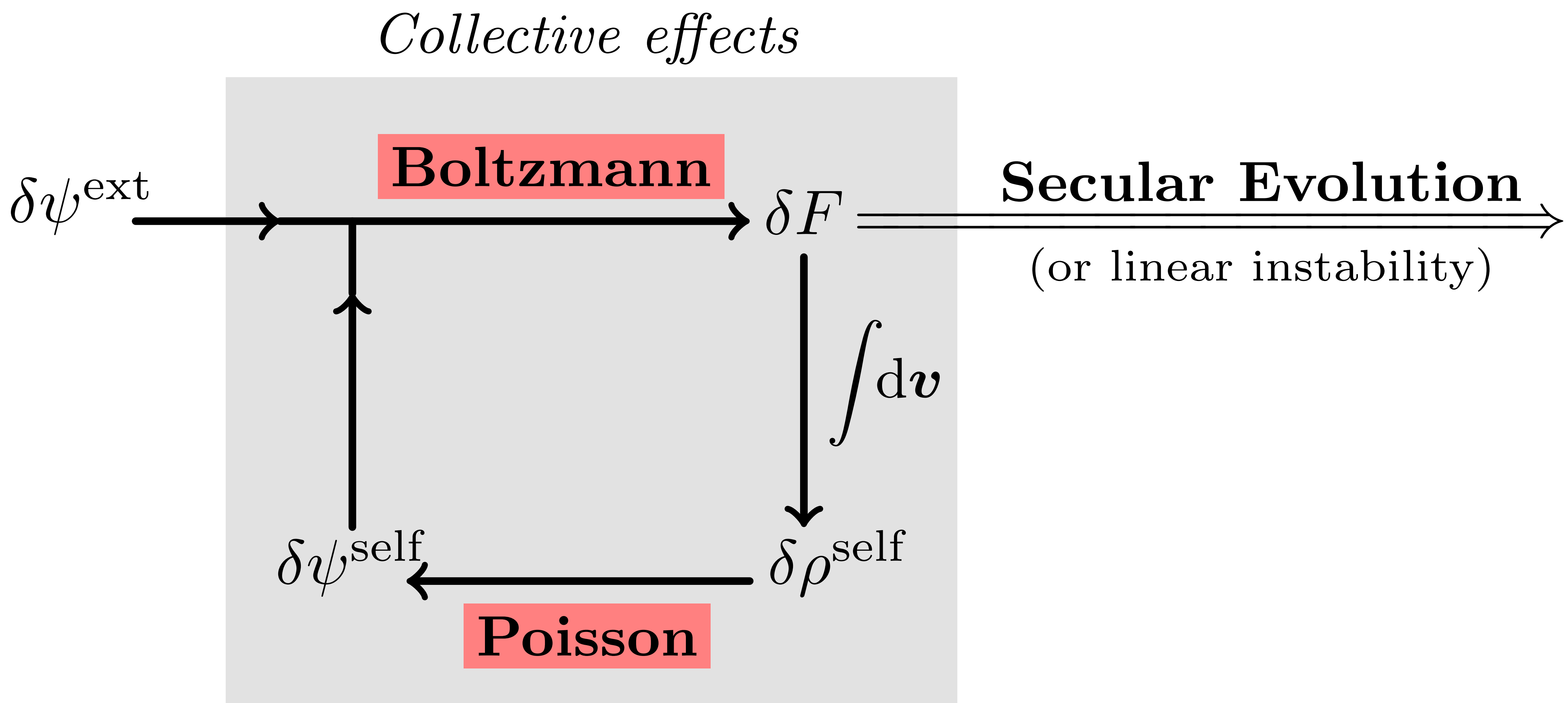




Small (dressed) resonant effect drive secular evolution (via orbital distorsion)

Galaxies are self-gravitating

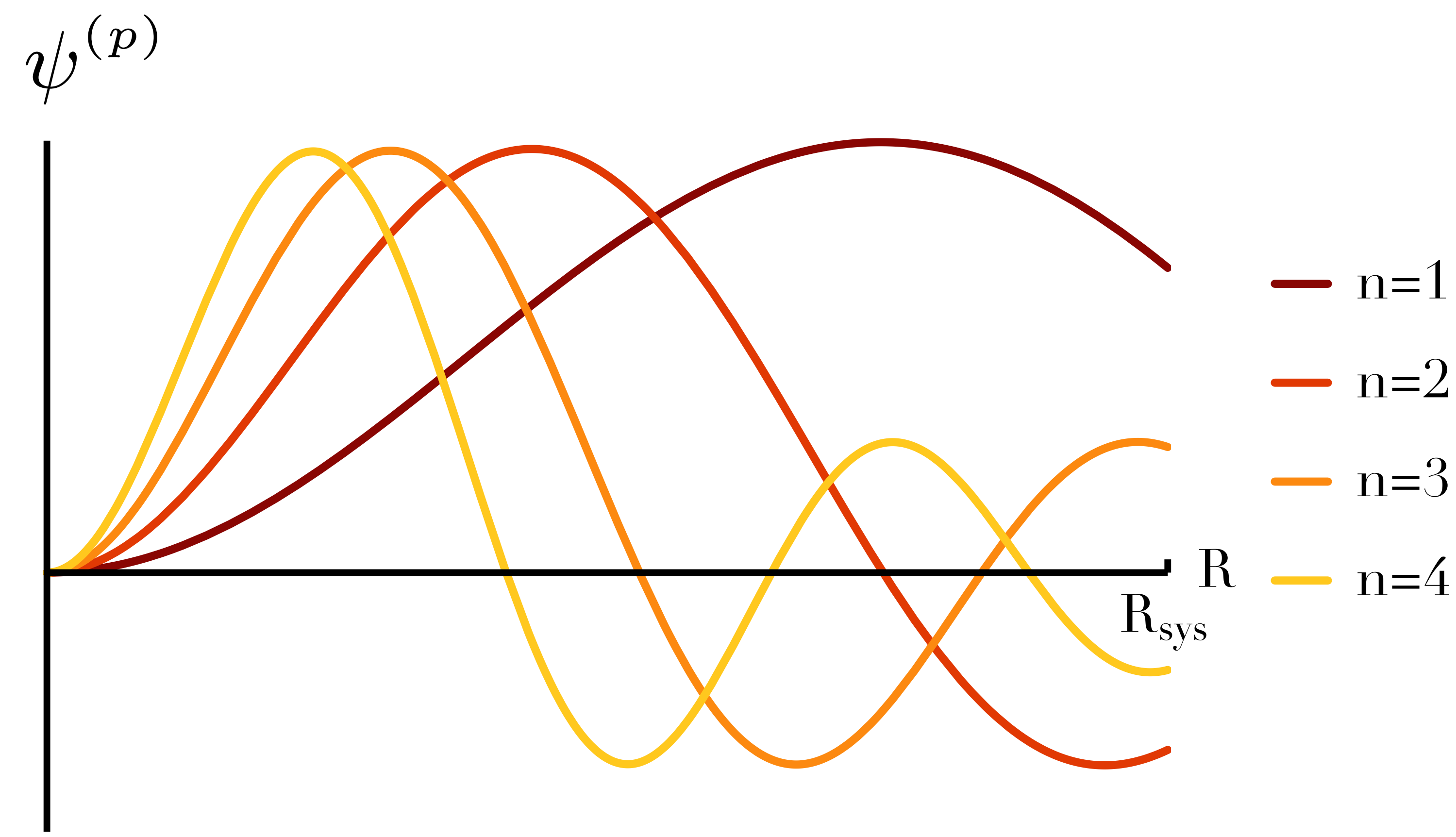
- **Self-gravitating amplification** (for linear response)



- **Matrix method** - (Kalnajs (1976)
  $\Longrightarrow$ **Representative basis** $(\psi^{(p)}, \rho^{(p)})$
  to solve Poisson once for all.

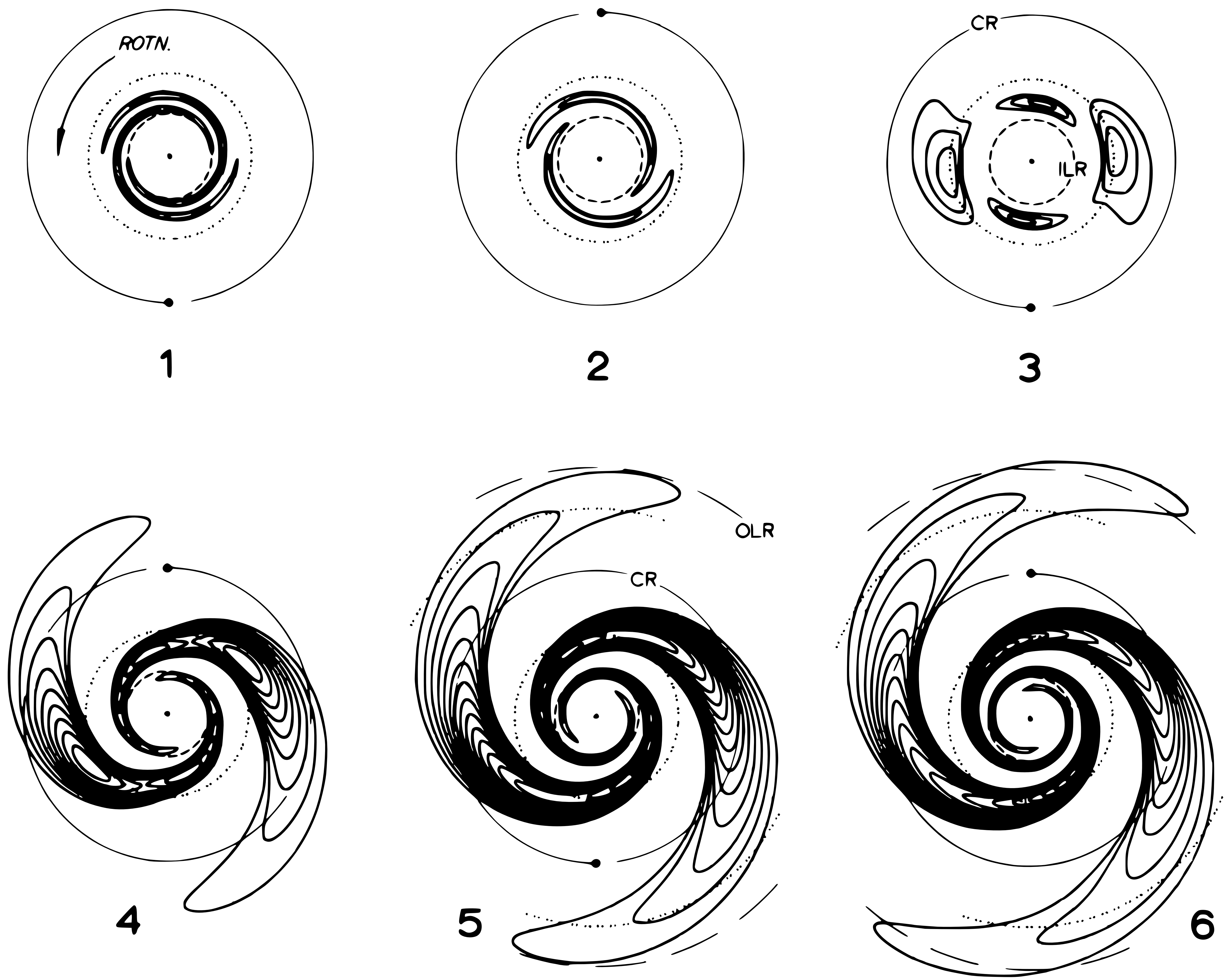
$$\begin{cases} \Delta\psi^{(p)} = 4\pi G \rho^{(p)} \,, \\ \int \mathrm{d}\boldsymbol{x}\, \psi^{(p)*}(\boldsymbol{x})\, \rho^{(q)}(\boldsymbol{x}) = -\delta_p^q \,. \end{cases}$$

also Weinberg (1989)



Galaxies are self-gravitating

- Discs strongly amplify perturbations, e.g. **swing amplification**



Toomre (1981)

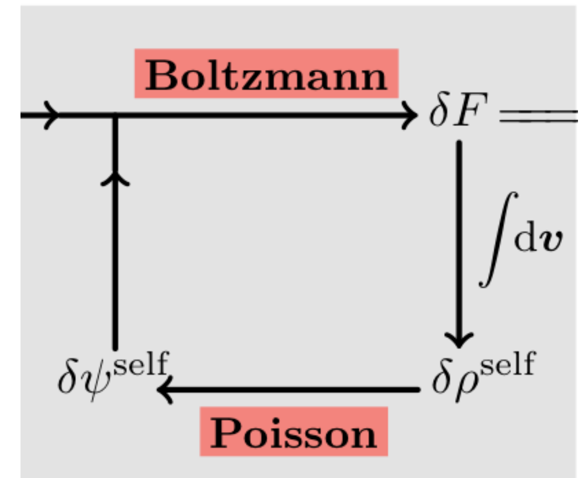
x100^2 amplification from gravitational wake for diffusion

Self-gravitating dressing

- Represent the **potential perturbations** on the basis

$$\begin{cases} \delta\psi^{\mathrm{ext}}(\boldsymbol{x},t) = \sum_p b_p(t)\,\psi^{(p)}(\boldsymbol{x}) & \text{Imposed } \textbf{external perturbation}. \\ \delta\psi^{\mathrm{self}}(\boldsymbol{x},t) = \sum_p a_p(t)\,\psi^{(p)}(\boldsymbol{x}) & \textbf{Amplified response} \text{ of the system.} \end{cases}$$

- Non-Markovian **amplification mechanism**



$$\boldsymbol{a}(t) = \int_{-\infty}^{t} \mathrm{d}\tau\, \mathbf{M}(t-\tau)\, \big[\boldsymbol{a}(\tau) + \boldsymbol{b}(\tau)\big]\,.$$

- **Dressing** of the perturbations

gravitational susceptibility

$$\boxed{\underbrace{\big[\widehat{\boldsymbol{a}} + \widehat{\boldsymbol{b}}\big](\omega)}_{\substack{\textbf{Total} \\ \text{perturbations}}} = \underbrace{\big[\mathbf{I} - \widehat{\mathbf{M}}(\omega)\big]^{-1}}_{\textbf{Dressing}} \cdot \underbrace{\widehat{\boldsymbol{b}}(\omega)}_{\substack{\textbf{External} \\ \text{perturbations}}}}$$

- System's **response matrix** (Kalnajs (1976))

$$\boxed{\widehat{\mathbf{M}}_{pq}(\omega) = (2\pi)^d \sum_{\boldsymbol{m}\in\mathbb{Z}^d} \int \mathrm{d}\boldsymbol{J}\, \frac{\boldsymbol{m}\cdot\partial F/\partial\boldsymbol{J}}{\omega - \boldsymbol{m}\cdot\boldsymbol{\Omega}}\, \psi_{\boldsymbol{m}}^{(p)*}(\boldsymbol{J})\, \psi_{\boldsymbol{m}}^{(q)}(\boldsymbol{J})\,.}$$

$\Longrightarrow$ Resonances at the **intrinsic frequencies:** $\omega = \boldsymbol{m}\cdot\boldsymbol{\Omega}\,.$

***Secularly**, gravitational susceptibility is **squared**!*

The dressed Fokker-Planck equation

- Describe the secular evolution driven by **external perturbations** for a system
  - **inhomogeneous**
  - **stable**
  - **self-gravitating**
  - **collisionless**
  - **perturbed**

- Some references: Kuzmin 1957
  - *Binney,Lacey (1980)*: No dressing
  - *Weinberg (2001)*: Spherical case
  - *Pichon, Aubert (2006)*: Environment effects
  - *Fouvry, Pichon, Prunet (2015)*: $2D$ WKB limit
  - *Fouvry, Pichon, Chavanis, Monk (2016)*: $3D$ WKB limit

See also Kuzmin 1957, B+T 2008, and references in Heyvearts 2017

Dressed Fokker-Planck equation

- **Dressed Fokker-Planck equation**

$$\boxed{\frac{\partial F(\boldsymbol{J},t)}{\partial t} = \frac{\partial}{\partial \boldsymbol{J}} \cdot \left[\sum_{\boldsymbol{m}} \boldsymbol{m} \; D_{\boldsymbol{m}}(\boldsymbol{J}) \; \boldsymbol{m} \cdot \frac{\partial F}{\partial \boldsymbol{J}} \right] .}$$

- **Dressed diffusion coefficients**

$$\boxed{D_{\boldsymbol{m}}(\boldsymbol{J}) = \frac{1}{2} \sum_{p,q} \psi^{(p)}_{\boldsymbol{m}}(\boldsymbol{J}) \, \psi^{(q)*}_{\boldsymbol{m}}(\boldsymbol{J}) \left[\left[\mathbf{I} - \mathbf{M}\right]^{-1} \cdot \widehat{\mathbf{C}} \cdot \left[\mathbf{I} - \widehat{\mathbf{M}}\right]^{-1} \right]_{pq} (\omega = \boldsymbol{m} \cdot \boldsymbol{\Omega}) \, .}$$

- Some properties:
  - $F(\boldsymbol{J},t)$: **Orbital distorsion** in action space.
  - $\partial / \partial \boldsymbol{J}_1 \cdot$: **Divergence of a flux**, i.e. conservation.
  - $\boldsymbol{m}_1$: **Discrete** Fourier vectors - **Anistropic** diffusion.
  - $D_{\boldsymbol{m}}(\boldsymbol{J})$: **Anisotropic diffusion coefficients.**
  - $\left[\mathbf{I} - \widehat{\mathbf{M}}\right]^{-1}$: **Self-gravitating dressing.**
  - $\widehat{\mathbf{C}}$: Power spectrum of **external perturbations.**
  - $\boldsymbol{m}_1 \cdot \boldsymbol{\Omega}_1$: Fluctuations **at resonance.**

$\Longrightarrow$ **Master equation for externally-induced orbital distortion.**

Dressed Fokker-Planck equation

- **Dressed Fokker-Planck equation**

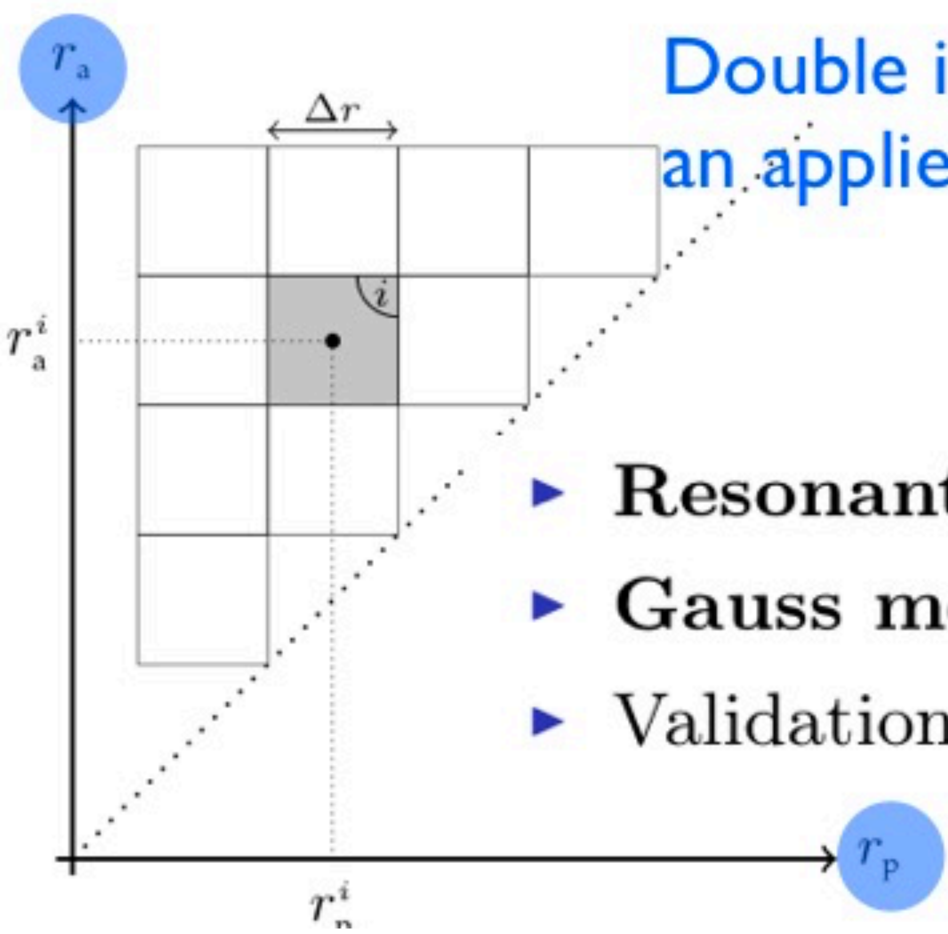
$$\boxed{\frac{\partial F(\boldsymbol{J},t)}{\partial t} = \frac{\partial}{\partial \boldsymbol{J}} \cdot \bigg[\sum_{\boldsymbol{m}} \boldsymbol{m} \, D_{\boldsymbol{m}}(\boldsymbol{J}) \; \boldsymbol{m} \cdot \frac{\partial F}{\partial \boldsymbol{J}} \bigg] .}$$

- **Dressed diffusion coefficients**

$$\boxed{D_{\boldsymbol{m}}(\boldsymbol{J}) = \frac{1}{2} \sum_{p,q} \psi_{\boldsymbol{m}}^{(p)}(\boldsymbol{J}) \, \psi_{\boldsymbol{m}}^{(q)*}(\boldsymbol{J}) \Big[\big[\mathbf{I} - \mathbf{M}\big]^{-1} \cdot \widehat{\mathbf{C}} \cdot \big[\mathbf{I} - \widehat{\mathbf{M}}\big]^{-1} \Big]_{pq} (\omega = \boldsymbol{m} \cdot \boldsymbol{\Omega}) \, .}$$

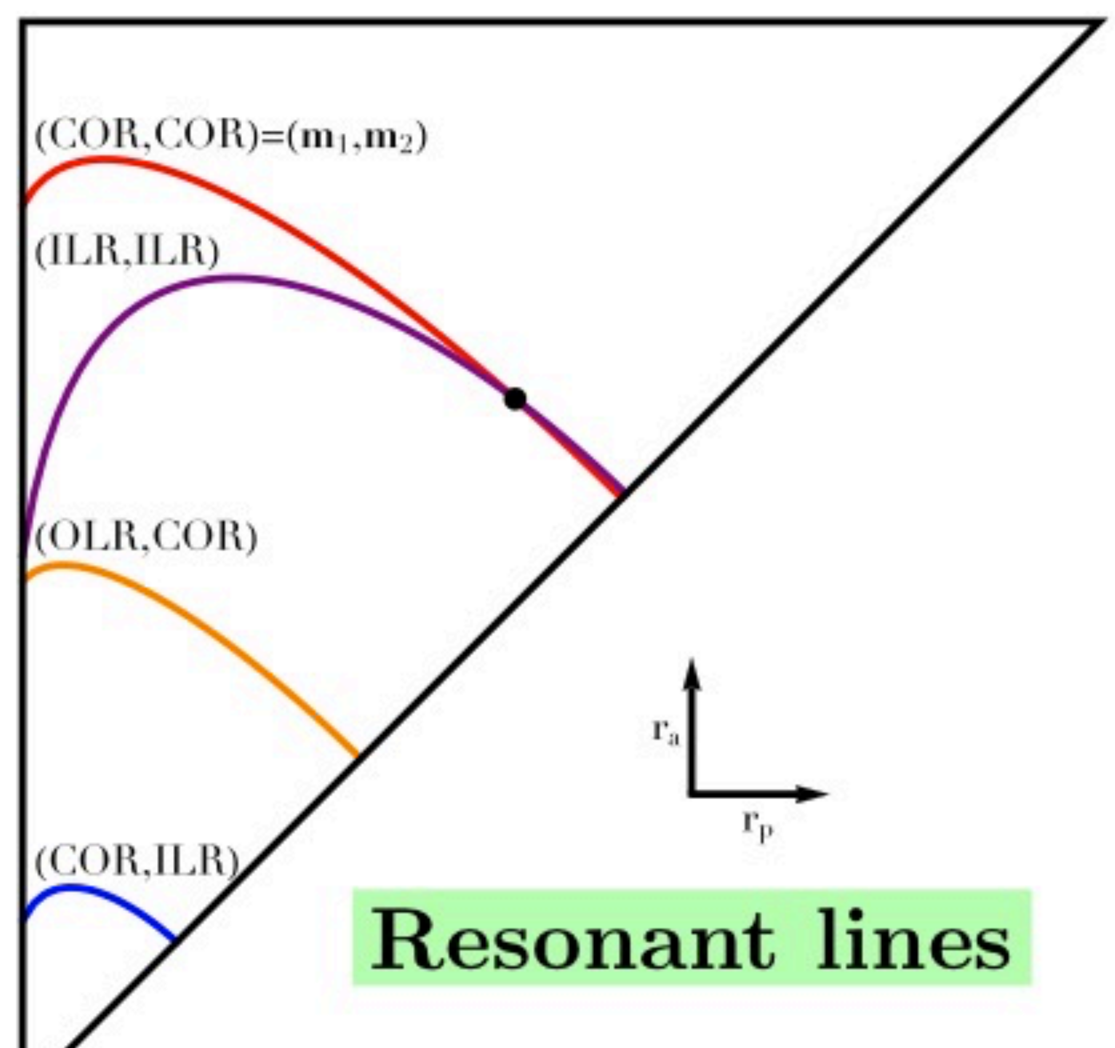
- Some properties:

$$\widehat{\mathbf{M}}_{pq}(\omega) \sim \sum_{\boldsymbol{m}} \int \mathrm{d}\boldsymbol{J} \, \frac{\boldsymbol{m} \cdot \partial F / \partial \boldsymbol{J}}{\omega - \boldsymbol{m} \cdot \boldsymbol{\Omega}} \, \psi_{\boldsymbol{m}}^{(p)*}(\boldsymbol{J}) \, \psi_{\boldsymbol{m}}^{(q)}(\boldsymbol{J}) \, .$$



Double integral over phase space:
an applied-math numerical **challenge**

- **Resonant poles** and **action integrals**.
- **Gauss method** with (r_p, r_a) as *actions*.
- Validation with **unstable modes**.



Fouvry et al. (2015)

The inhomogeneous Balescu-Lenard equation

- Describe the secular evolution driven by **finite$-N$ effects** for a system
  - **inhomogeneous**
  - **stable**
  - **self-gravitating**
  - **isolated**
  - **discrete**

- Some references:
  - *Balescu (1960), Lenard (1960)*: Plasma case
  - *Weinberg (1993)*: Homogeneous approximation
  - *Heyvaerts (2010)*: Angle-Action - BBGKY *see also Luciani Pellat (1987)*
  - *Chavanis (2012)*: Angle-Action - Klimontovitch
  - *Fouvry, Pichon, Chavanis (2015)*: $2D$ WKB limit
  - *Fouvry, Pichon, Magorrian, Chavanis (2015)*: $2D$ with full amplification
  - *Fouvry, Pichon, Chavanis, Monk (2016)*: $3D$ WKB limit
  - *Fouvry, Pichon, Chavanis, (2017):* Kepler solution

see also Polyachenko & Shukhman 1982; Luciani & Pellat 1987; Mynick 1988; Chavanis 2013

The inhomogeneous Balescu-Lenard equation

- Describe the secular evolution driven by **finite$-N$ effects** for a system
  - **inhomogeneous**
  - **stable**
  - **self-gravitating**
  - **isolated**
  - **discrete**

- Some references:

  - Changes in $\mathbf{J}$ causes $f(\mathbf{J})$ to change
  - So mean-field model evolves
  - Traditional theory computes rate of evolution by summing Kepler scatterings over pairs of stars
  - Recent work shows this is fundamentally mistaken

see also Polyachenko & Shukhman 1982; Luciani & Pellat 1987; Mynick 1988; Chavanis 2013

The idea behind **resonant relaxation** (in one cartoon).

Resonant encounters

- **Resonance condition** $\delta_{\mathrm{D}}(\boldsymbol{m}_1 \cdot \boldsymbol{\Omega}_1 - \boldsymbol{m}_2 \cdot \boldsymbol{\Omega}_2) \Longrightarrow$ **Distant encounters.**

Here ● and ● resonate
in some rotating frame

The two (*blue* and *red*) sets of orbits satisfy the resonance condition **m**1·**Ω1=m2·Ω2**, and therefore will interact consistently, driving a significant distortion of their shapes.

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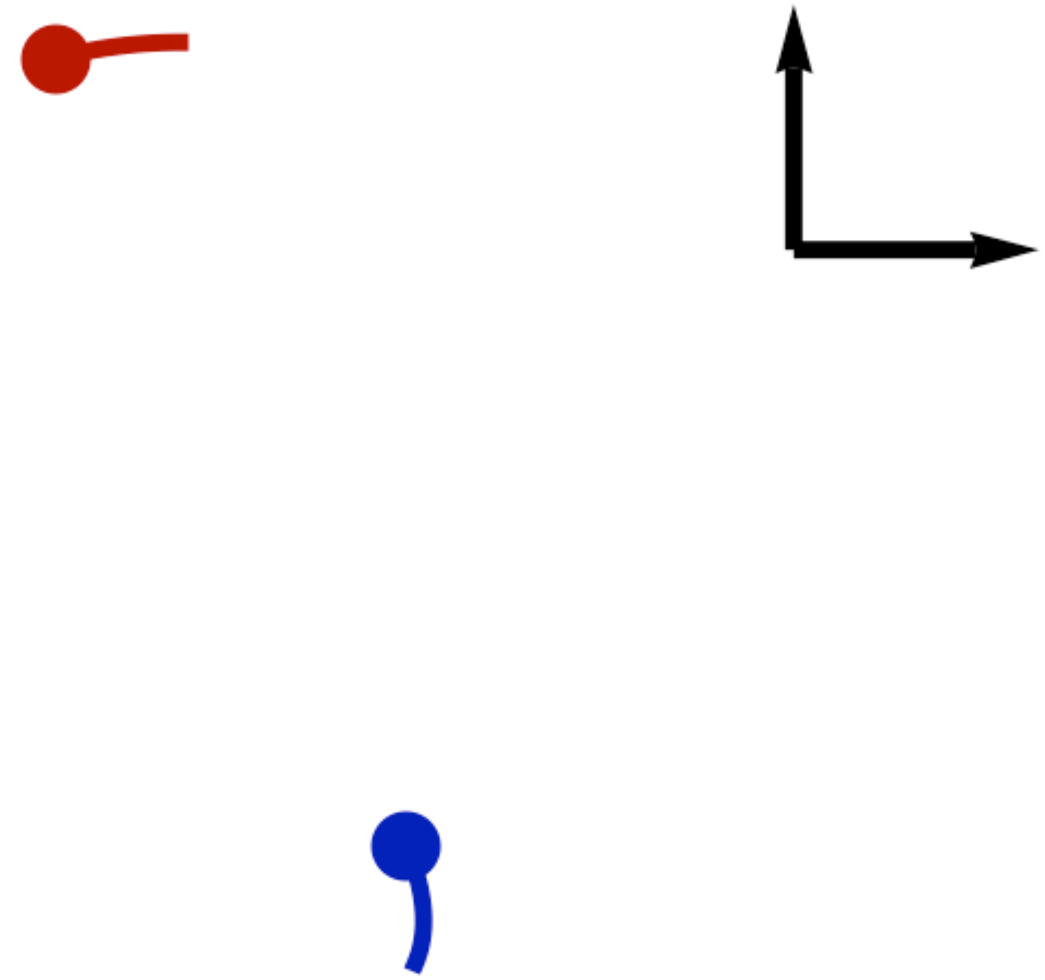

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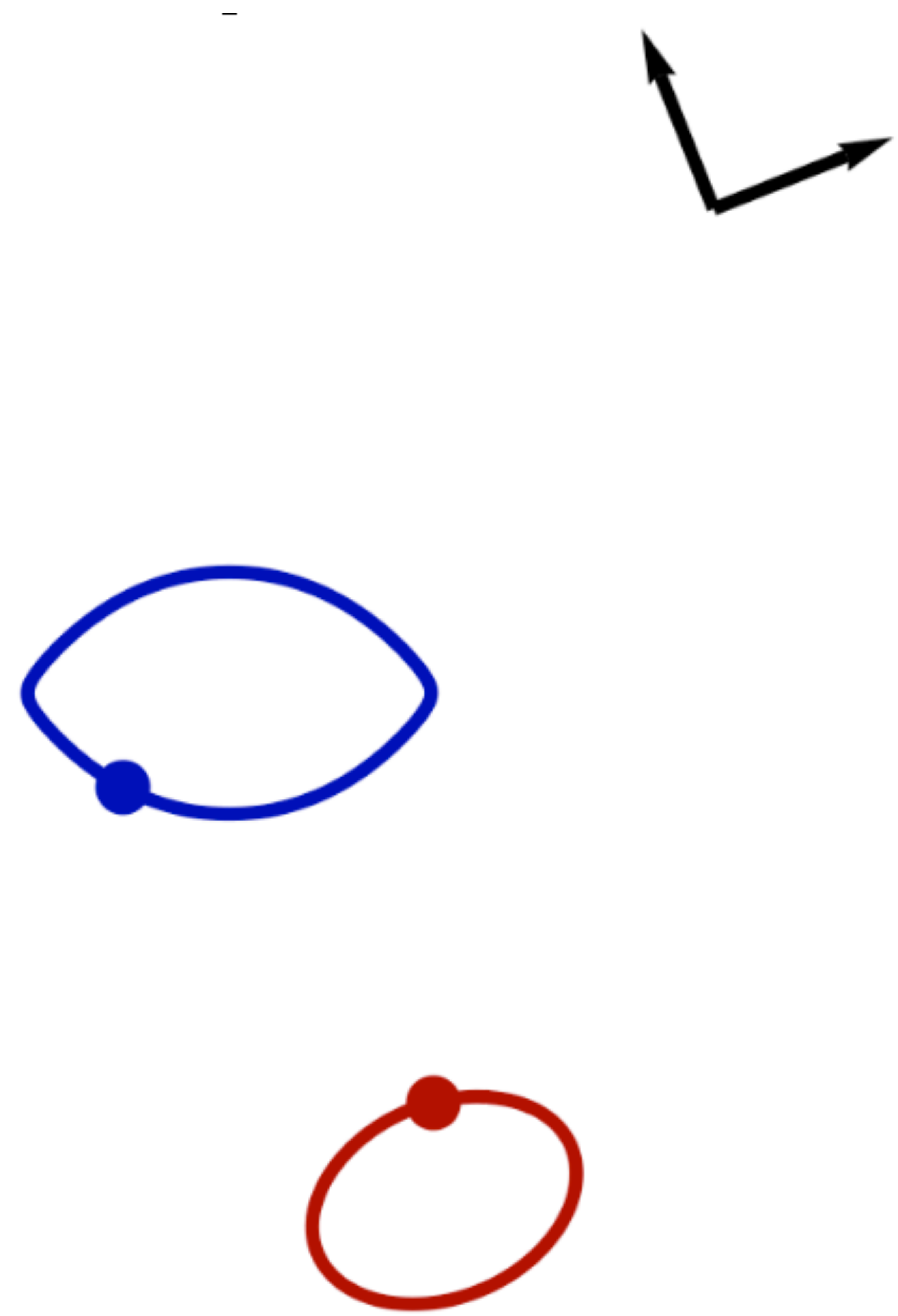

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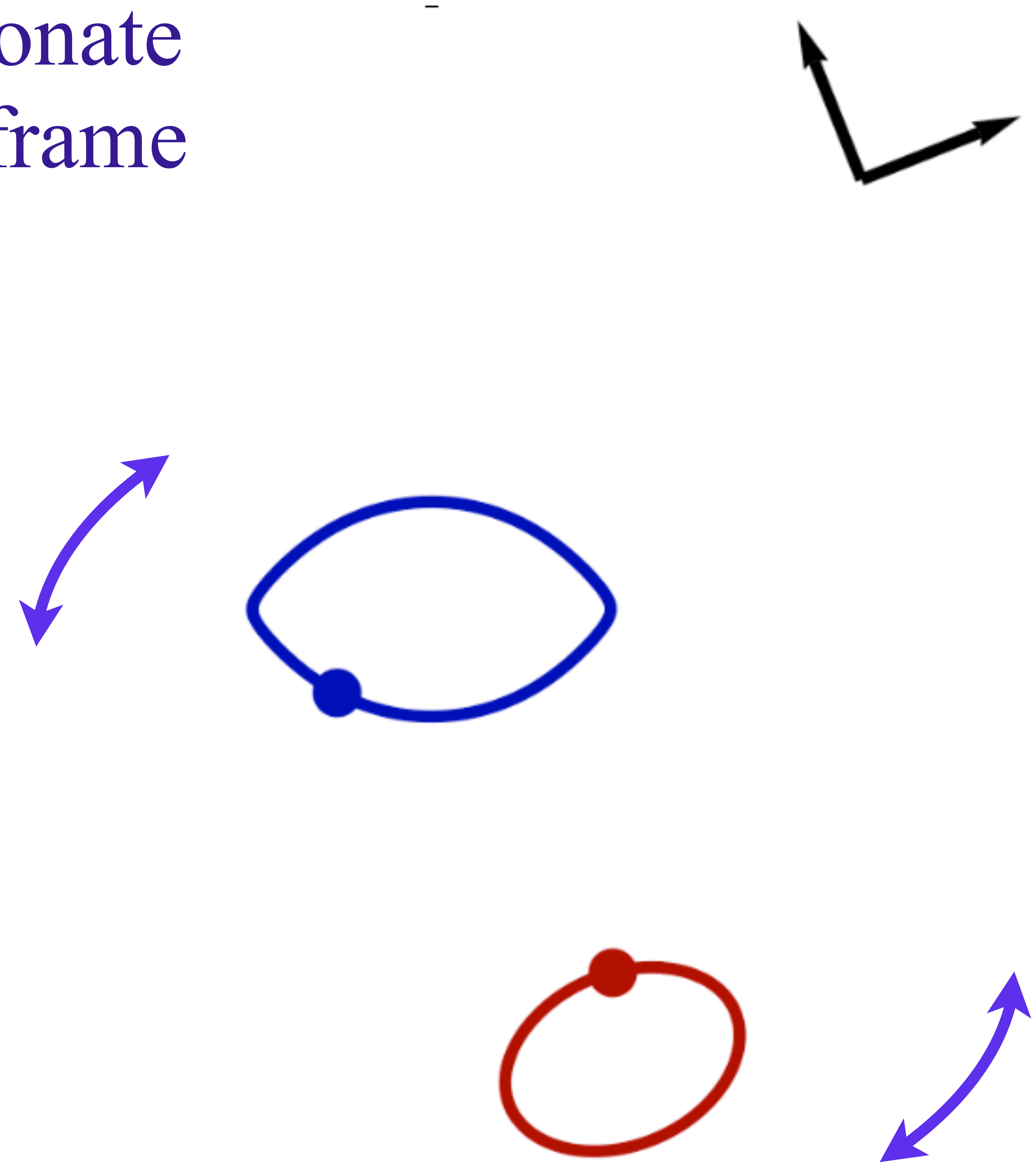

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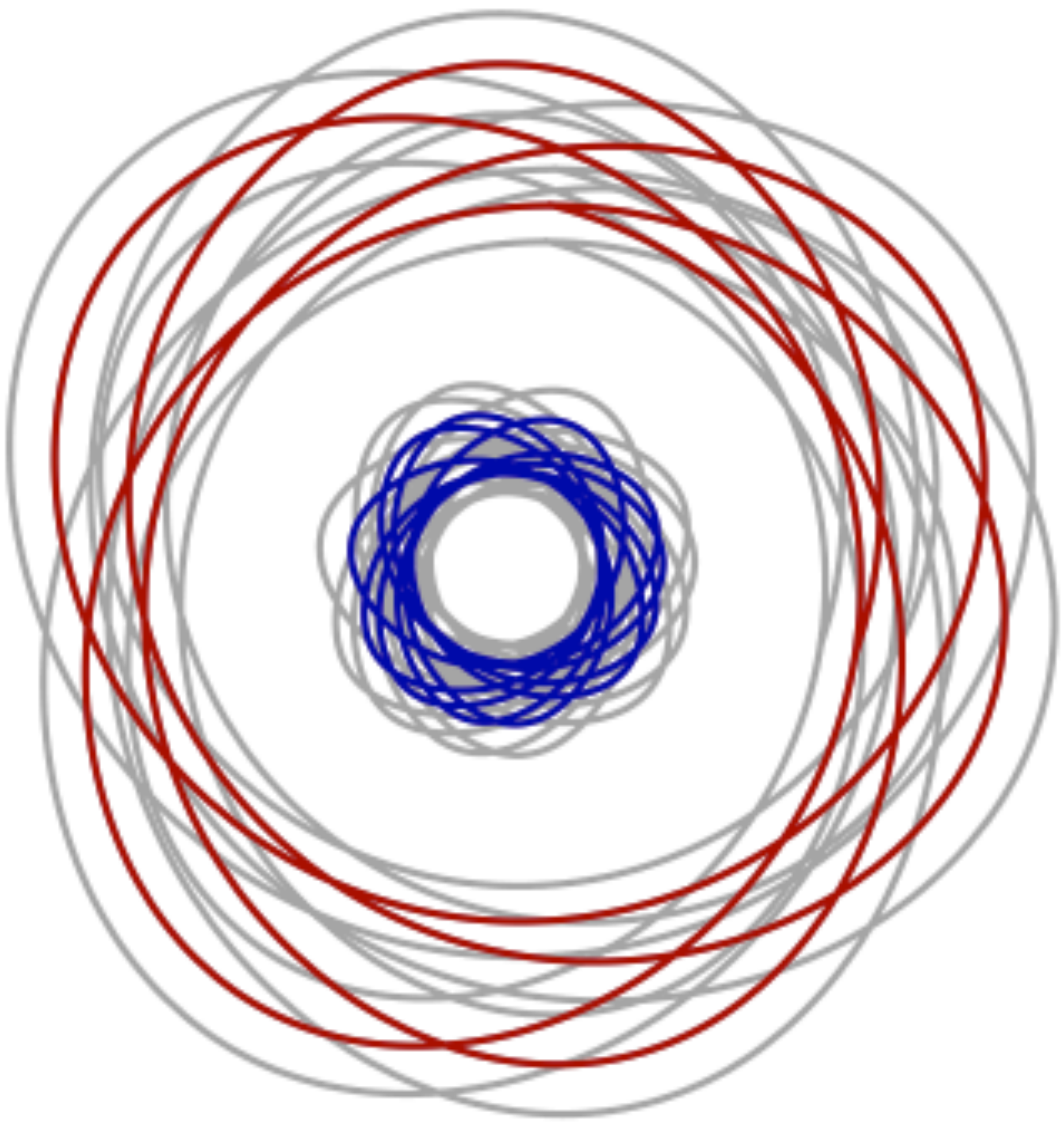
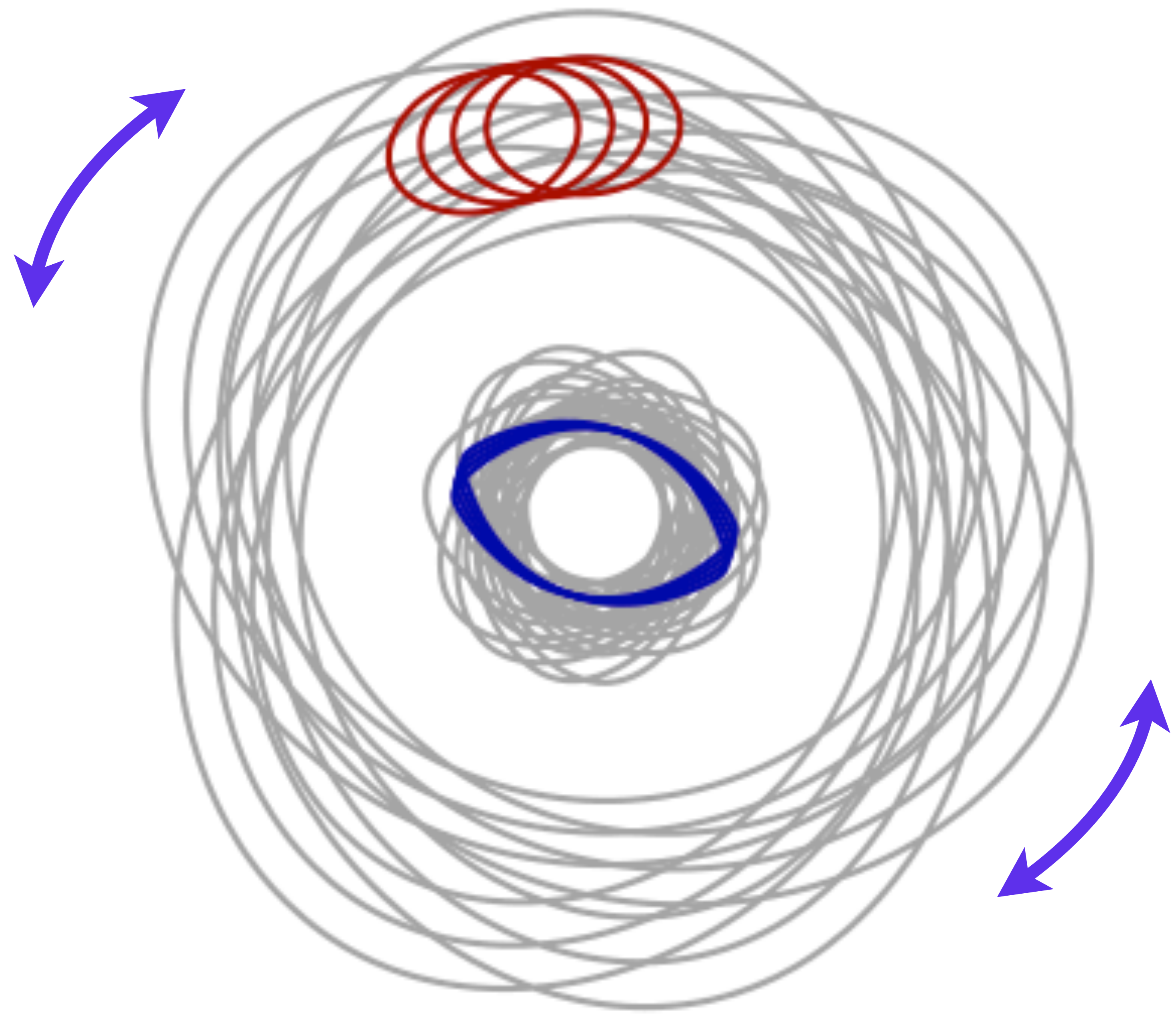
in some rotating frame

Through resonances
departure from axial symmetry





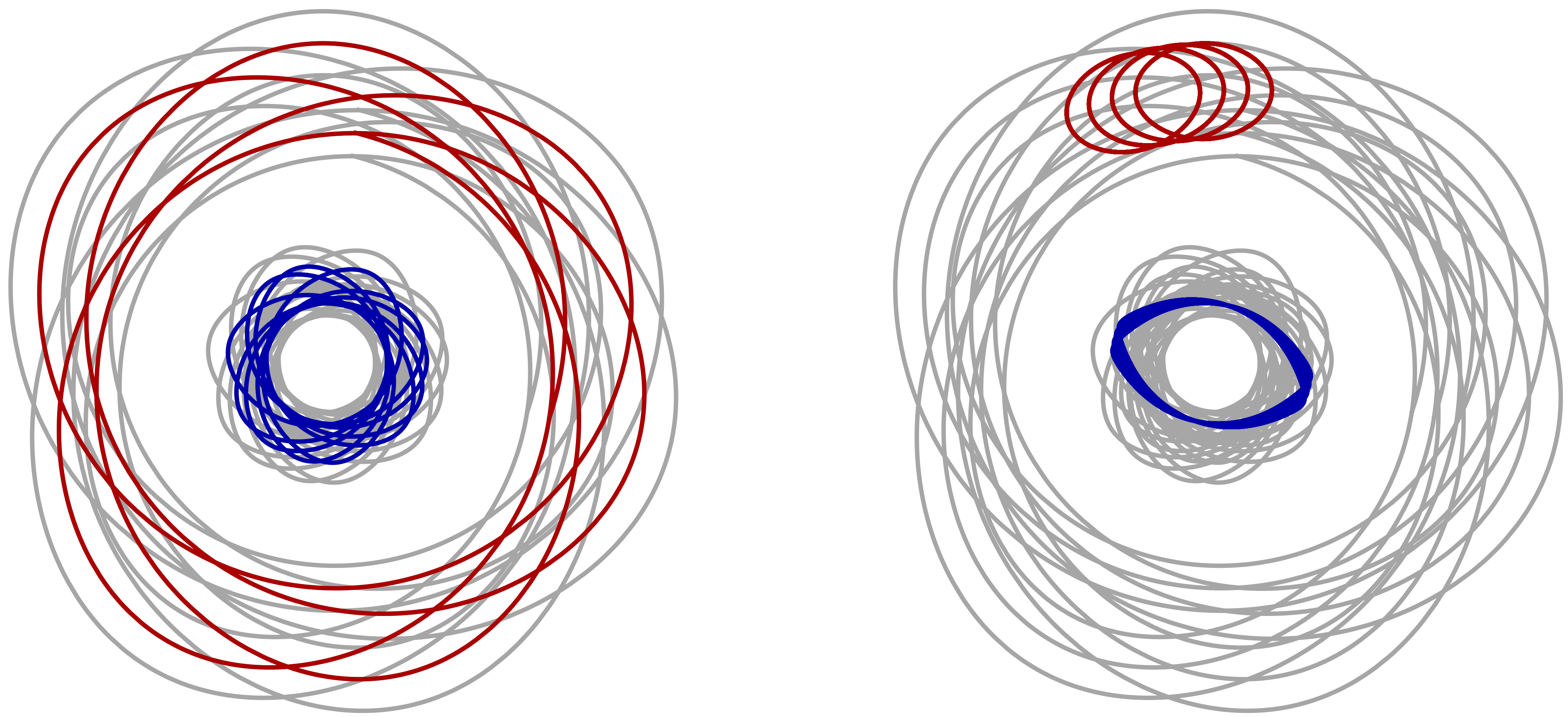
No Torque

resonance drives recurrence

Net Torque

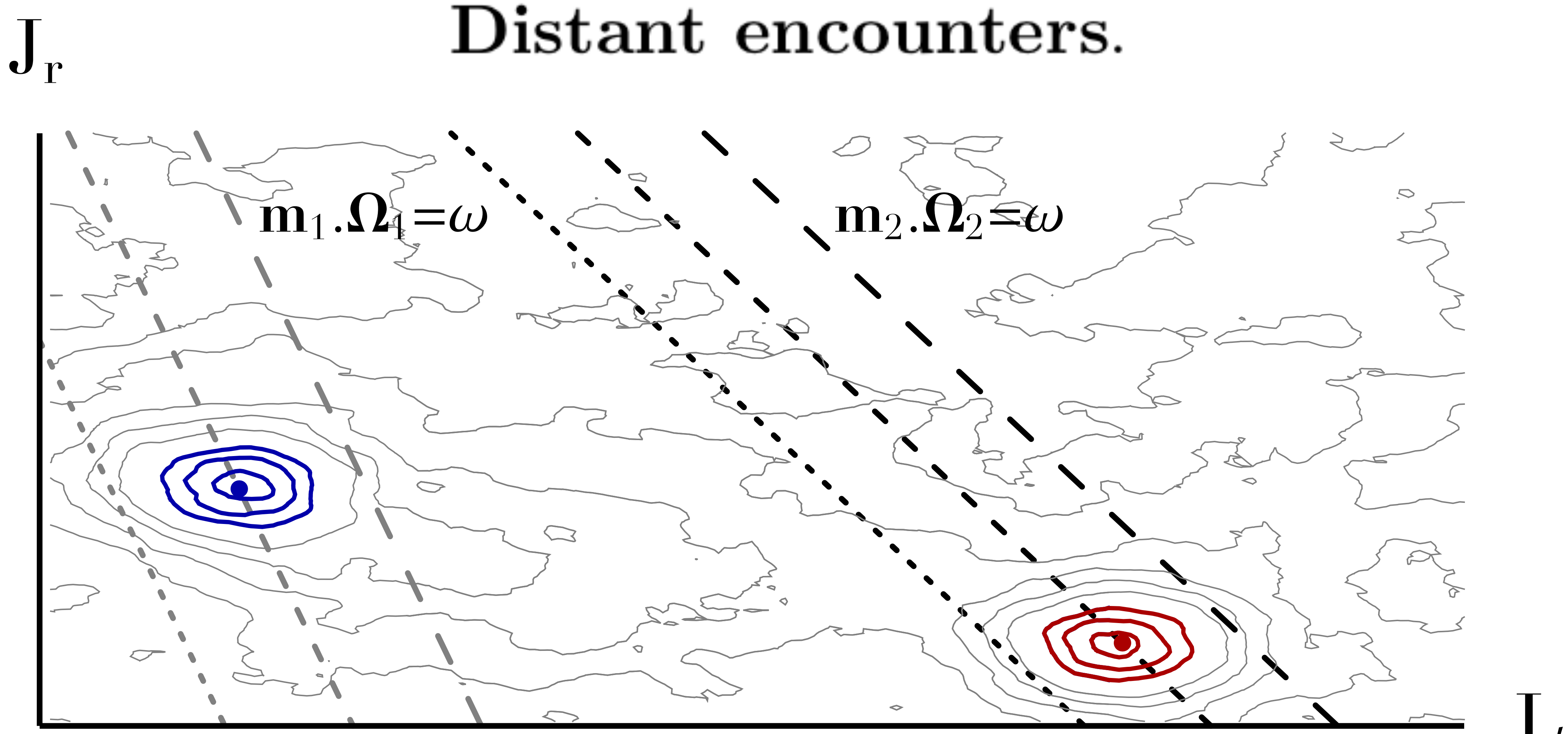
The idea behind secular evolution: shot noise **fluctuations resonate!**

- **Resonance condition:** $\delta_{\rm D}(\boldsymbol{m}_1\cdot\boldsymbol{\Omega}_1-\boldsymbol{m}_2\cdot\boldsymbol{\Omega}_2)$



Distant encounters.

Map of fluctuations in orbital space



Fouvry et al. (2015)

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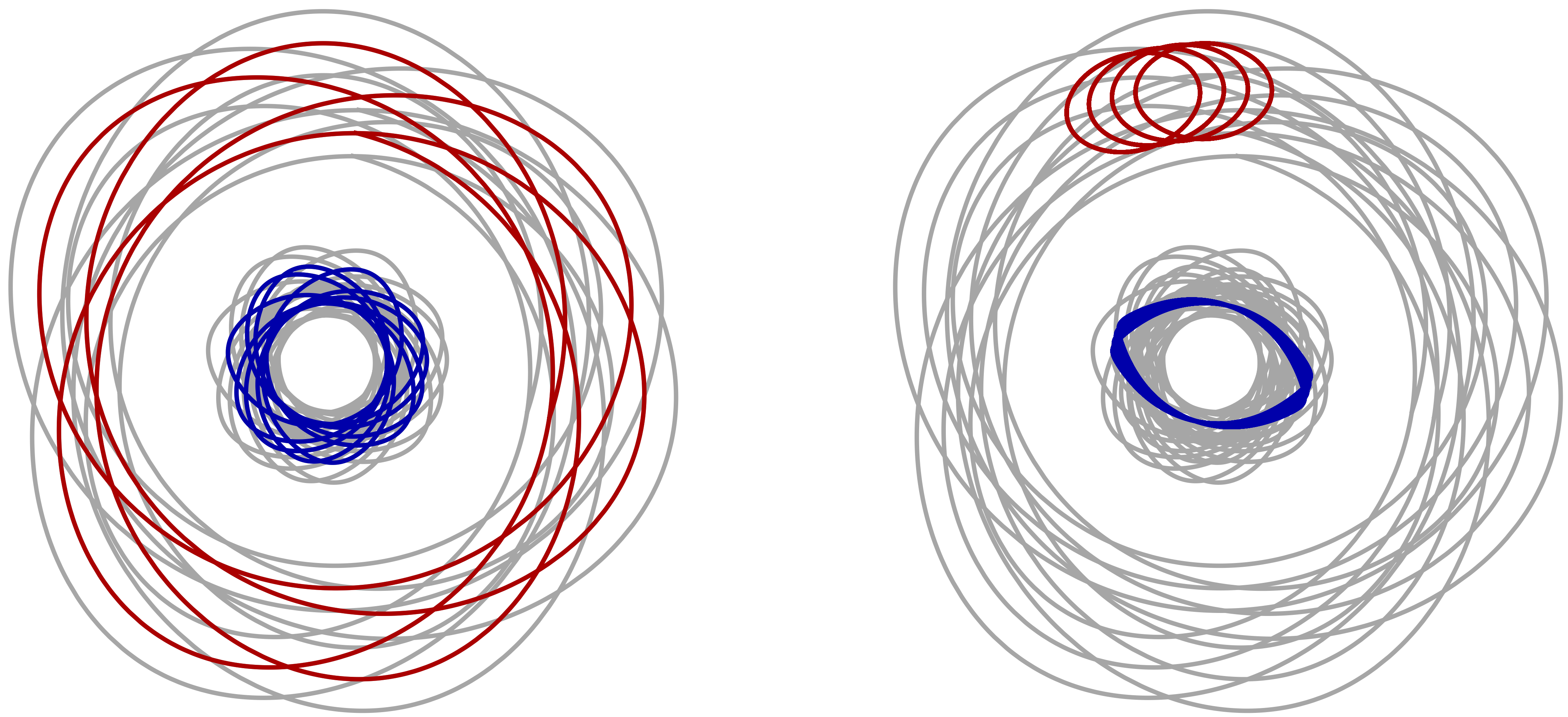

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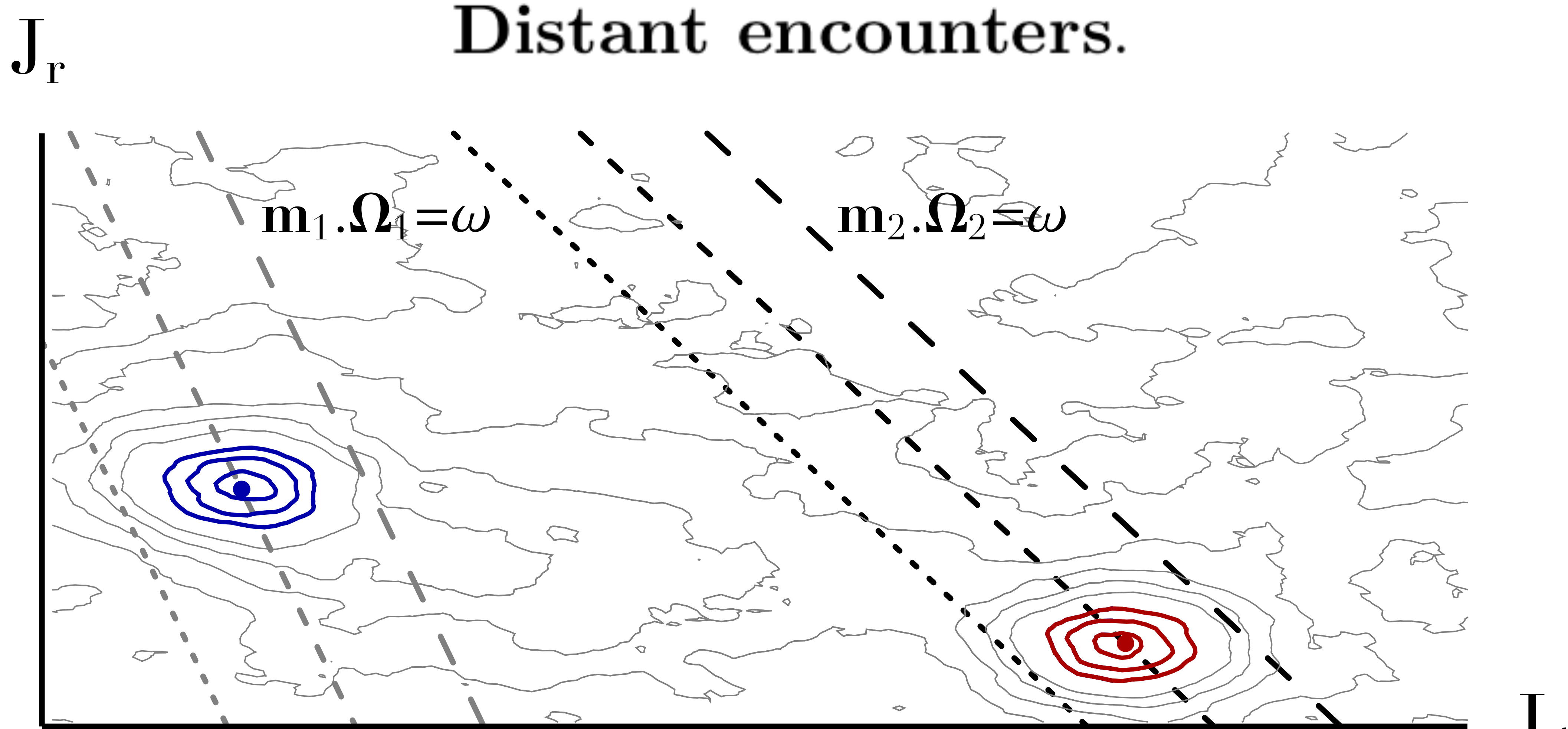

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Small recurrent (resonant) effects drive secular evolution (via orbital distorsion)

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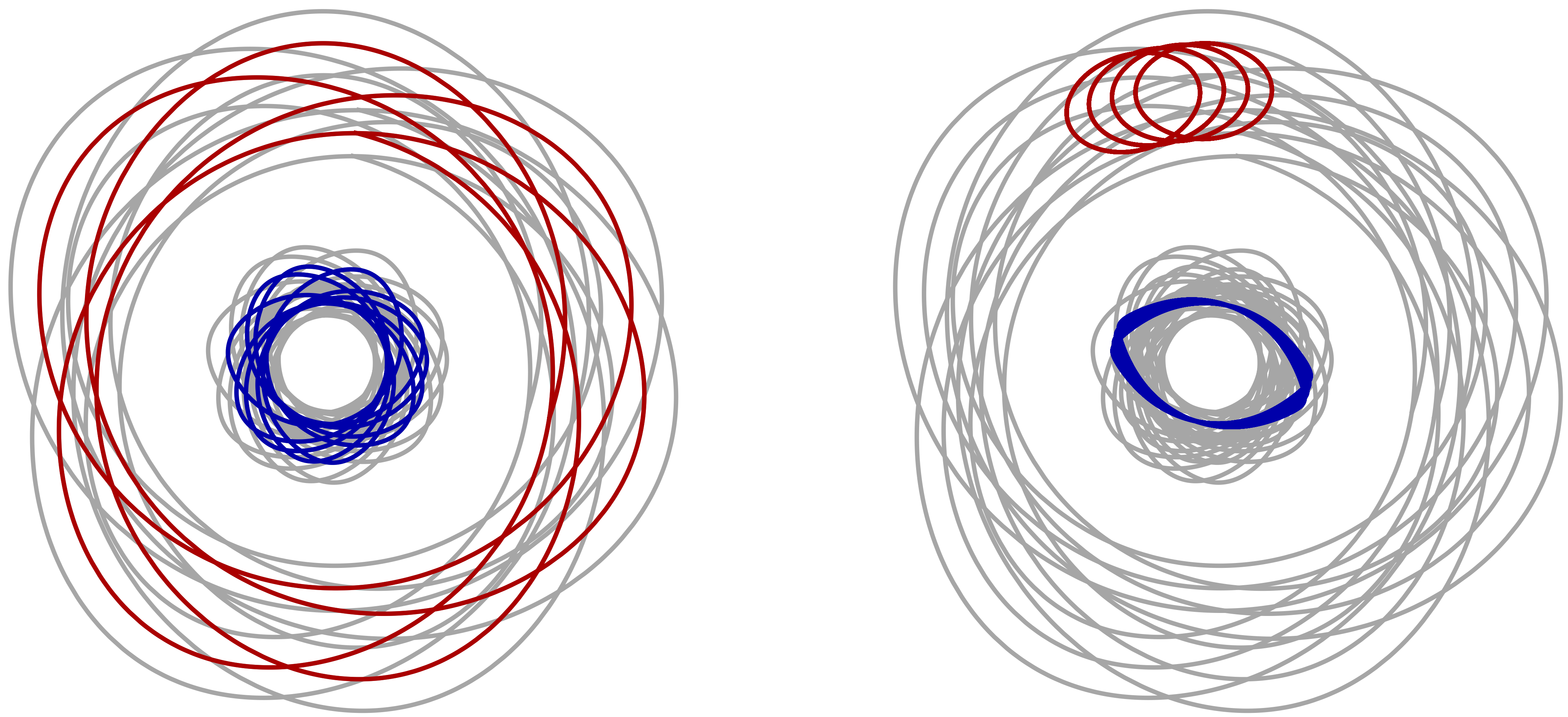

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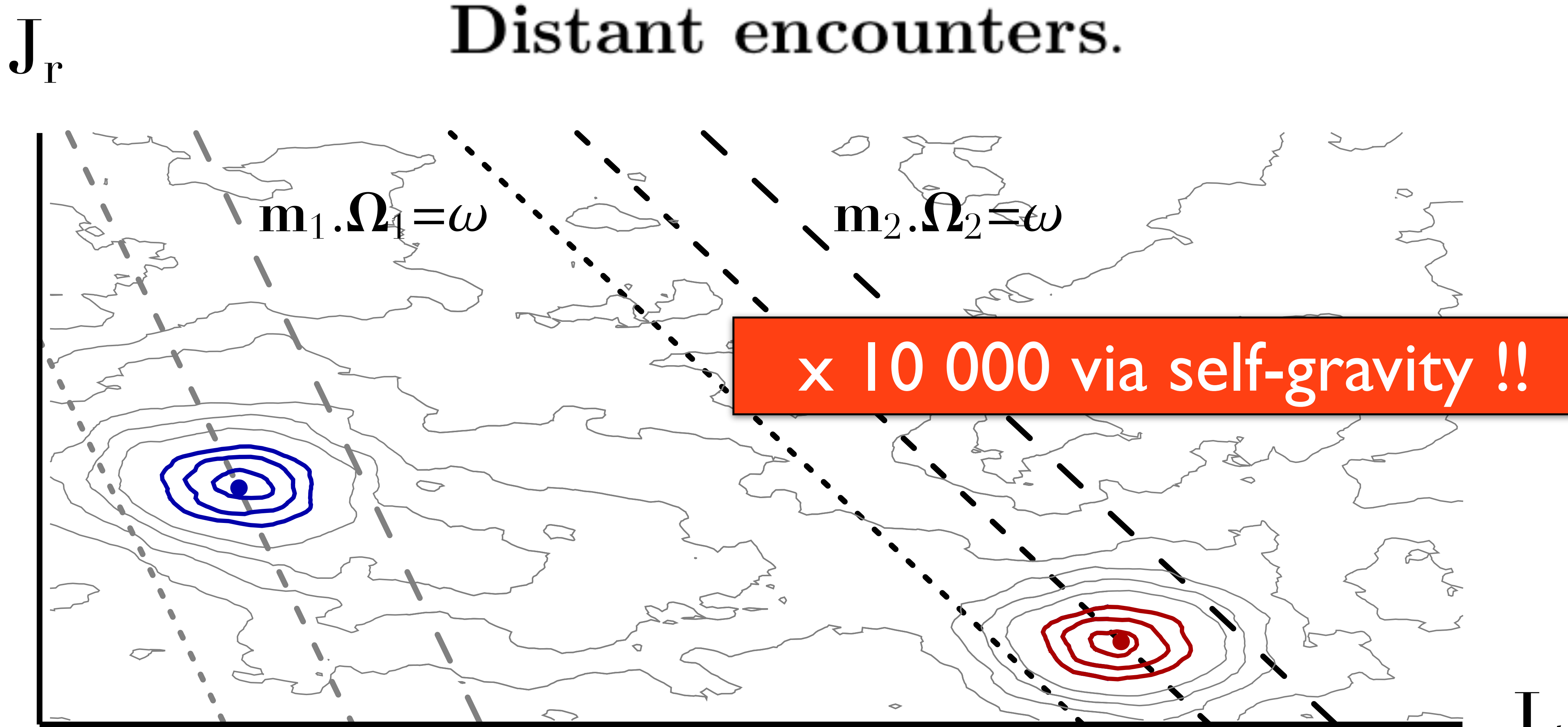

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Inhomogeneous Balescu-Lenard equation

- **Inhomogeneous Balescu-Lenard equation**
  Heyvaerts (2010), Chavanis (2012)

$$\boxed{\frac{\partial F(\boldsymbol{J}_1,t)}{\partial t} = \pi(2\pi)^d \frac{M_{\text{tot}}}{N} \frac{\partial}{\partial \boldsymbol{J}_1} \cdot \left[\sum_{\boldsymbol{m}_1,\boldsymbol{m}_2} \boldsymbol{m}_1 \int \mathrm{d}\boldsymbol{J}_2 \, \frac{\delta_{\mathrm{D}}(\boldsymbol{m}_1\cdot\boldsymbol{\Omega}_1 - \boldsymbol{m}_2\cdot\boldsymbol{\Omega}_2)}{|\mathcal{D}_{\boldsymbol{m}_1,\boldsymbol{m}_2}(\boldsymbol{J}_1,\boldsymbol{J}_2,\boldsymbol{m}_1\cdot\boldsymbol{\Omega}_1)|^2} \left[\boldsymbol{m}_1\cdot\frac{\partial}{\partial \boldsymbol{J}_1} - \boldsymbol{m}_2\cdot\frac{\partial}{\partial \boldsymbol{J}_2}\right] F(\boldsymbol{J}_1,t)\, F(\boldsymbol{J}_2,t) \right].}$$

- Some properties:
  - $F(\boldsymbol{J},t)$: **Orbital distorsion** in action space.
  - $1/N$: Driven by **finite**$-N$ **effects**.
  - $\partial/\partial\boldsymbol{J}_1\cdot$: **Divergence of a flux**, i.e. conservation.
  - $\boldsymbol{m}_1$: **Discrete** Fourier vectors - **Anistropic** diffusion.
  - δ_{D}: **Resonance condition** for distant encounters.
  - $1/\mathcal{D}_{\boldsymbol{m}_1,\boldsymbol{m}_2}$: **Self-gravitating dressing** (squared).
  - $\boldsymbol{m}_1\cdot\boldsymbol{\Omega}_1$: Secular diffusion **at resonance**.

$\Longrightarrow$ **Master equation for self-induced orbital distortion.**

Individual stochastic diffusion

- Self-consistent diffusion of the **system as a whole**

  $\Longrightarrow$ **Anisotropic Balescu-Lenard equation**

$$\boxed{\frac{\partial \overline{F}}{\partial \tau} = \frac{\partial}{\partial \boldsymbol{J}^{\mathrm{s}}} \cdot \left[\boldsymbol{A}(\boldsymbol{J}, \tau)\, \overline{F}(\boldsymbol{J}, \tau) + \boldsymbol{D}(\boldsymbol{J}, \tau) \cdot \frac{\partial \overline{F}}{\partial \boldsymbol{J}^{\mathrm{s}}} \right] .}$$

  $\boldsymbol{A}(\overline{F})$ drift vector, $\boldsymbol{D}(\overline{F})$ diffusion tensor.

- **Individual dynamics** of a wire at position $\boldsymbol{\mathcal{J}}(\tau)$

  $\Longrightarrow$ **Stochastic Langevin equation** - (Risken (1996))

$$\boxed{\frac{\mathrm{d}\boldsymbol{\mathcal{J}}}{\mathrm{d}\tau} = \boldsymbol{h}(\boldsymbol{\mathcal{J}}, \tau) + \boldsymbol{g}(\boldsymbol{\mathcal{J}}, \tau) \cdot \boldsymbol{\Gamma}(\tau) .} \qquad \mathbf{g}(\mathcal{J}, \tau) \propto \sqrt{\mathbf{D}}$$

  $\boldsymbol{h}$ and $\boldsymbol{g}$ vector and tensor, and $\boldsymbol{\Gamma}$ stochastic Langevin forces.

  $\Longrightarrow$ **Dual equation**, whose ensemble average gives back BL.

- In the Langevin's rewriting, **particles are dressed orbits.**

  $\Longrightarrow$ Huge gains in timesteps for integration.

Difficulties of the Balescu-Lenard equation

- Balescu-Lenard equation

$$\frac{\partial F(\boldsymbol{J}_1,t)}{\partial t} = \pi(2\pi)^d \frac{M_{\text{tot}}}{N} \frac{\partial}{\partial \boldsymbol{J}_1}\cdot\bigg[\sum_{\boldsymbol{m}_1,\boldsymbol{m}_2} \boldsymbol{m}_1 \int \mathrm{d}\boldsymbol{J}_2 \, \frac{\delta_{\mathrm{D}}(\boldsymbol{m}_1\cdot\boldsymbol{\Omega}_1 - \boldsymbol{m}_2\cdot\boldsymbol{\Omega}_2)}{|\mathcal{D}_{\boldsymbol{m}_1,\boldsymbol{m}_2}(\boldsymbol{J}_1,\boldsymbol{J}_2,\boldsymbol{m}_1\cdot\boldsymbol{\Omega}_1)|^2} \left[\boldsymbol{m}_1\cdot\frac{\partial}{\partial \boldsymbol{J}_1} - \boldsymbol{m}_2\cdot\frac{\partial}{\partial \boldsymbol{J}_2}\right] F(\boldsymbol{J}_1,t)\, F(\boldsymbol{J}_2,t)\bigg].$$

- **Dressed susceptibility coefficients**

$$\frac{1}{\mathcal{D}_{\boldsymbol{m}_1,\boldsymbol{m}_2}(\boldsymbol{J}_1,\boldsymbol{J}_2,\omega)} = \sum_{p,q} \psi^{(p)}_{\boldsymbol{m}_1}(\boldsymbol{J}_1) \left[\mathbf{I} - \widehat{\mathbf{M}}(\omega)\right]^{-1}_{pq} \psi^{(q)\,*}_{\boldsymbol{m}_2}(\boldsymbol{J}_2).$$

Difficulties:

- **Inhomogeneous system**
  - **Angle-action** $(\boldsymbol{x},\boldsymbol{v}) \mapsto (\boldsymbol{\theta},\boldsymbol{J})$.
- **Long-range system**
  - **Basis elements** $\psi^{(p)}$.
- **Self-gravitating system**
  - **Response matrix** $\widehat{\mathbf{M}}(\omega)$.
- **Resonant encounters**
  - **Resonance** $\delta_{\mathrm{D}}(\boldsymbol{m}_1\cdot\boldsymbol{\Omega}_1 - \boldsymbol{m}_2\cdot\boldsymbol{\Omega}_2)$.

Balescu-Lenard - Global approach

- Balescu-Lenard equation

$$\frac{\partial F(\boldsymbol{J}_1,t)}{\partial t} = \pi(2\pi)^d \frac{M_{\mathrm{tot}}}{N} \frac{\partial}{\partial \boldsymbol{J}_1} \cdot \bigg[\sum_{\boldsymbol{m}_1,\boldsymbol{m}_2} \boldsymbol{m}_1 \int \mathrm{d}\boldsymbol{J}_2 \, \frac{\delta_{\mathrm{D}}(\boldsymbol{m}_1\cdot\boldsymbol{\Omega}_1 - \boldsymbol{m}_2\cdot\boldsymbol{\Omega}_2)}{|\mathcal{D}_{\boldsymbol{m}_1,\boldsymbol{m}_2}(\boldsymbol{J}_1,\boldsymbol{J}_2,\boldsymbol{m}_1\cdot\boldsymbol{\Omega}_1)|^2} \bigg[\boldsymbol{m}_1 \cdot \frac{\partial}{\partial \boldsymbol{J}_1} - \boldsymbol{m}_2 \cdot \frac{\partial}{\partial \boldsymbol{J}_2} \bigg] F(\boldsymbol{J}_1,t) \, F(\boldsymbol{J}_2,t) \bigg] .$$

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Difficulties:

integrate twice over phase space

- **Inhomogeneous system**
  - **Angle-action** $(\boldsymbol{x},\boldsymbol{v}) \mapsto (\boldsymbol{\theta},\boldsymbol{J}) \Longrightarrow$ $2D$ discs are **explicitly integrable.**
- **Long-range system**
  - **Basis elements** $\psi^{(p)} \Longrightarrow$ **Global** basis elements.
- **Self-gravitating system**
  - **Response matrix** $\widehat{\mathbf{M}}(\omega) \Longrightarrow$ Numerical **linear theory**.
- **Resonant encounters**
  - **Resonance** $\delta_{\mathrm{D}}(\boldsymbol{m}_1\cdot\boldsymbol{\Omega}_1 - \boldsymbol{m}_2\cdot\boldsymbol{\Omega}_2) \Longrightarrow$ Integrate along **resonant lines.**

Balescu-Lenard - Global approach

- Balescu-Lenard equation

$$\frac{\partial\, F(\boldsymbol{J}_1,t)}{\partial t} = \pi(2\pi)^d \frac{M_{\mathrm{tot}}}{N} \frac{\partial}{\partial \boldsymbol{J}_1}\cdot\bigg[\sum_{\boldsymbol{m}_1,\boldsymbol{m}_2} \boldsymbol{m}_1 \!\int\! \mathrm{d}\boldsymbol{J}_2 \, \frac{\delta_{\mathrm{D}}(\boldsymbol{m}_1\cdot\boldsymbol{\Omega}_1 - \boldsymbol{m}_2\cdot\boldsymbol{\Omega}_2)}{|\mathcal{D}_{\boldsymbol{m}_1,\boldsymbol{m}_2}(\boldsymbol{J}_1,\boldsymbol{J}_2,\boldsymbol{m}_1\cdot\boldsymbol{\Omega}_1)|^2} \bigg[\boldsymbol{m}_1\cdot\frac{\partial}{\partial \boldsymbol{J}_1} - \boldsymbol{m}_2\cdot\frac{\partial}{\partial \boldsymbol{J}_2}\bigg] F(\boldsymbol{J}_1,t)\, F(\boldsymbol{J}_2,t)\bigg] .$$

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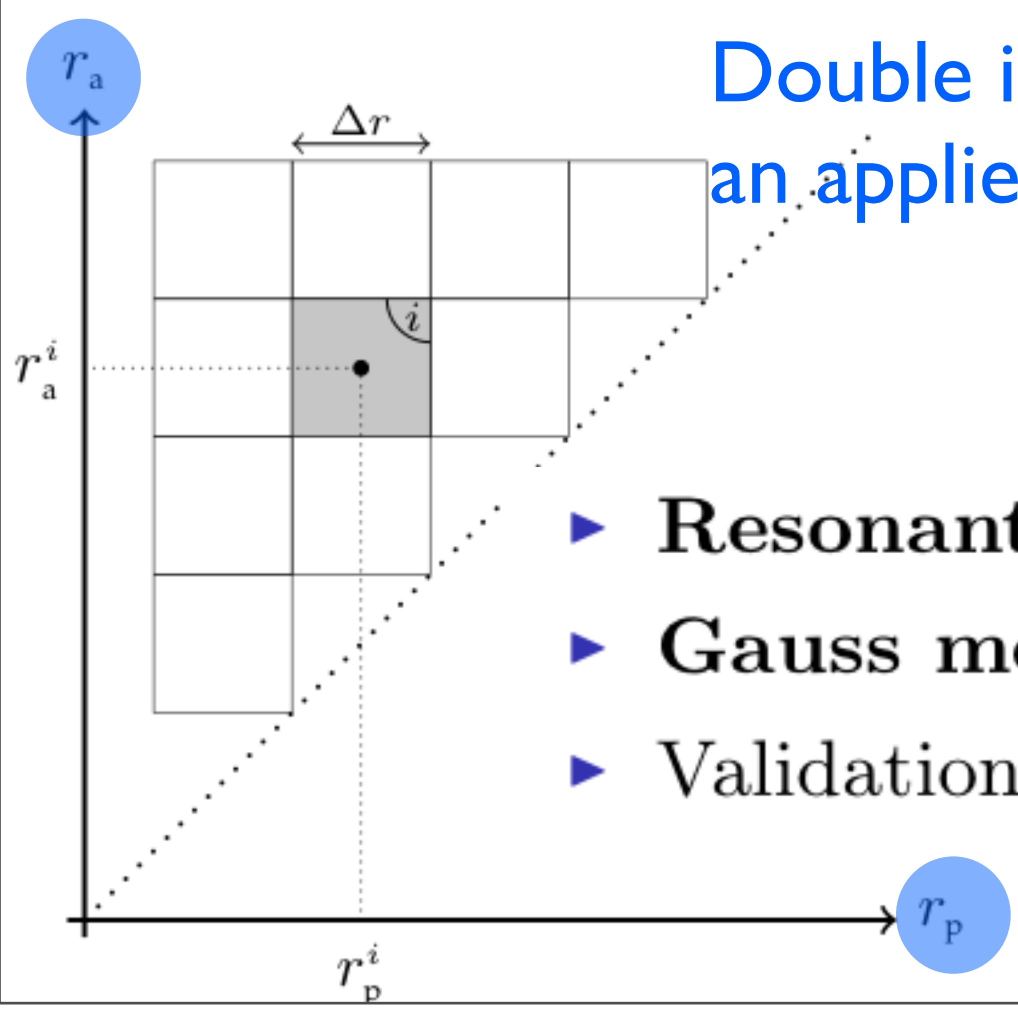
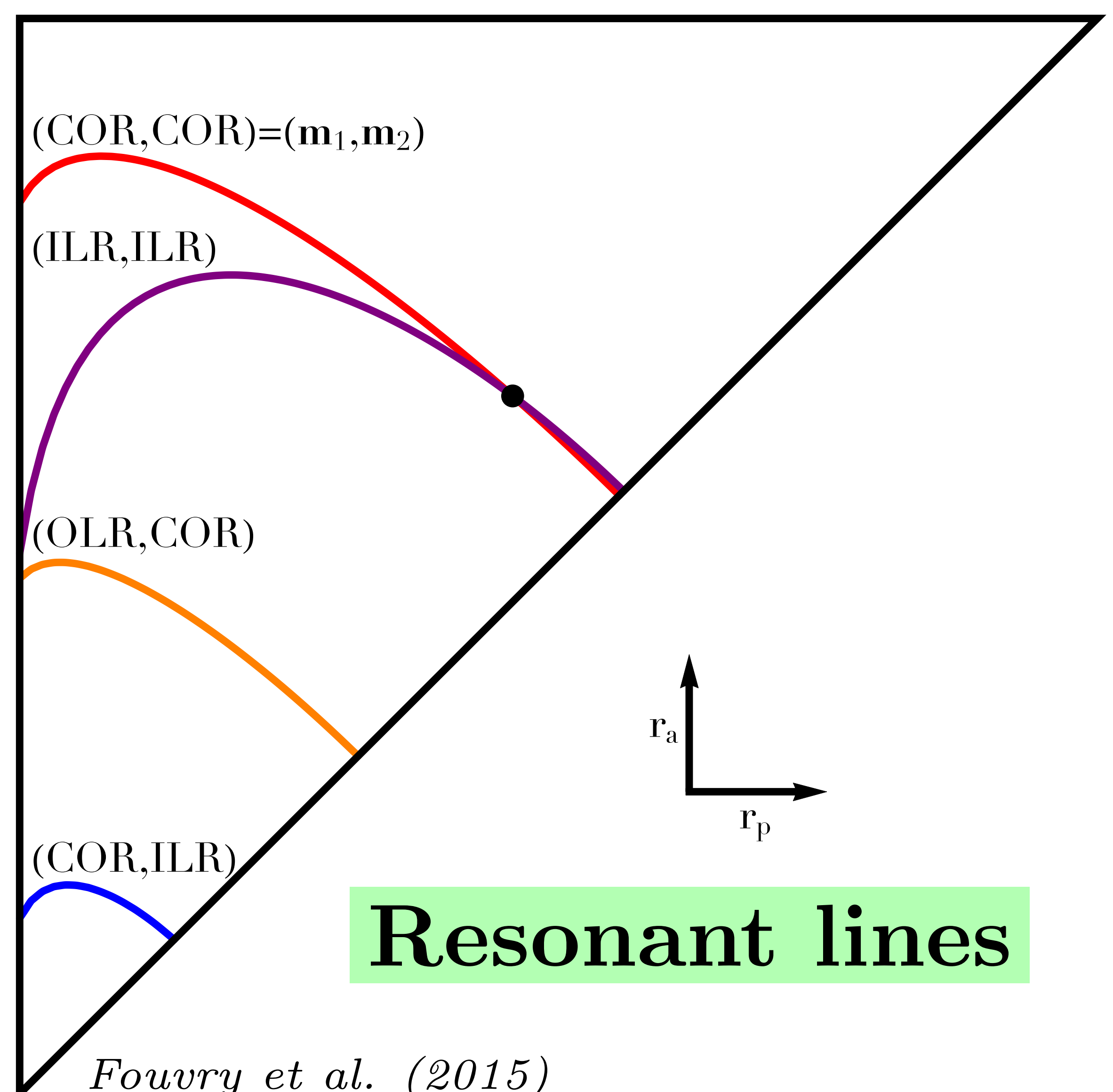
$$\widehat{\mathbf{M}}_{pq}(\omega) \sim \sum_{\boldsymbol{m}} \int \mathrm{d}\boldsymbol{J} \, \frac{\boldsymbol{m}\cdot\partial F/\partial \boldsymbol{J}}{\omega - \boldsymbol{m}\cdot\boldsymbol{\Omega}} \, \psi^{(p)*}_{\boldsymbol{m}}(\boldsymbol{J})\, \psi^{(q)}_{\boldsymbol{m}}(\boldsymbol{J}) .$$

Double integral over phase space:
an applied-math numerical **challenge**



- **Resonant poles** and **action integrals**.
- **Gauss method** with $(r_{\mathrm{p}}, r_{\mathrm{a}})$ as *actions*.
- Validation with **unstable modes**.

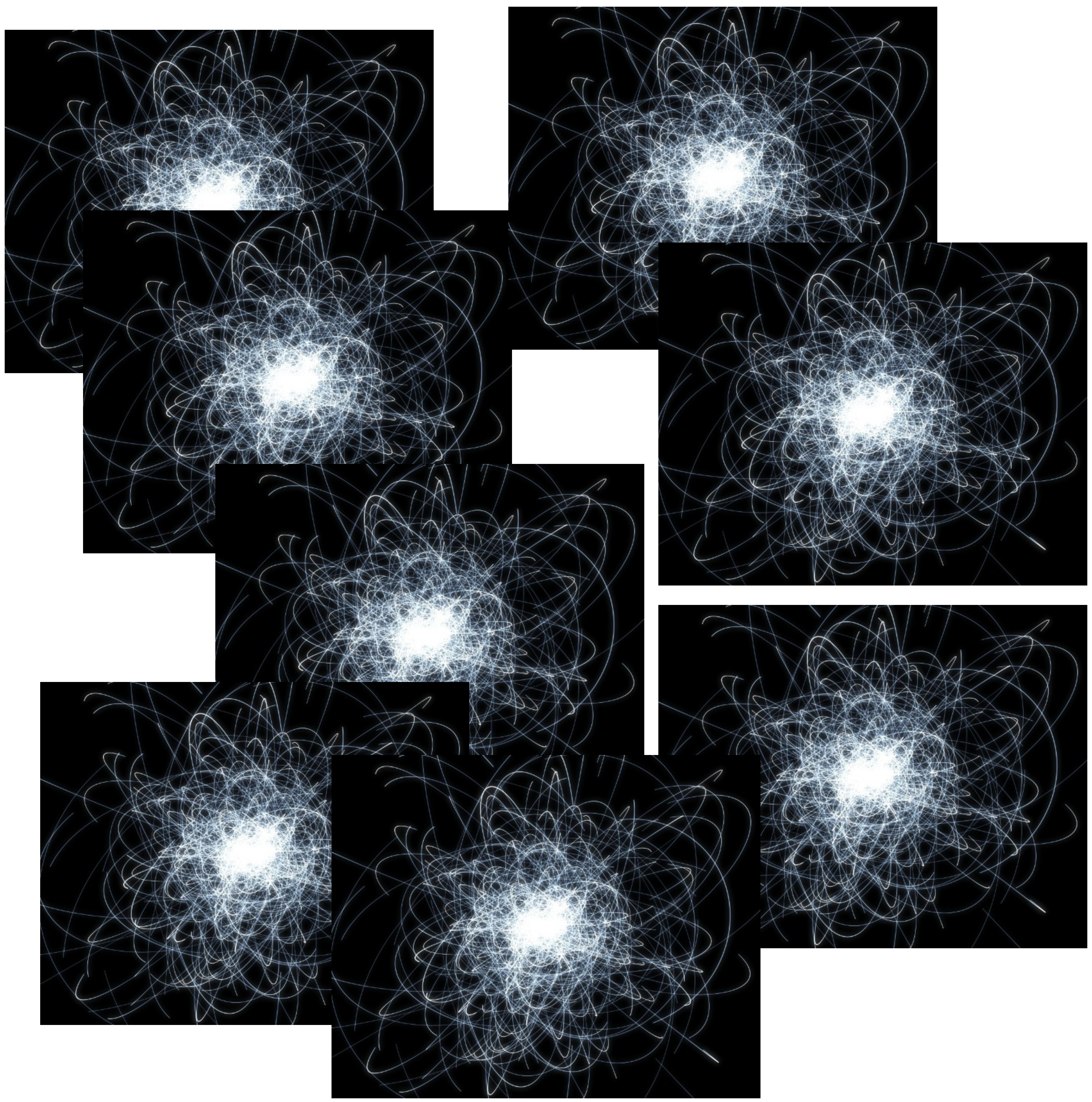


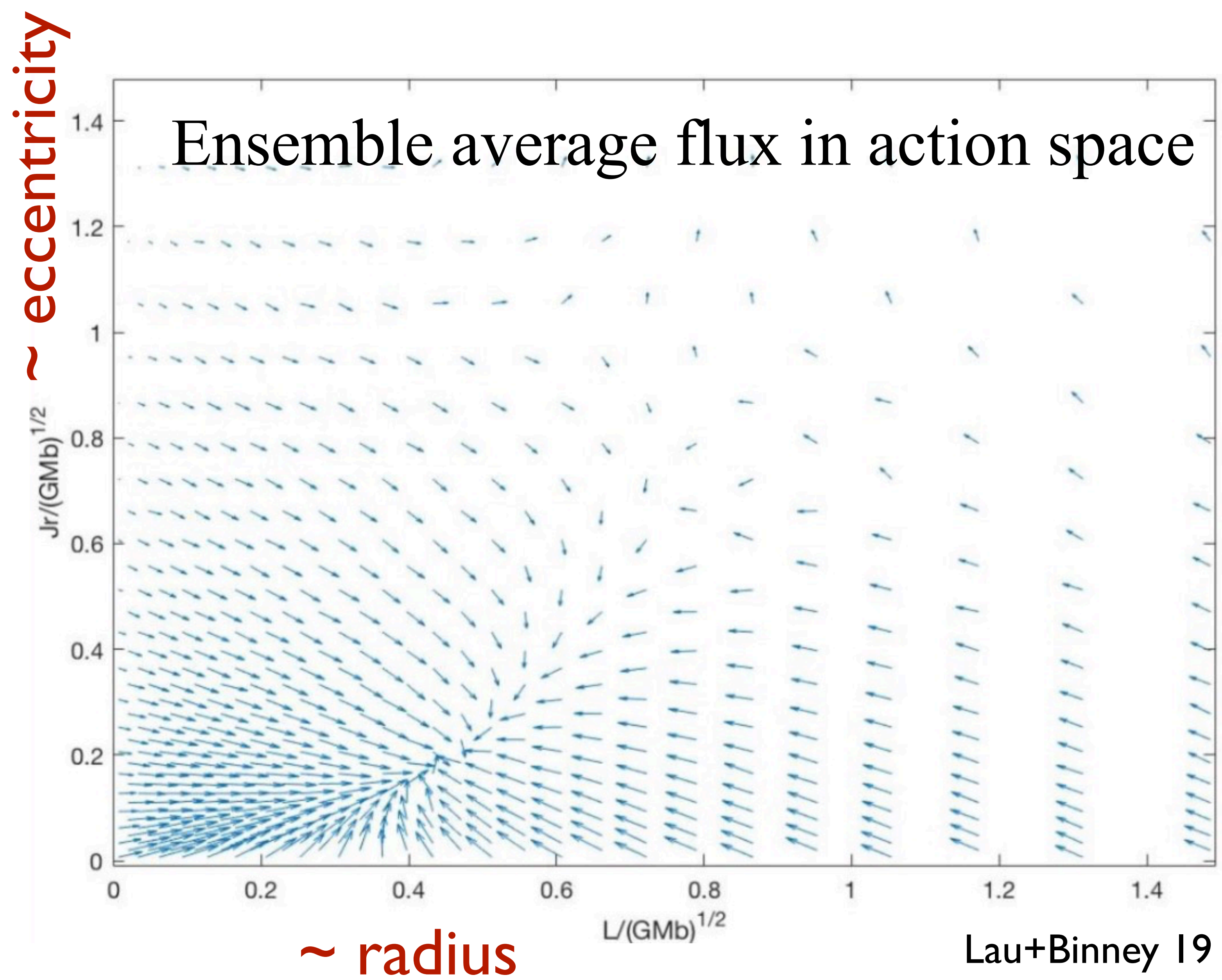
Resonant lines

N-body alternative

How: Ensemble average *rate of change* of action to compute **secular flux**

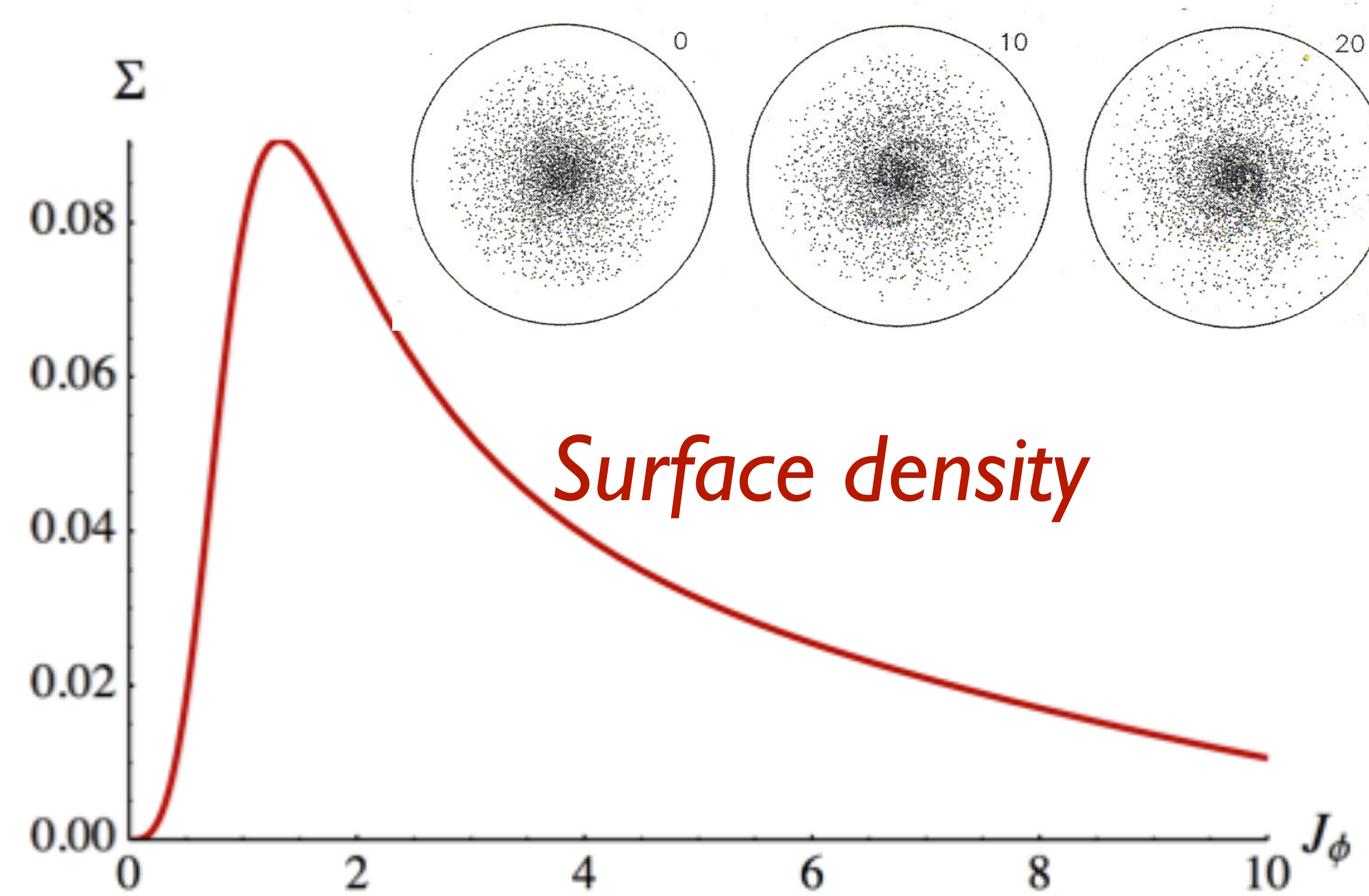
$$\boxed{\frac{\partial f}{\partial t} + \nabla \cdot \mathcal{F} = 0 \qquad \text{with} \qquad \mathcal{F} = \left\langle \frac{\Delta \mathbf{I}}{\Delta t} \right\rangle}$$



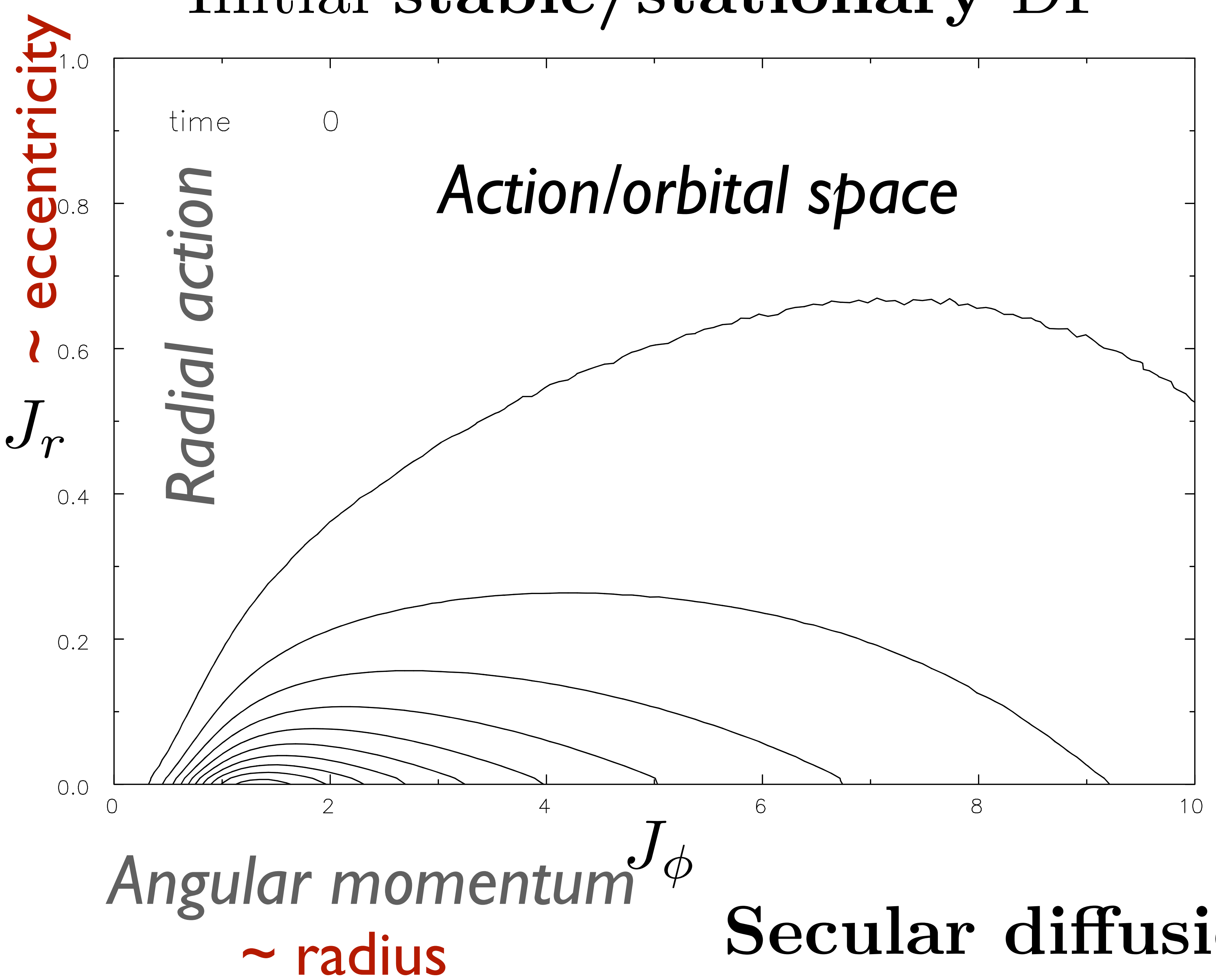


An example of secular evolution

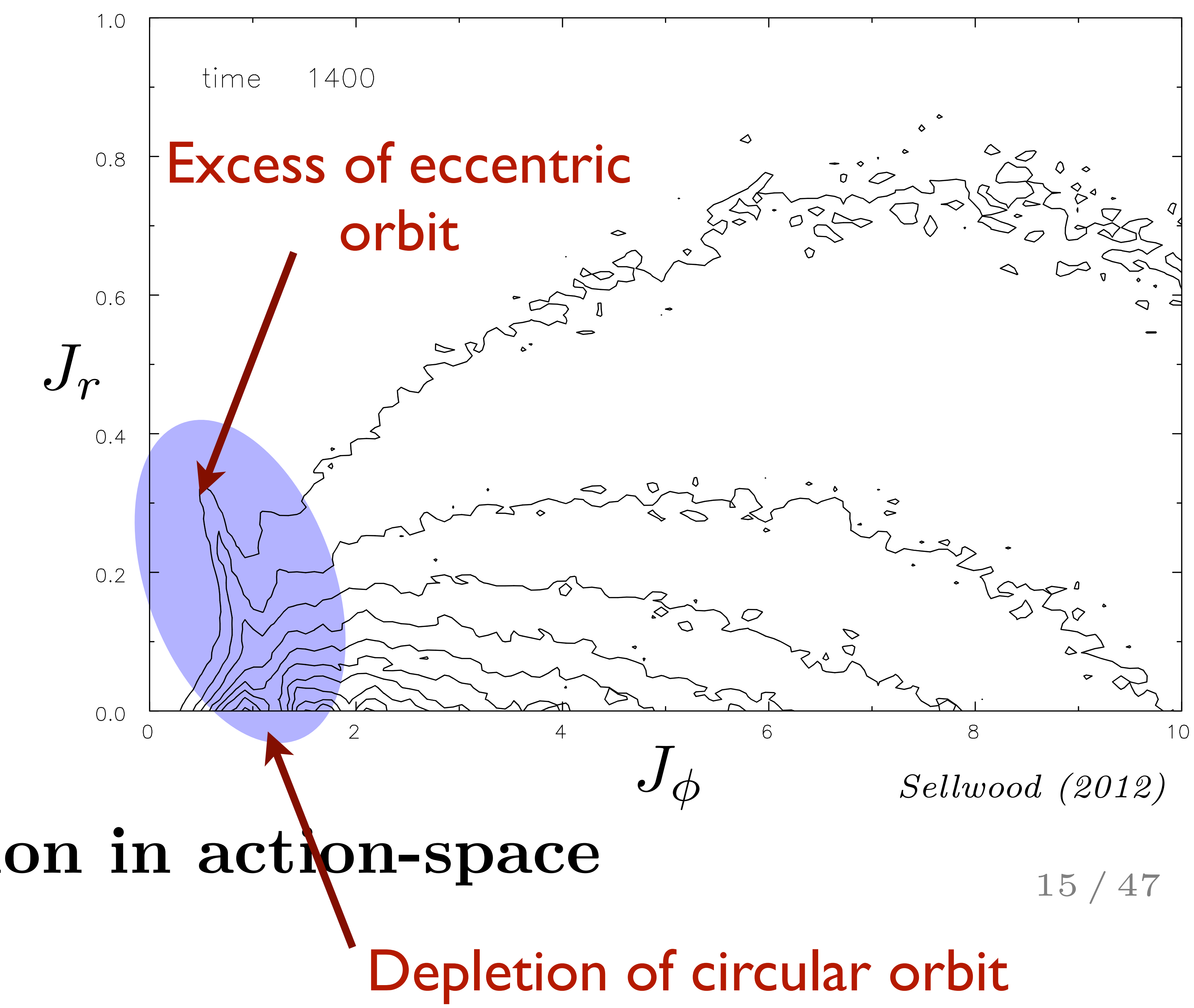
- Sellwood's 2012 numerical experiment
  - Stationnary **stable** tapered *Mestel disc*
  - $N-$body code with 500M particles
  - Appearance of **transient spiral waves**
  - Archetype of **radial migration**



Initial **stable/stationary** DF



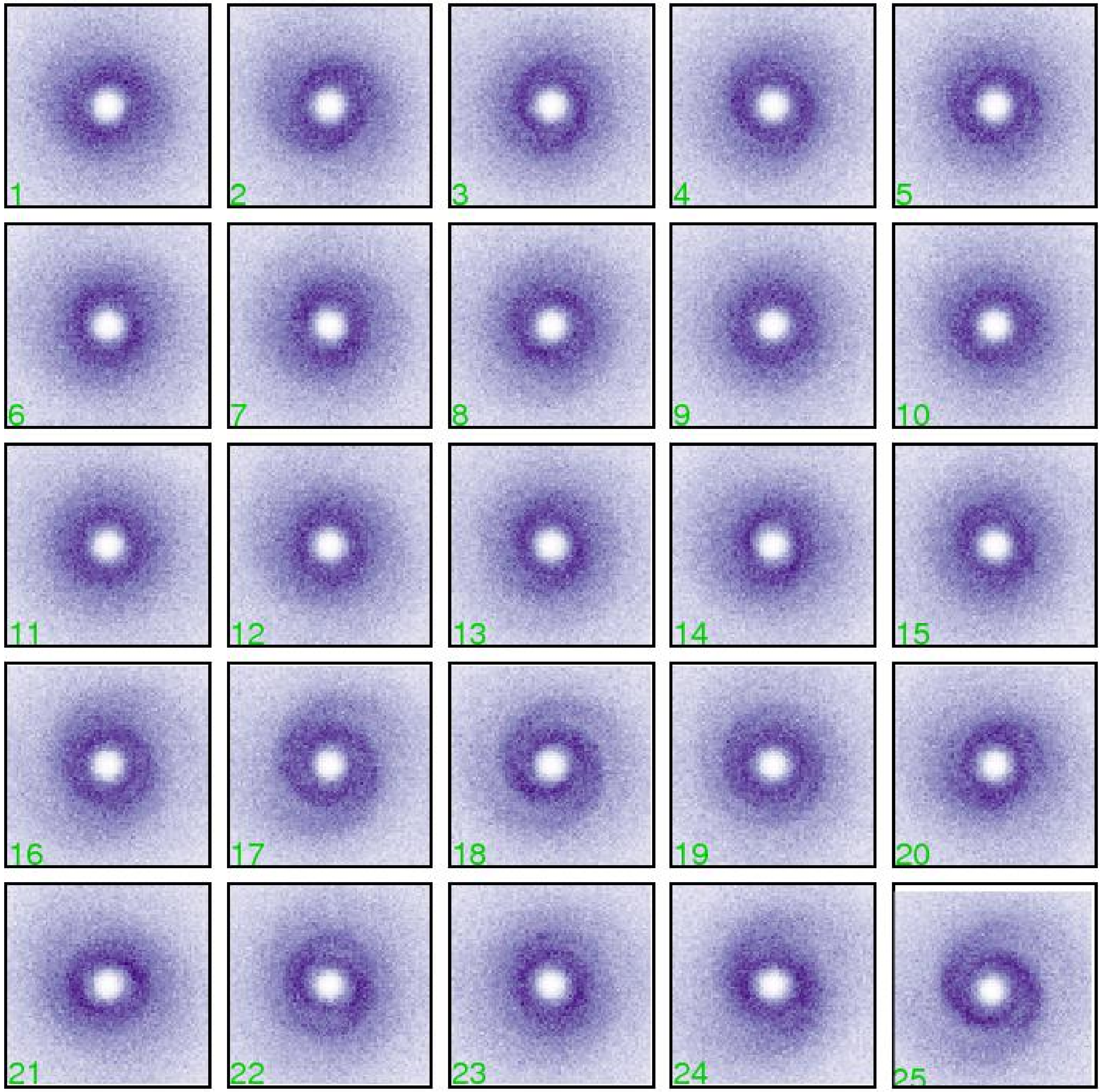
Evolved DF

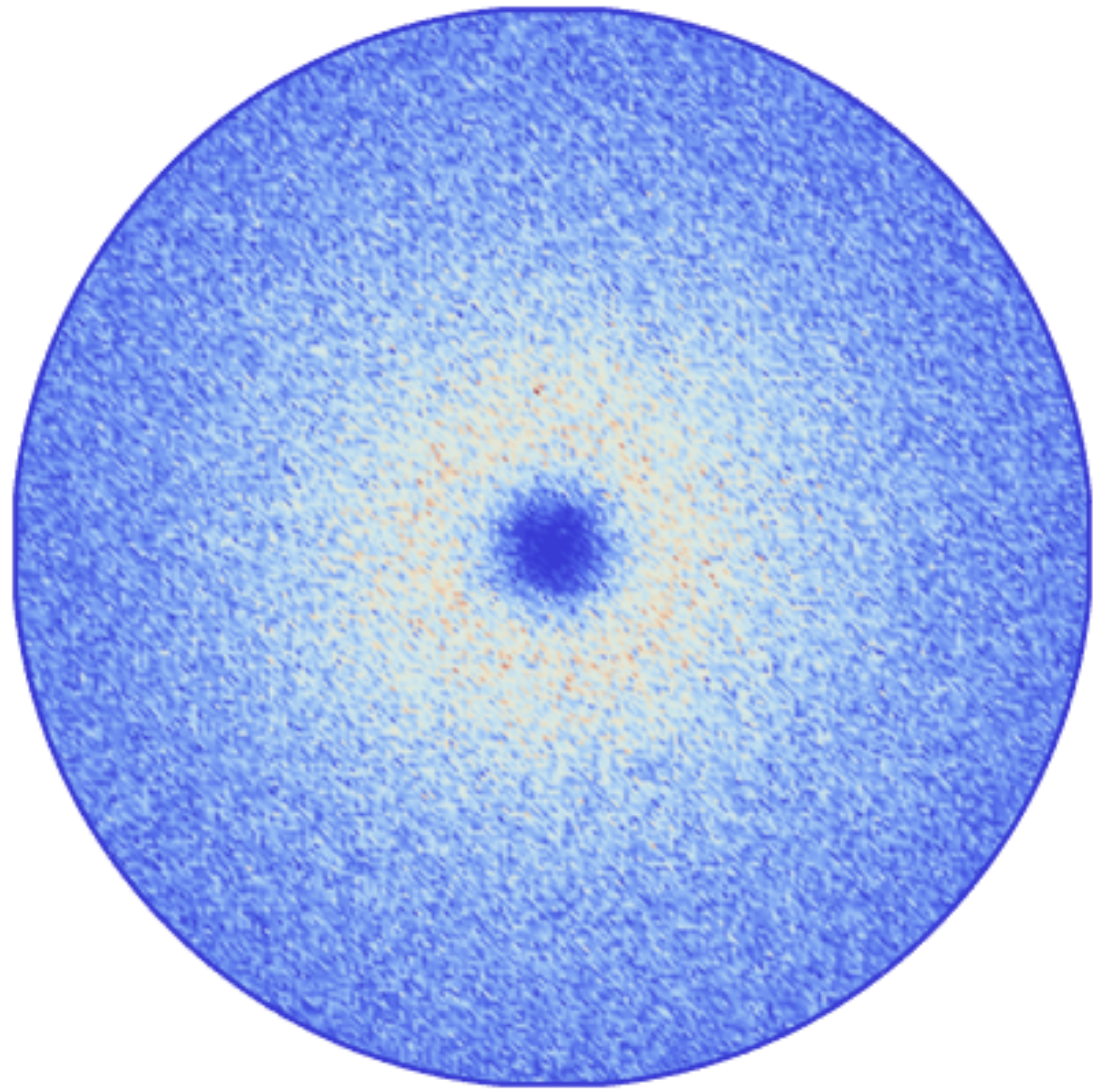


Secular diffusion in action-space

An example of secular evolution

- In **configuration space**



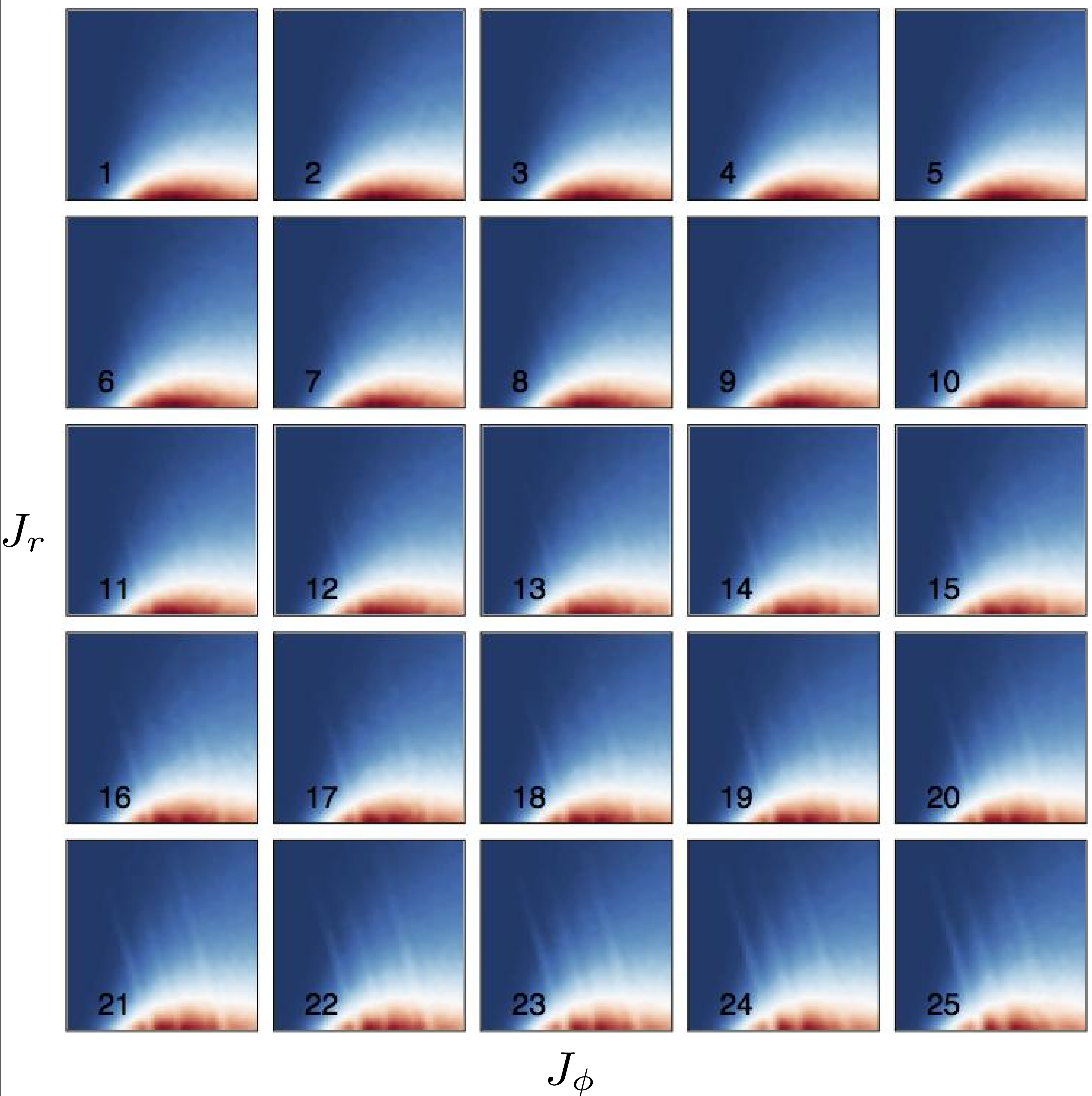


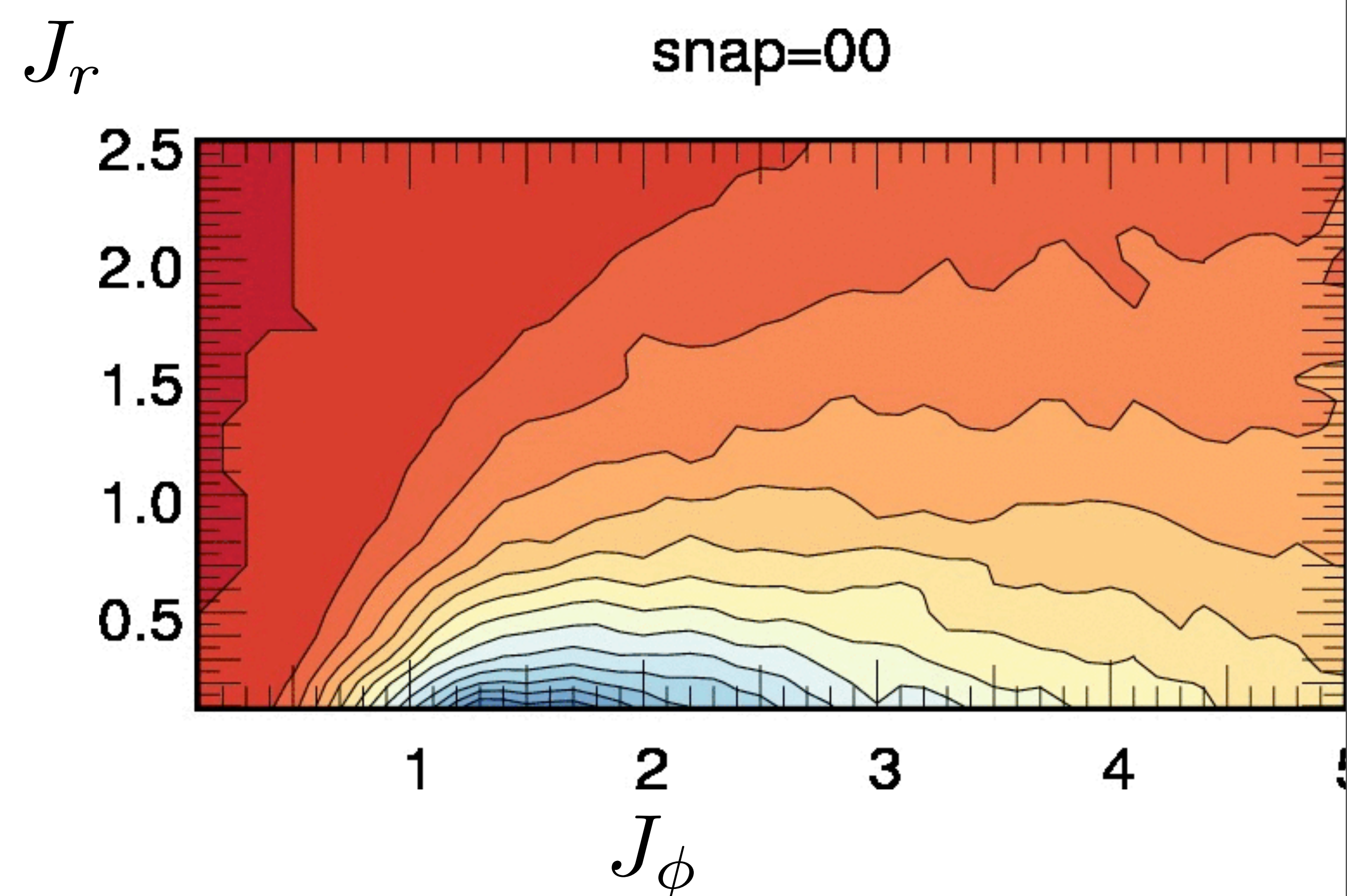
- Spontaneous appearance of uncorrelated **transient spiral waves**.

?

An example of secular evolution

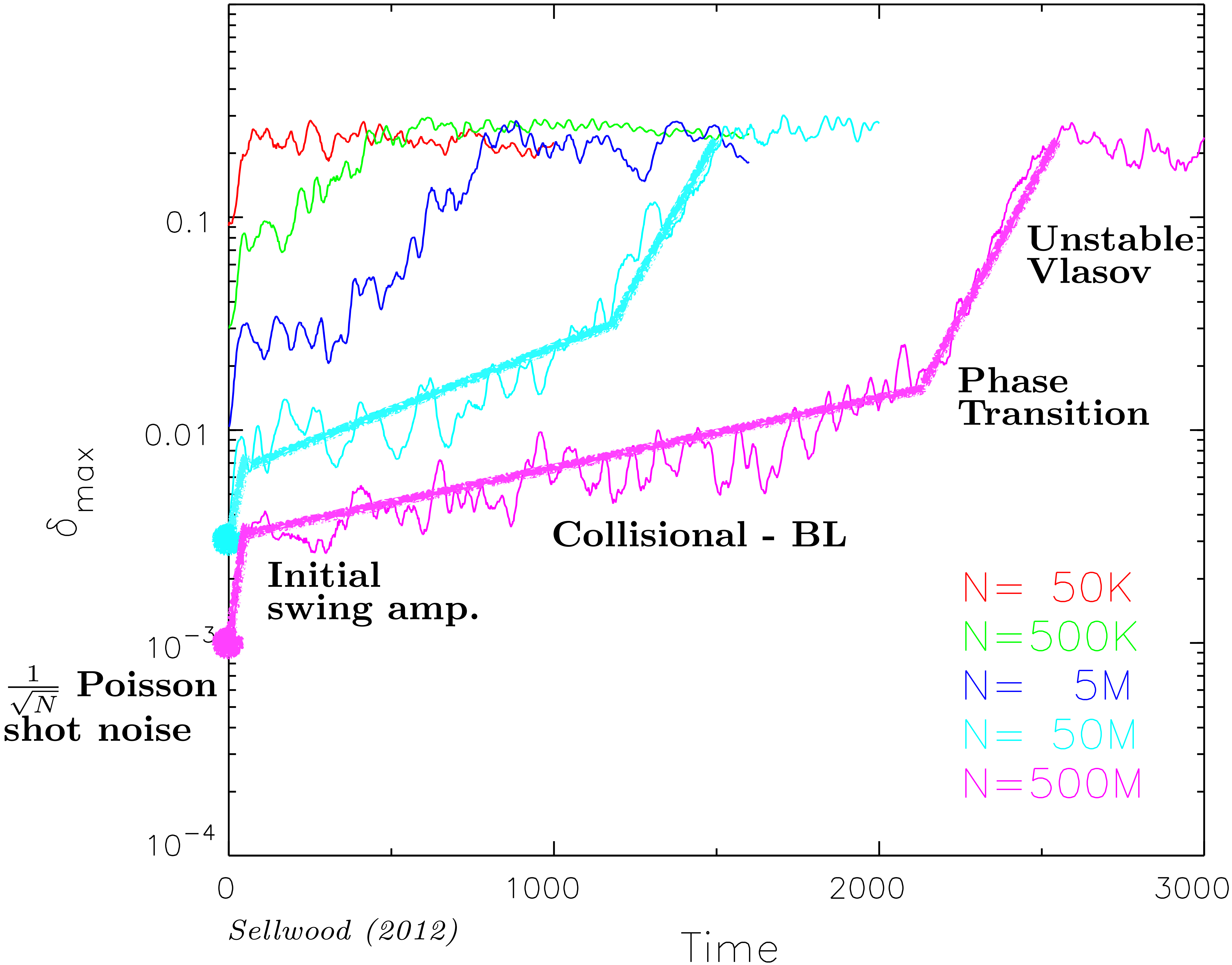
- In **orbital space**





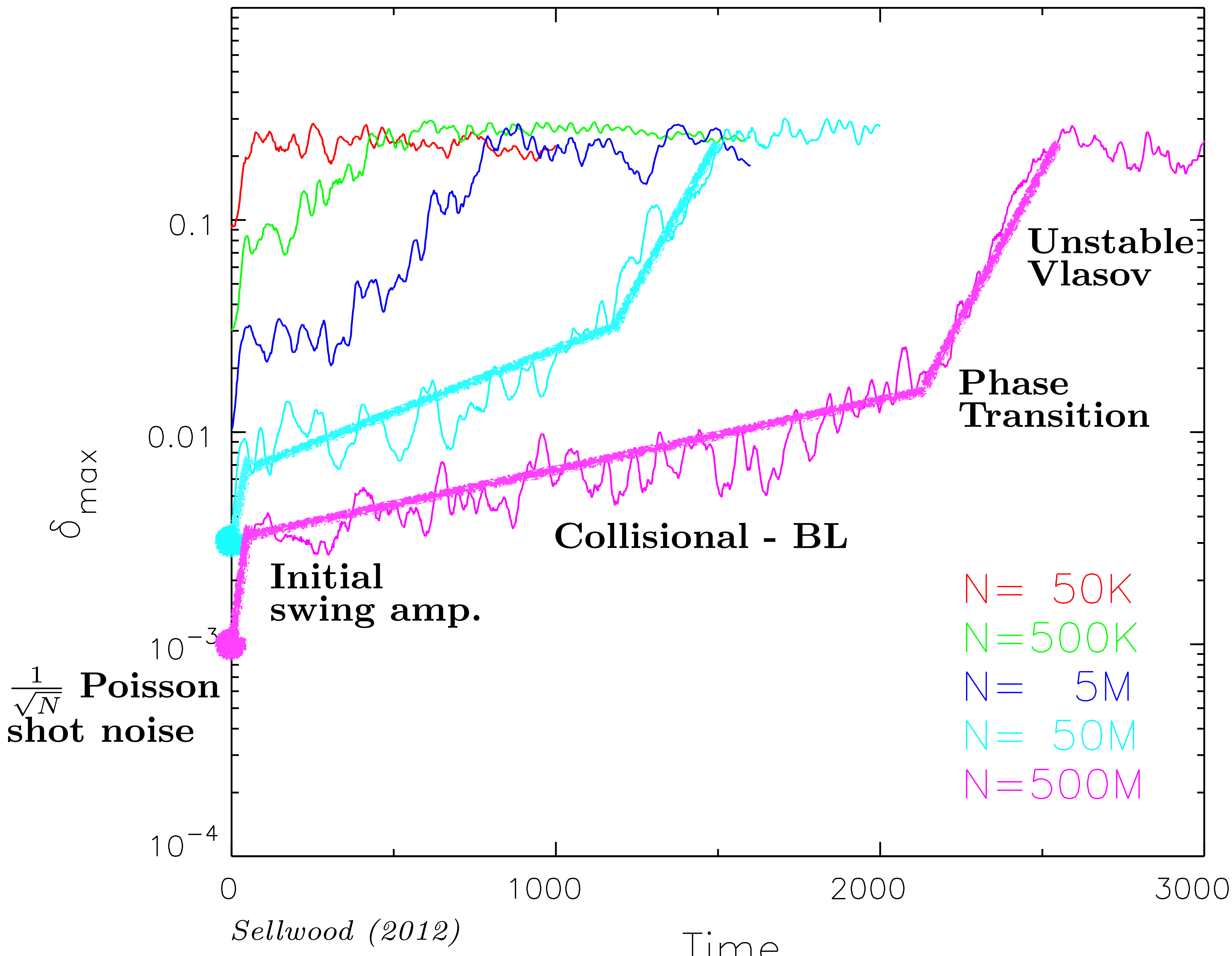
- Long-term appearance of a dominant narrow **resonant ridge**.

The fate of secular evolution



Sellwood (2012)

The fate of secular evolution



stellar discs are so (dynamically) unlikely they will drive themselve out of equibriium

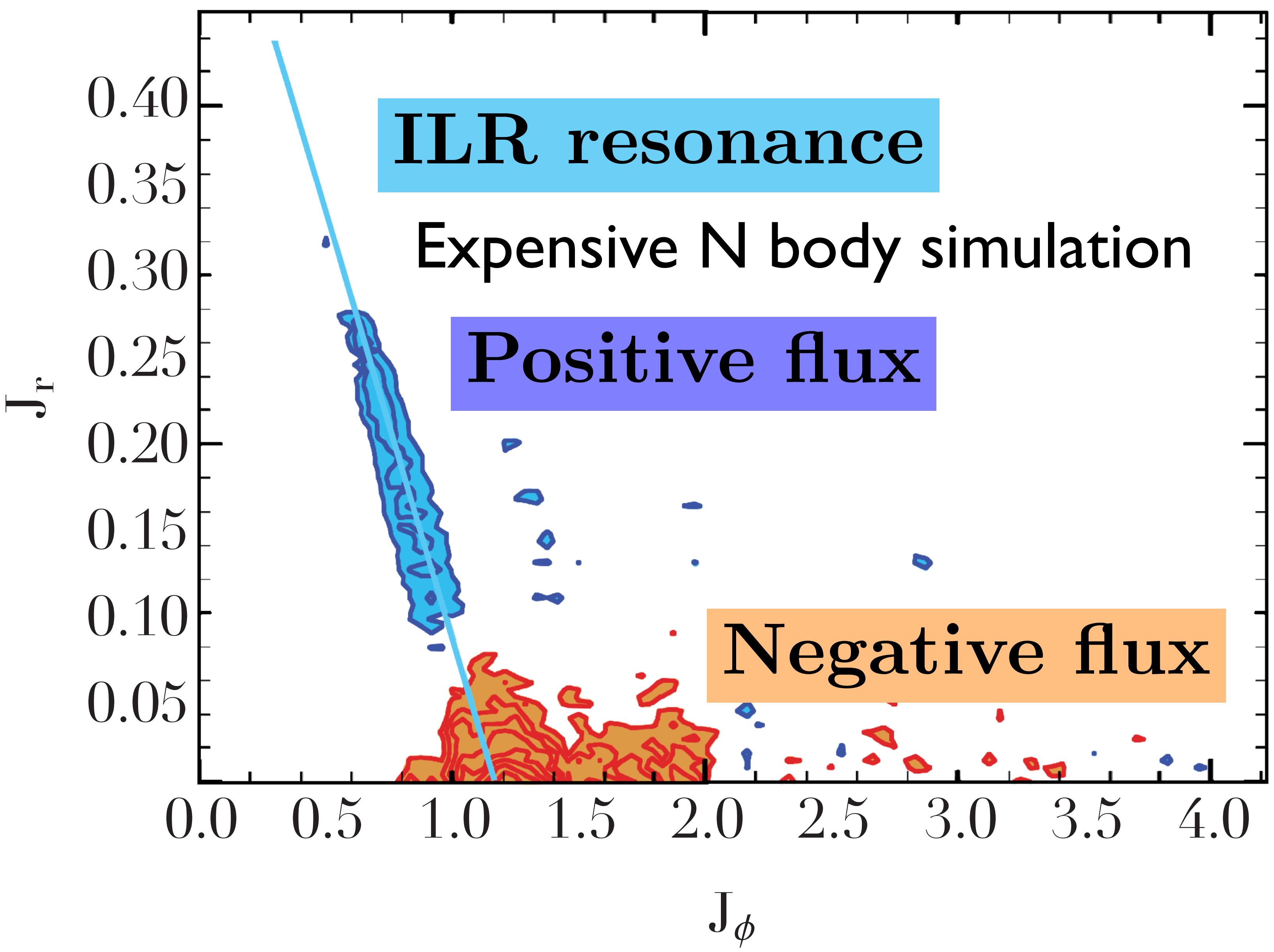
Balescu-Lenard Global - Application

1/3

- **Diffusion flux** in action space

$$\frac{\partial F}{\partial t} = \mathrm{div}\big(\boldsymbol{\mathcal{F}}_{\mathrm{tot}}(\boldsymbol{J})\big)\,.$$

- Predicted contours for $\mathrm{div}\big(\boldsymbol{\mathcal{F}}_{\mathrm{tot}}\big)(t\!=\!0^{+})$



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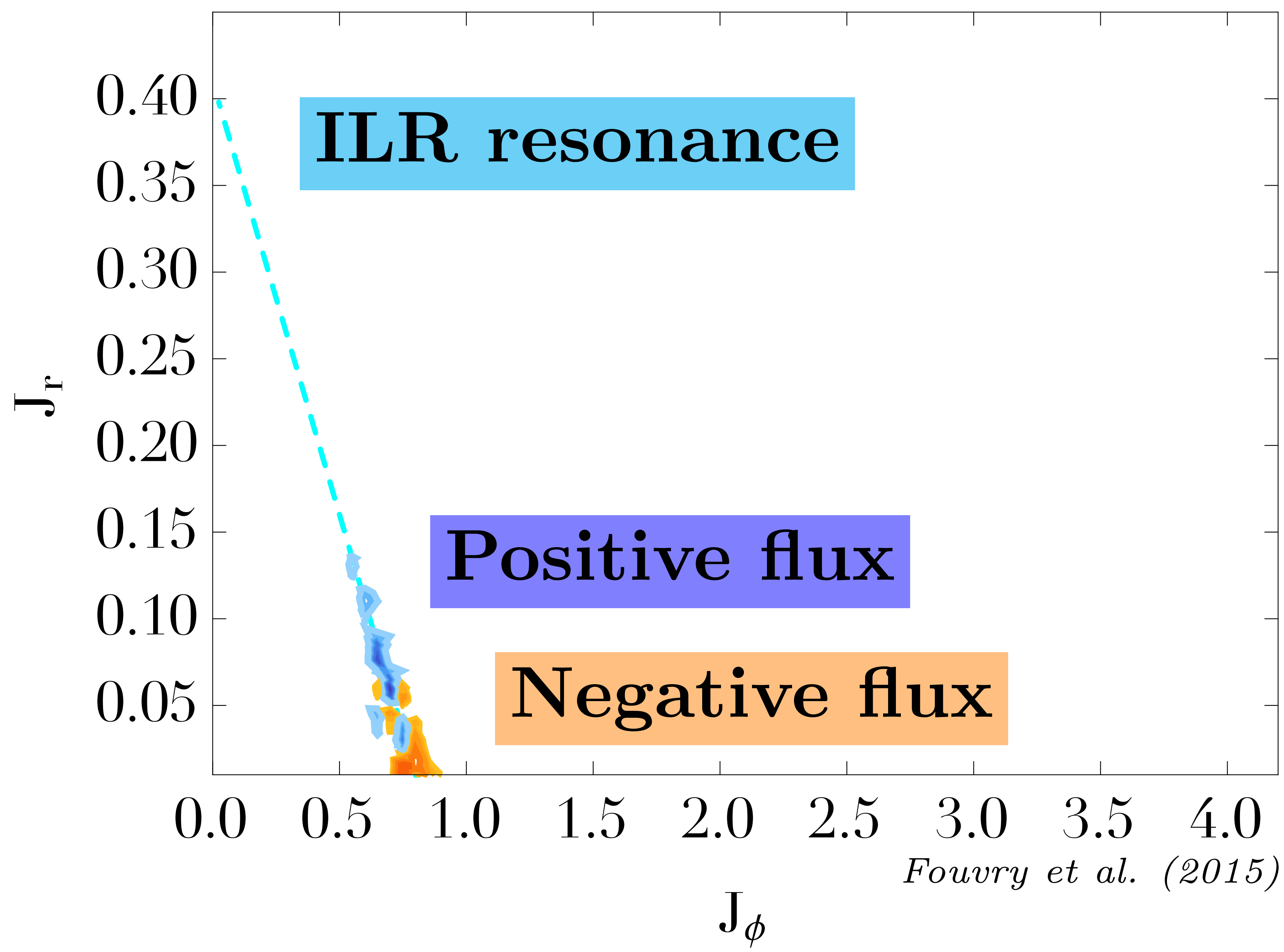

Balescu-Lenard

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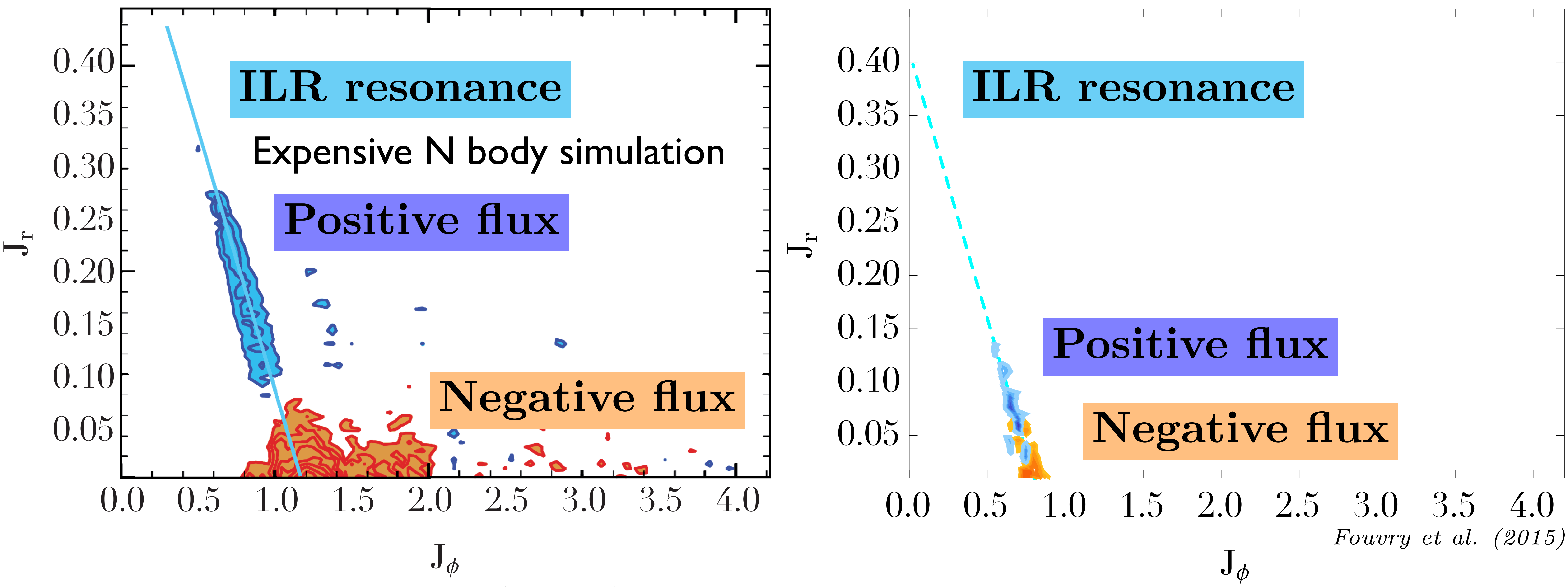
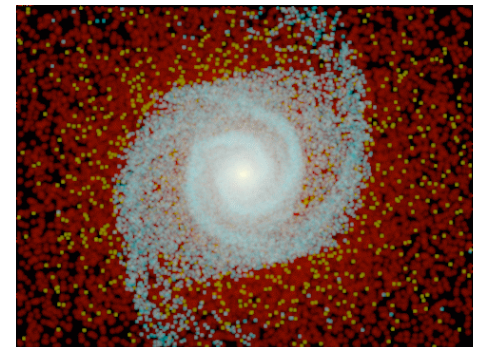


Sellwood (2012) **Balescu-Lenard**

Solves a long standing puzzle of galactic dynamics: **uncorrelated** swing-amplified spiral sequences *secularly* induce formation of very ***specific*** families of distorted (churned *and* blurred) orbits forming a resonant *ridge*.

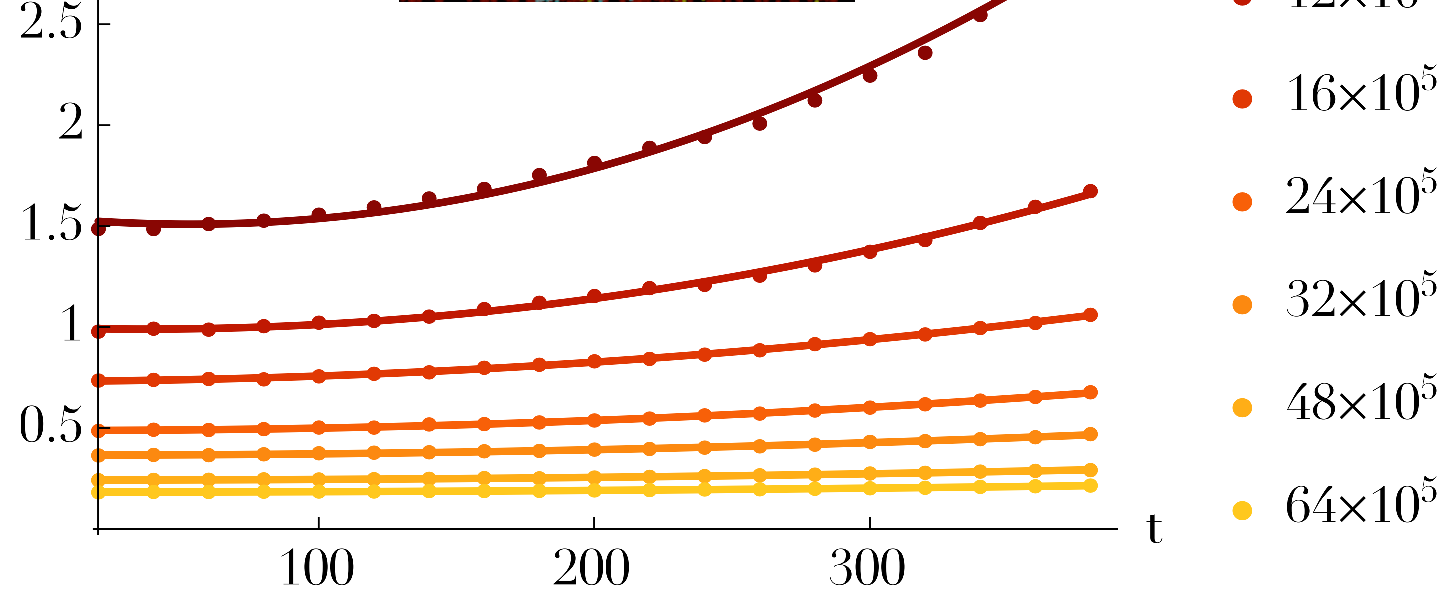
Scaling with N

2/3



- Two entangled sources of **fluctuations**
  - Unavoidable **Poisson shot noise**
  - Irreversible **secular evolution**
- Quantify the **amount of evolution**

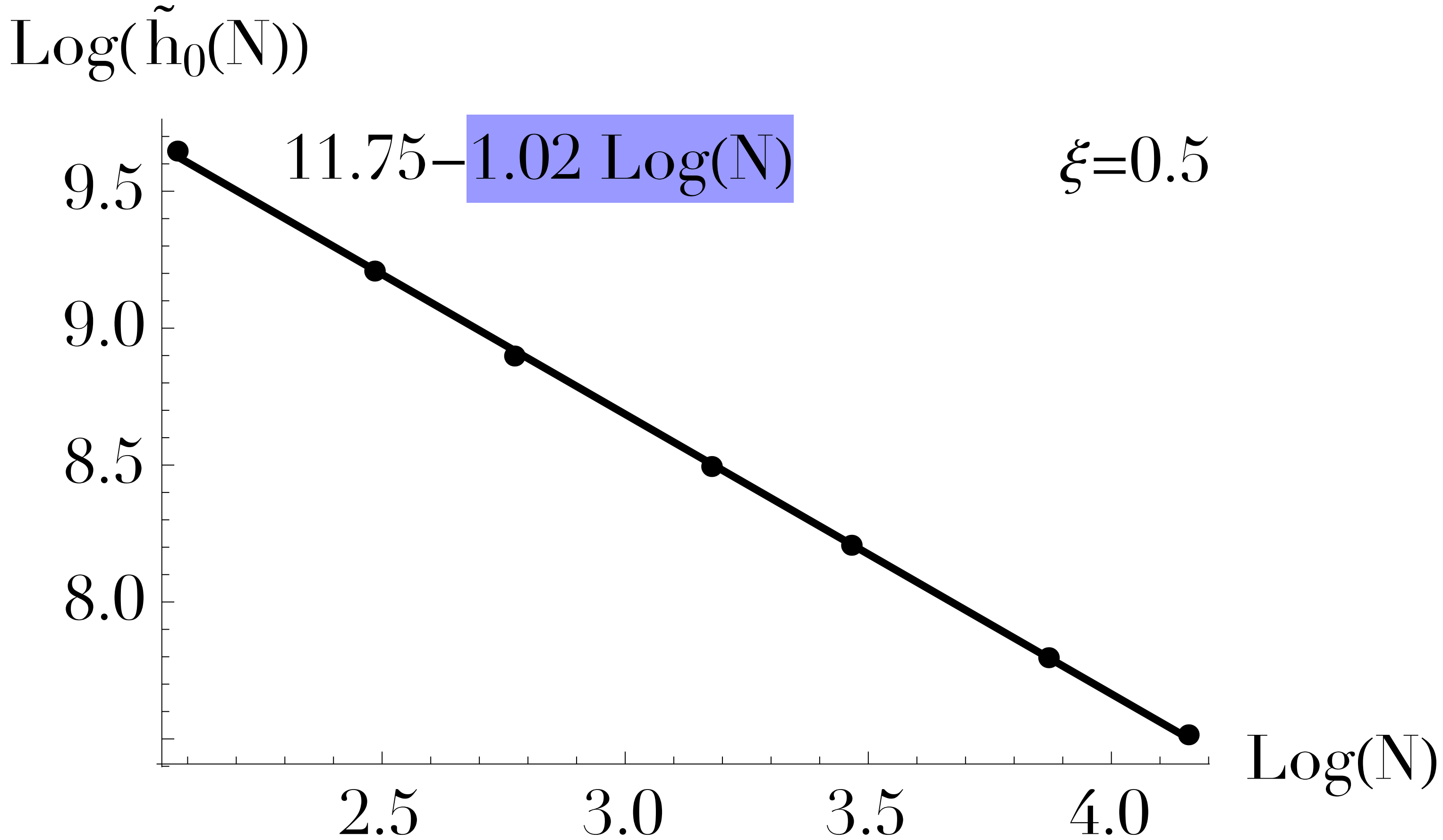
$$\tilde{h}(t,N) = \left\langle \int \mathrm{d}\boldsymbol{J} \left[F(t,\boldsymbol{J},N) - \langle F(0,\boldsymbol{J},N) \rangle \right]^{2} \right\rangle$$

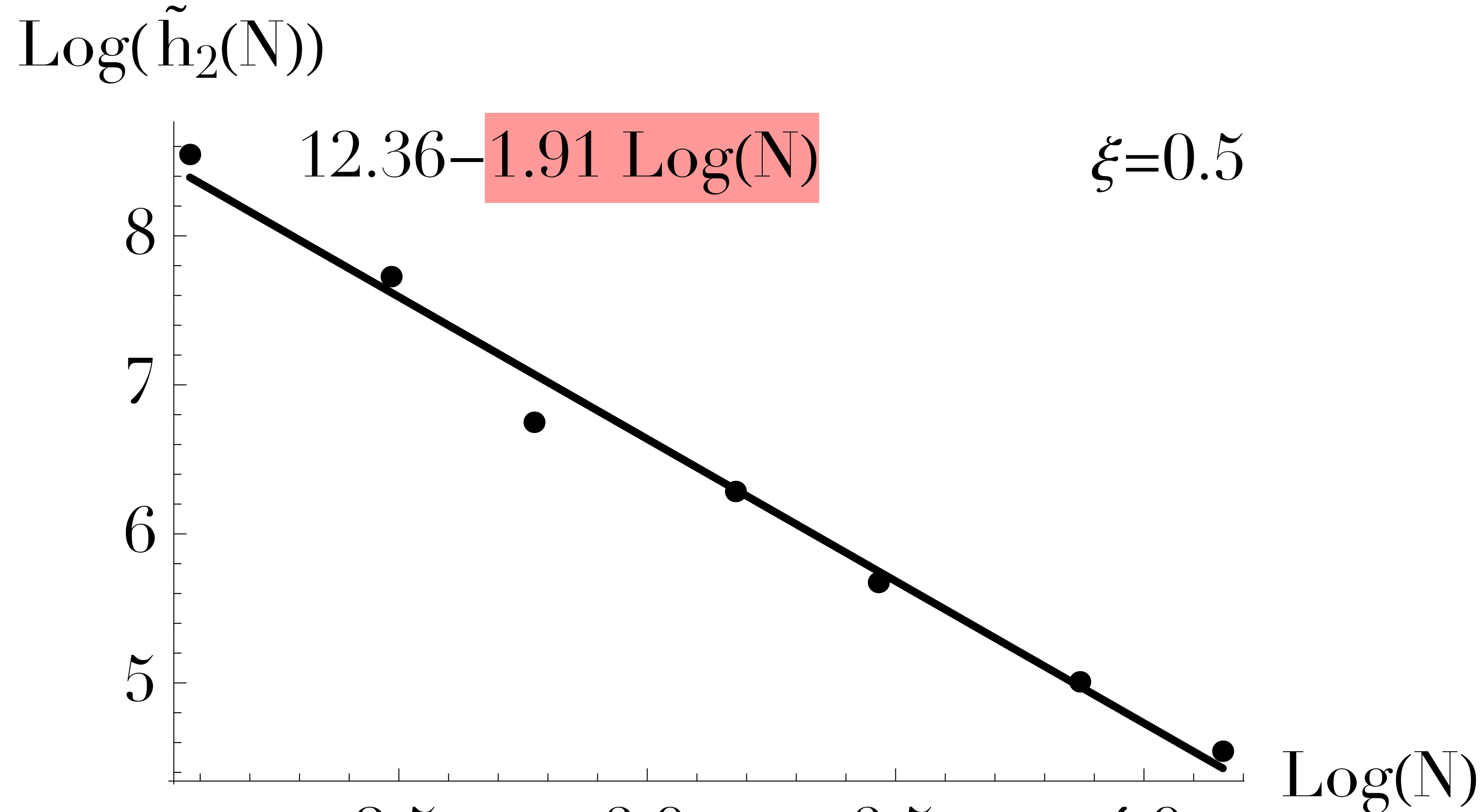


- Initial behaviour

$$\tilde{h}(t,N) \simeq \tilde{h}_0^N + t\,\tilde{h}_1^N + \frac{t^2}{2}\,\tilde{h}_2^N \Longrightarrow \begin{cases} \tilde{h}_0^N \propto N^{-1} \; (\textbf{Poisson shot noise}) \\ \tilde{h}_1^N = 0 \\ \tilde{h}_2^N \propto N^{-2} \; (\textbf{Collisional scaling}) \end{cases}$$

- $N-$**body measurements**





Process displays characteristic scaling with N and cosmic time

Balescu-Lenard Global - Timescale

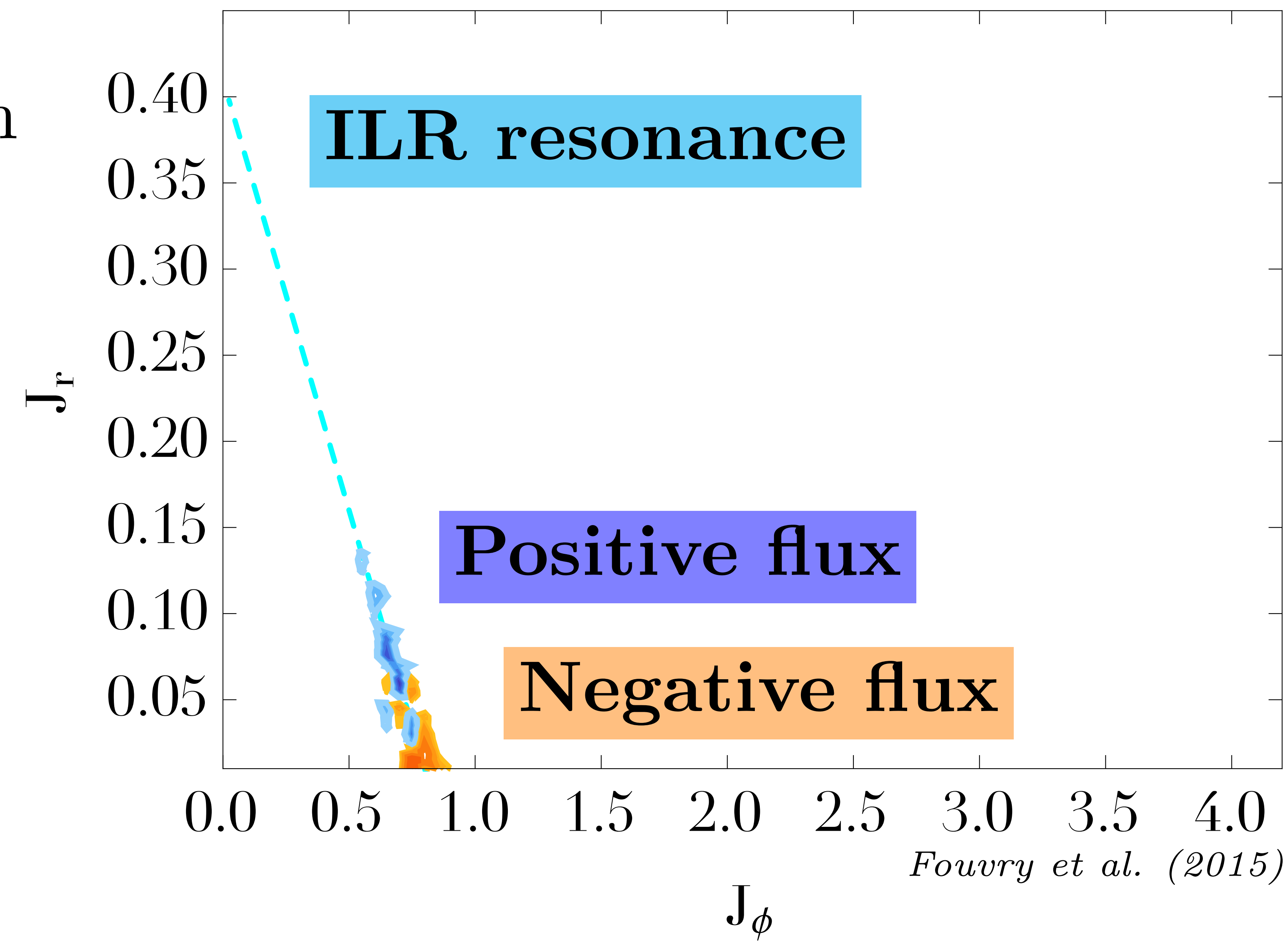
- Normalised Balescu-Lenard equation

$$\frac{\partial F}{\partial t} = \frac{1}{N} C_{\mathrm{BL}}[F] \implies \boxed{\frac{\partial F}{\partial \tau} = C_{\mathrm{BL}}[F] \quad \text{with} \quad \tau = \frac{t}{N}}\,.$$

- Comparison with S12's simulation

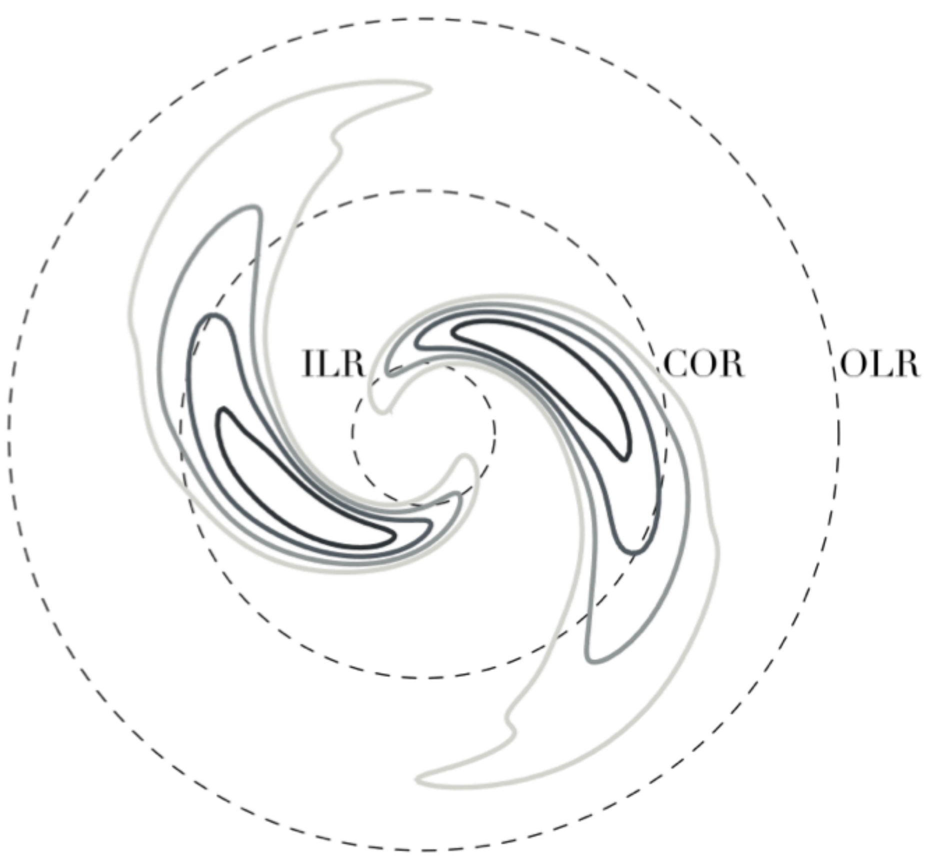
$$\frac{\Delta \tau_{\mathrm{S12}}}{\Delta \tau_{\mathrm{BL}}} \sim 1\,.$$

$$\implies \boxed{\textbf{Appropriate timescales.}}$$



Fouvry et al. (2015)

Balescu-Lenard



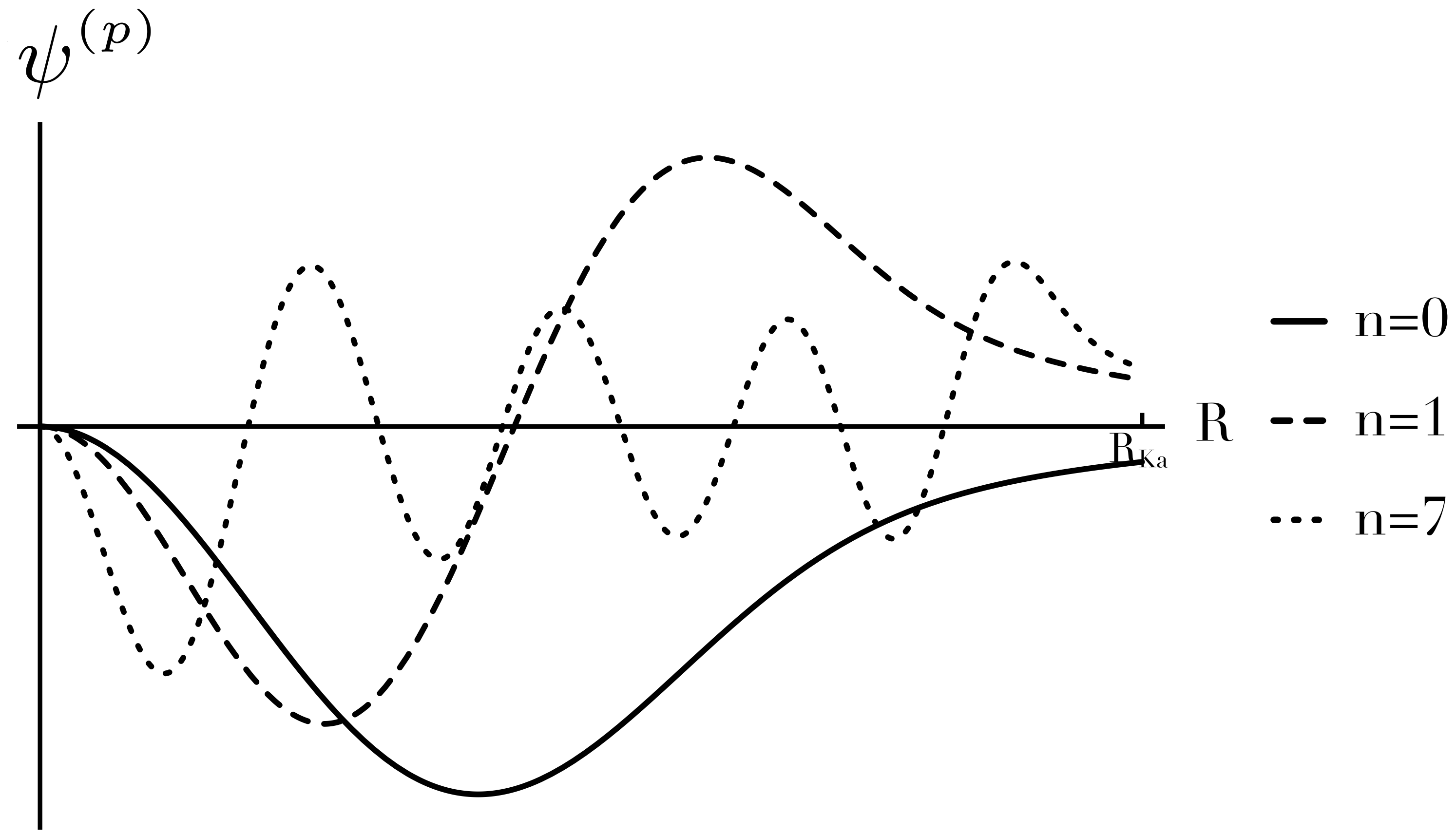
*Secularly, gravitational susceptibility is **squared**!*

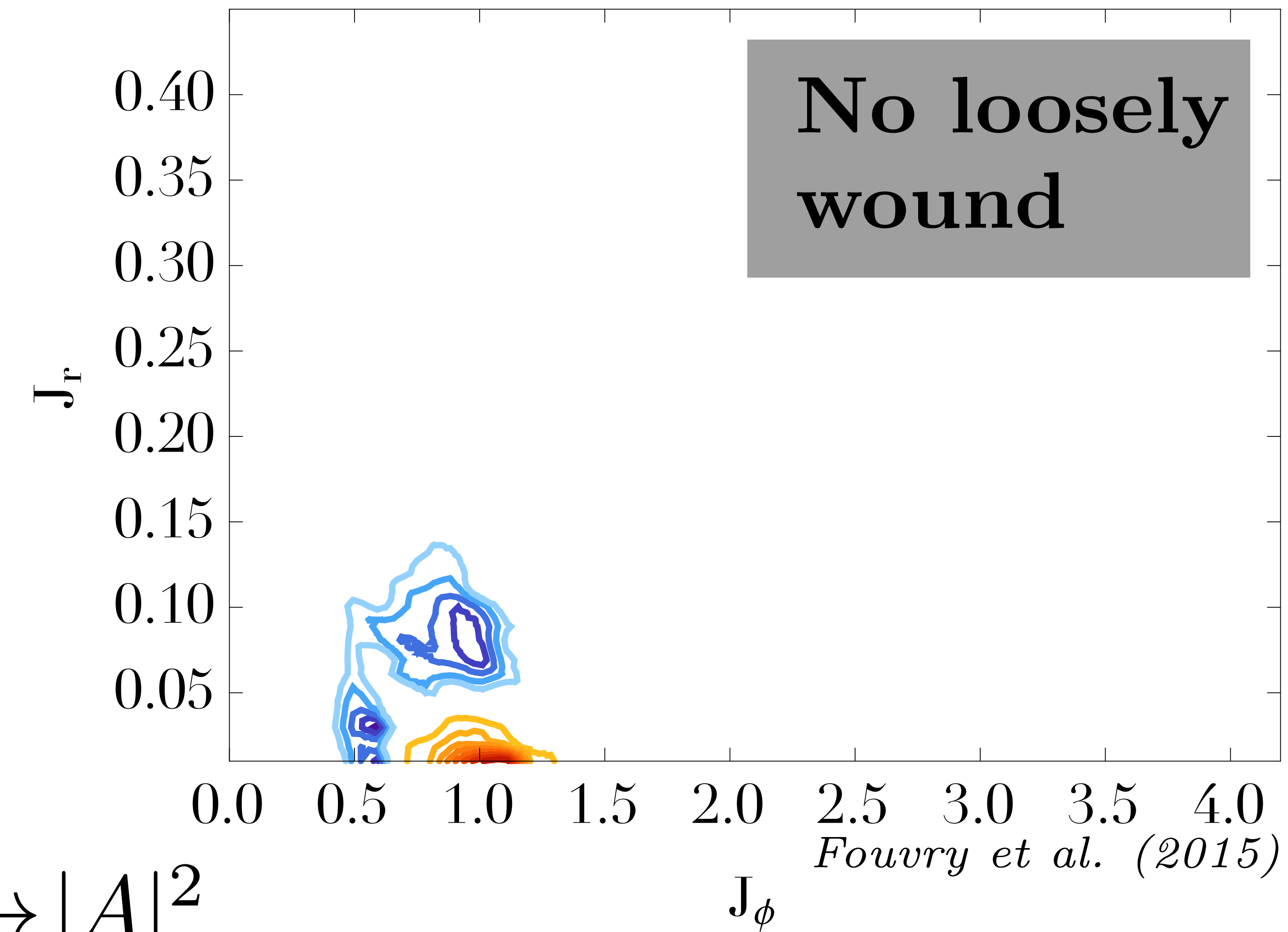
In Mestel disc 1 $M_\odot$ polarize 10^2 $M_\odot$

$10^4 = (10^2)^2 =$ **10 000** !!

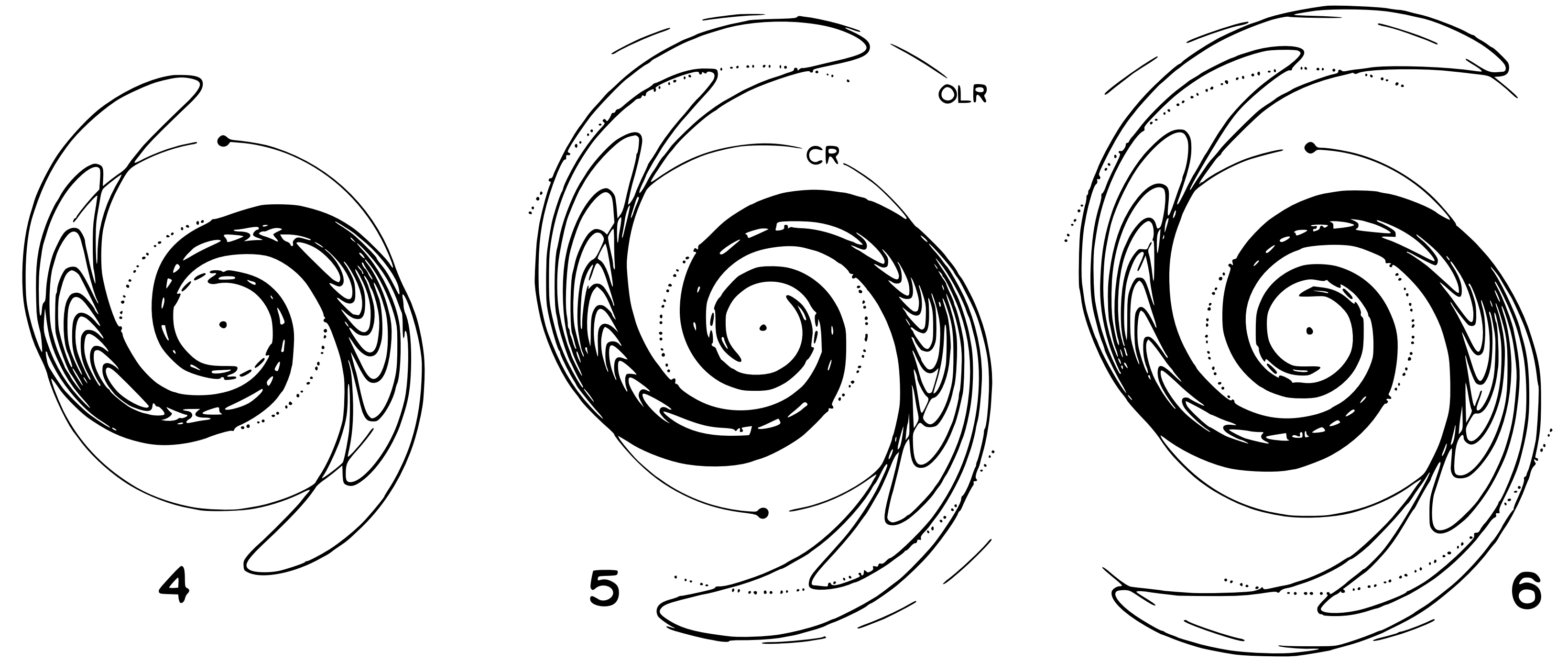
The role of swing amplification

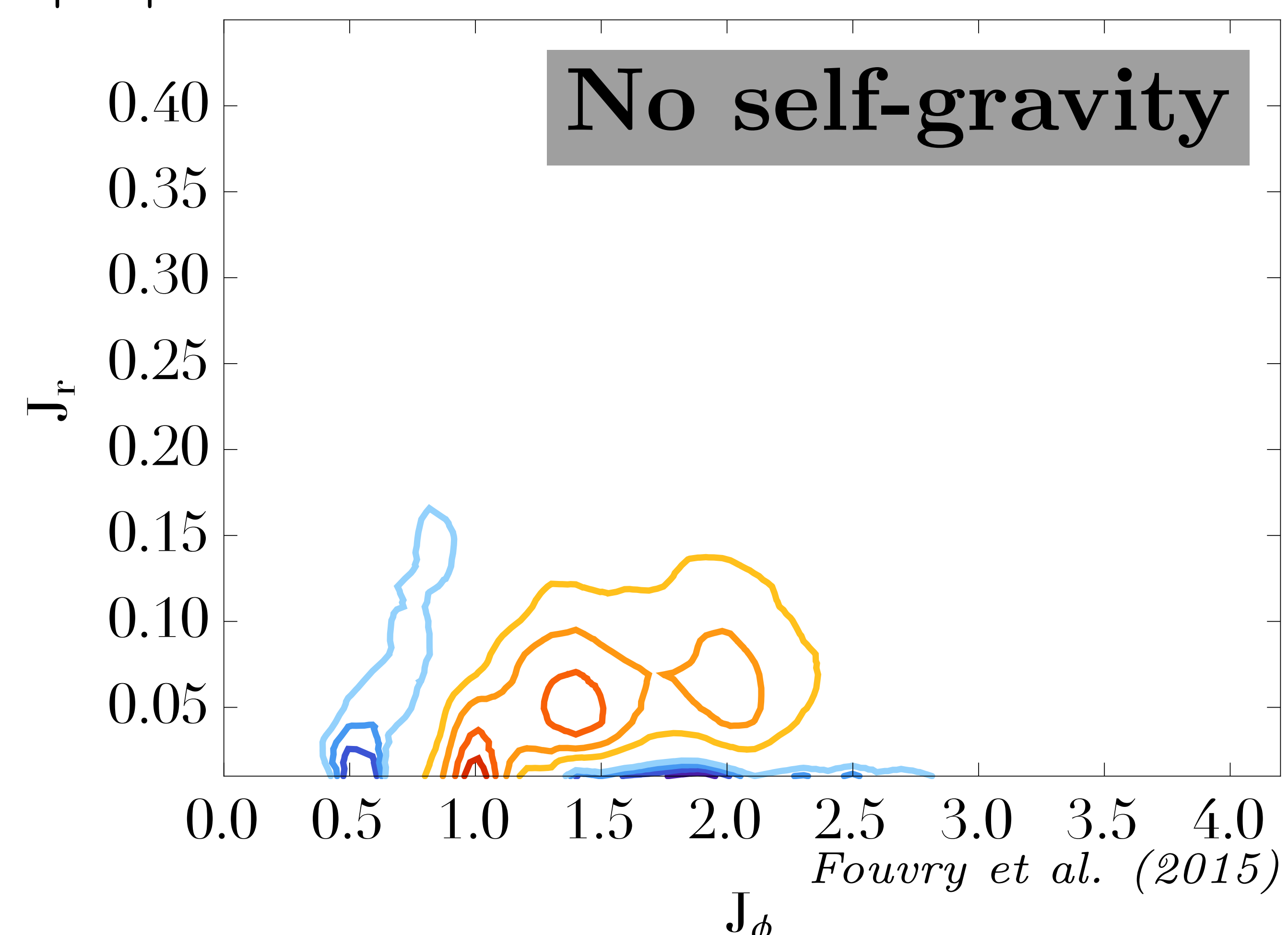
- Removing **loosely wound** basis elements





- Turning off **collective effects** $1/|\mathcal{D}|^2 \rightarrow |A|^2$



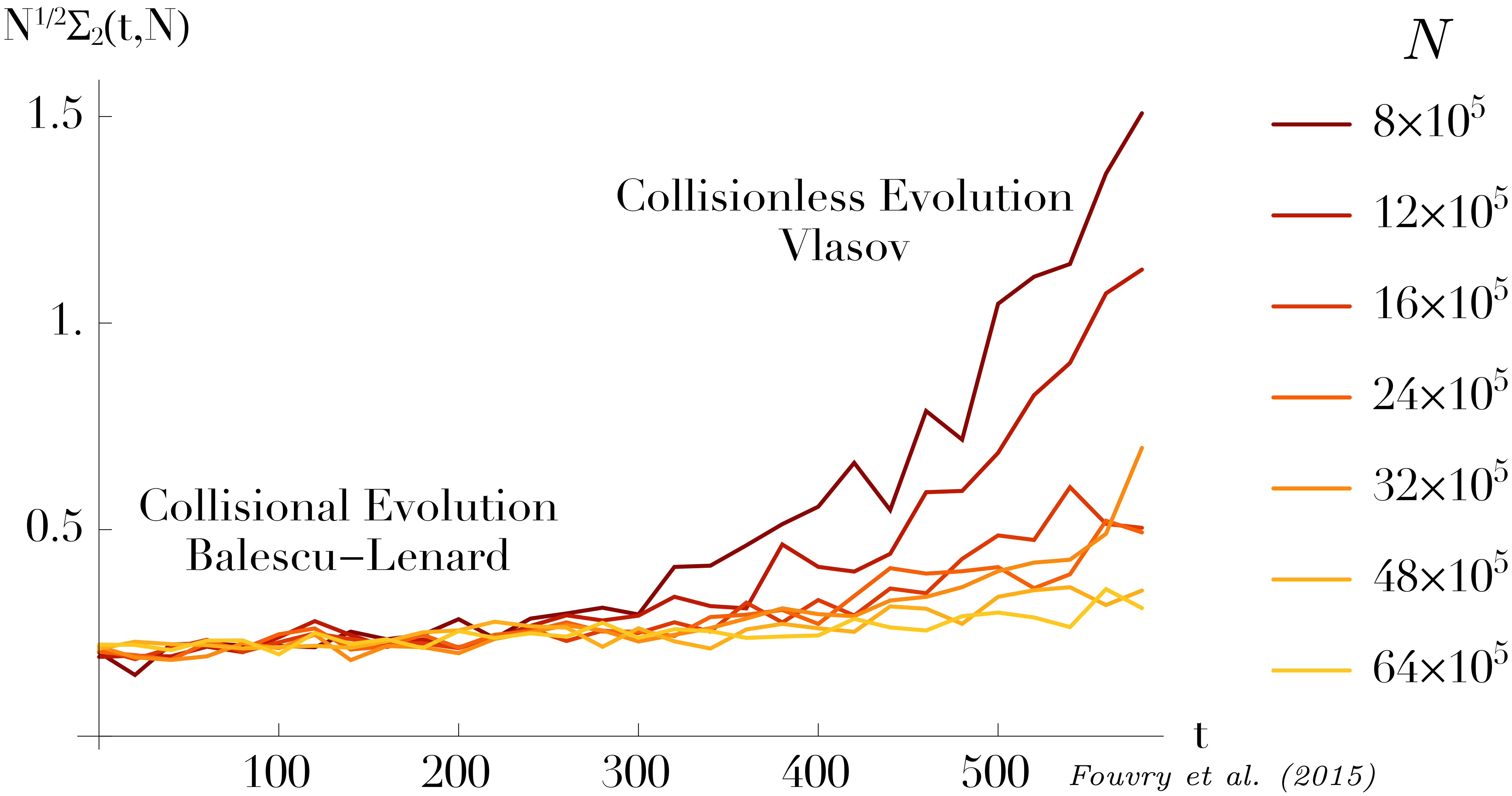


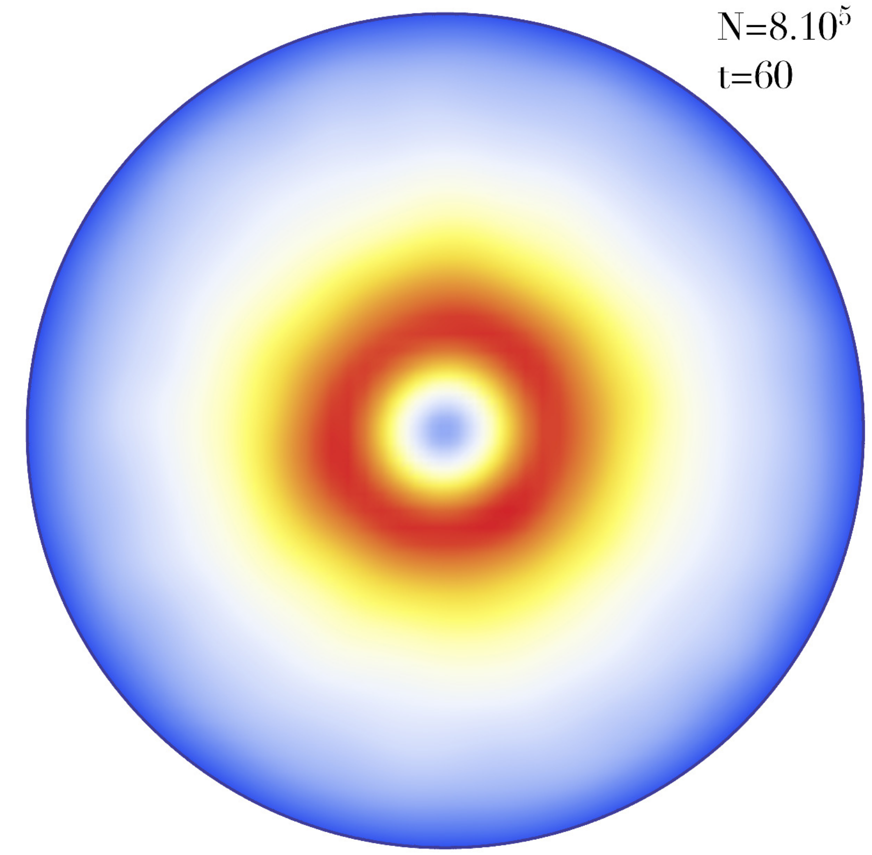
$\Longrightarrow$ **Self-gravitating amplification of loosely wound perturbations**

Proof in importance of self-gravity+ flexibility of Kinetic formalism

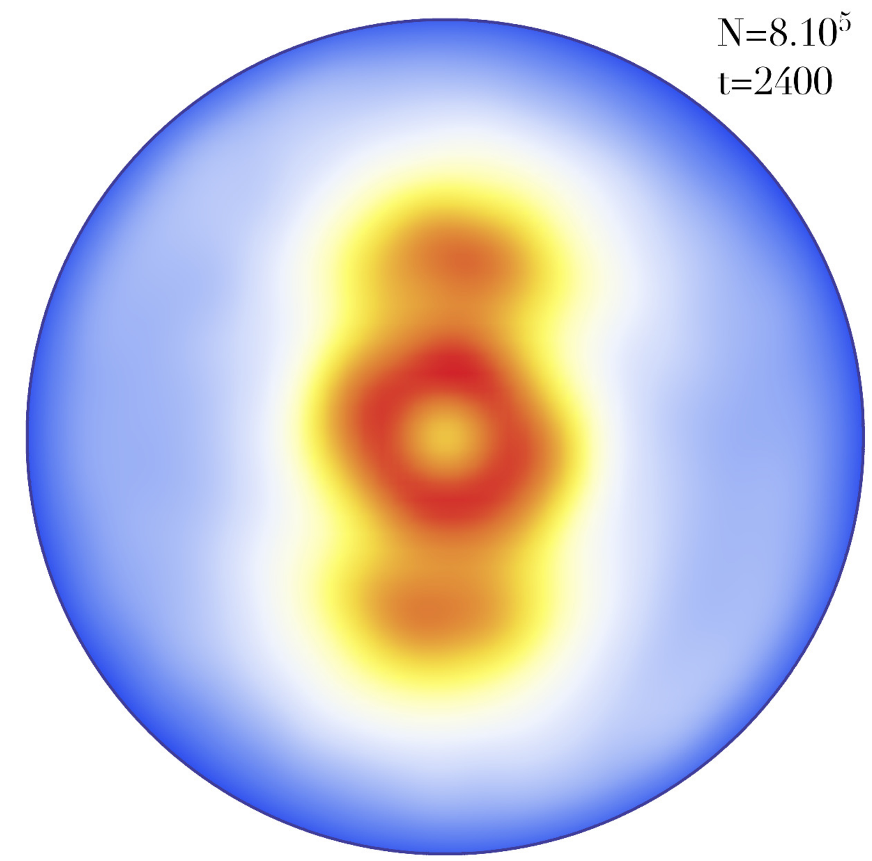
Late time evolution

- **Phase transition: BL $\Longrightarrow$ Vlasov.**





Initial times



Late times

- 2-body (resonant) relaxation $\Longrightarrow$ small-scale structures in the DF

Destabilisation at the collisionless level

'Radial migration' drives the system to a new state of equilibrium which turns out to be unstable: the system then **redistributes** AM on a dynamical timescale

The WKB approach

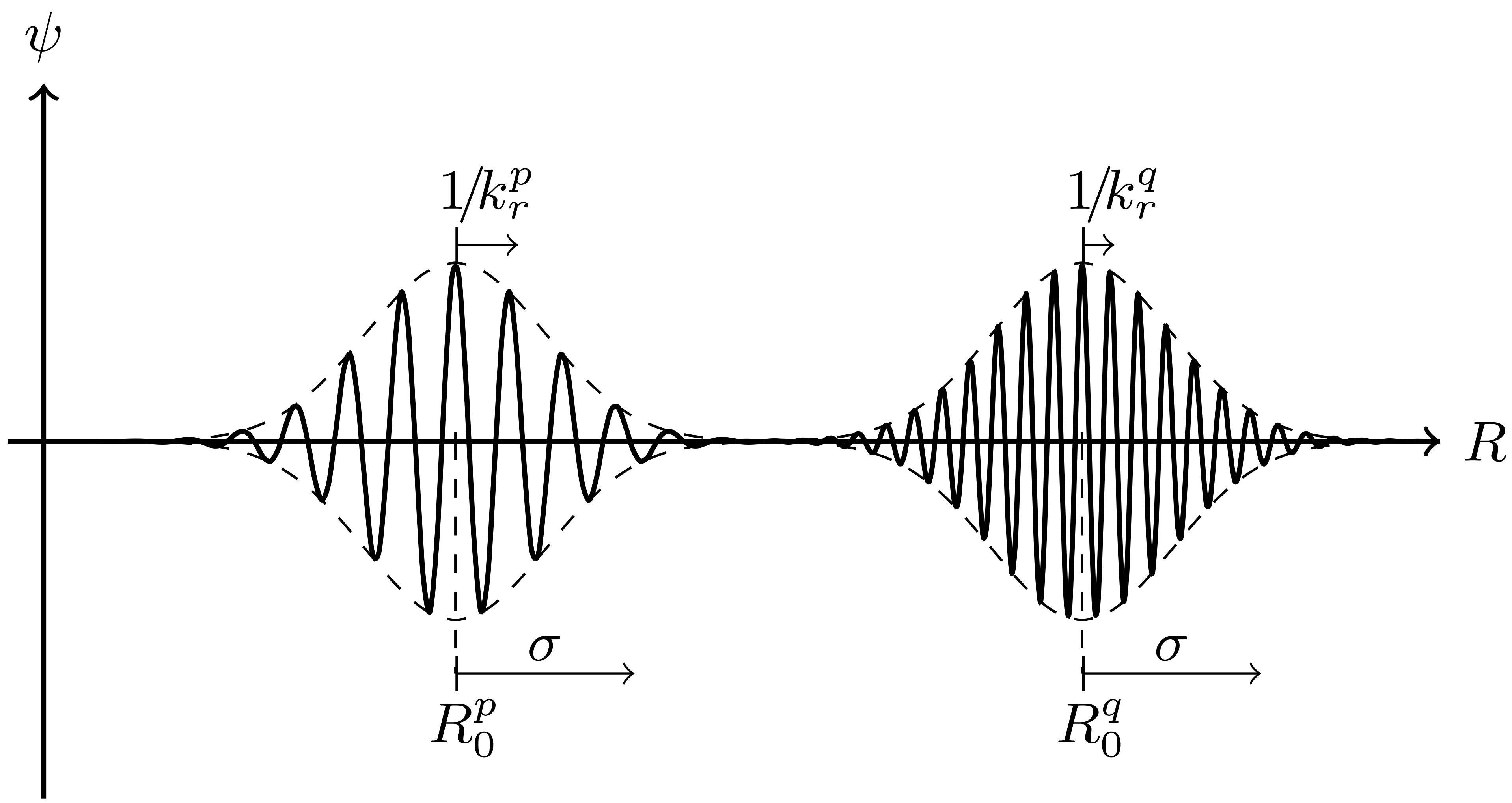
- Difficulty: $\delta\psi \overset{\textbf{Poisson}}{\underset{\textit{non-local}}{\Longleftrightarrow}} \delta\rho = \int \mathrm{d}\boldsymbol{v}\, \delta F\,.$

  $\Rightarrow$ Restriction to **tightly wound perturbations** (WKB approximation).

- New **wavelet basis**

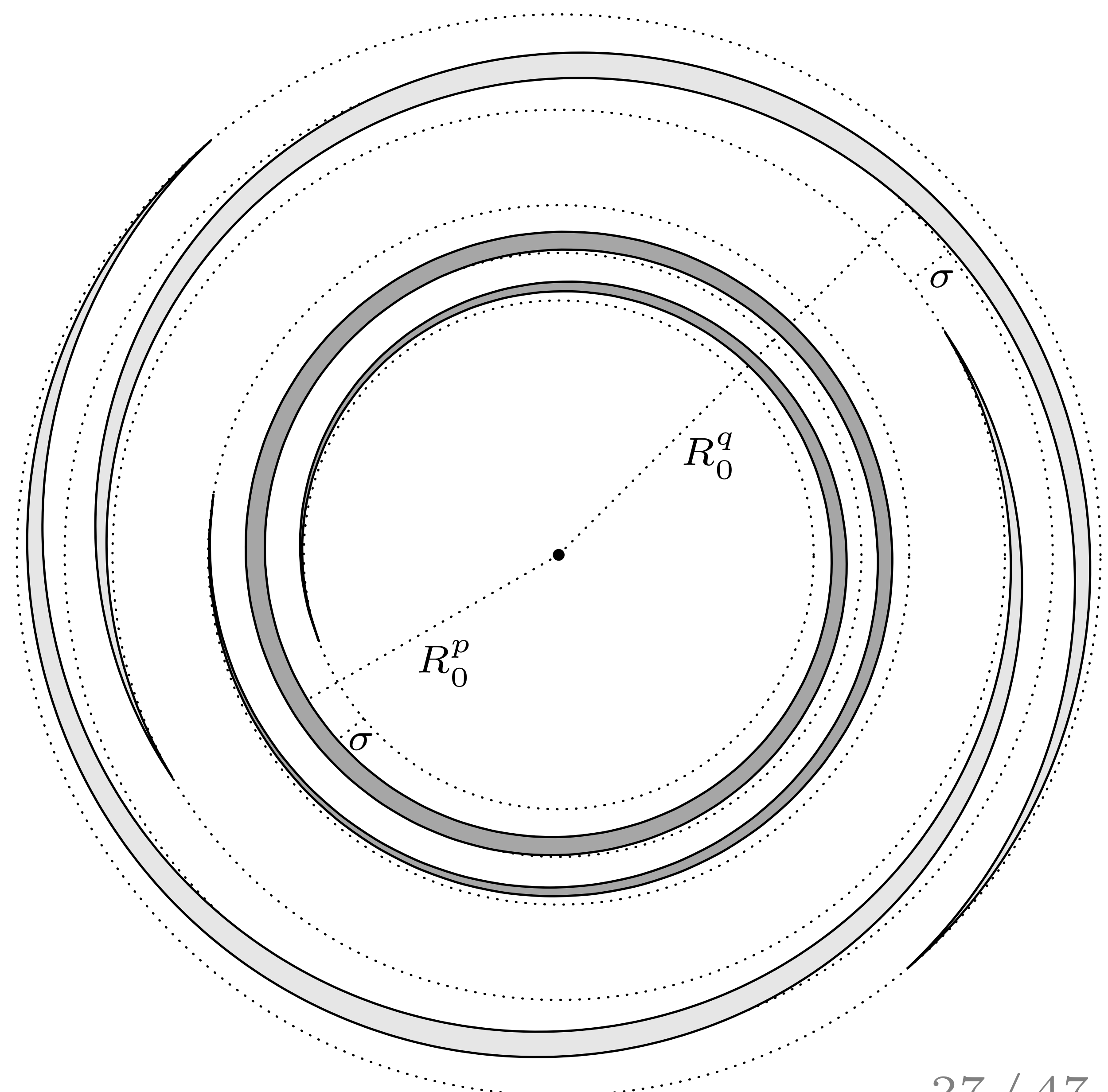
$$\boxed{\psi^{(p)} = \psi^{[k_r, k_\phi, R_0]}(R,\phi) = \mathcal{A}\,\mathrm{e}^{\mathrm{i}(k_r R + k_\phi \phi)}\,\exp\!\left[-\frac{(R-R_0)^2}{2\sigma^2}\right].}$$

$\Longrightarrow$ **Explicit, biorthogonal and both local and global**.



Fouvry et al. (2015)

radial view



2D view

The WKB calculation

- **Diagonal** response matrix

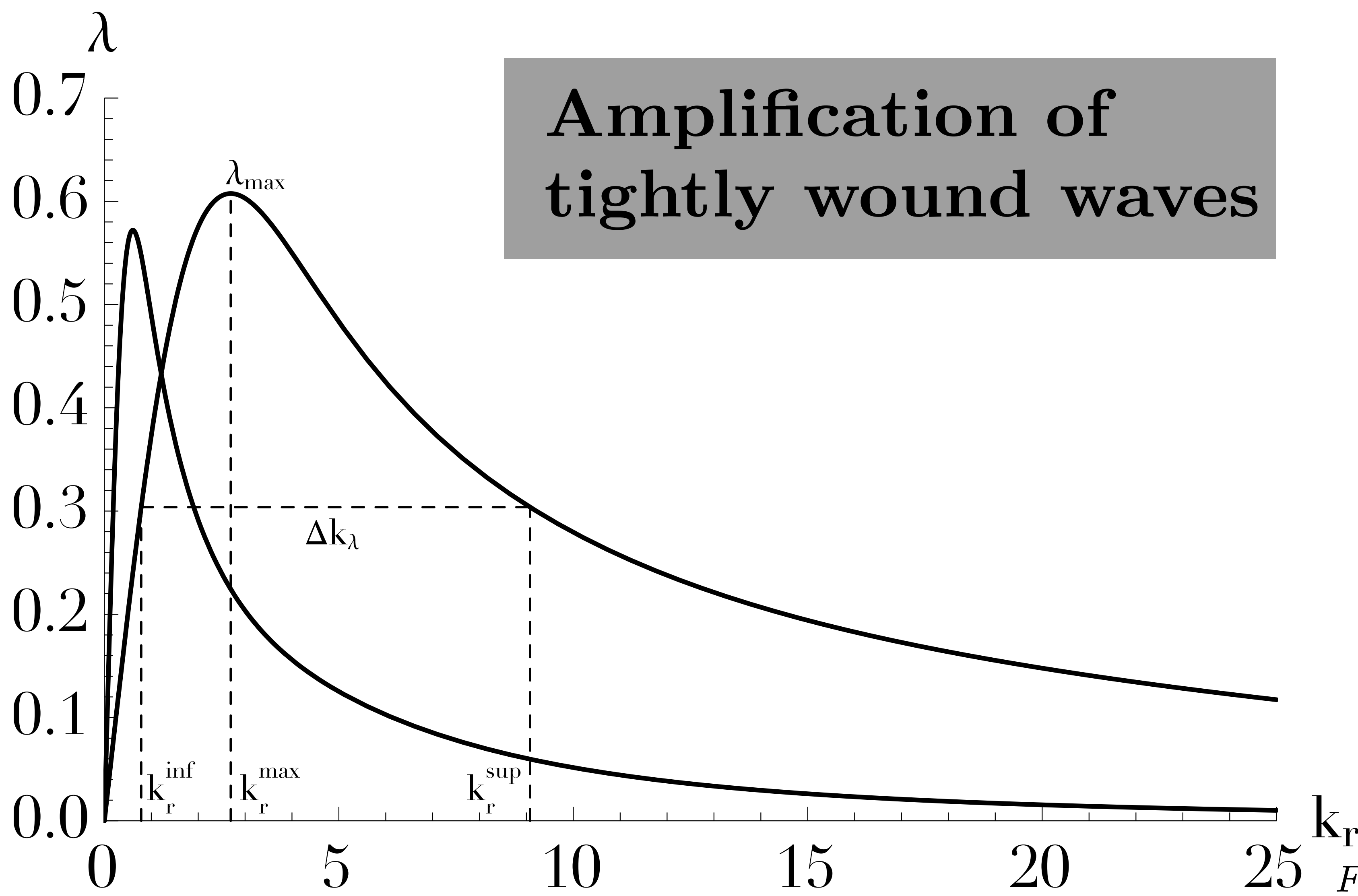
$$\boxed{\widehat{\mathbf{M}}_{pq} = \widehat{\mathbf{M}}_{[k_r^p, k_\phi^p, R_0],[k_r^q, k_\phi^q, R_0]} = \delta_{k_r^p}^{k_r^q}\, \delta_{k_\phi^p}^{k_\phi^q}\, \lambda_{[k_r^p, k_\phi^p, R_0]}\,.}$$

$$\boxed{\lambda(k_r, k_\phi, R_0) = \frac{2\pi G \xi \Sigma}{\kappa^2 (1 - s^2)} \mathcal{F}(s, \chi)}\,.$$

$$\begin{cases} * \, s = \dfrac{\omega - k_\phi \Omega}{\kappa}\,, \\ * \, \chi = \dfrac{\sigma_r^2 k_r^2}{\kappa^2}\,, \\ * \, \mathcal{F}(s,\chi) \quad \text{(reduction factor)}\,. \end{cases}$$

Kalnajs (65), Lin&Shu(66)

Amplification of tightly wound waves



Fouvry et al. (2015)

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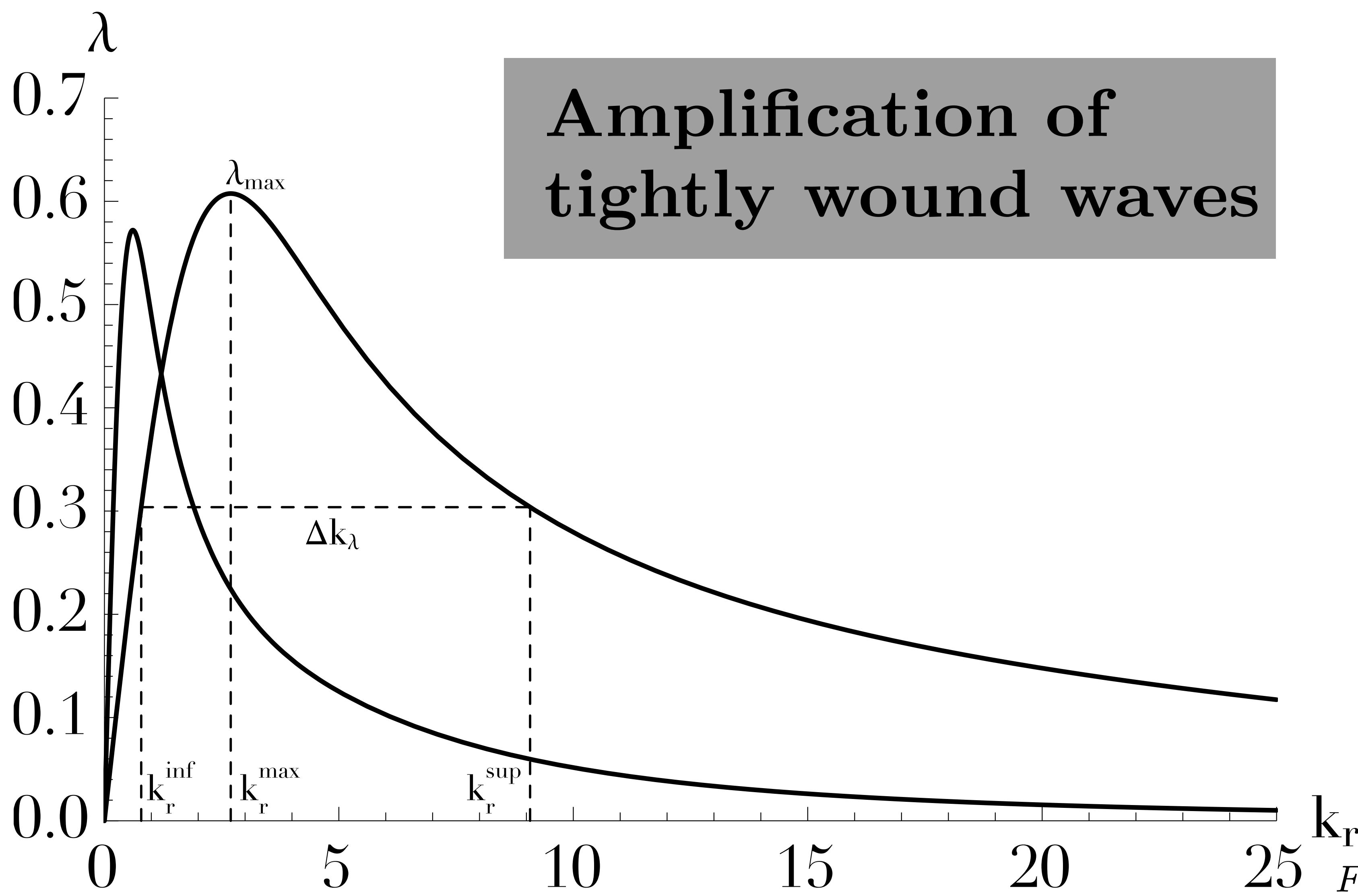
(recovers the Toomre-Lin-Shu dispersion relation).

- Restriction to **local resonances**: $\delta_{\mathrm{D}}(\boldsymbol{m}_1 \cdot \boldsymbol{\Omega}_1 - \boldsymbol{m}_2 \cdot \boldsymbol{\Omega}_2)$

$$\begin{cases} \boldsymbol{m}_1 \cdot \boldsymbol{\Omega}_1(R_1) - \boldsymbol{m}_2 \cdot \boldsymbol{\Omega}_2(R_2) = 0 \\ |R_1 - R_2| \leq (\mathrm{few})\sigma \end{cases} \Longrightarrow \begin{cases} \boldsymbol{m}_2 = \boldsymbol{m}_1 \,, \\ R_2 = R_1 \,. \end{cases}$$



Fouvry et al. (2015)

The WKB calculation

- **Diagonal** response matrix

$$\boxed{\widehat{\mathbf{M}}_{pq} = \widehat{\mathbf{M}}_{[k_r^p, k_\phi^p, R_0],[k_r^q, k_\phi^q, R_0]} = \delta_{k_r^p}^{k_r^q} \, \delta_{k_\phi^p}^{k_\phi^q} \, \lambda_{[k_r^p, k_\phi^p, R_0]} \, .}$$

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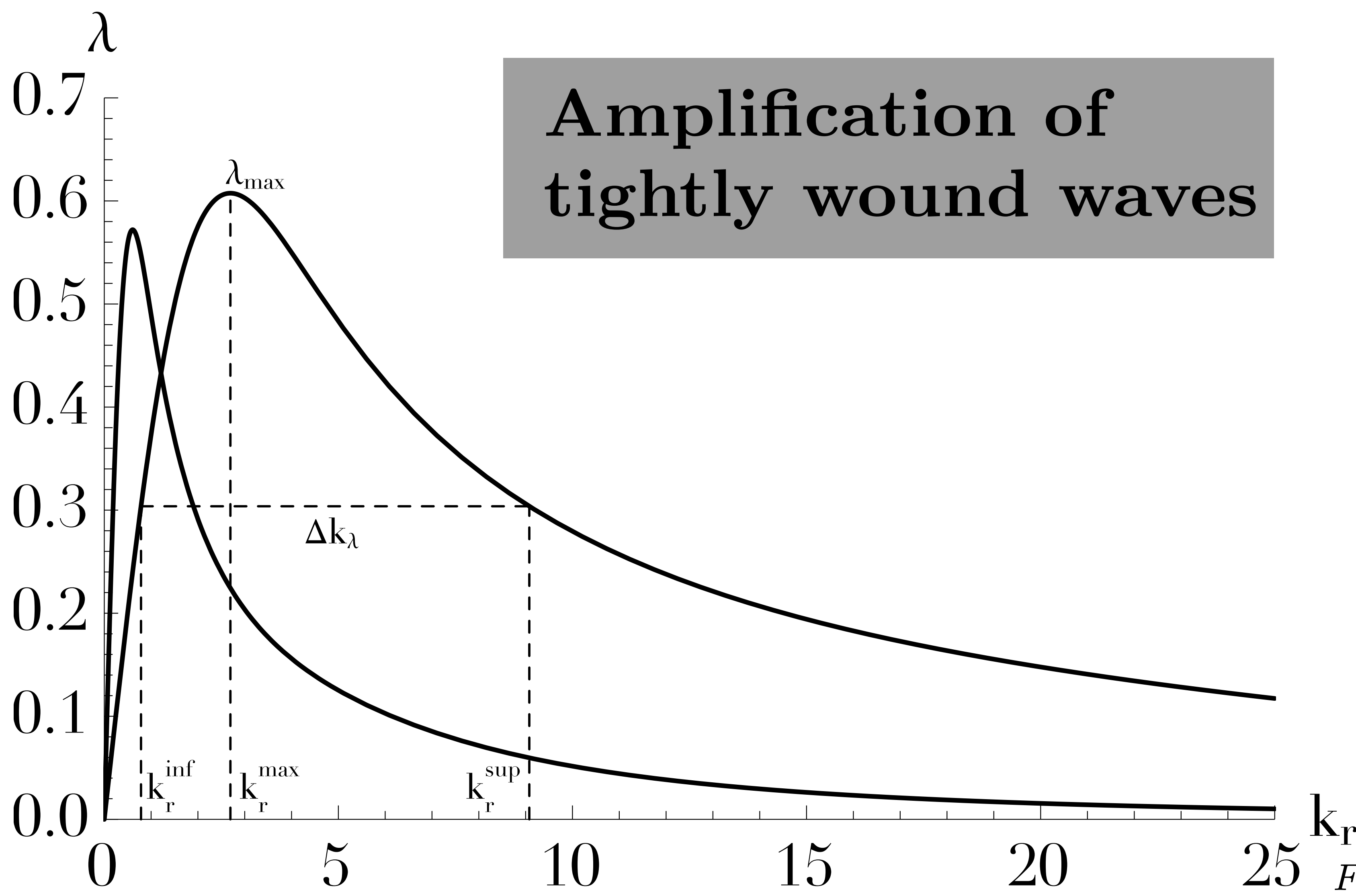
- **Explicit quadratures** for the dressed diffusion flux



Amplification of tightly wound waves

Fouvry et al. (2015)

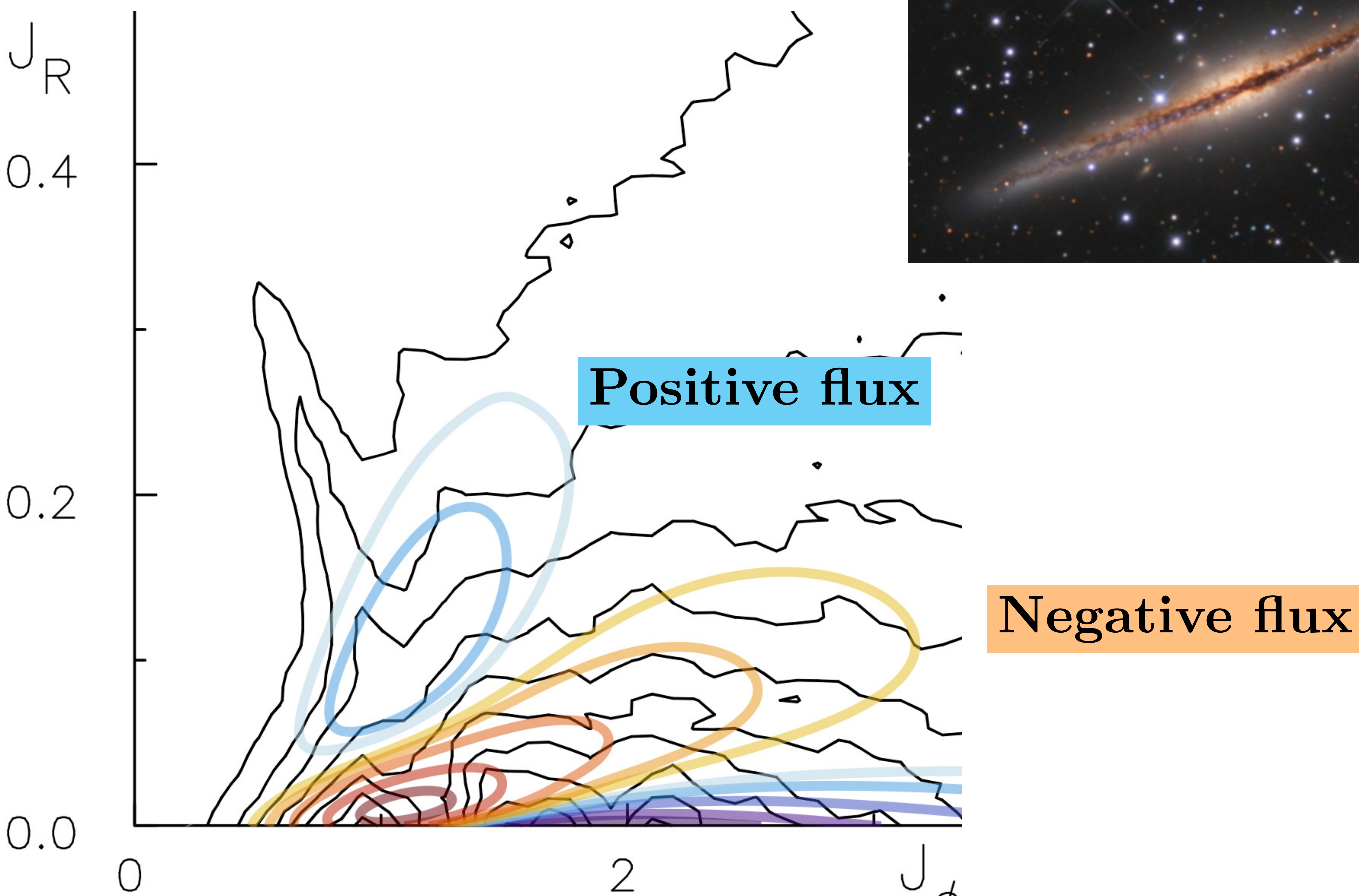
Balescu-Lenard WKB - Application

- **Diffusion flux** in action space

Radial migration + disc heating

$$\frac{\partial F}{\partial t} = \mathrm{div}\big(\boldsymbol{\mathcal{F}}_{\mathrm{tot}}(\boldsymbol{J})\big)\,.$$

- Predicted contours for $\mathrm{div}\big(\boldsymbol{\mathcal{F}}_{\mathrm{tot}}\big)(t\!=\!0^{+})$



Fouvry et al. (2015)

The effect of the Halo mass

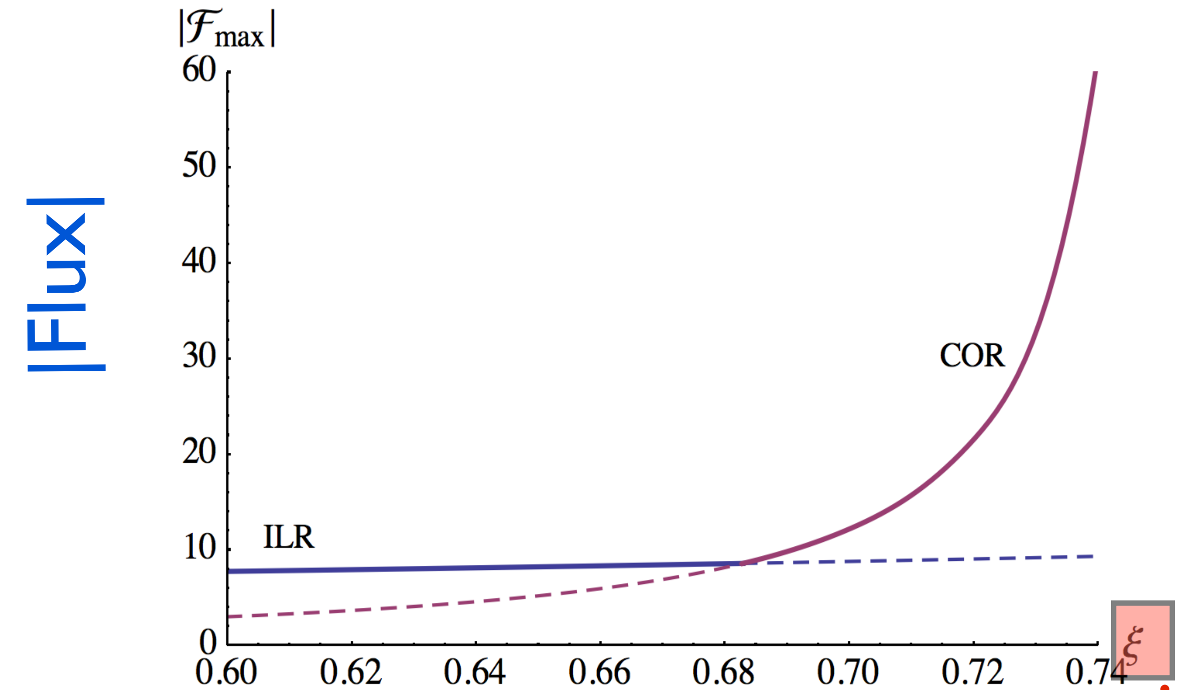
- Importance of the response eigenvalues

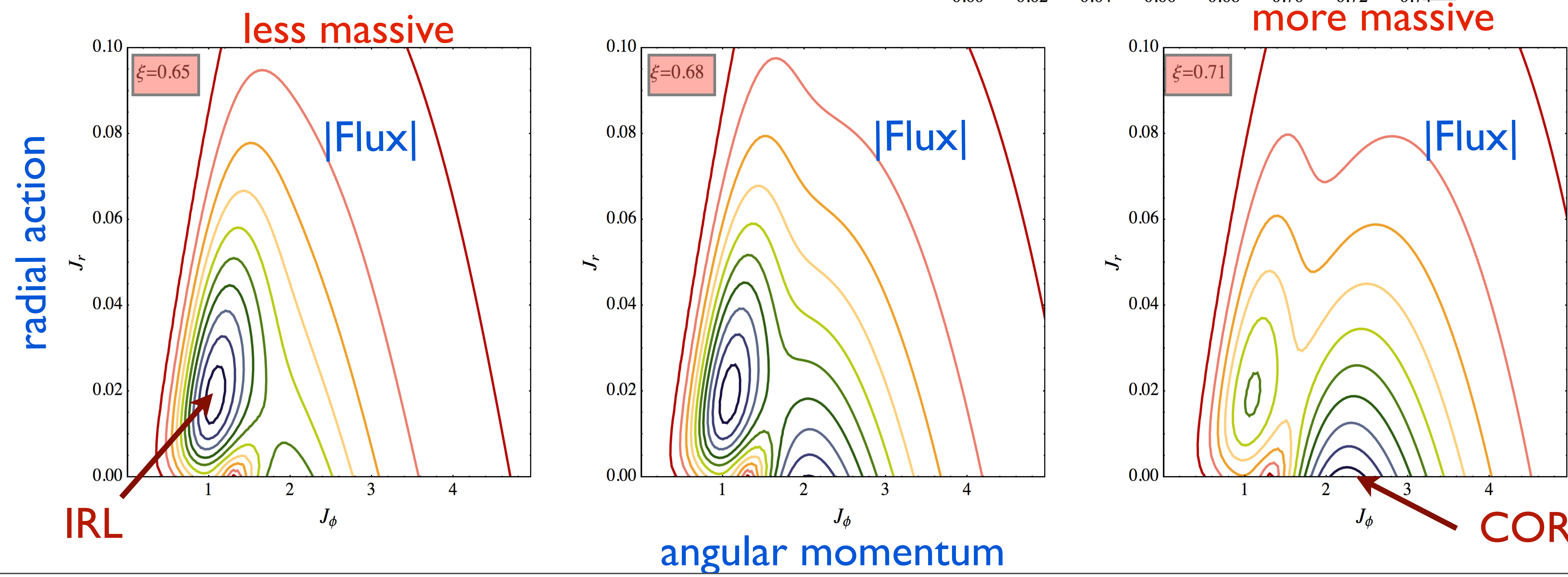
$$\boxed{\lambda(k_r, k_\phi, R_0) = \frac{2\pi G \boxed{\xi} \Sigma}{\kappa^2(1-s^2)}\mathcal{F}(s,\chi)}$$

$$\begin{cases} * \, s = \dfrac{\omega - k_\phi \Omega}{\kappa} \, , \\ * \, \chi = \dfrac{\sigma_r^2 k_r^2}{\kappa^2} \, , \\ * \, \mathcal{F}(s,\chi) \quad (\text{reduction factor}) \, . \end{cases}$$

- Increase of the active fraction ξ
  - Transition: ILR $\longrightarrow$ COR
  - *heating* $\longrightarrow$ *radial migration*

Blurring Churning

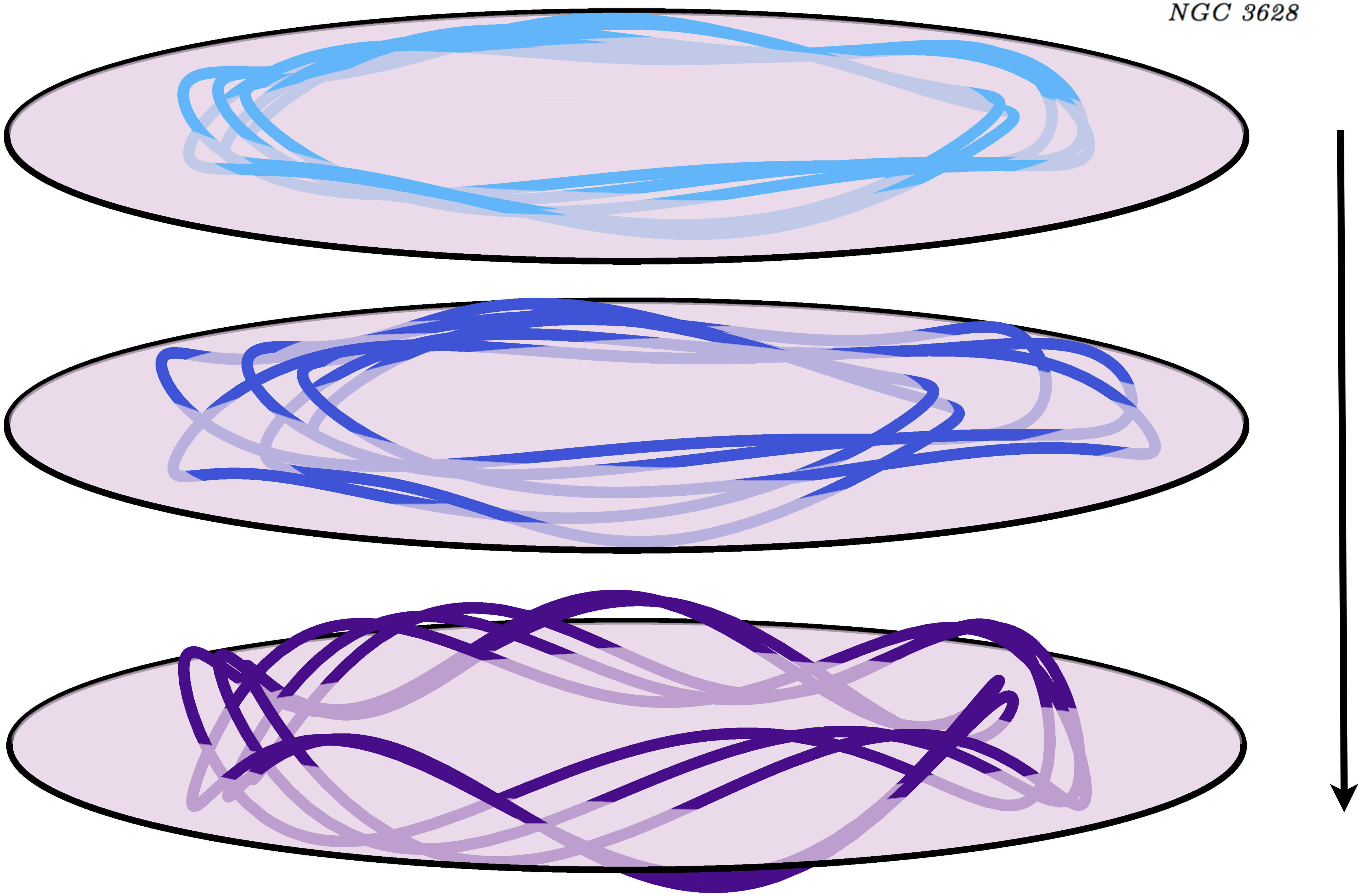




Secular disc thickening ?

NGC 3628

vertical ridges

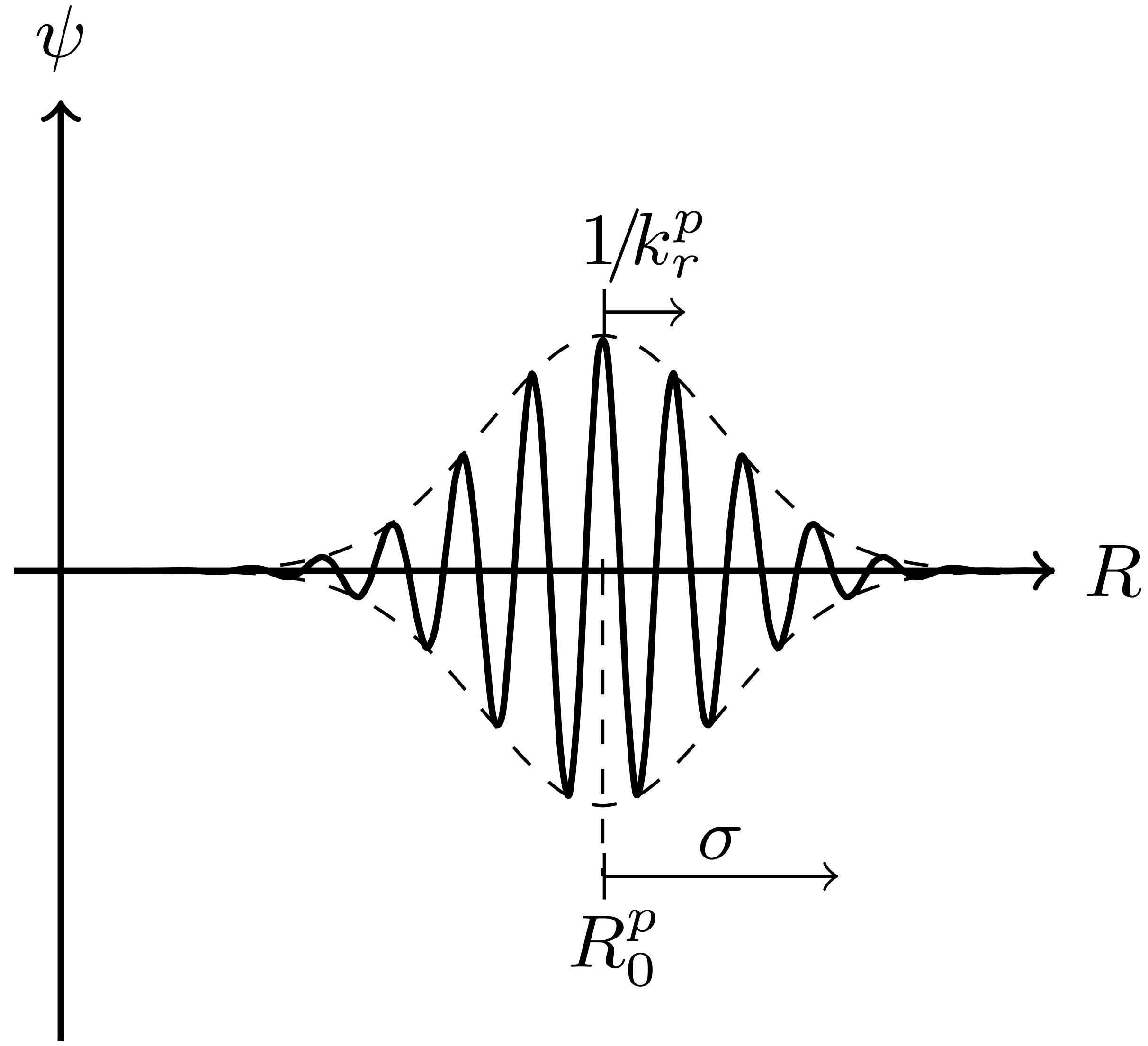


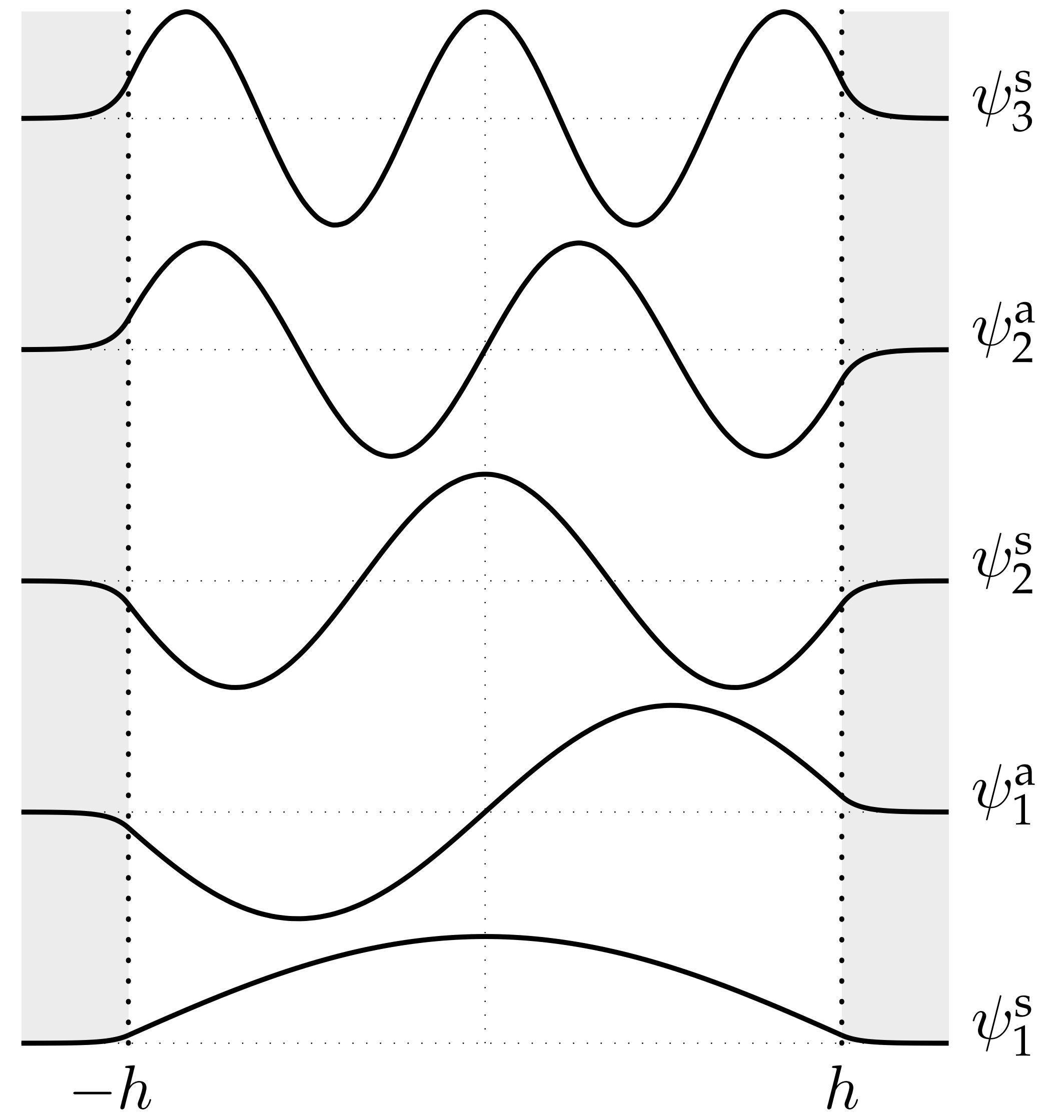
Vertical distorsion of orbit

Treat self consistently thickening due to spiral waves and GMC deflections

Stellar disc thickening

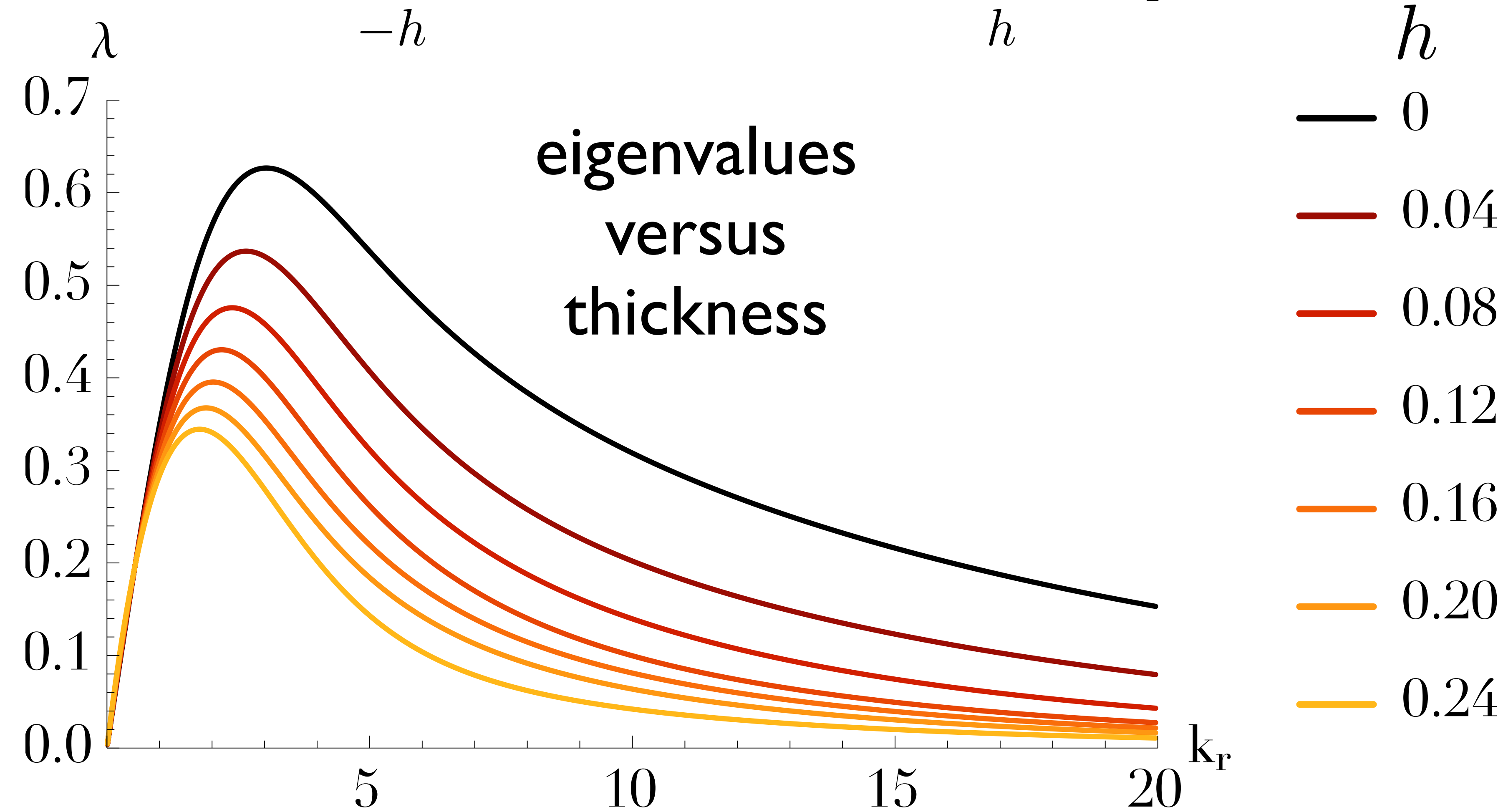
- **Thick discs** are not guaranteed to be integrable
  $\Longrightarrow$ Generalisation of the **WKB approximation**





- Same results hold
  - **Biorthogonal basis**
  - **Diagonal $\widehat{\mathbf{M}}$**
  - **Local resonances**
  - New **thickened** Q factor

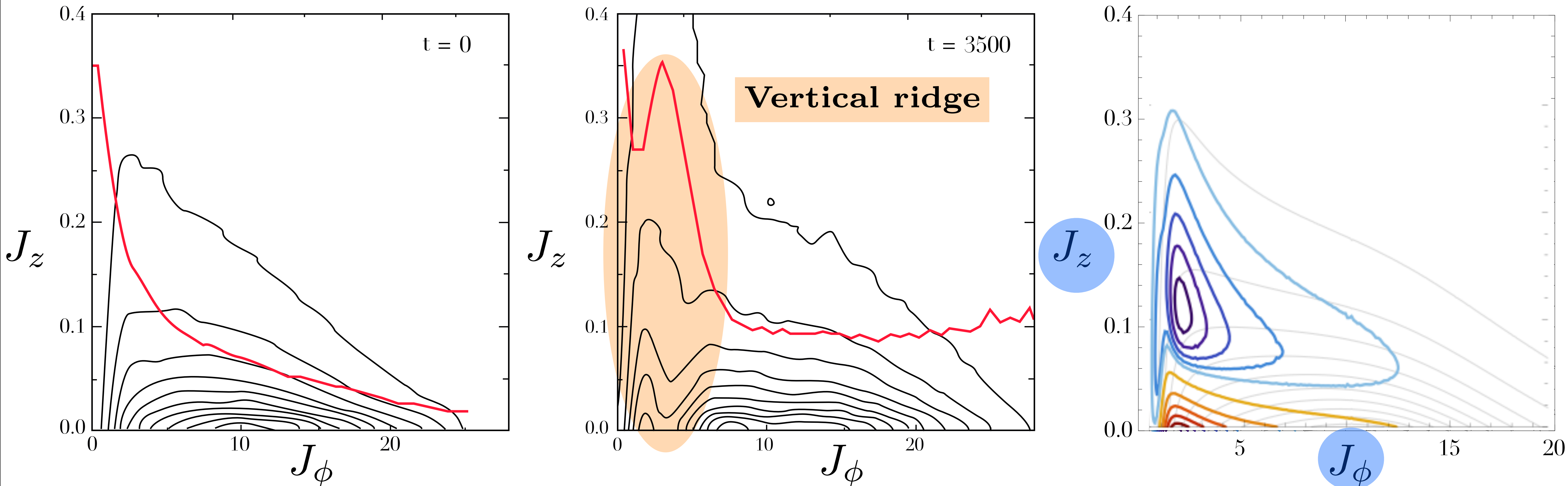
$$\boxed{Q_{\mathrm{thick}} = Q_{\mathrm{thin}} \exp\left[1.61 \frac{\sigma_z/\nu}{\sigma_r/\kappa}\right]}$$



Fouvry et al. (2016)

Stellar disc thickening

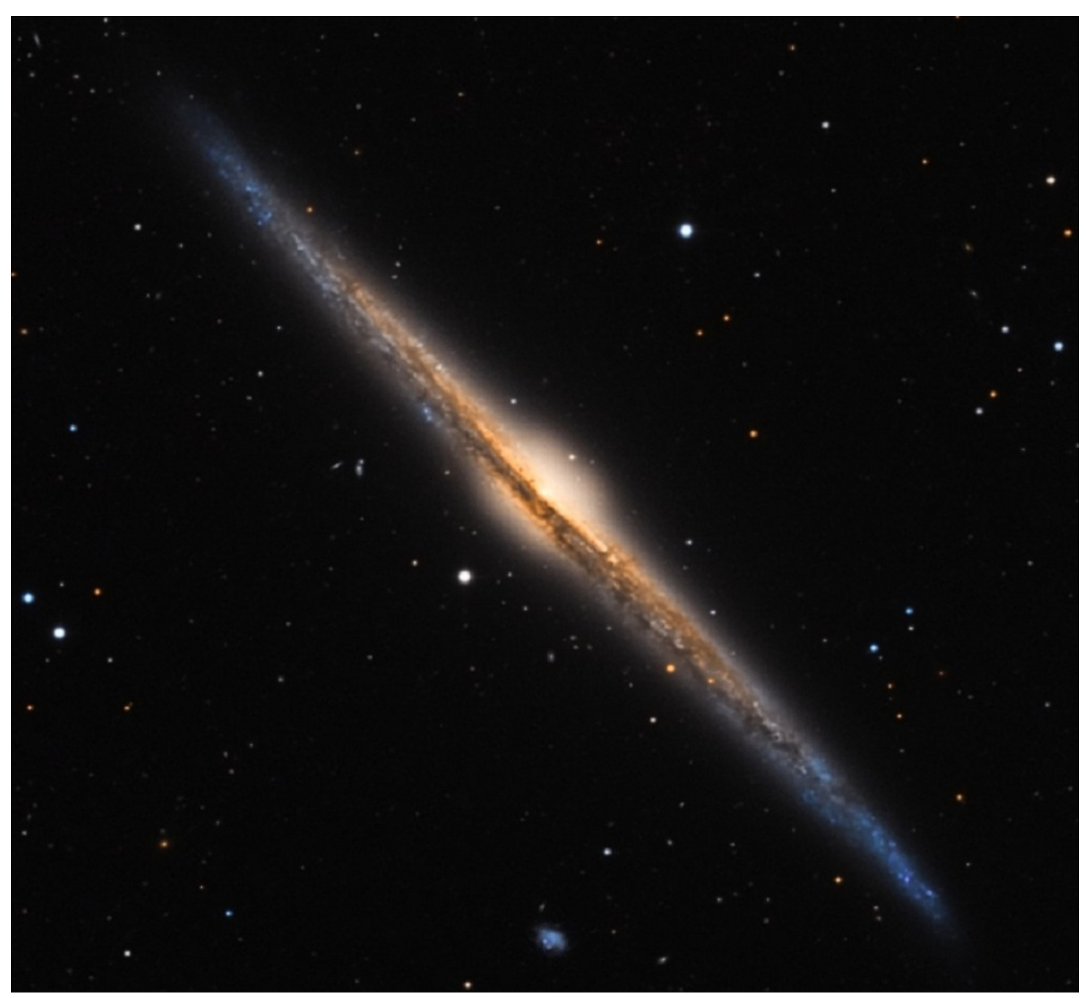
- Spontaneous thickening via **Poisson shot noise/GMCs**



Initial times *Solway (2012)*

Late times (N−body) *Solway (2012)*

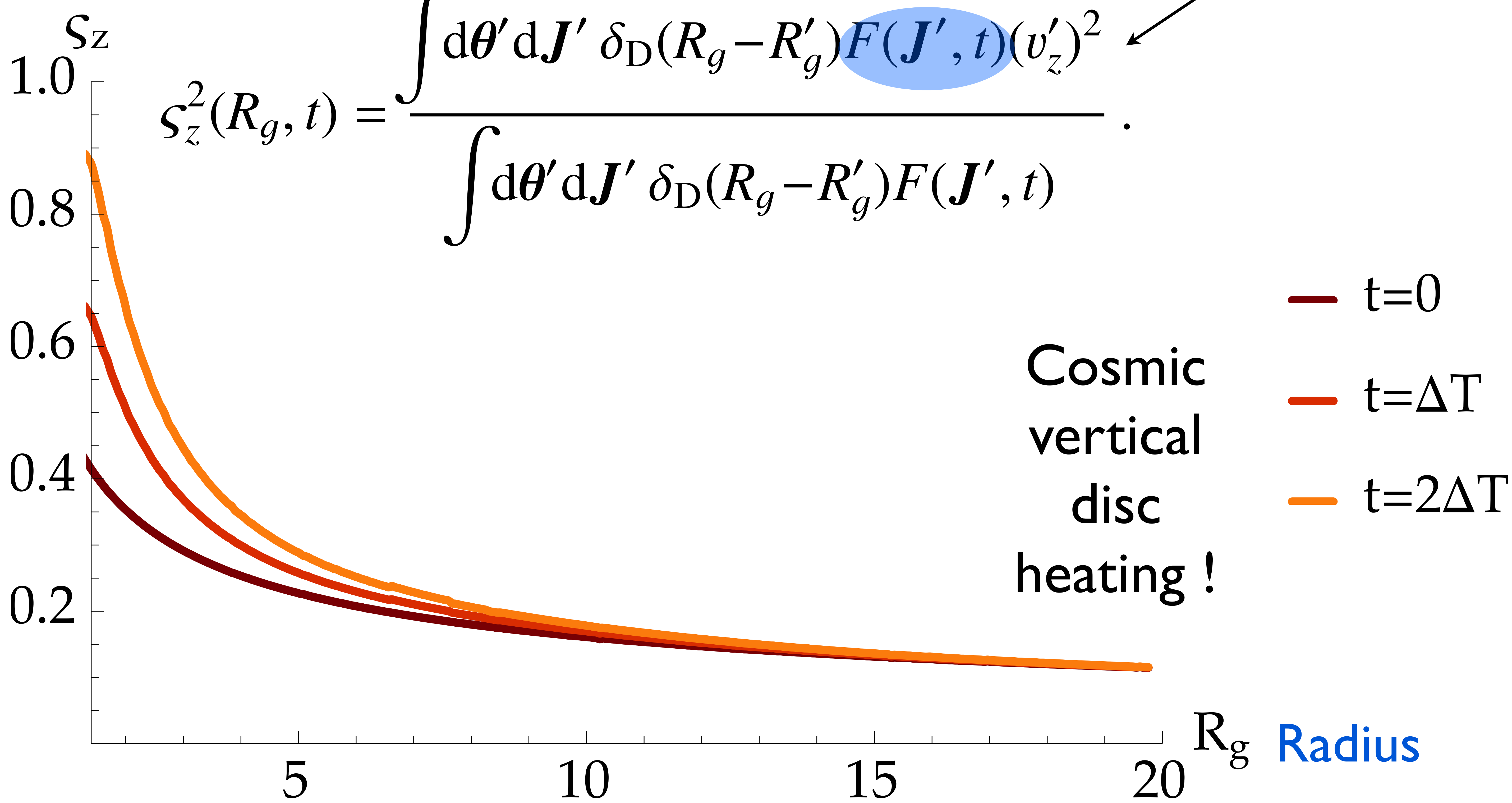
BL - WKB *Fouvry et al. (2016)*



Stellar disc thickening

vertical velocity

$$\varsigma_z^2(R_g,t) = \frac{\int \mathrm{d}\boldsymbol{\theta}'\,\mathrm{d}\boldsymbol{J}'\;\delta_{\mathrm{D}}(R_g - R_g')\,F(\boldsymbol{J}',t)\,(v_z')^2}{\int \mathrm{d}\boldsymbol{\theta}'\,\mathrm{d}\boldsymbol{J}'\;\delta_{\mathrm{D}}(R_g - R_g')\,F(\boldsymbol{J}',t)}\,.$$



Cosmic vertical disc heating !

Illustration of the increase of the velocity dispersion $\varsigma_z(R_g,t)$

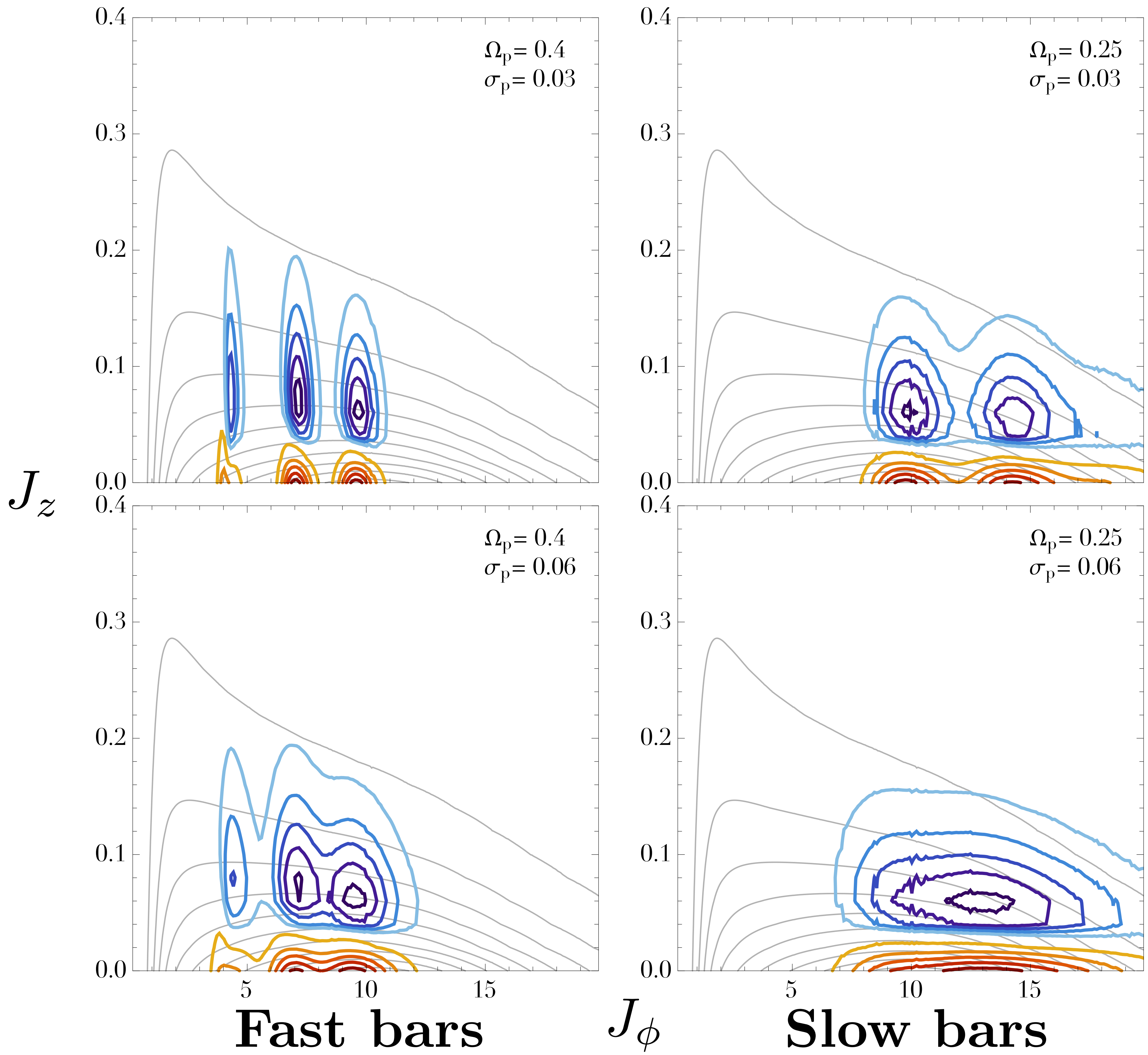
Disc thickening - Bars

- The effect of **central decaying bars**

$$\widehat{C}[J_\phi,\omega] \propto \boxed{A_{\mathrm{b}}(J_\phi)}\, \exp\!\left[-\frac{(\omega - \boxed{m_{\mathrm{p}}\Omega_{\mathrm{p}}})^2}{2\,\boxed{\sigma_{\mathrm{p}}^2}}\right] \Longrightarrow \begin{cases} \boxed{A_{\mathrm{b}}}: \text{ Bar profile} \\ \boxed{\Omega_{\mathrm{p}}}: \text{ Pattern speed} \\ \boxed{\sigma_{\mathrm{p}}}: \text{ Decay frequency} \end{cases}$$

Long-lived bars

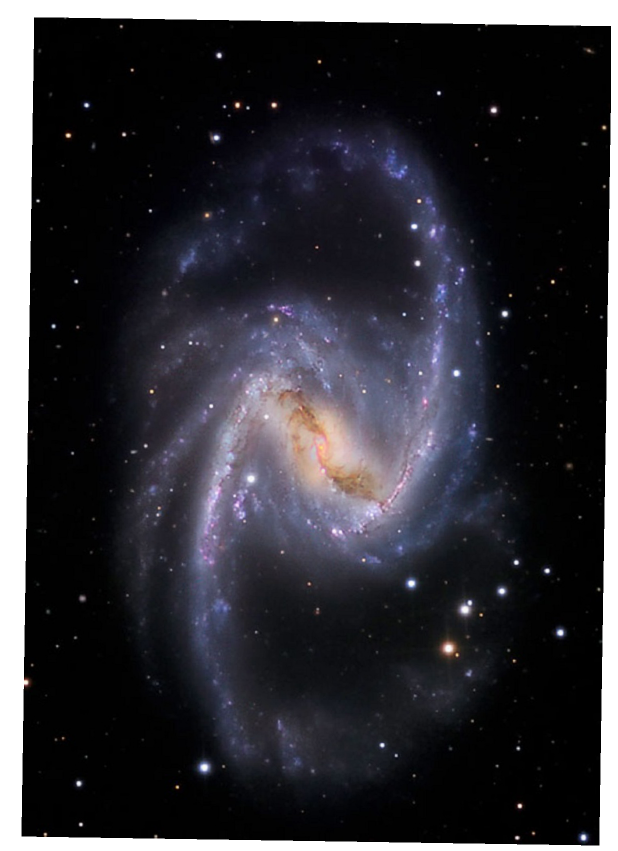
Short-lived bars



Fast bars **Slow bars**

Positive flux

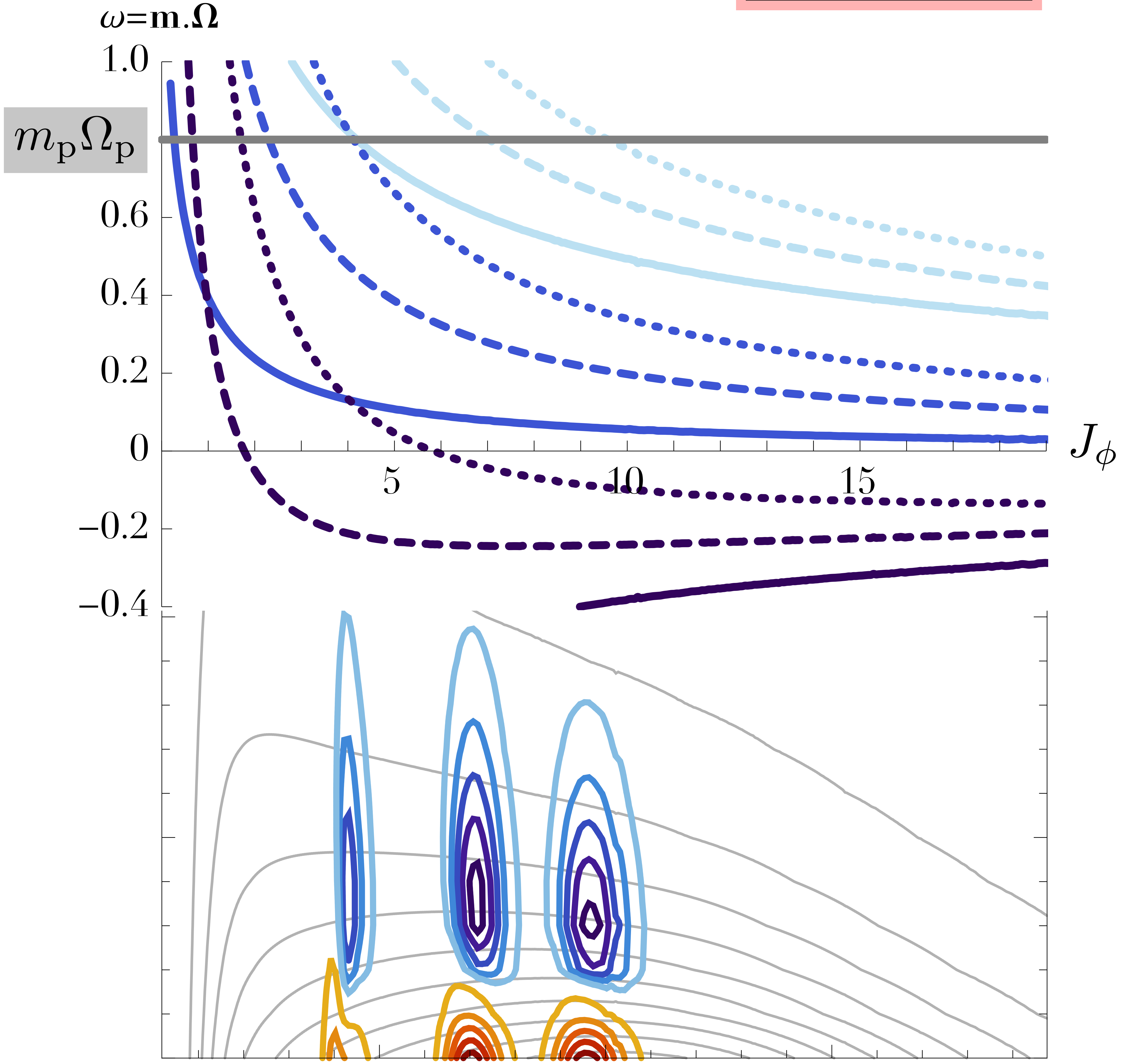
Negative flux

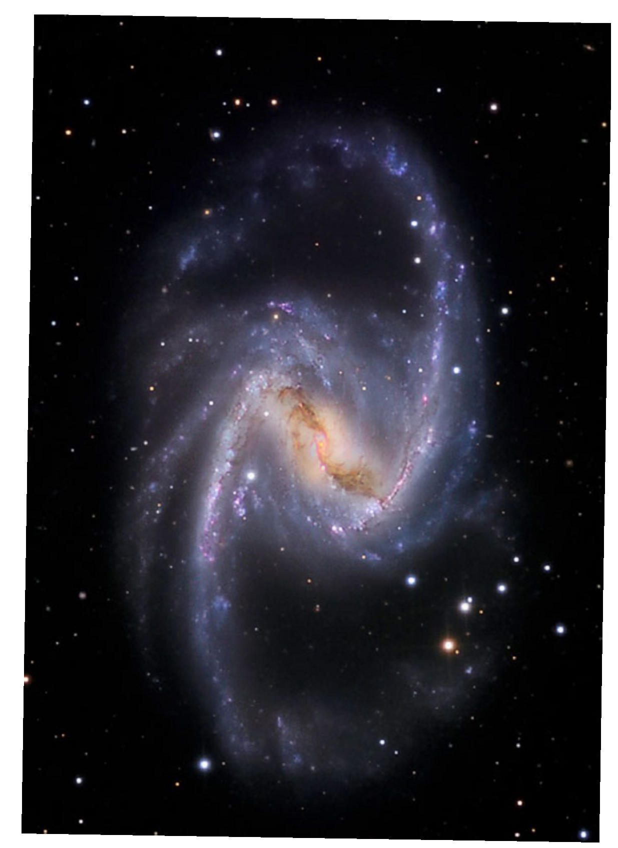


The role of resonances

- Perturbations evaluated **at resonance**: $\boxed{\widehat{C}(\omega\!=\!\boldsymbol{m}\cdot\boldsymbol{\Omega})}$

$\boldsymbol{m}=(m_\phi, m_r, m_z)$





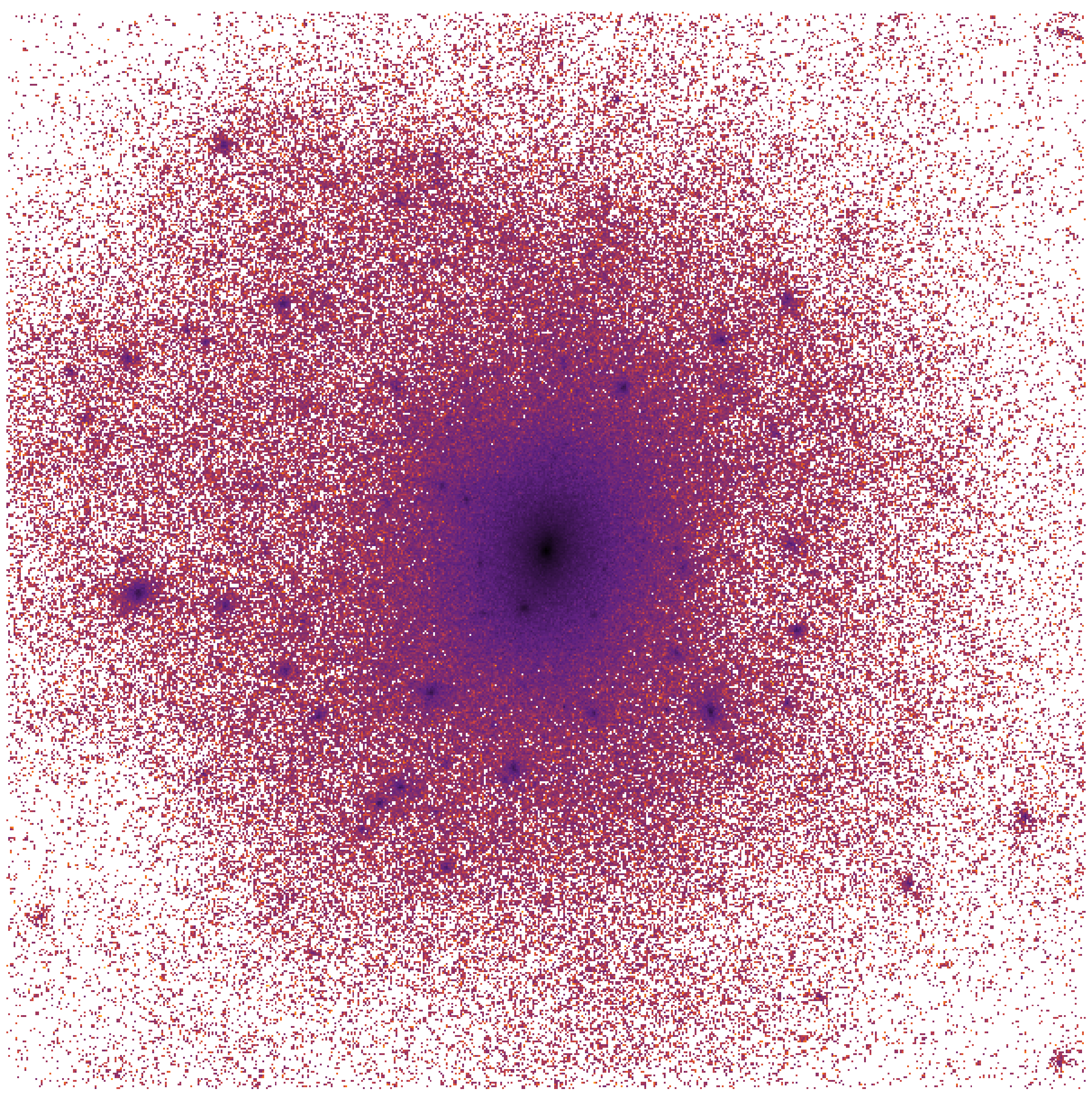
Wednesday, 3April, 19

The role of external perturbations

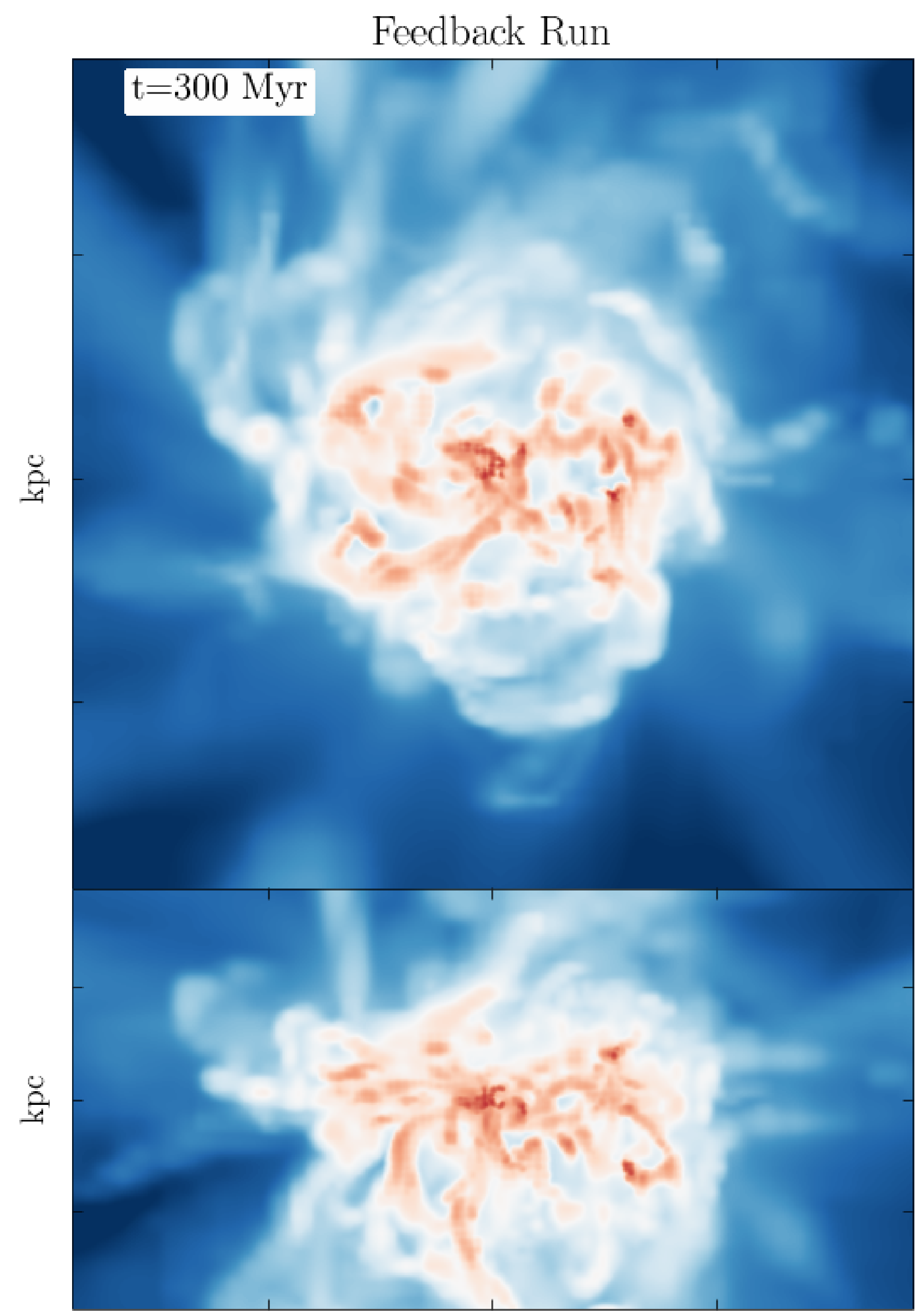
- Diffusion sourced by **stochastic fluctuations**

$$\delta\psi^{\mathrm{ext}}(\boldsymbol{x},t)=\sum_p b_p(t)\,\psi^{(p)}(\boldsymbol{x}) \quad\Longrightarrow\quad \boxed{C_{pq}(t_1-t_2)=\big\langle b_p(t_1)\,b_q^*(t_2)\big\rangle\,.}$$

- Exemple of perturbations: **Halo** $\Longleftrightarrow$ **Disc**.
  - **Dark matter clumps**
    Halo $\rightsquigarrow$ Disc
  - **Supernova feedback**
    Disc $\rightsquigarrow$ Halo



500 kpc



Rebekka Bieri - IAP

Method: quasi-linear theory

Why Kinetic Theory?	
Universal	Applied throughout science.
Statistical	Mean response & fluctuations/ Parameter space exploration.
Modular/flexible	Add/remove physical processes: metals+sinks+mass spectrum.
Very large timesteps	Gravitational polarization built in.
Non-linear	Nurture/nature split.

CONCLUSIONS

- From **linear response** to **secular evolution**.

$$\boxed{\textbf{Stellar dynamics enters the cosmic framework.}}$$

- Frameworks for the effects of **external and internal** perturbations.

$$\boxed{\textbf{Nature vs. Nurture}}$$

- First implementation of **Balescu-Lenard** in (astro)physics
- Approach complementary to $N-$body and Monte Carlo methods

BL = master equation describing *self-consistently* resonant relaxation

Bring Home Messages

What: Quasi linear theory = stellar version of dissipation-fluctuation theorem
= How do ***orbital structure*** of galaxies ***diffuse*** away from mean field locked trajectory.

How: time decoupling + matrix method (long range : non local + resonances)

Why:

✳Non-linear: qualify perturbation properties as well as equilibrium:
(address nature - nurture conundrum or probe DH).

✳Break the pb by scale (cf zoom simulation) **or** by component (BL - FP);

✳Statistical $\equiv$ ensemble average of sims (cf cosmology)

✳Captures climate not weather;

✳Correct description of Chandrasekhar friction

✳Theory (can be parametrised, expressed in WKB limit, switch off gravity etc..).

Cons:

✳Time decoupling, doesn't capture today's weather;

✳Assumes integrability (but...)

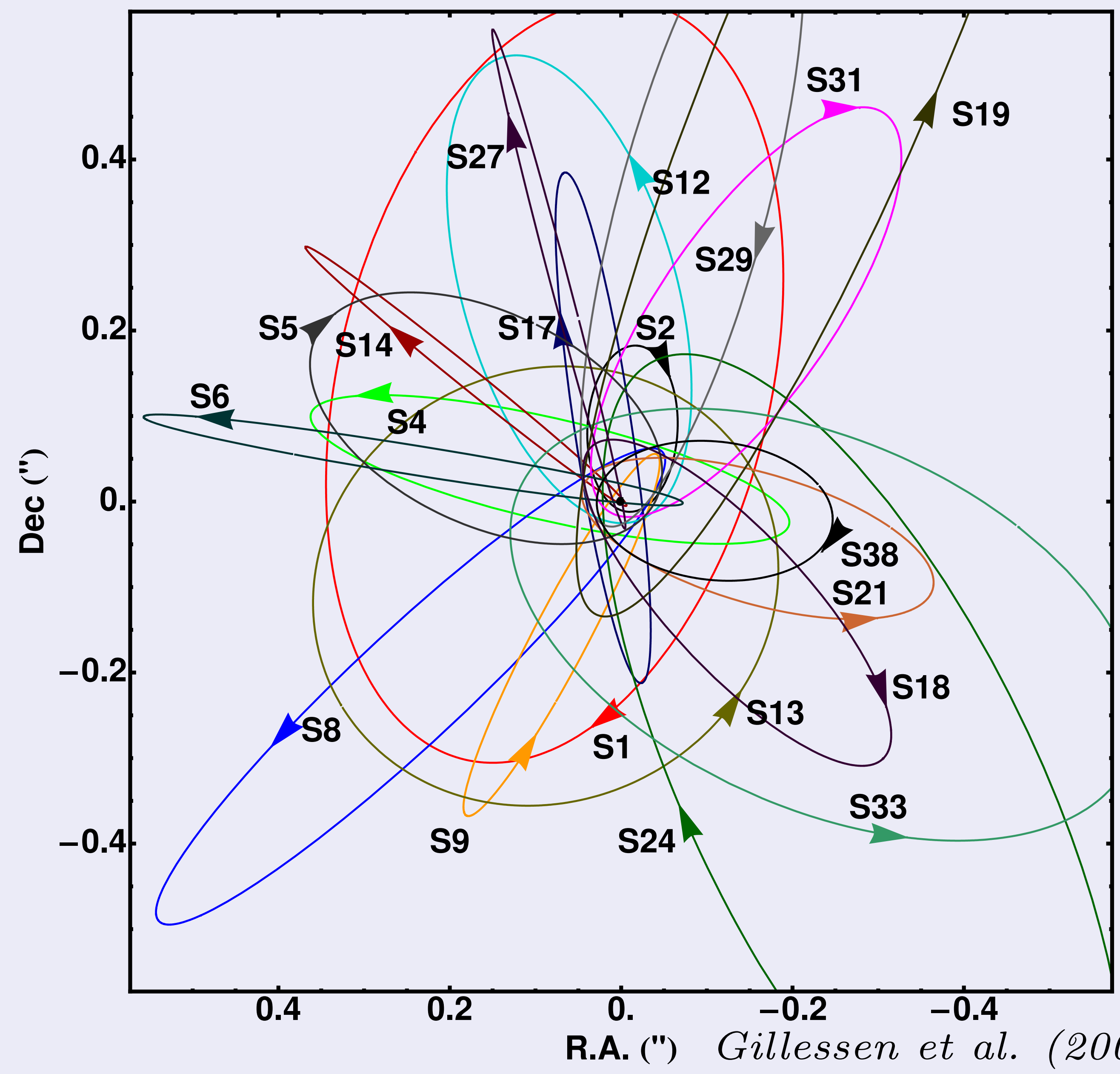
✳Technically not trivial to implement in its full glory (but ...)

For gaia : account for NLs+ resonances for stream+ thick disc+ bars+cusp-core

Application 2

The case of quasi-Keplerian systems

- Describe the secular evolution driven by **finite$-N$ effects** for a **quasi-Keplerian system**
  - **inhomogeneous**
  - **dynamically degenerate**
  - **stable**
  - **self-gravitating**
  - **discrete**



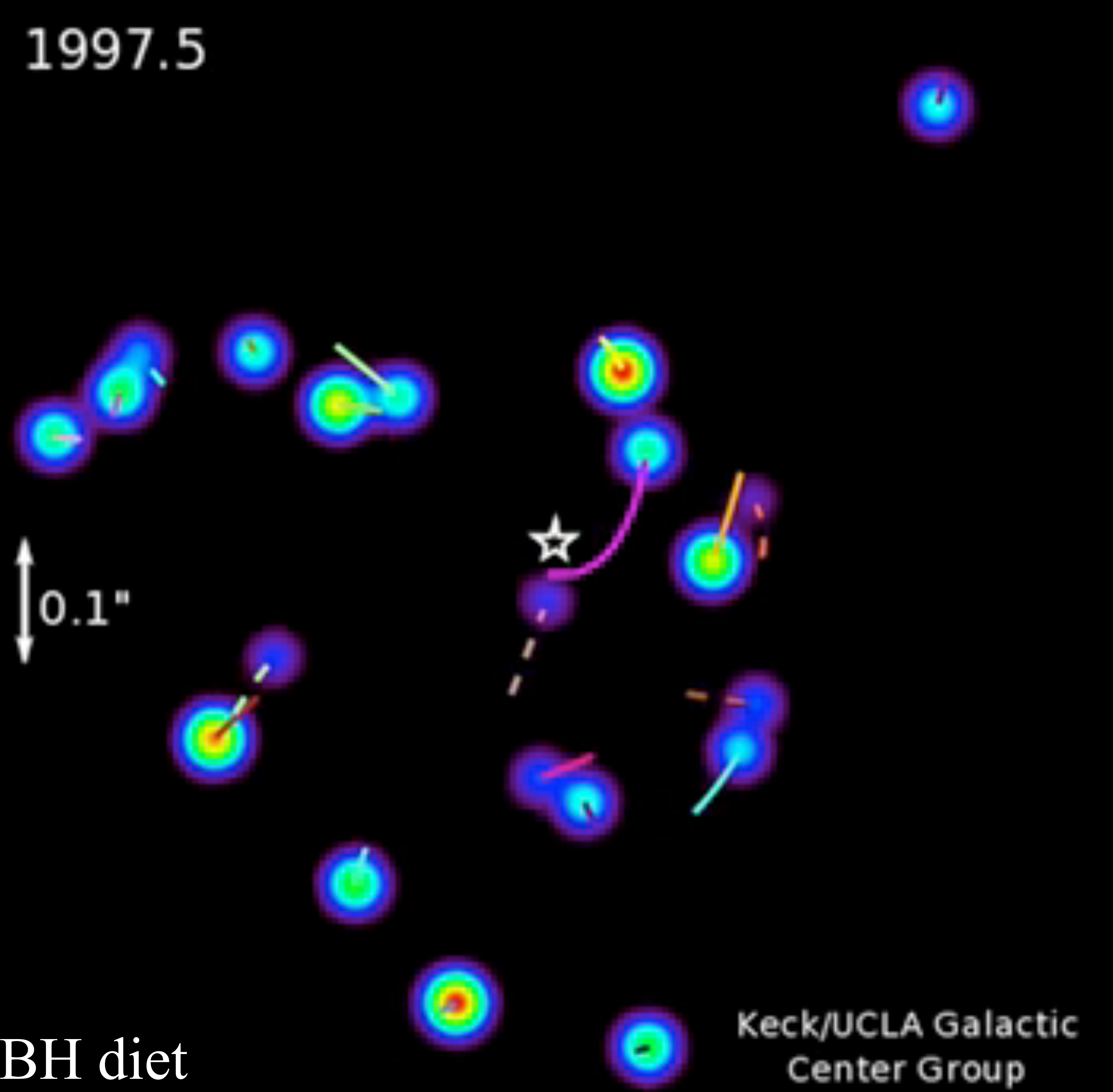
Gillessen et al. (2009)

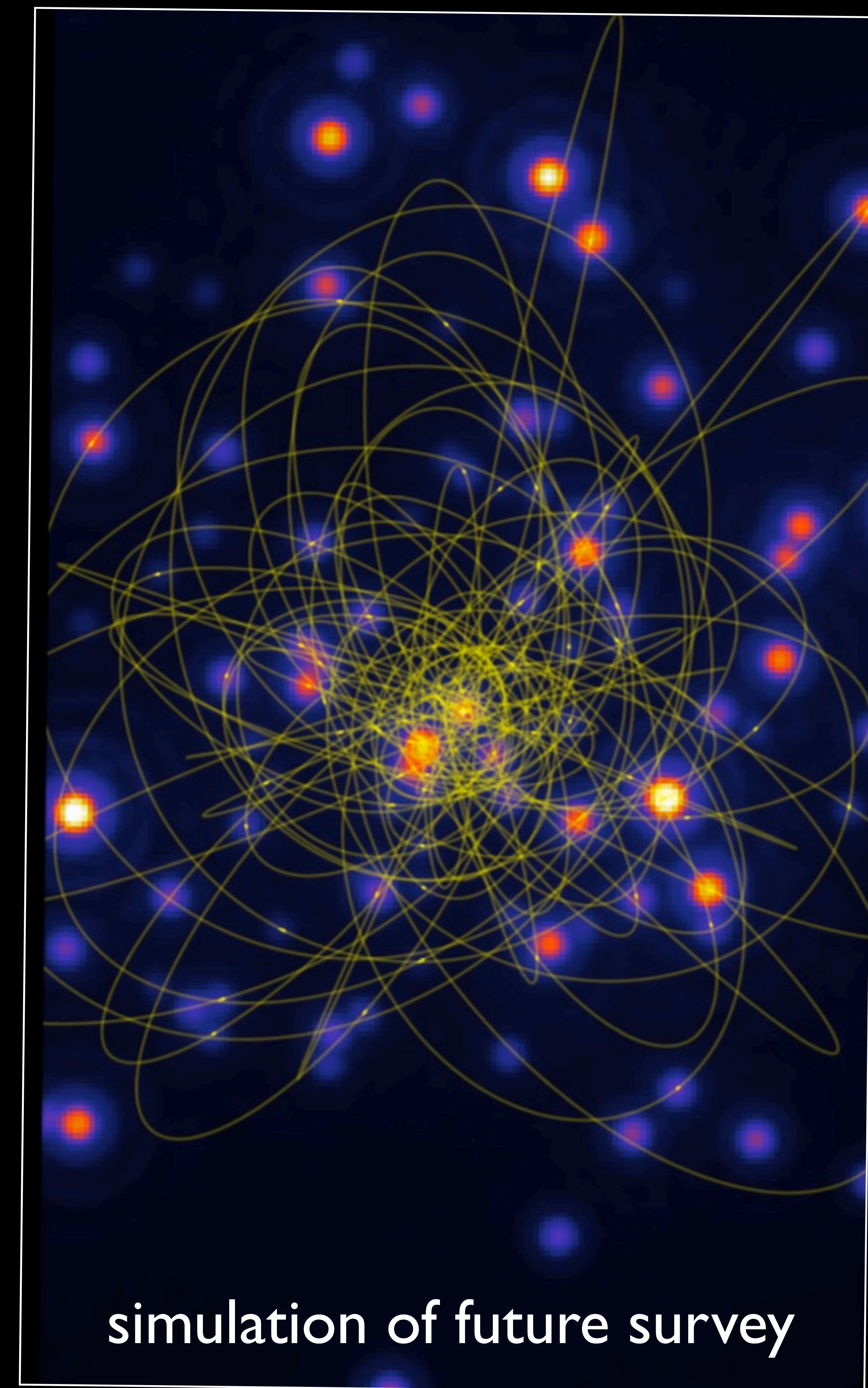
- **How efficiently are BHs fed?**

  Last parsec pb? EMRI? TDE??

- Some references:
  - *Rauch, Tremaine (1996)*: Resonant relaxation
  - *Meritt et al. (2011)*: Schwarzschild barrier
  - *Bar-Or, Alexander (2014, 2016)*: $\eta-$formalism
  - *Sridhar, Touma (2016)*: Gilbert's method for Landau
  - *Fouvry, Pichon, Magorrian (2016)*: BBGKY approach
  - *Fouvry, Pichon, Chavanis (2017)*: First Implementation

Galactic Center stellar cluster





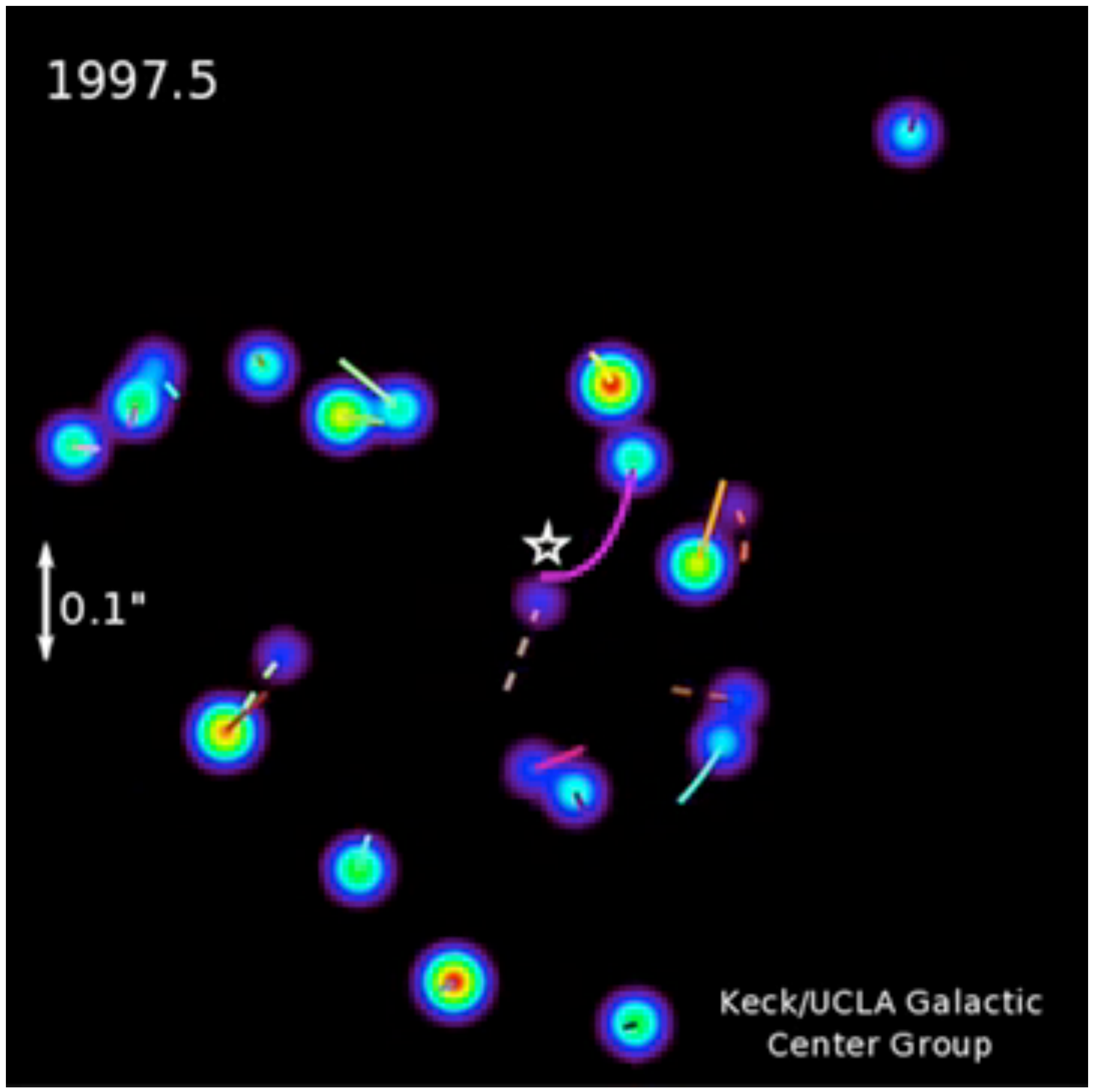
- BH diet
- BH spin up
- Cluster dark component

q-K systems are dynamically degenerate

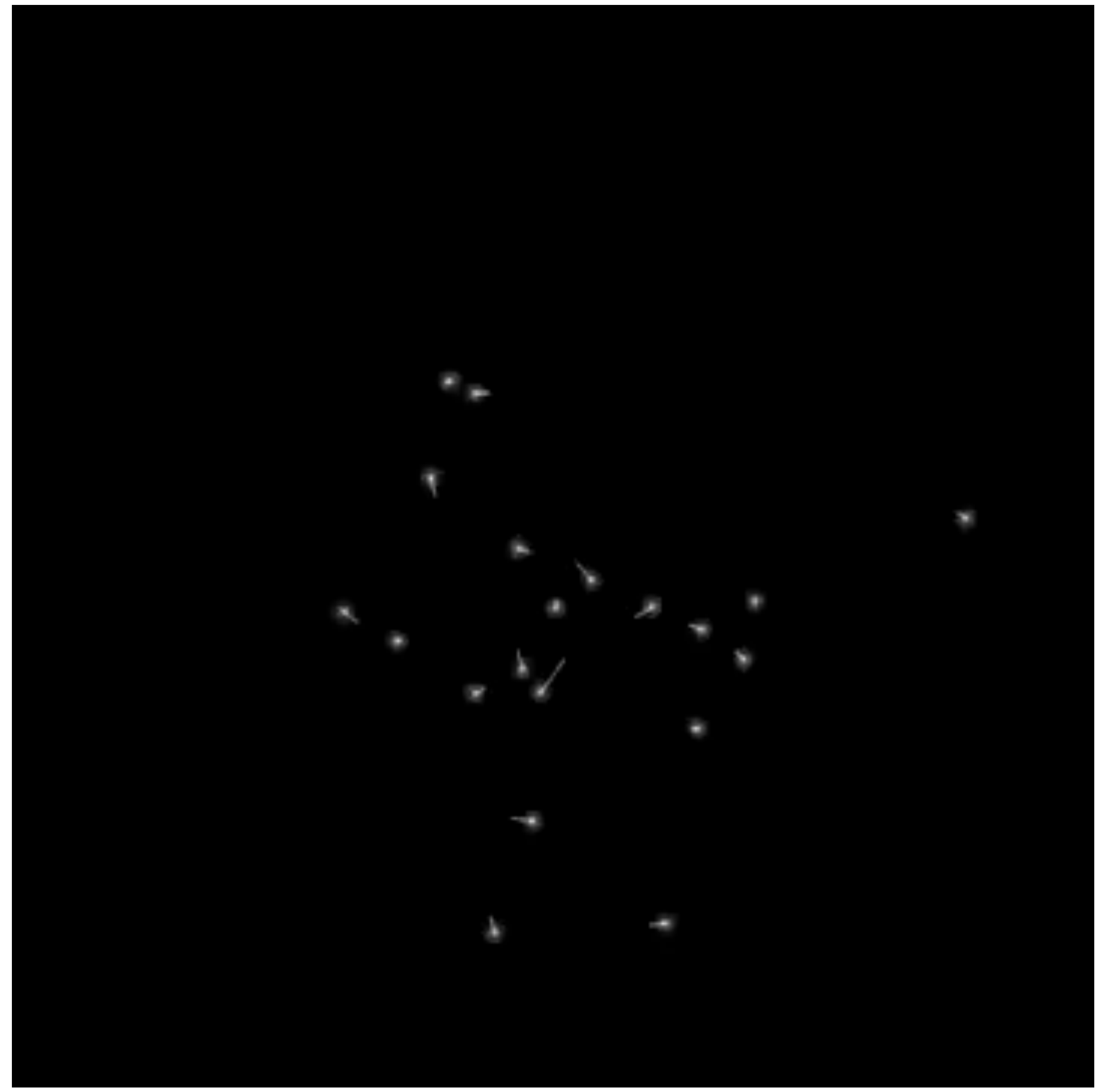
- SMBH dominates the potential: $\varepsilon = M_\star / M_\bullet \ll 1$

  $\Longrightarrow$ Keplerian orbits are **closed**.

  Dynamical degeneracy: $\forall \mathbf{J}, \mathbf{n} \cdot \mathbf{\Omega}_{\mathrm{Kep}}(\mathbf{J}) = 0\,.$



KECK Observations



N**-body simulations** *(B. Bar-Or)*

- **Orbit-Average**: $\boxed{\textbf{Stars} \Longrightarrow \textbf{Wires}}$

Quasi-Keplerian systems

- BH dominates the dynamics: $\varepsilon = M_\star / M_\bullet \ll 1$

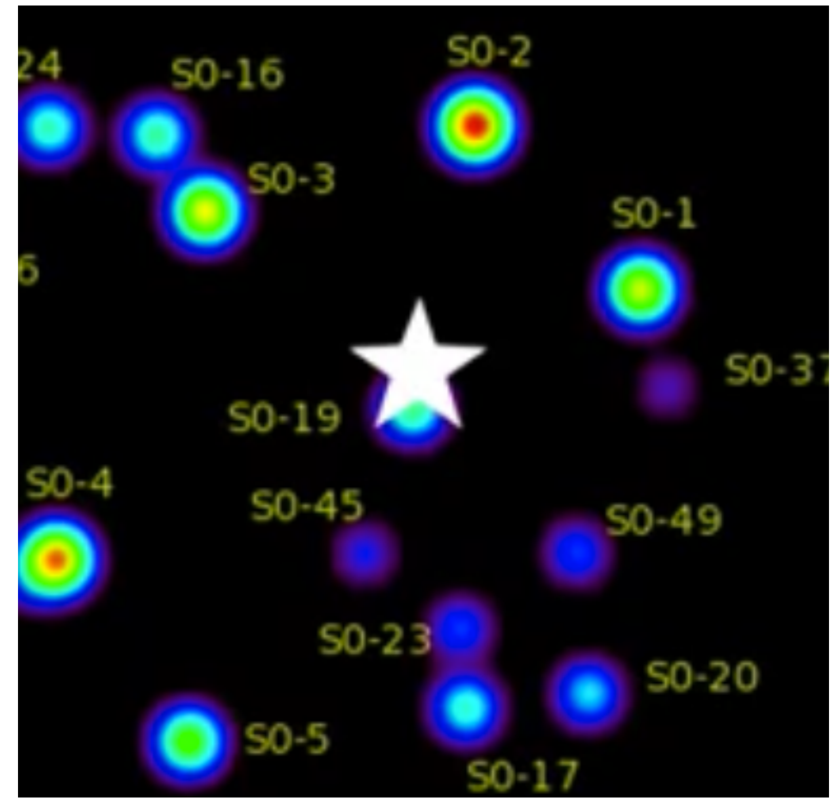
  $\Longrightarrow$ Keplerian orbits are **closed**.

- **Dynamical degeneracy**: $\forall \boldsymbol{J}, \boldsymbol{n} \cdot \boldsymbol{\Omega}_{\rm Kep}(\boldsymbol{J}) = 0\,.$

  $\Longrightarrow$ **Delaunay variables**

$$\boldsymbol{J} = \big(\underbrace{I = J_r + L}_{\textbf{Fast}\ J^{\rm f}}\,,\ \underbrace{L\,,\ L_z}_{\textbf{Slow}\ \boldsymbol{J}^{\rm s}}\big)\ ;\ \boldsymbol{\theta} = \big(\underbrace{\theta^{\rm f}}_{\text{Kep. phase}}\,,\ \underbrace{\boldsymbol{\theta}^{\rm s}}_{\textbf{Int. of motion}}\big)$$

$$\boldsymbol{\Omega}_{\rm Kep} = (\Omega_{\rm Kep}, 0, 0)\,.$$

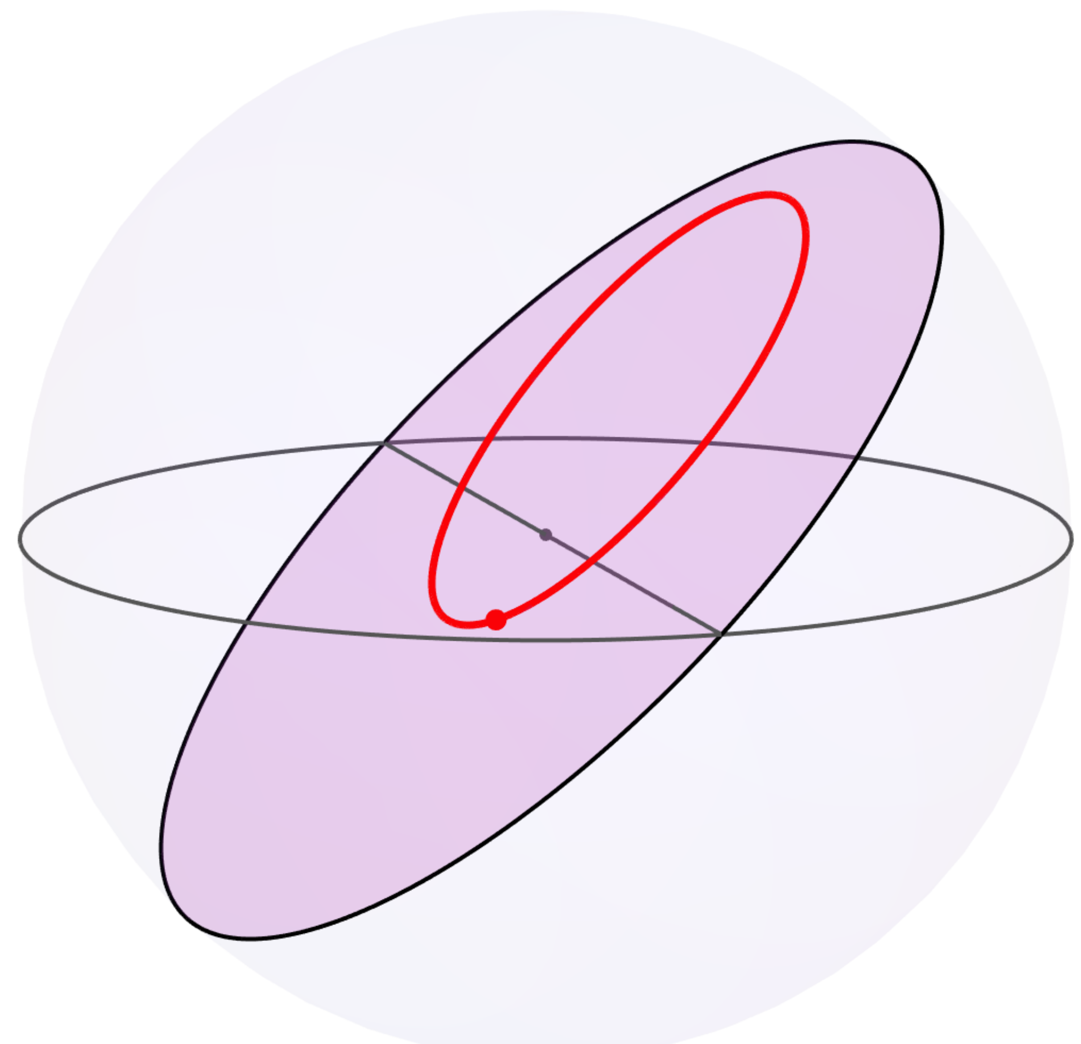


- Orbits characterised by **wires' coordinates**

$$\boldsymbol{\mathcal{E}} = (\boldsymbol{J}, \boldsymbol{\theta}^{\rm s})\,.$$

- System **phase-mixed** w.r.t. the Kep. phase

$$F(\boldsymbol{J}, \boldsymbol{\theta}) \simeq \overline{F}(\boldsymbol{\mathcal{E}})\,.$$



- Keplerian wires **precess** in $\boldsymbol{\theta}^{\rm s}$

$$\boldsymbol{\Omega}^{\rm s} = \frac{\partial \overline{\Phi}_{\rm prec}}{\partial \boldsymbol{J}^{\rm s}} = \frac{\partial [\overline{\Phi}_{\rm self} + \overline{\Phi}_{\rm rel} + \overline{\Phi}_{\rm ext}]}{\partial \boldsymbol{J}^{\rm s}}\,.$$

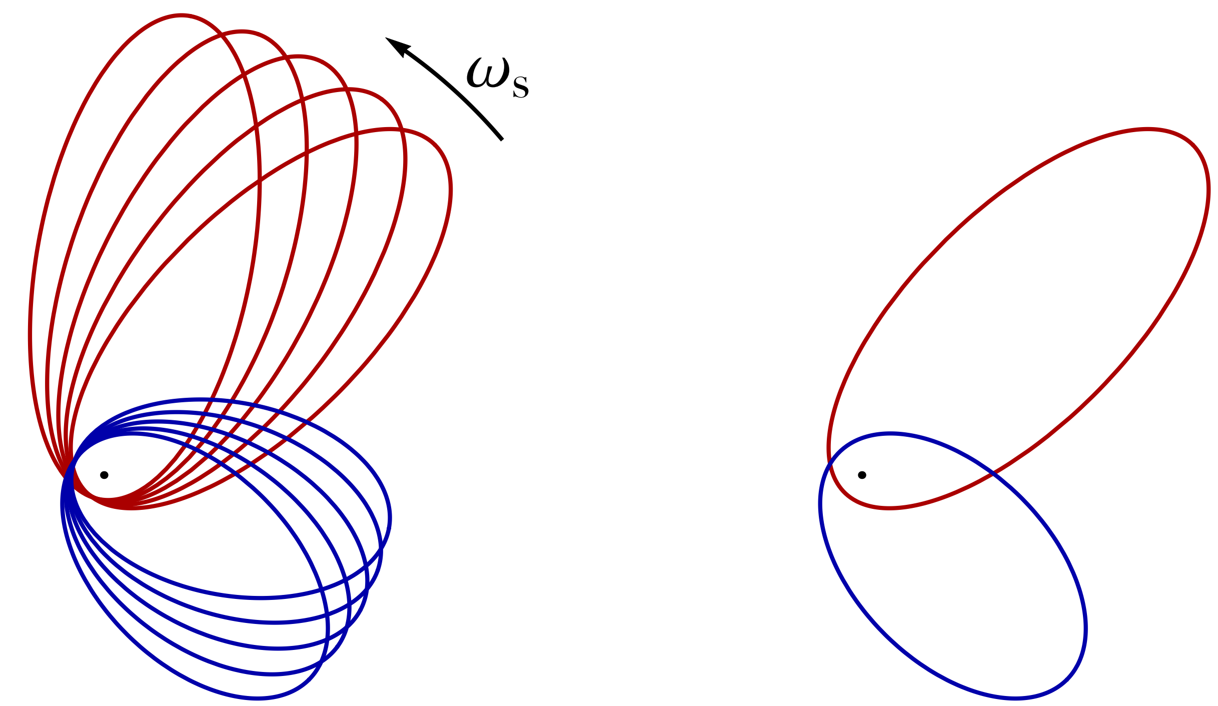
Disc has mass

SMBH relativistic correction

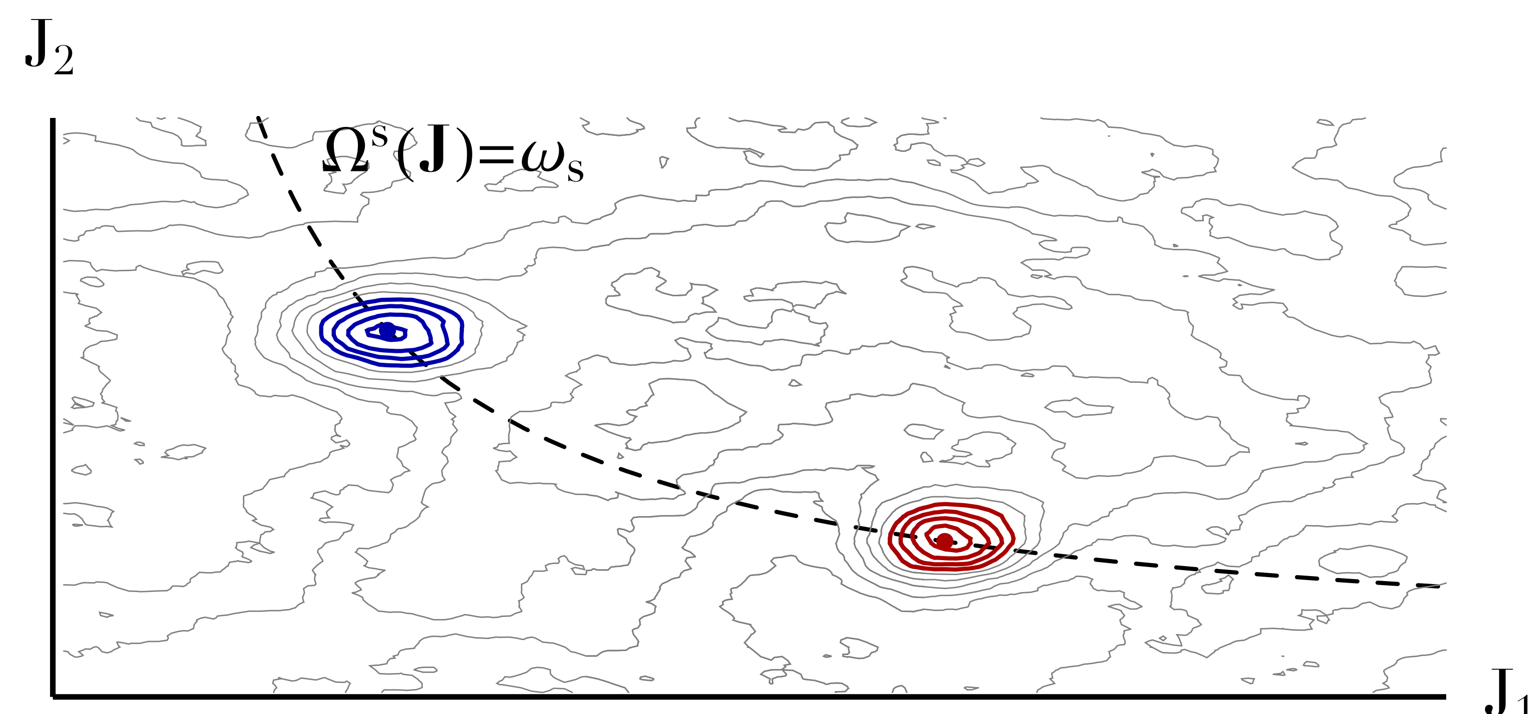
The degenerate Balescu-Lenard equation

- **The master equation of resonant relaxation**

$$\frac{\partial \overline{F}(\boldsymbol{J},\tau)}{\partial \tau} = \frac{1}{N} \frac{\partial}{\partial \boldsymbol{J}_1^{\mathrm{s}}} \cdot \Bigg[\sum_{\boldsymbol{m}_1^{\mathrm{s}},\boldsymbol{m}_2^{\mathrm{s}}} \boldsymbol{m}_1^{\mathrm{s}} \int \mathrm{d}\boldsymbol{J}_2 \, \frac{\delta_{\mathrm{D}}(\boldsymbol{m}_1^{\mathrm{s}} \cdot \boldsymbol{\Omega}_1^{\mathrm{s}} - \boldsymbol{m}_2^{\mathrm{s}} \cdot \boldsymbol{\Omega}_2^{\mathrm{s}})}{|\mathcal{D}_{\boldsymbol{m}_1^{\mathrm{s}},\boldsymbol{m}_2^{\mathrm{s}}}(\boldsymbol{J}_1,\boldsymbol{J}_2,\boldsymbol{m}_1^{\mathrm{s}} \cdot \boldsymbol{\Omega}_1^{\mathrm{s}})|^2} \times \left[\boldsymbol{m}_1^{\mathrm{s}} \cdot \frac{\partial}{\partial \boldsymbol{J}_1^{\mathrm{s}}} - \boldsymbol{m}_2^{\mathrm{s}} \cdot \frac{\partial}{\partial \boldsymbol{J}_2^{\mathrm{s}}} \right] \overline{F}(\boldsymbol{J}_1,\tau) \, \overline{F}(\boldsymbol{J}_2,\tau) \Bigg] .$$



- Some properties:
  - $\overline{F}(\boldsymbol{J},\tau)$: **Orbital distortion.**
  - $\partial \tau$: $\tau = t\, M_\star / M_\bullet$, BH dominance.
  - $1/N$: $1/N$ **resonant relaxation.**
  - $\partial / \partial \boldsymbol{J}_1^{\mathrm{s}} \cdot$: **Adiabatic conservation.**
  - δ_{D}: **Resonance on precessions.**
  - $1/\mathcal{D}_{\boldsymbol{m}_1^{\mathrm{s}},\boldsymbol{m}_2^{\mathrm{s}}}$: **Self-gravity.**



Fouvry et al. (2016)

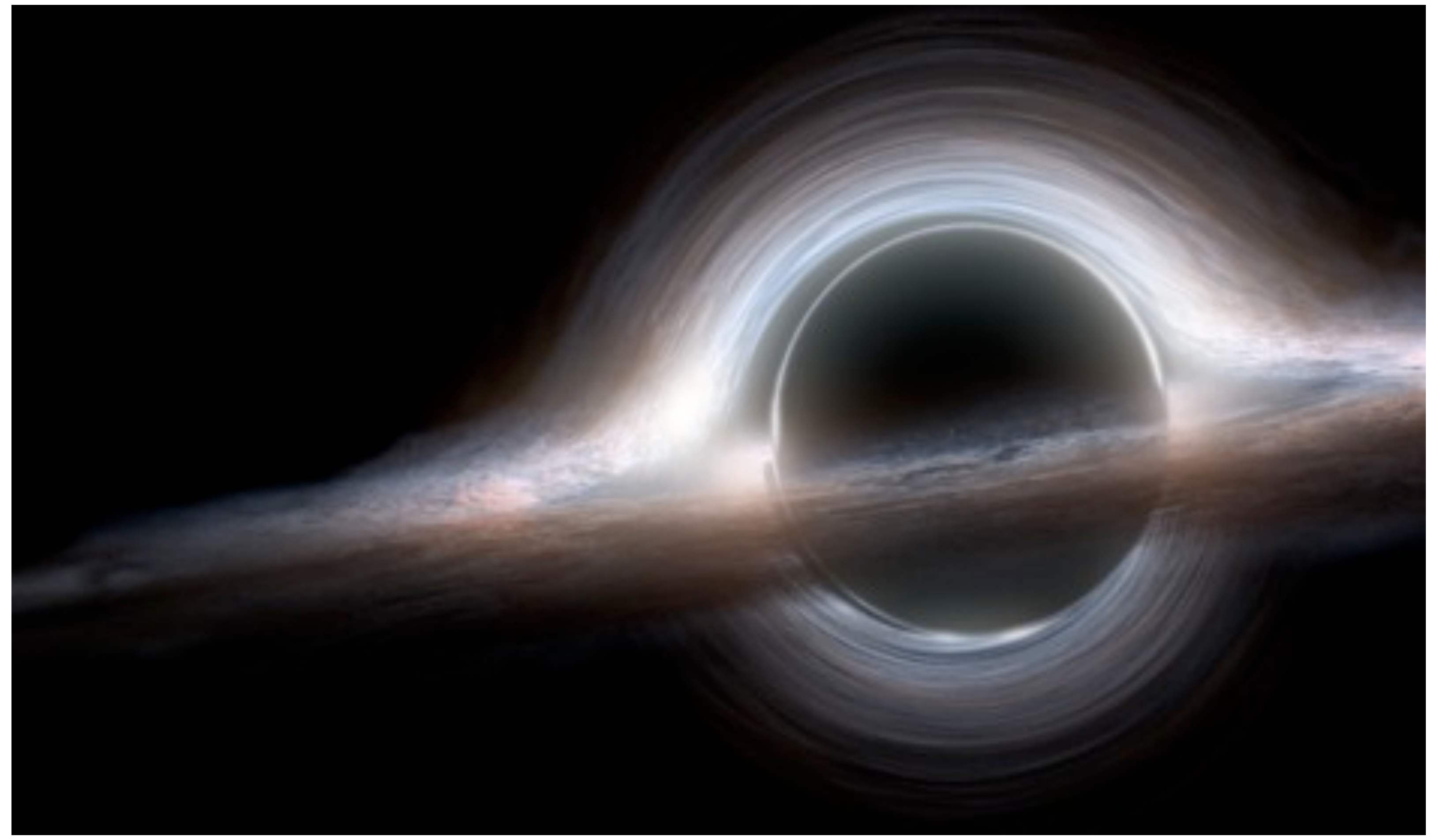
using averaging over fast angle

$$\overline{F}(\boldsymbol{J},\boldsymbol{\theta}^{\mathrm{s}}) = \int \frac{\mathrm{d}\boldsymbol{\theta}^{\mathrm{d}}}{(2\pi)^{d-k}} F(\boldsymbol{J},\boldsymbol{\theta}^{\mathrm{s}},\boldsymbol{\theta}^{\mathrm{d}}) .$$

→ avoid Dirac of zero!

Physical origin of Schwarzschild barrier

One PN and 1.5PN relativistic correction



$$\mathbf{\Omega}^{\mathrm{s}} = \frac{\partial[\overline{\Phi}+\overline{\Phi}_{\mathrm{a}}]}{\partial \boldsymbol{J}^{\mathrm{s}}}.$$

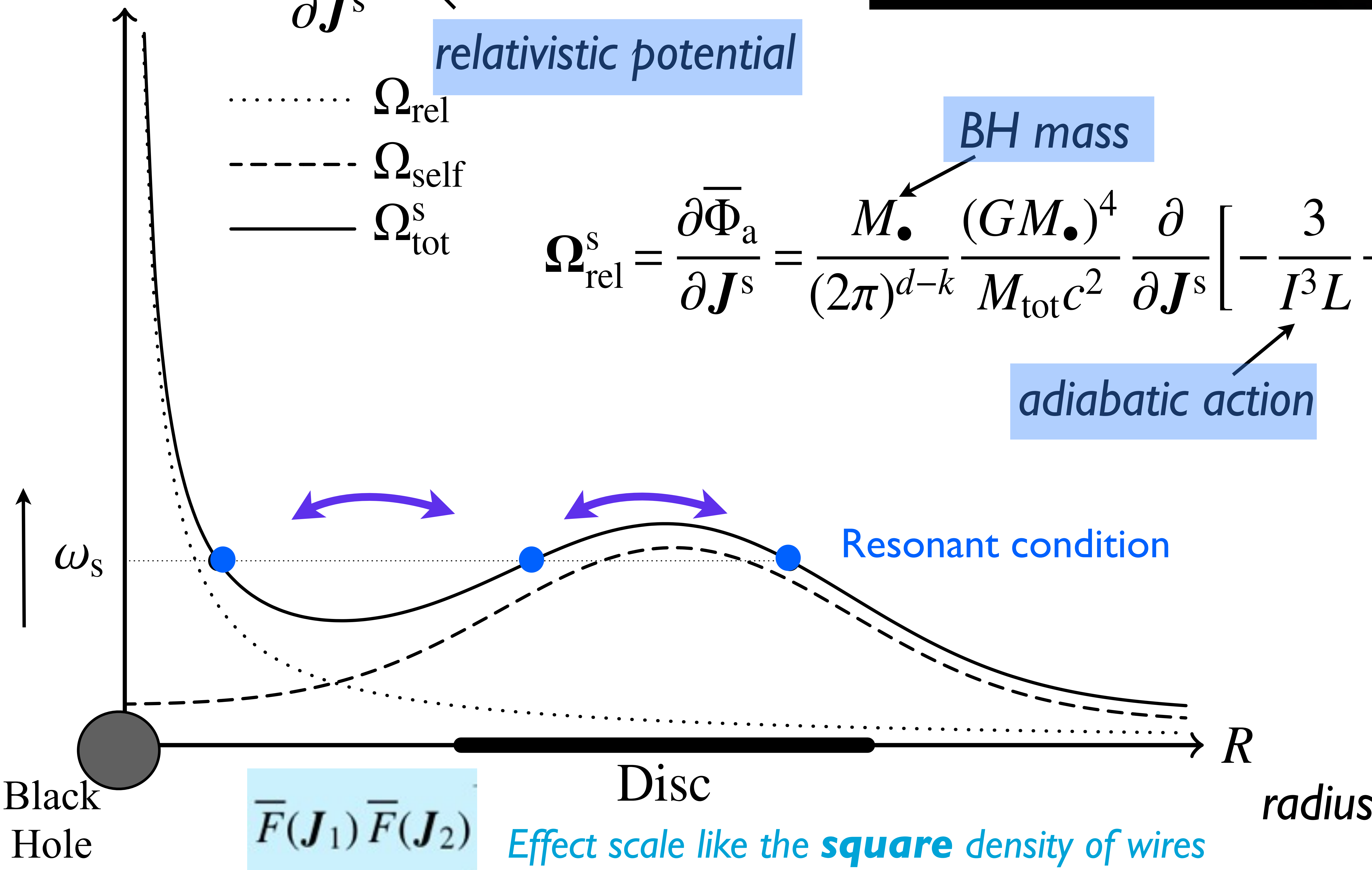
relativistic potential

$$\mathbf{\Omega}^{\mathrm{s}}_{\mathrm{rel}} = \frac{\partial \overline{\Phi}_{\mathrm{a}}}{\partial \boldsymbol{J}^{\mathrm{s}}} = \frac{M_\bullet}{(2\pi)^{d-k}} \frac{(GM_\bullet)^4}{M_{\mathrm{tot}}c^2} \frac{\partial}{\partial \boldsymbol{J}^{\mathrm{s}}} \left[-\frac{3}{I^3 L} + \frac{2GM_\bullet}{c} \frac{sL_z}{I^3 L^3} \right].$$

BH mass

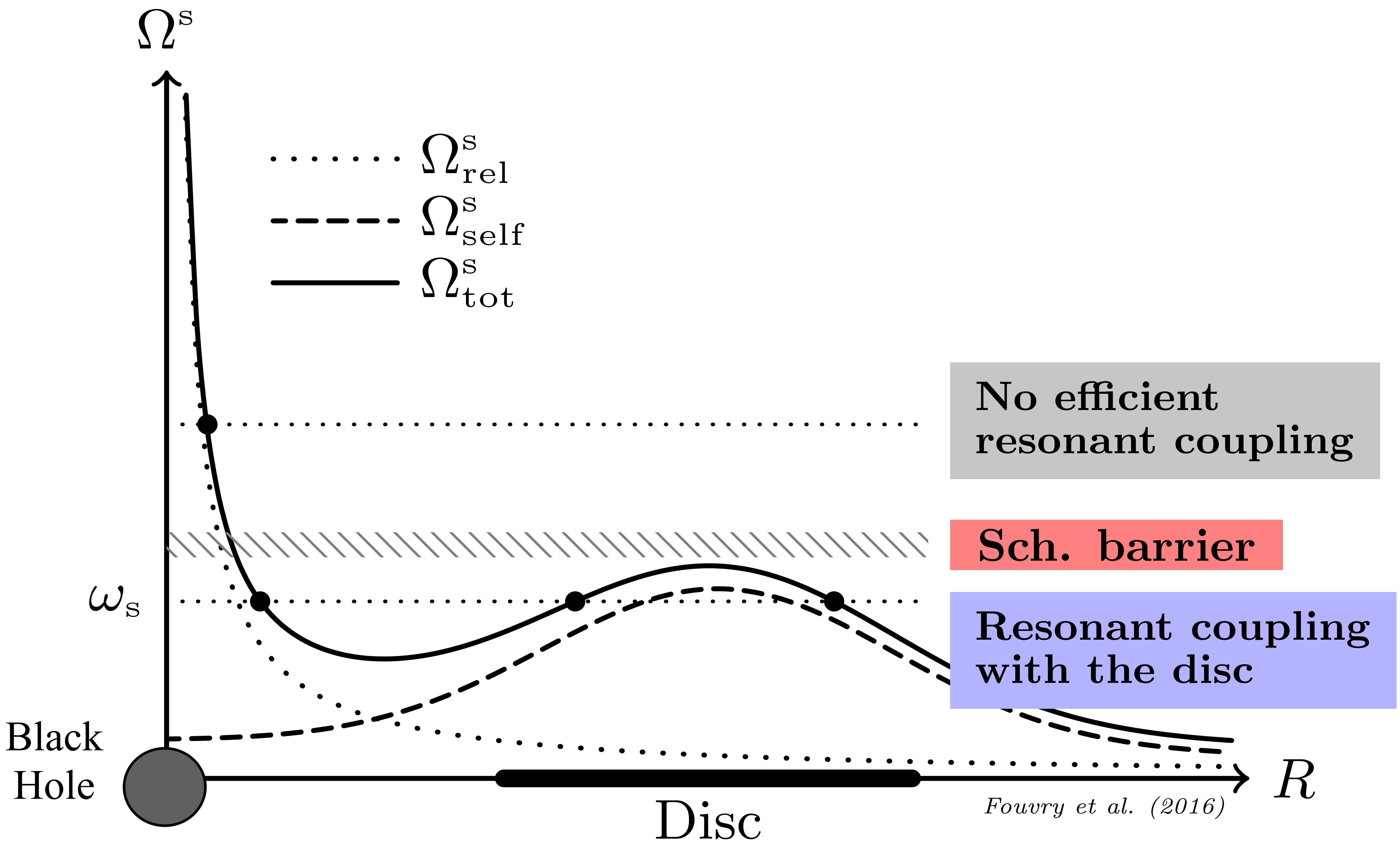
BH spin

adiabatic action



Balescu-Lenard and Schwarzschild barrier

- Precession frequencies: **self-consistent** + **relativistic** - (Kocsis et al. (2011))



- When stars move inward
  - **Precession frequencies increase**.
  - Resonant coupling with the disc becomes impossible.
  - Schwarzschild barrier - (Merritt et al. (2011), Bar-Or et al. (2016)).

Individual stochastic diffusion

- Self-consistent diffusion of the **system as a whole**

  $\Longrightarrow$ **Anisotropic Balescu-Lenard equation**

$$\boxed{\frac{\partial \overline{F}}{\partial \tau} = \frac{\partial}{\partial \boldsymbol{J}^{\mathrm{s}}} \cdot \left[\boldsymbol{A}(\boldsymbol{J},\tau)\, \overline{F}(\boldsymbol{J},\tau) + \boldsymbol{D}(\boldsymbol{J},\tau) \cdot \frac{\partial \overline{F}}{\partial \boldsymbol{J}^{\mathrm{s}}} \right].}$$

  $\boldsymbol{A}(\overline{F})$ drift vector, $\boldsymbol{D}(\overline{F})$ diffusion tensor.

- **Individual dynamics** of a wire at position $\boldsymbol{\mathcal{J}}(\tau)$

  $\Longrightarrow$ **Stochastic Langevin equation** - (Risken (1996))

$$\boxed{\frac{\mathrm{d}\boldsymbol{\mathcal{J}}}{\mathrm{d}\tau} = \boldsymbol{h}(\boldsymbol{\mathcal{J}},\tau) + \boldsymbol{g}(\boldsymbol{\mathcal{J}},\tau) \cdot \boldsymbol{\Gamma}(\tau)\,.} \qquad \mathbf{g}(\mathcal{J},\tau) \propto \sqrt{\mathbf{D}}$$

  $\boldsymbol{h}$ and $\boldsymbol{g}$ vector and tensor, and $\boldsymbol{\Gamma}$ stochastic Langevin forces.

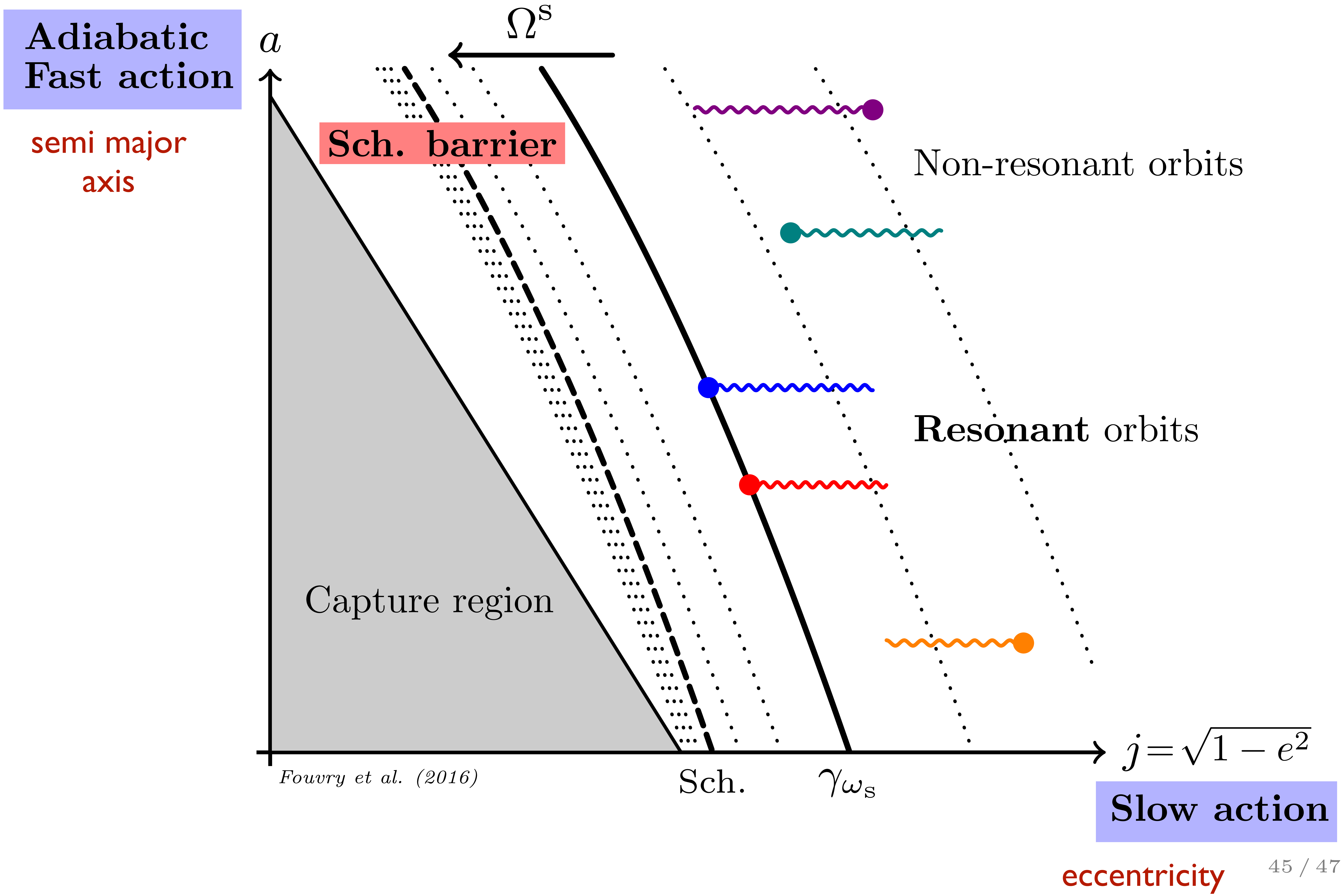
  $\Longrightarrow$ **Dual equation**, whose ensemble average gives back BL.

- In the Langevin's rewriting, **particles are dressed orbits.**

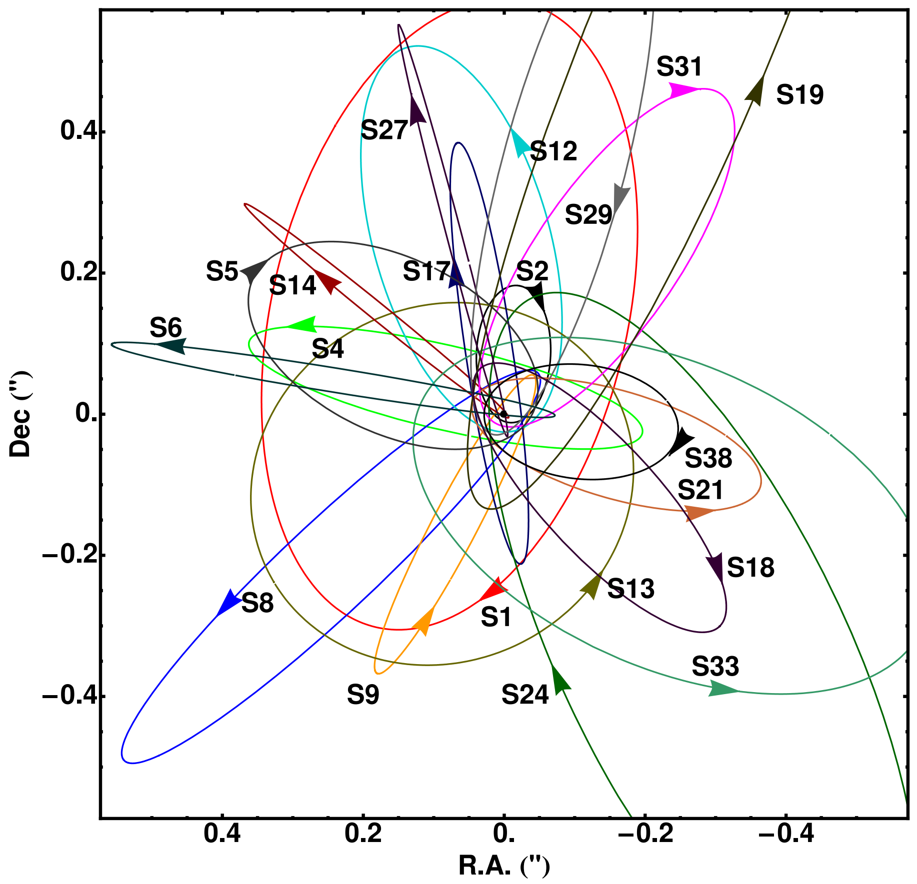
  $\Longrightarrow$ Huge gains in timesteps for integration.

Langevin view of the Schwarzschild barrier

- **Langevin stochastic diffusion** of orbits



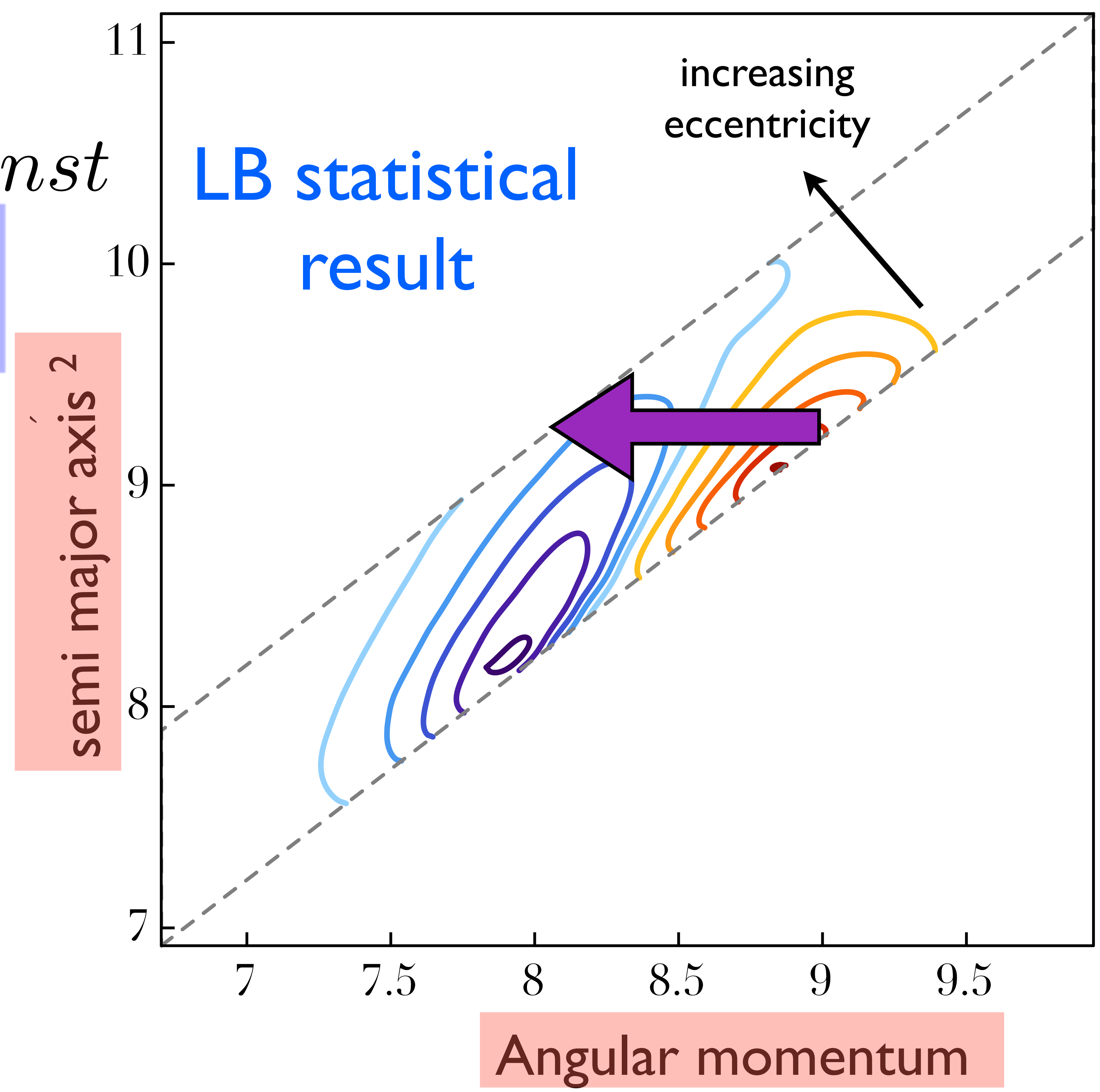
Resonant relaxation near SMBH



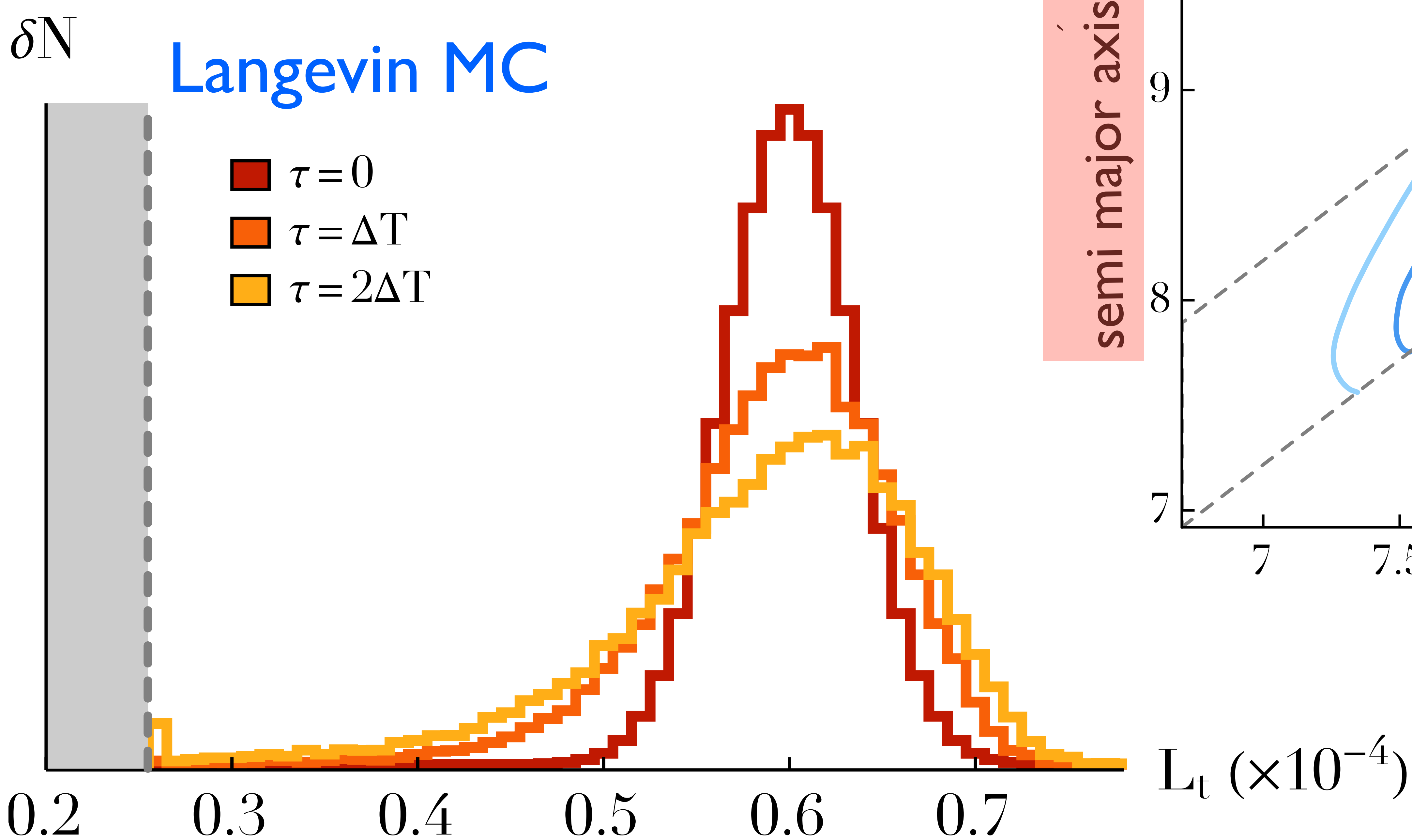
$J_r + L = const$

Adiabatic Fast action

Flux map in action space



Slow action



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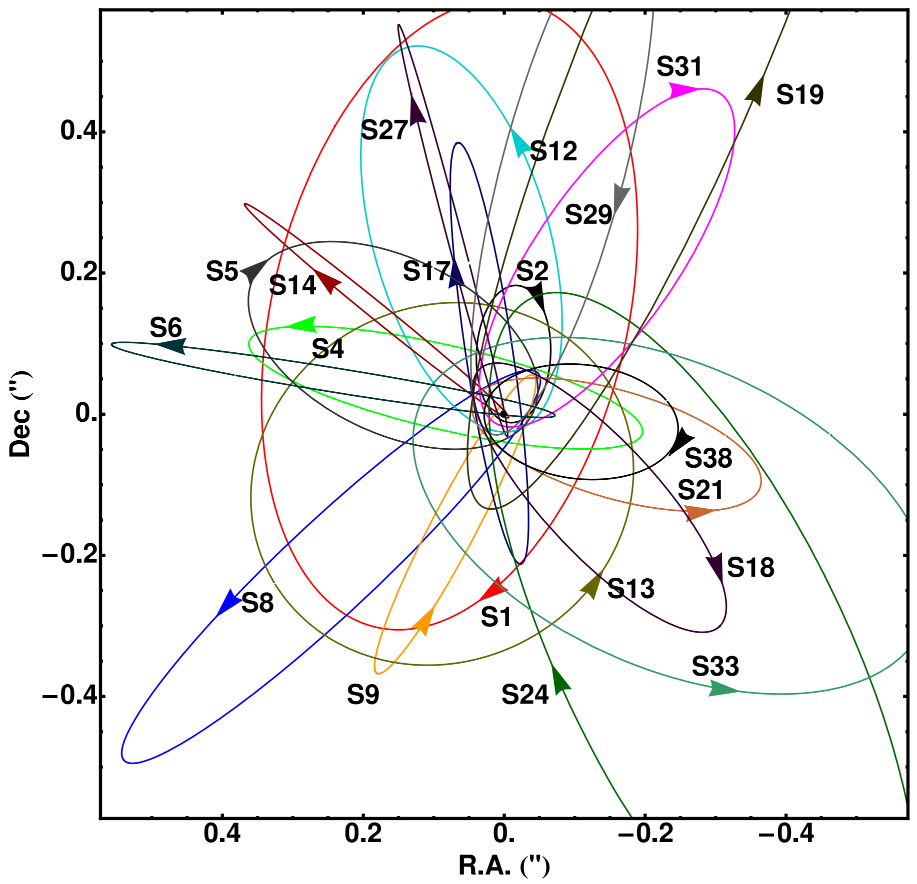

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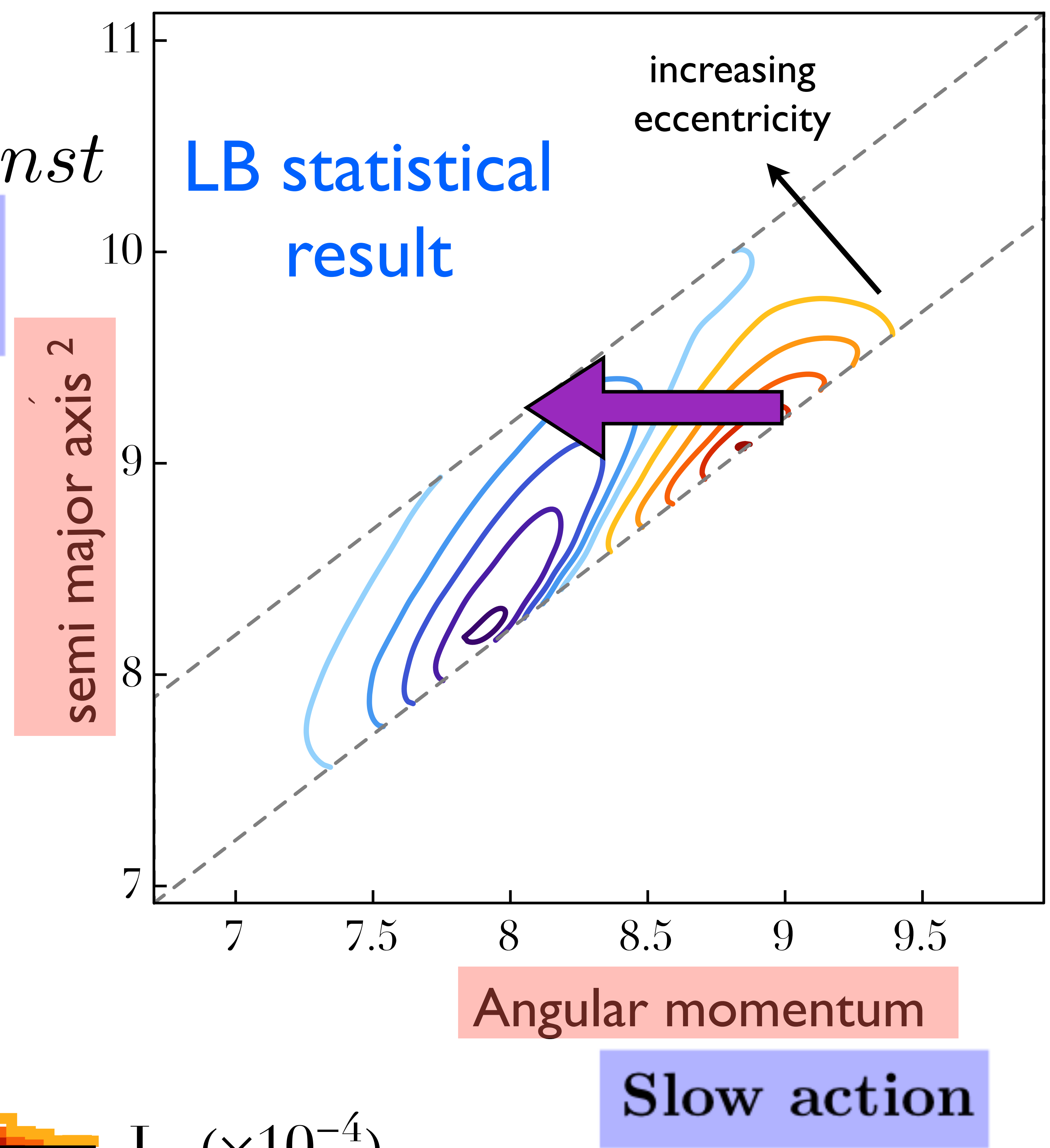
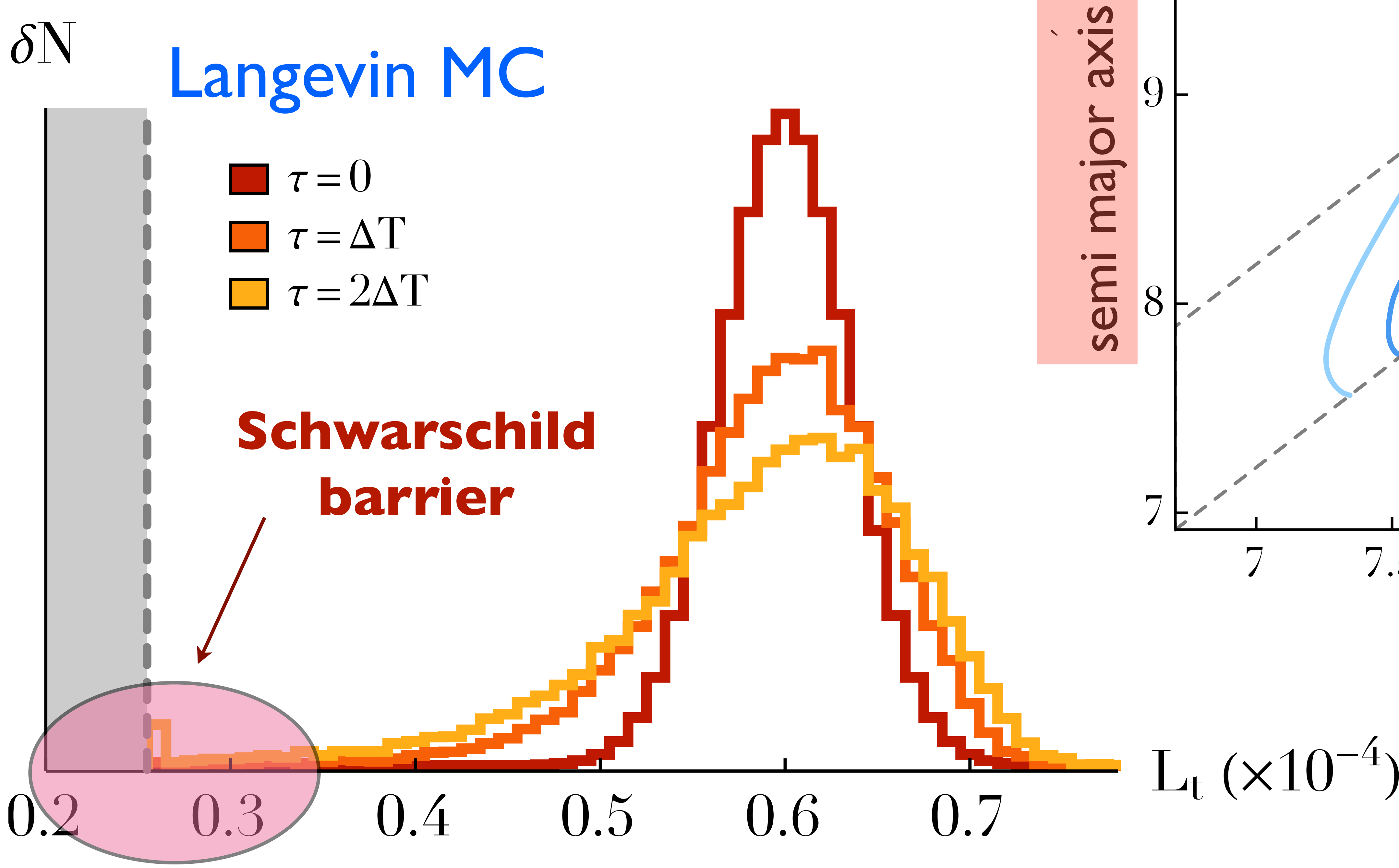
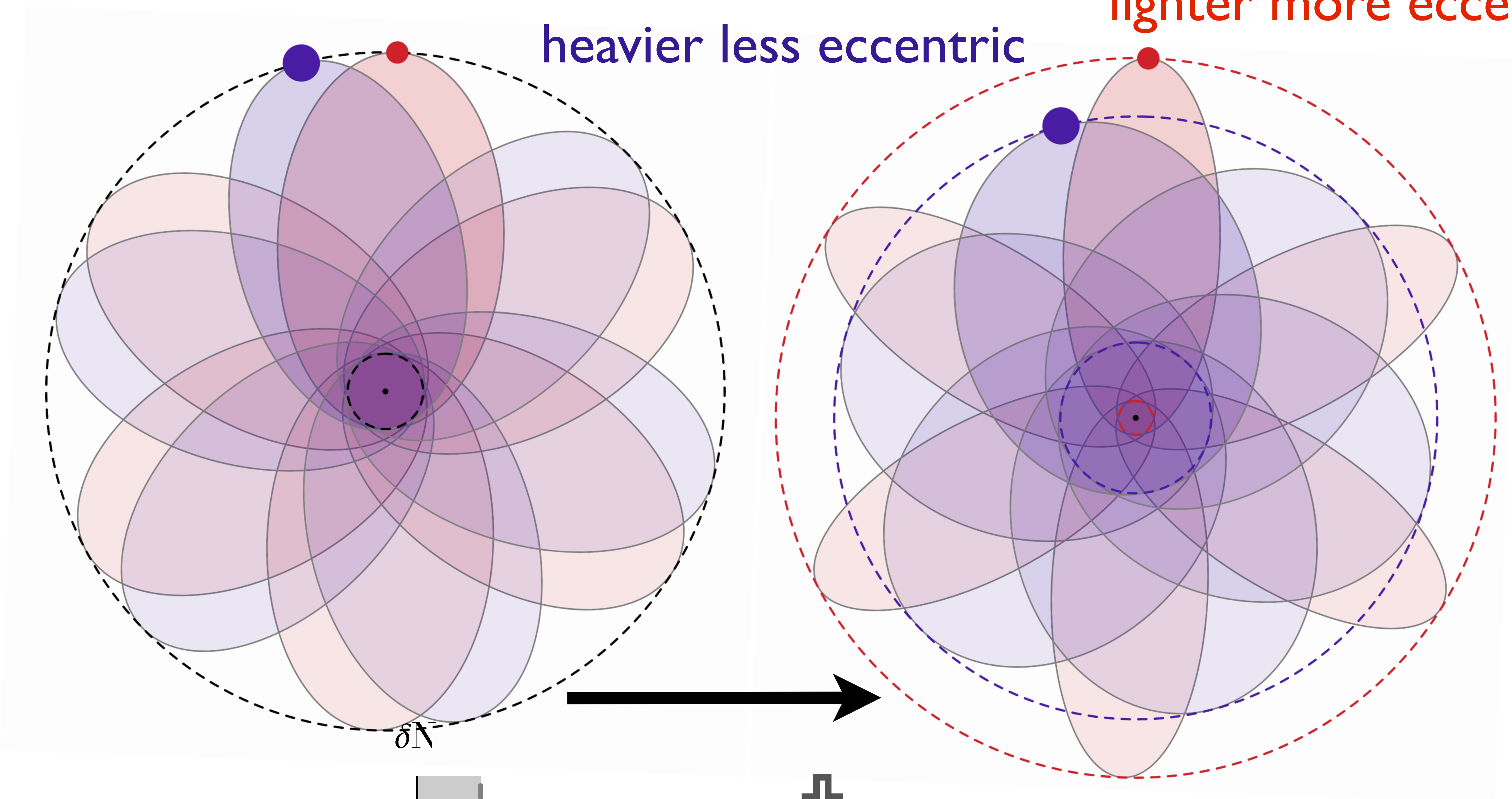
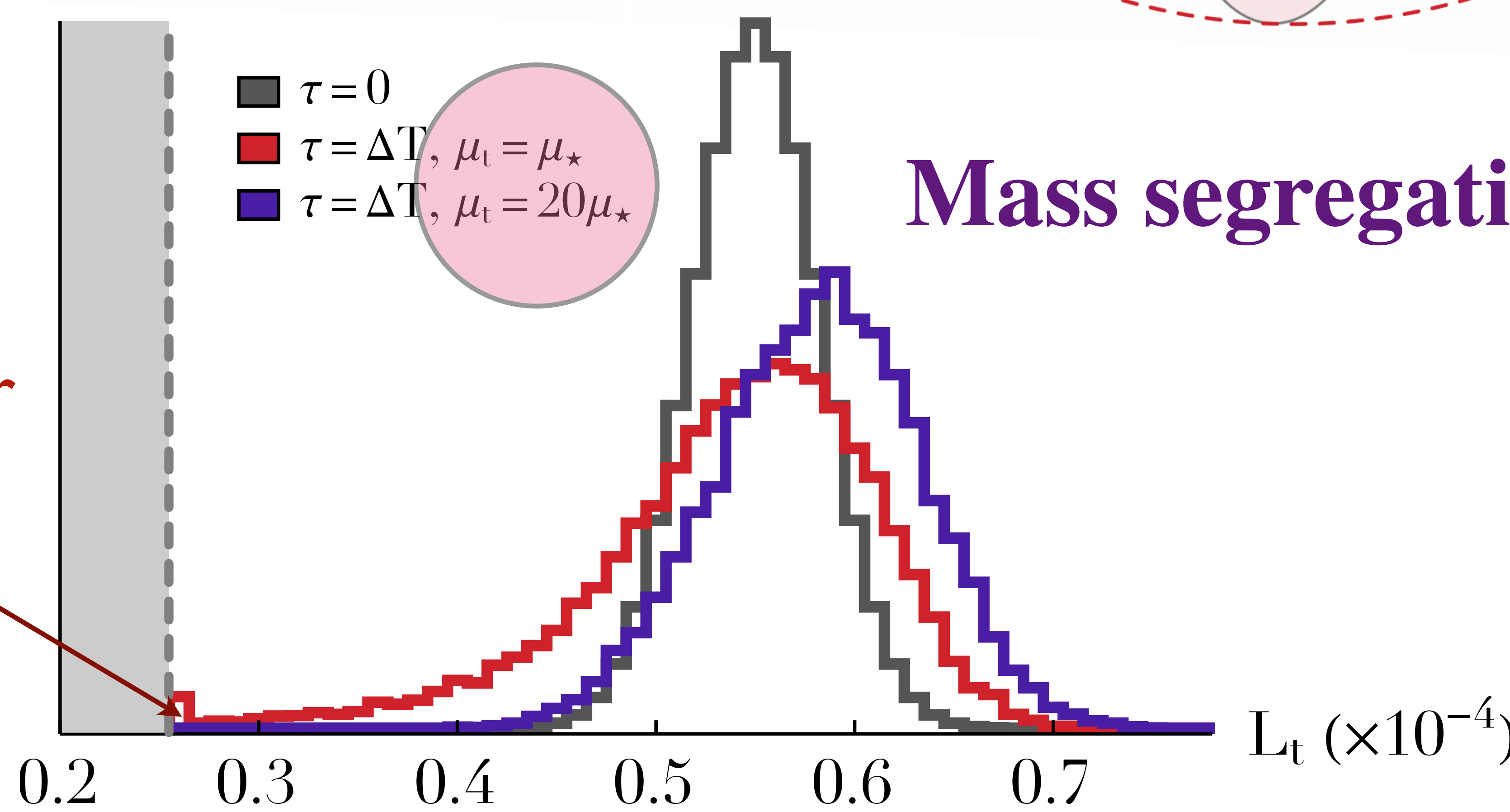




Multi species ?



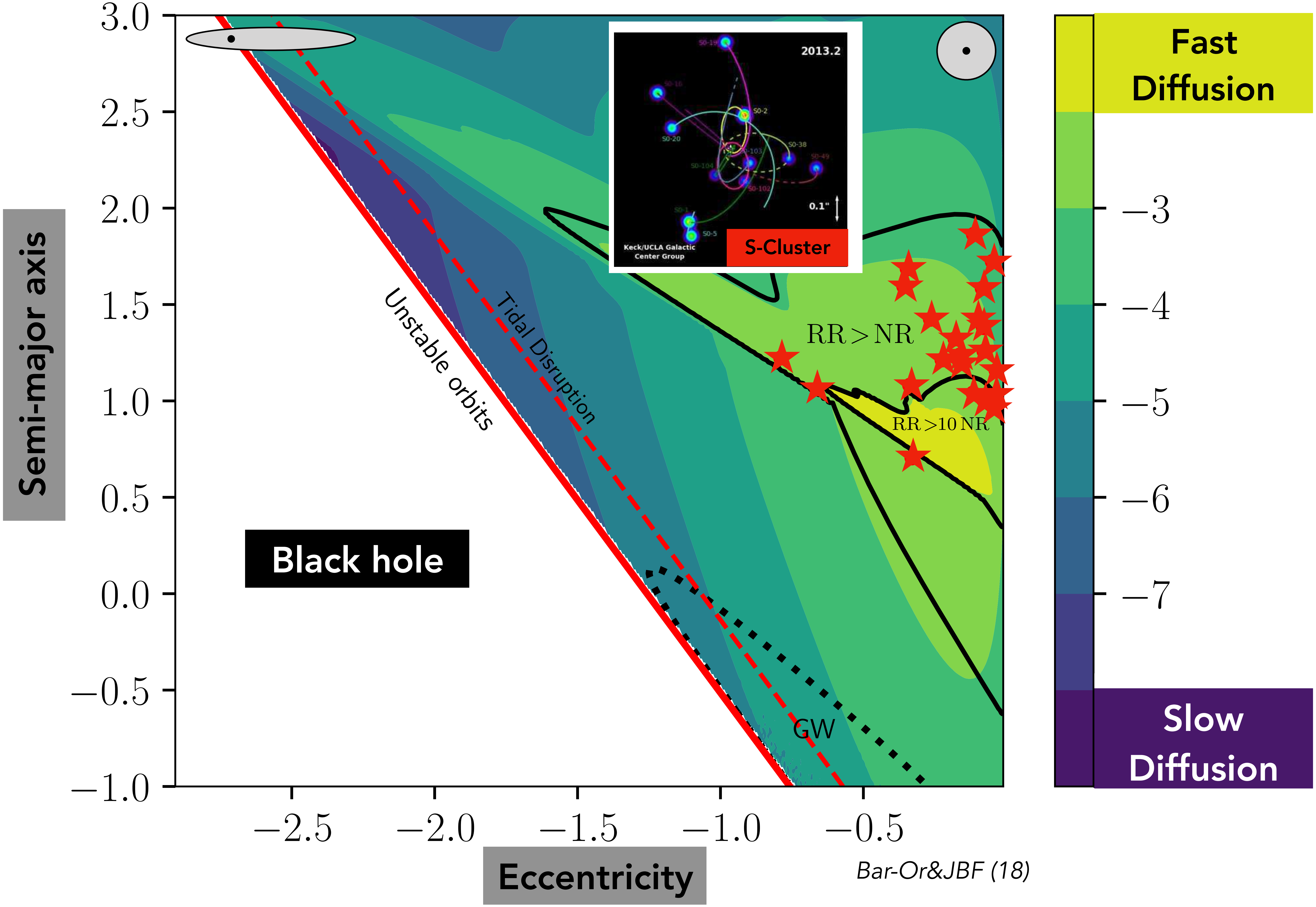
Schwarschild barrier



Mass segregation near SMBH

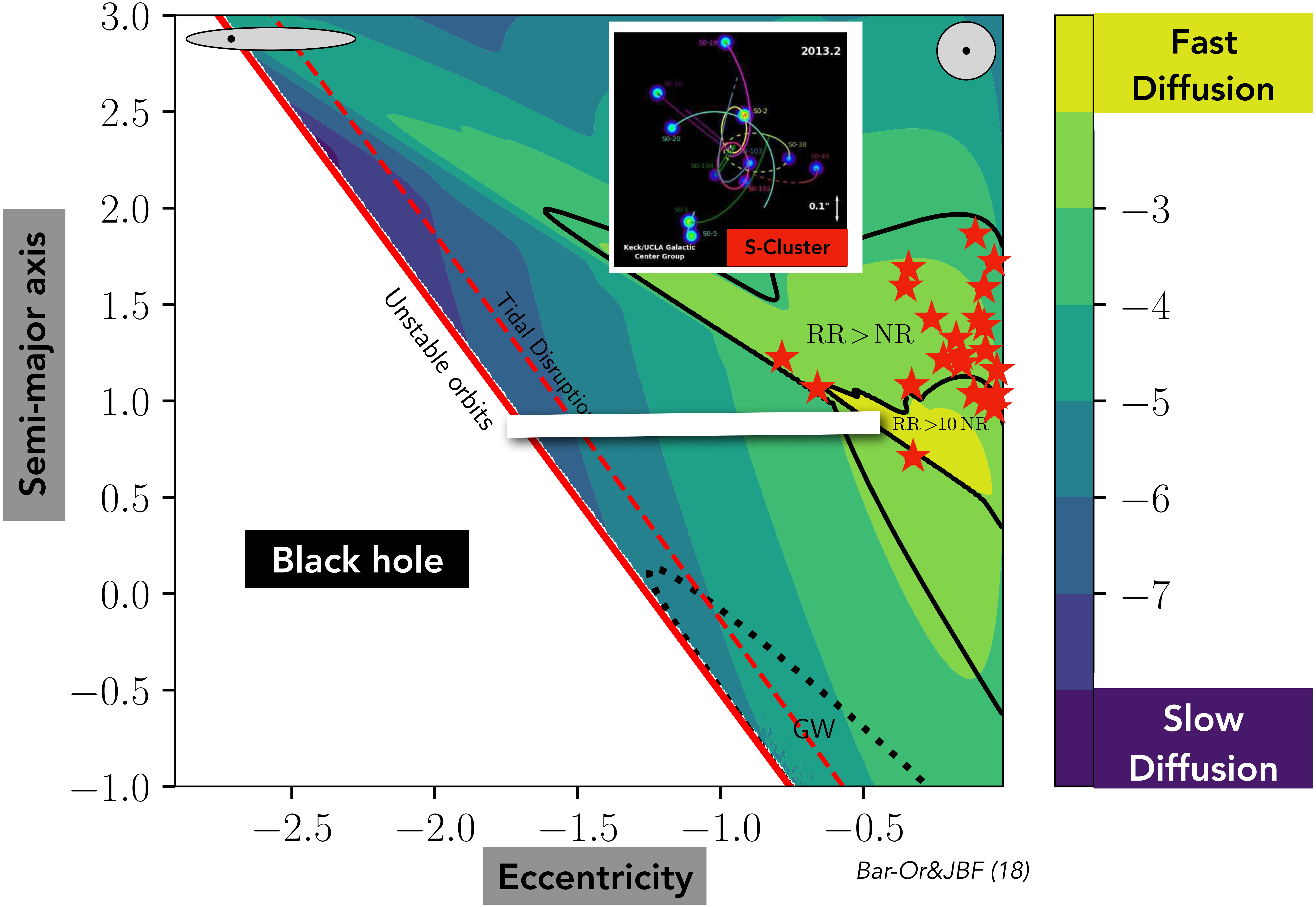
Resonant relaxation segregates adiabatically ***lighter*** stars towards SMBH

Scalar Resonant Relaxation in Galactic Nuclei



Diffusion coefficients in ***orbital space***

Scalar Resonant Relaxation in Galactic Nuclei



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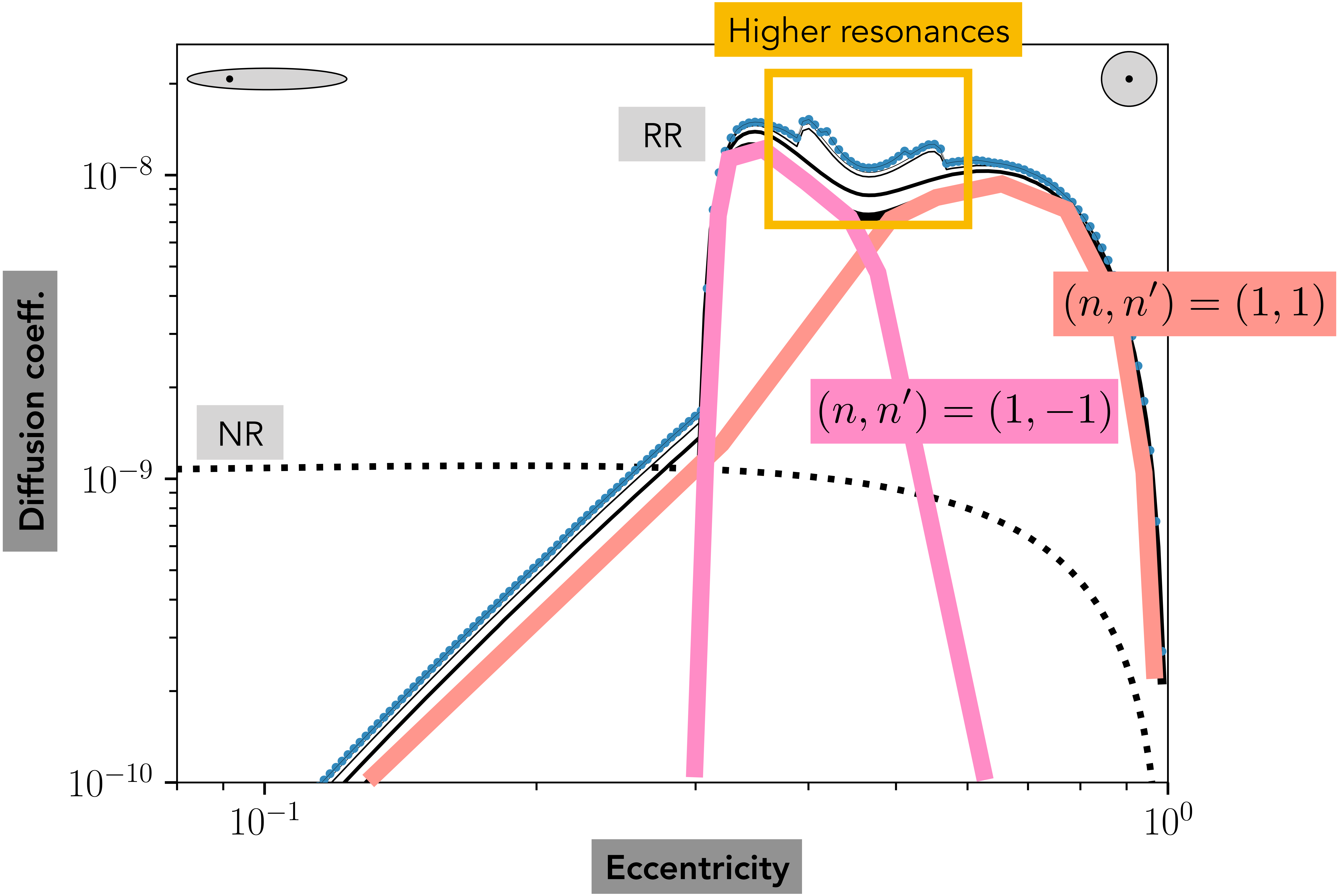
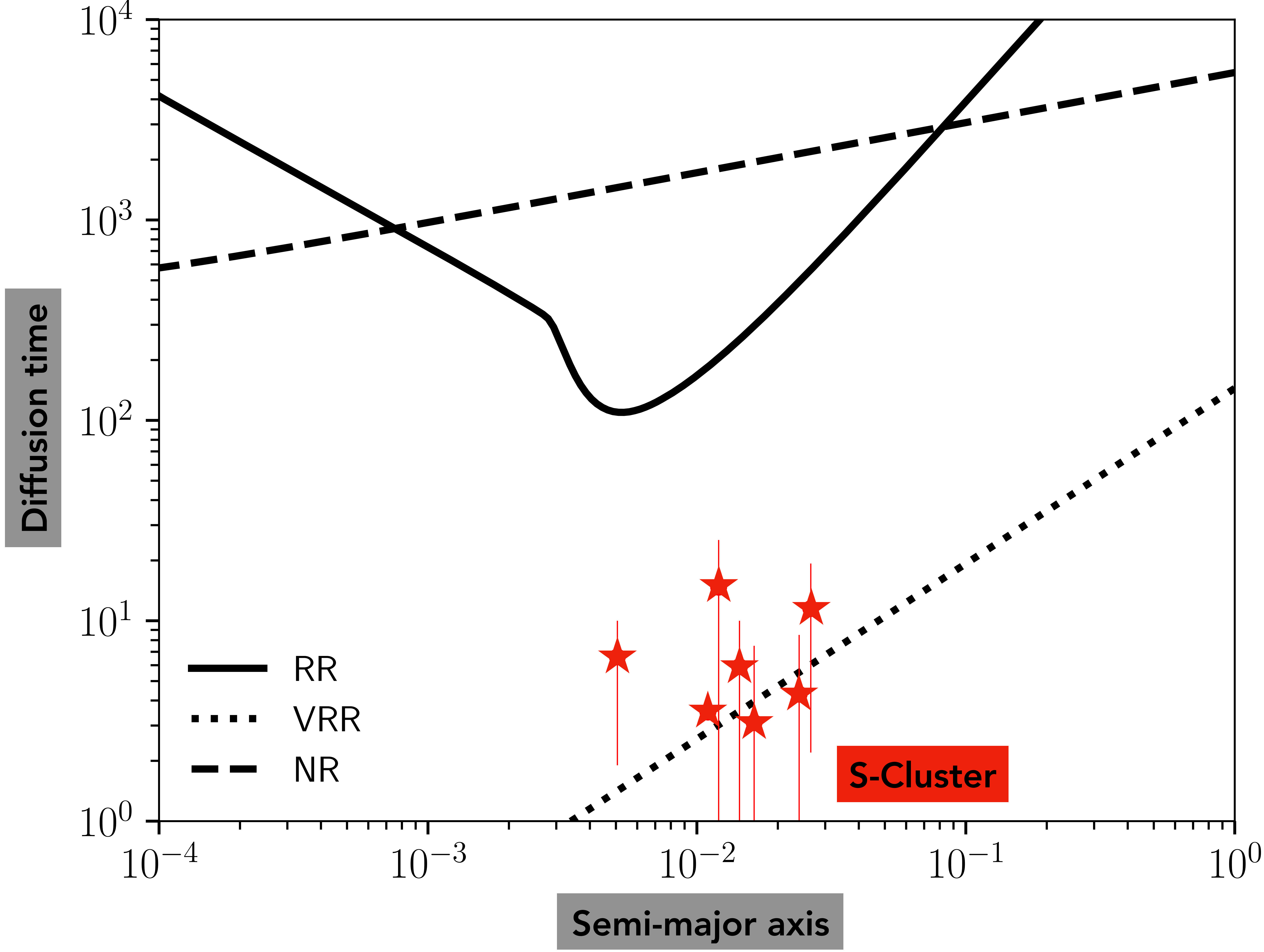
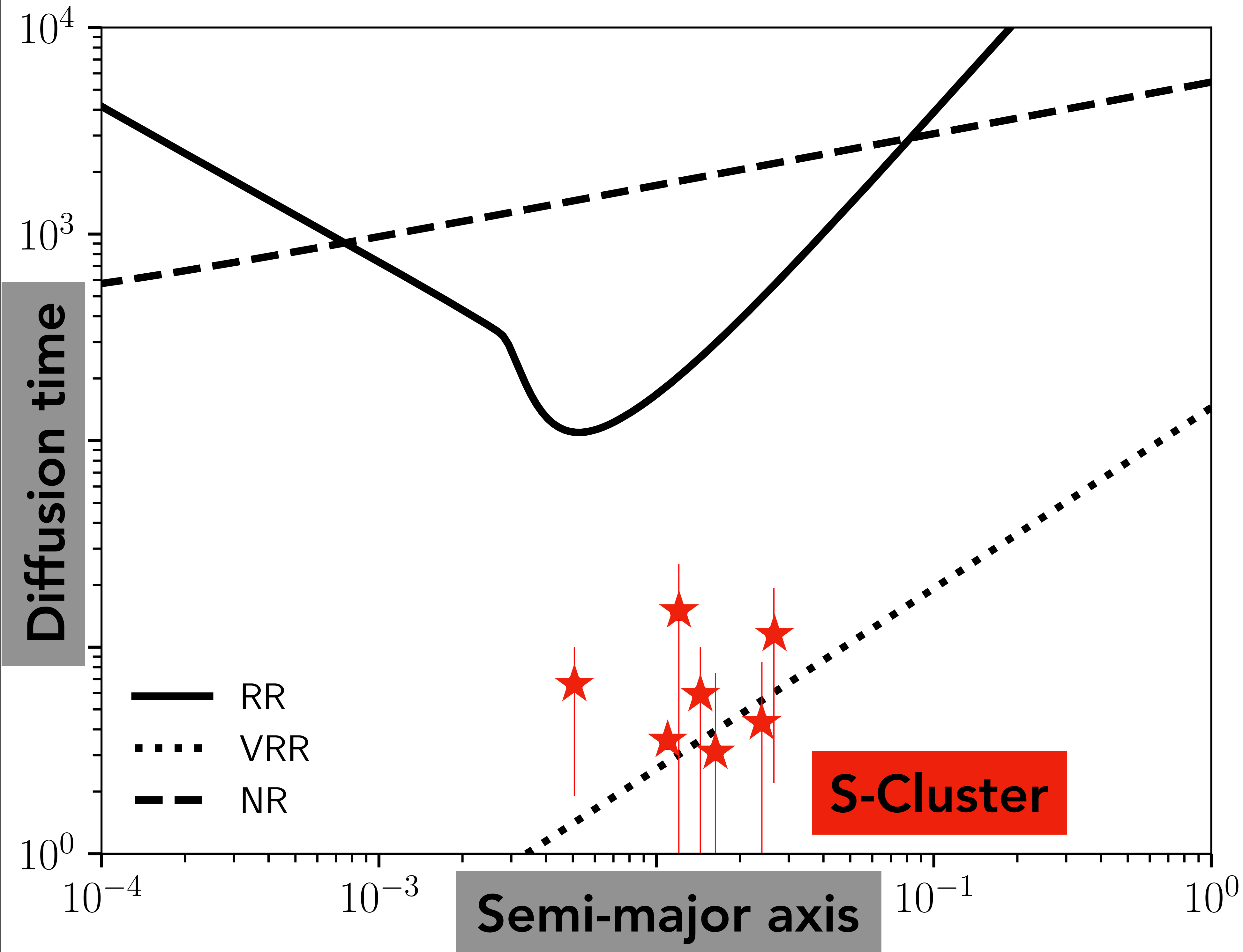


Typical diffusion timescales





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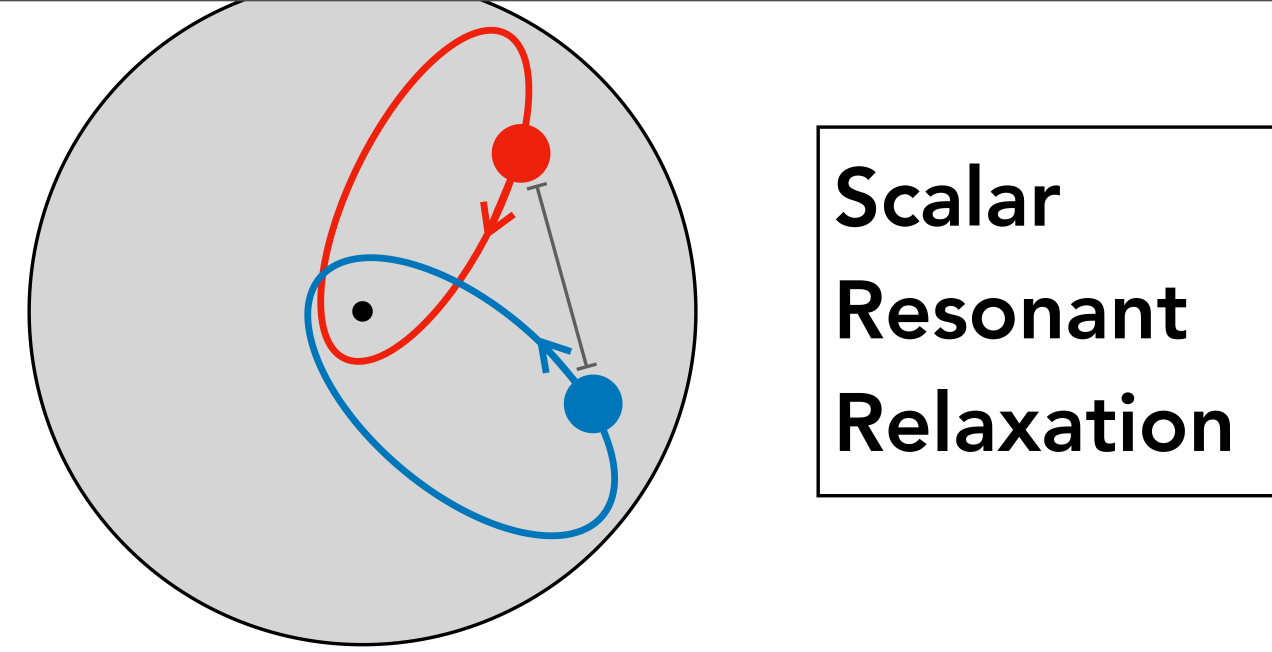
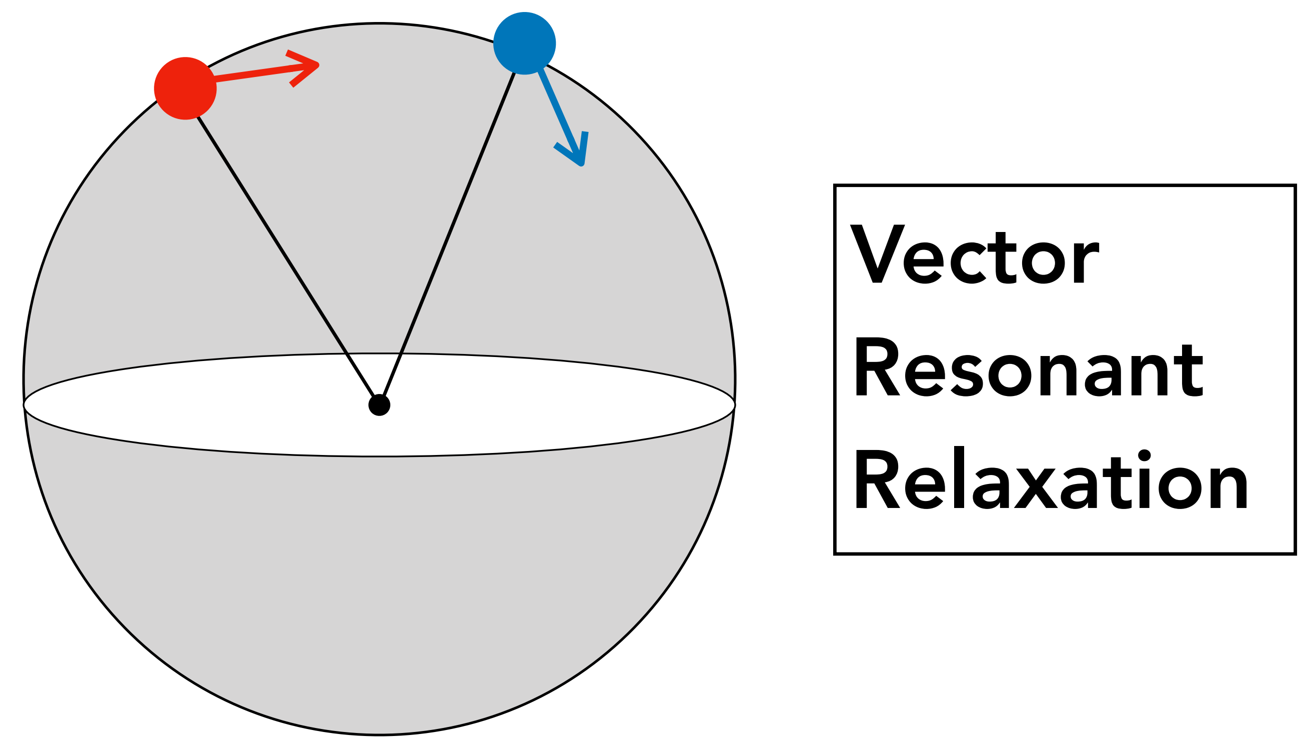
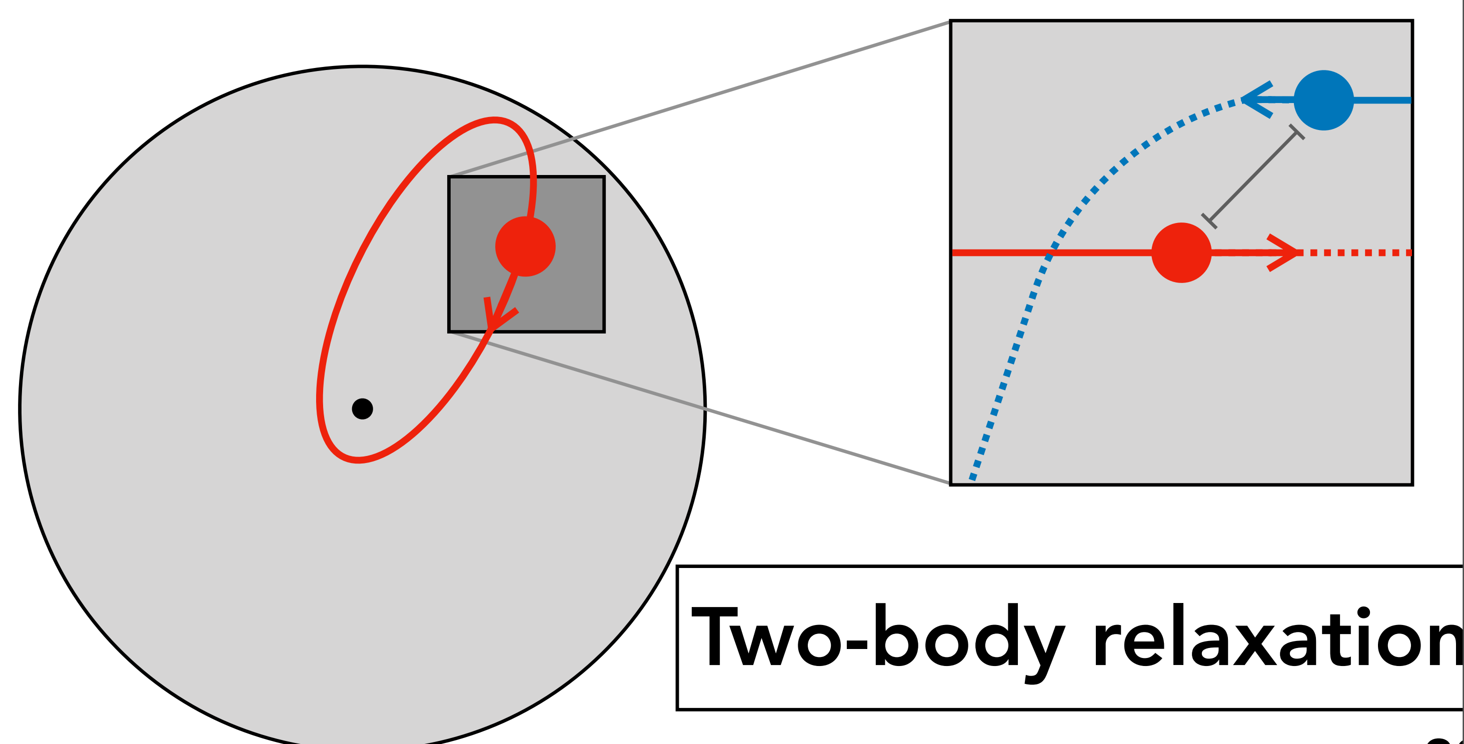






Context

How to feed a **supermassive black hole**?

Stellar diffusion in galactic centers

+ ***Origin and structure*** *of SgrA**
+ *Relaxation in* ***eccentricity***, ***orientation***

Sources of **gravitational waves**

+ *BHs-binary mergers*
+ *EMRIs, TDE*

CONCLUSIONS

- From **linear response** to **secular evolution**.

$$\boxed{\textbf{Stellar dynamics enters the cosmic framework}.}$$

- Frameworks for the effects of **external and internal** perturbations.

$$\boxed{\textbf{Nature vs. Nurture}}$$

- First implementation of **Balescu-Lenard** in (astro)physics
- Approach complementary to $N-$body and Monte Carlo methods

BL = master equation describing *self-consistently* resonant relaxation

Resonance 101

frame frequency

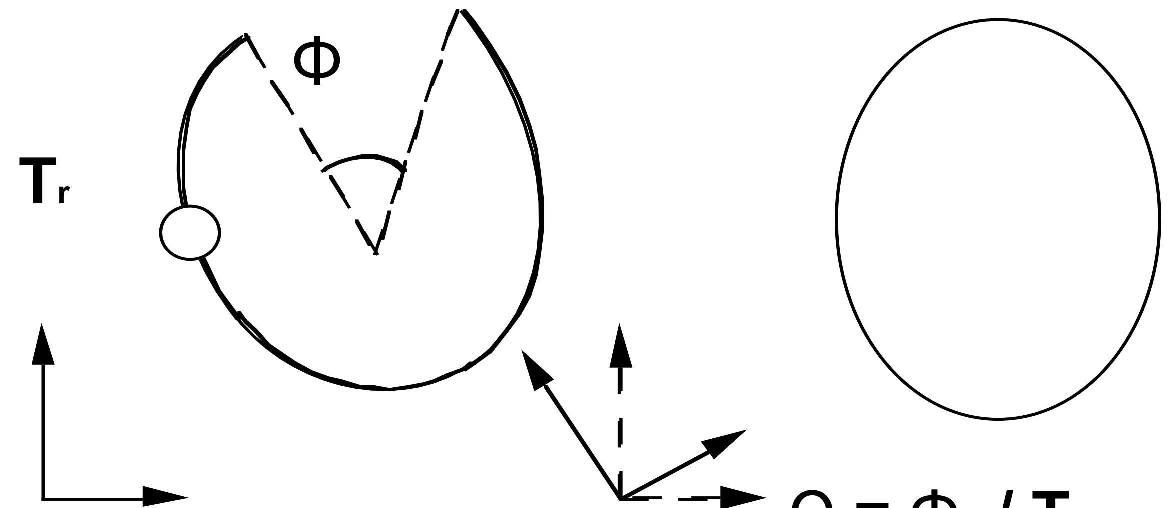
- **What is a resonance ?**

angular frequency

$$\Omega - \Omega_0 = \frac{\ell}{m}\kappa$$

epicyclic radial frequency

- **Where is a star resonant ?**



$\Omega_0 = \Phi \ / \ T_r$

• Resonant Orbital Stream =
ensemble of star describing a given *resonant* orbit
(reduction into dust)

quasi resonant streams = relative motion
tumbling stream

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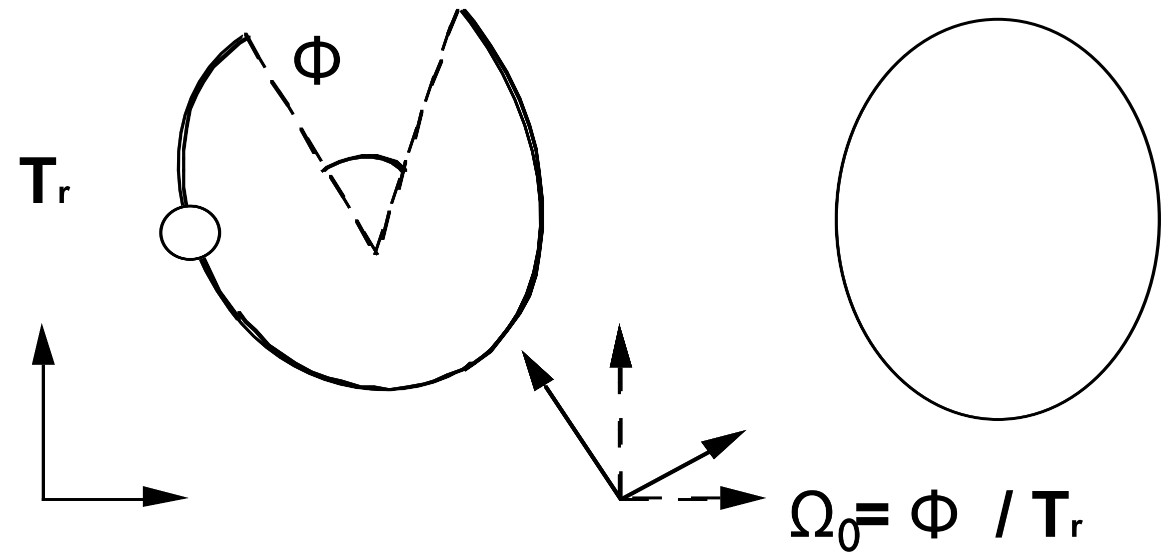

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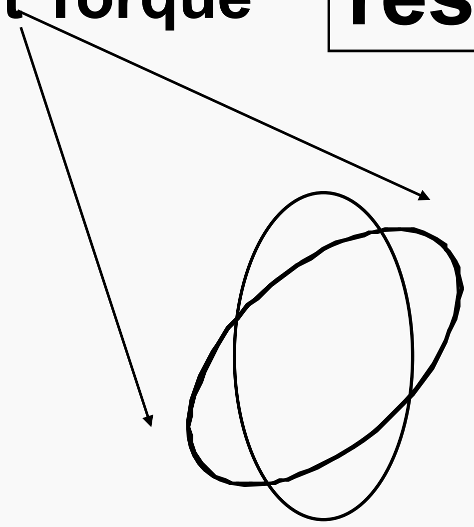
Net Torque $\boxed{\textbf{resonant interaction ?}}$ **No net torque**



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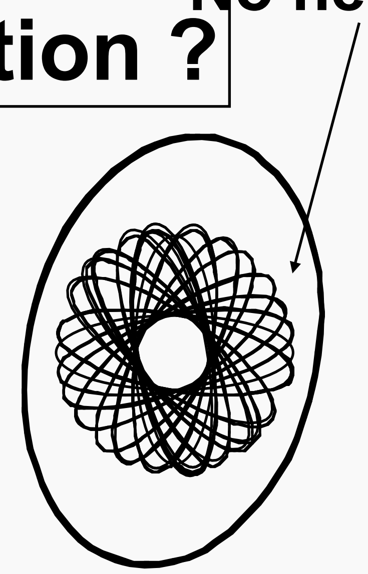

Non Resonant Orbital Stream
large relative motion = large inertia

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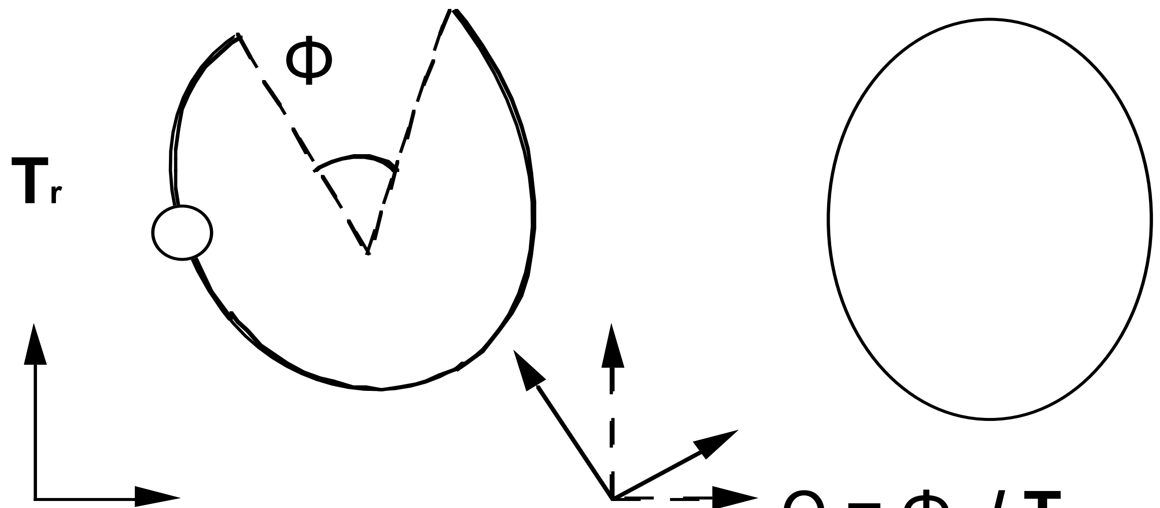

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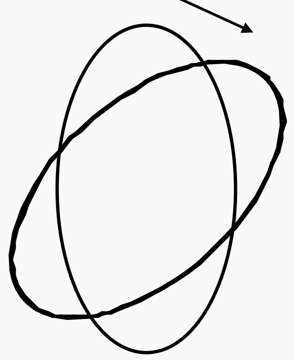

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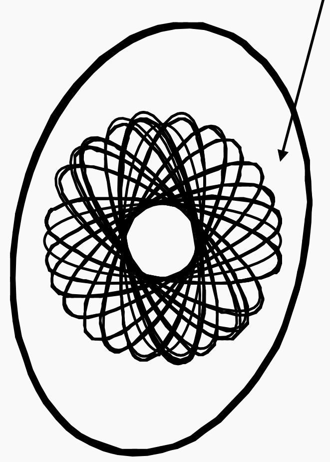

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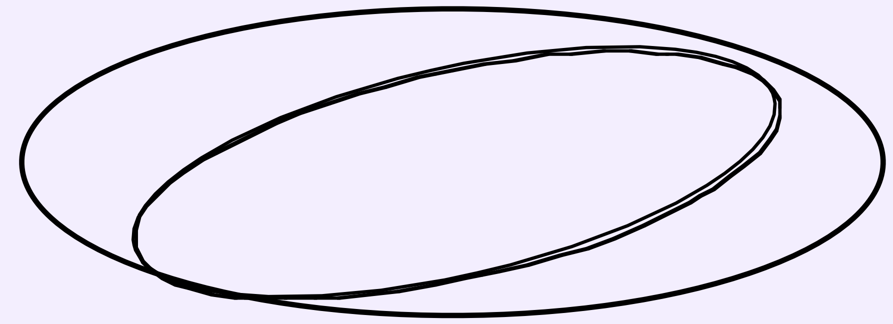
- **Gravity maximizes overlap**



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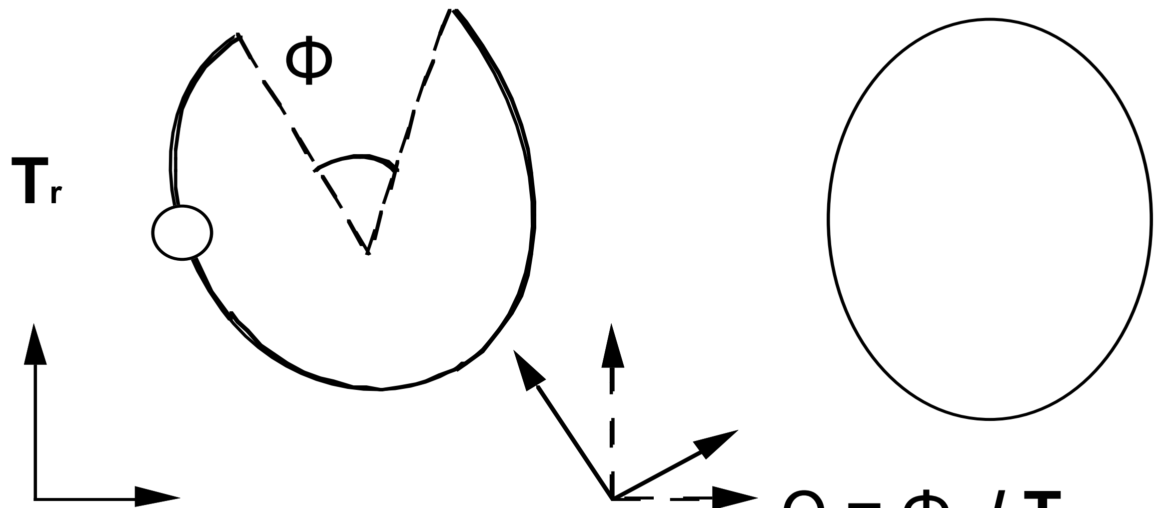

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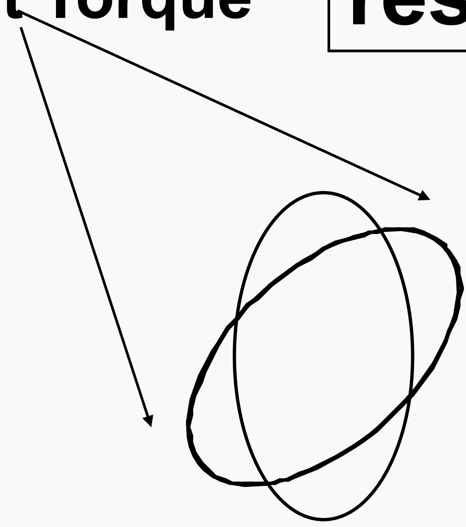
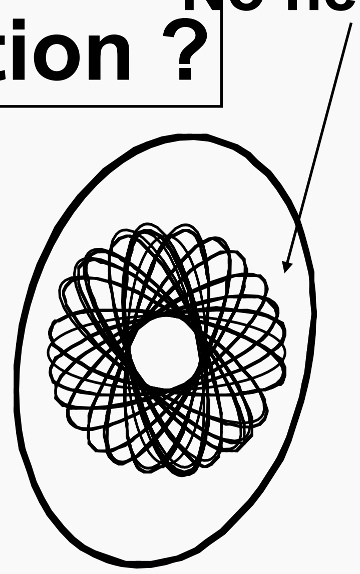


No net torque



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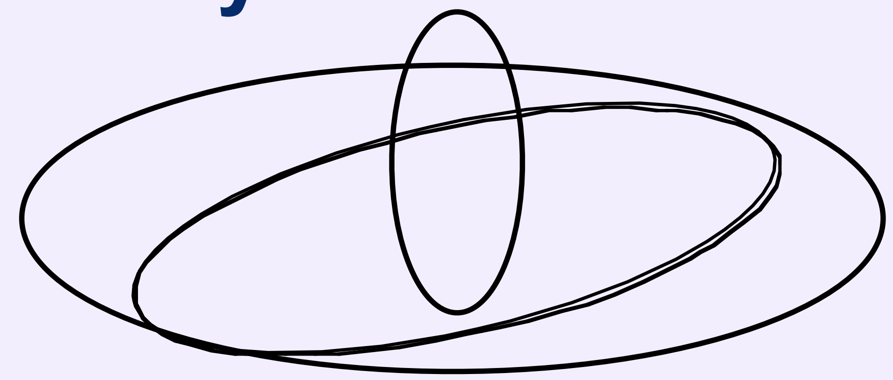

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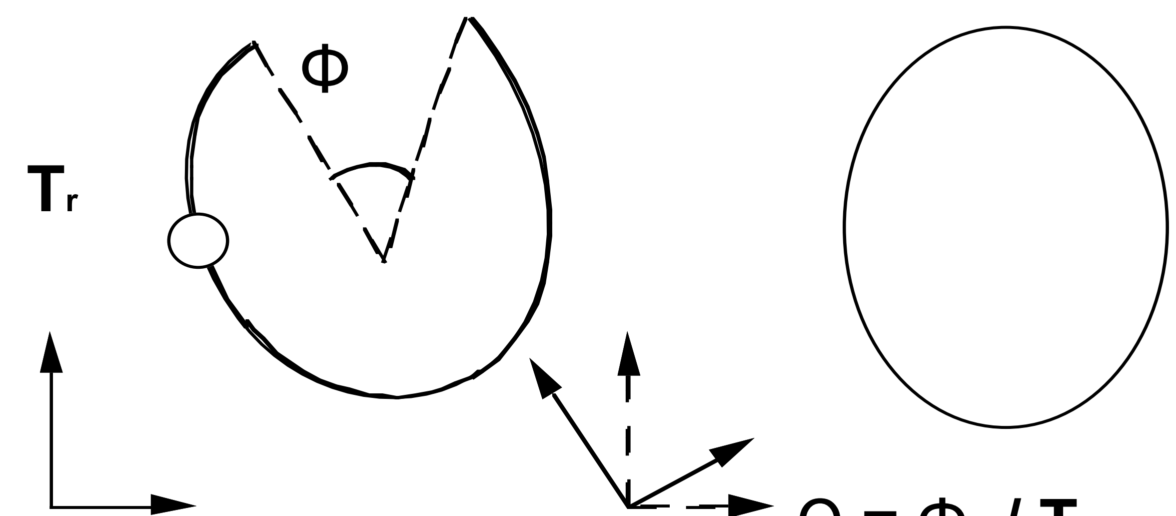

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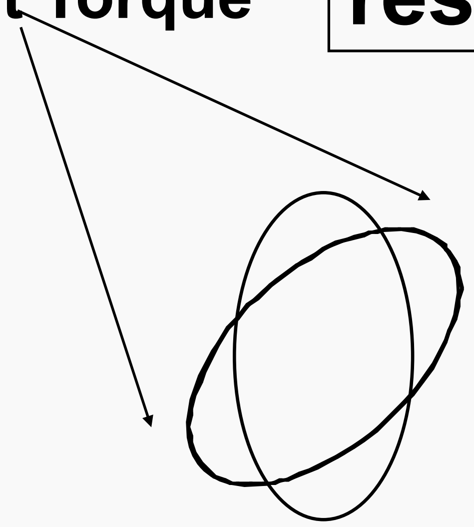

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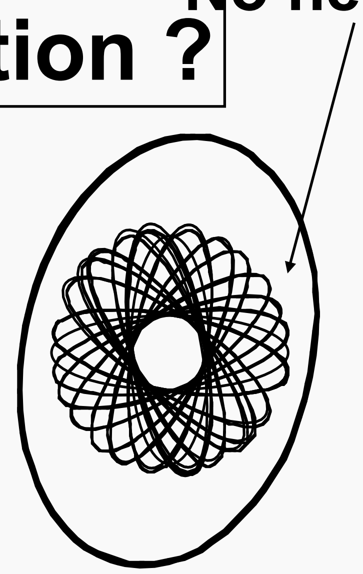

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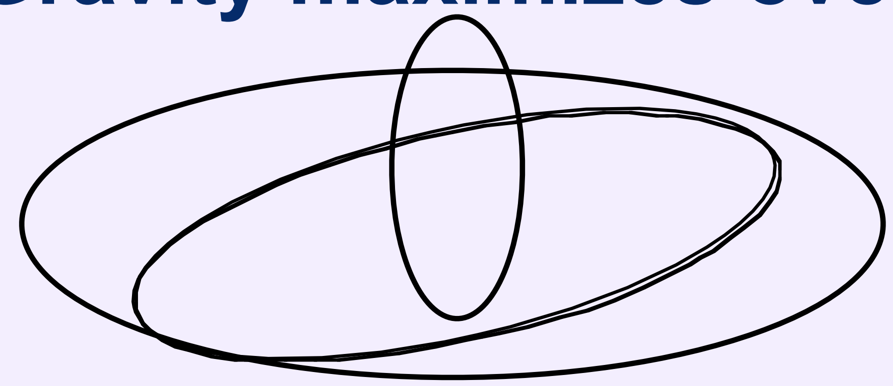


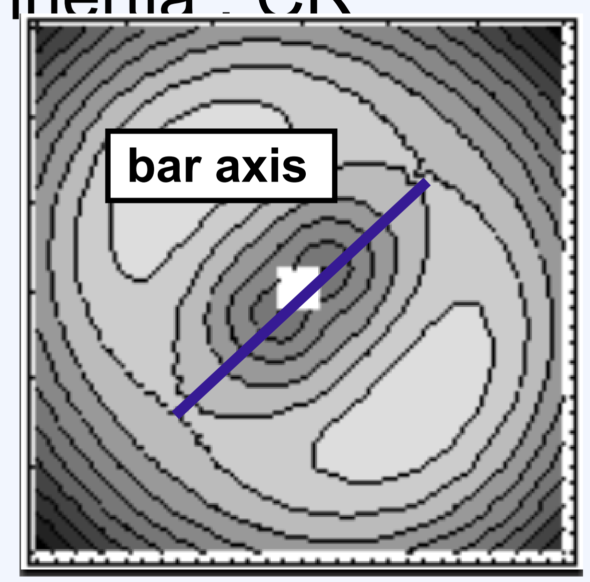
- **Inertia may repel**

How does a given stream reacts to a given torque ?

- **Cooperative Streams:**
  positive moment of intertia : ILR
- **Donkey Streams:**
  negative moment of inertia : CR

Corotation Resonance:
Donkey

Effective potential
rotating frame Ω

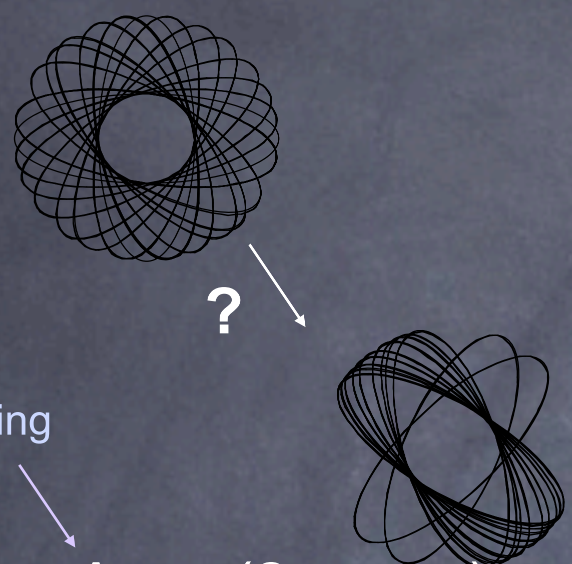


$$\dot{v} = -\nabla\psi_{eff} + 2\Omega \times v$$

Gyroscopic behaviour - orbit remains near equipotential: away from the bar

Tumbling orbit instability (a.k.a. HMF)

- Phase Portrait



Coupling

$$\Psi_{12} = A\cos(2\phi_1 - 2\phi_2)$$

Tumbling orbit instability (a.k.a. HMF)

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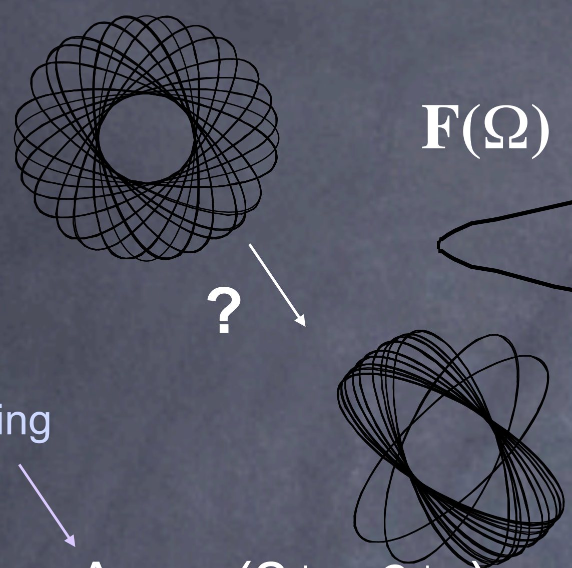

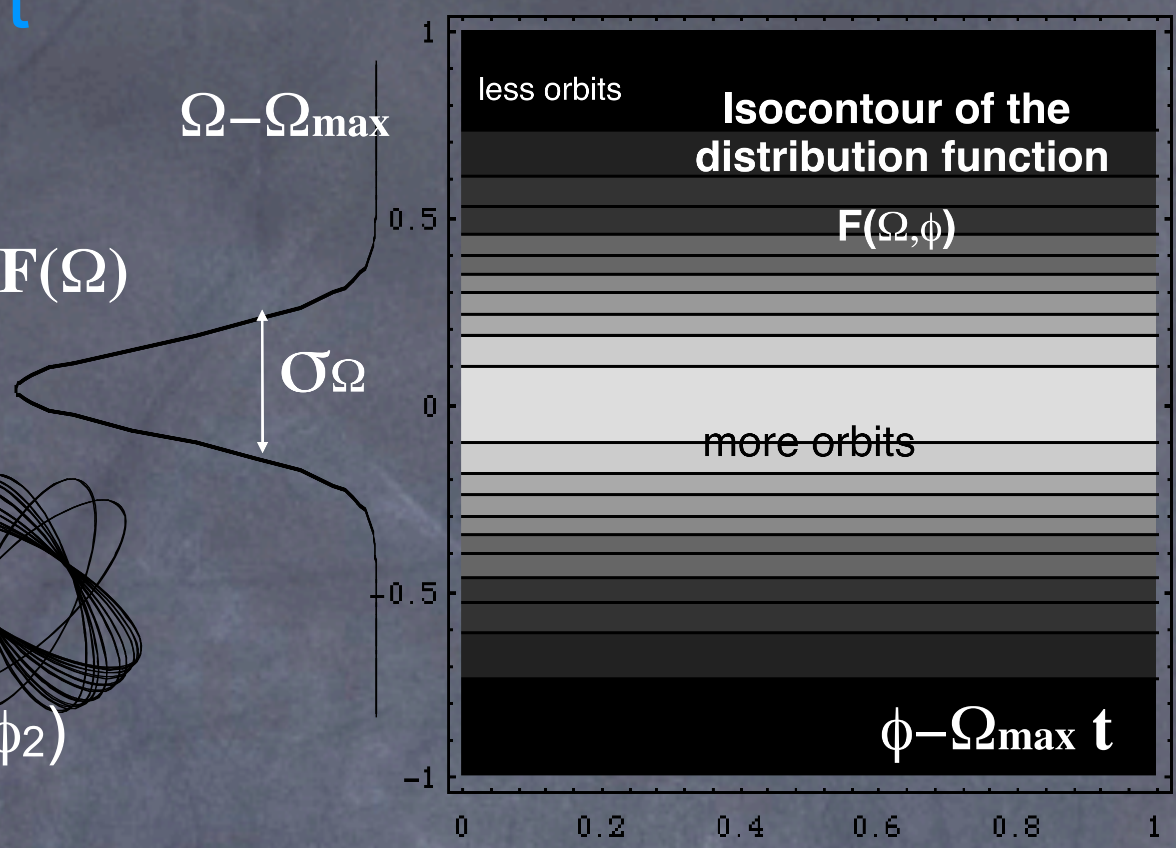

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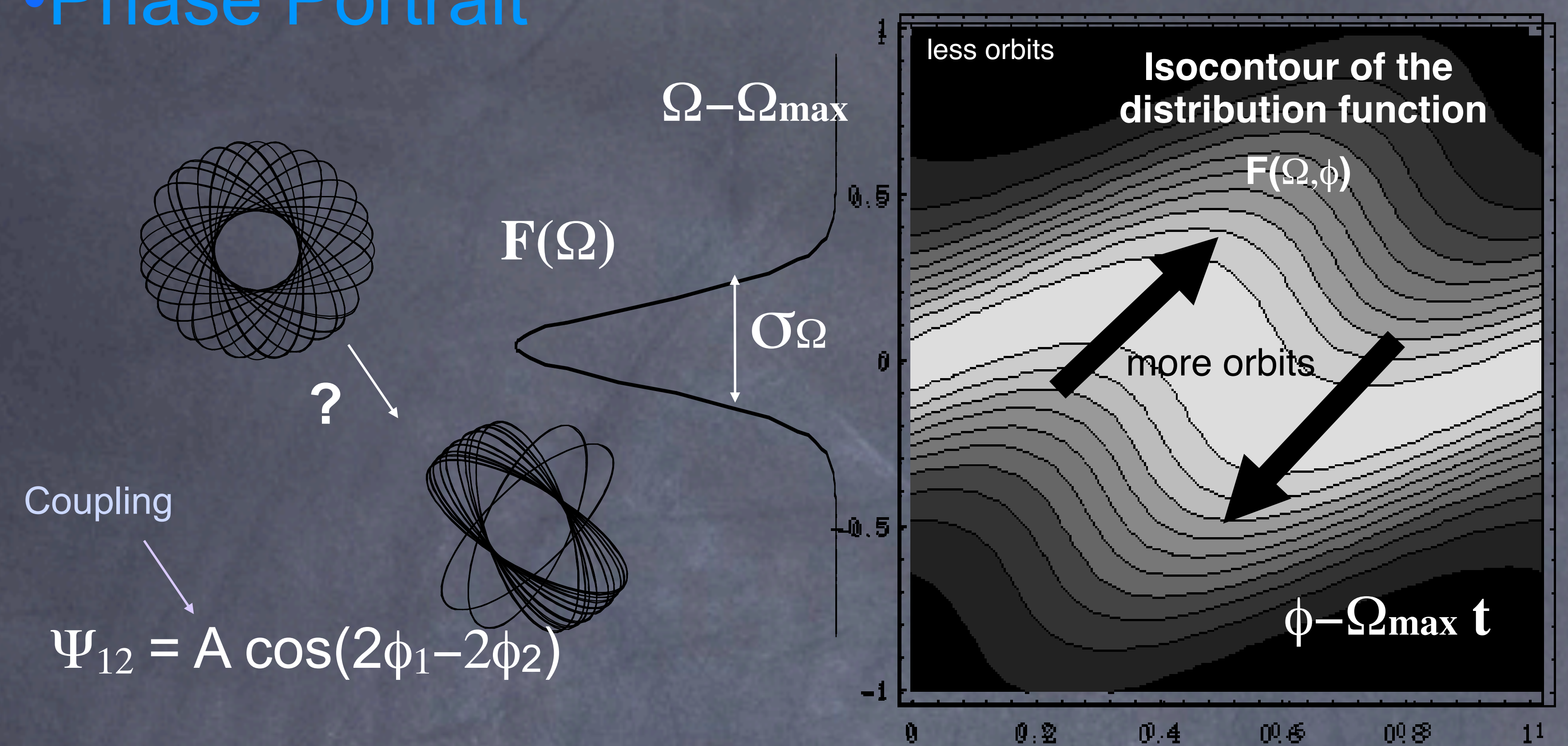

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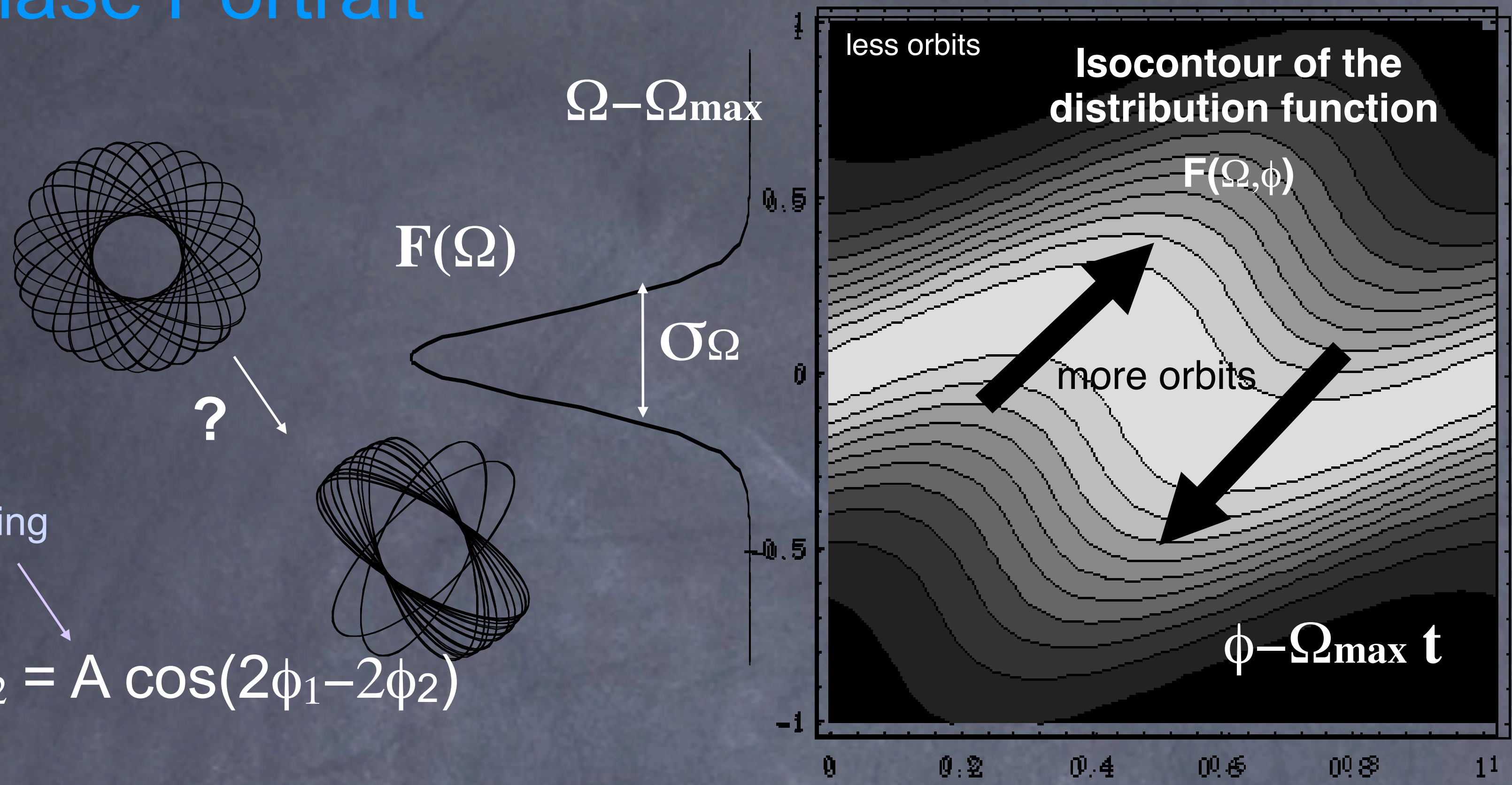

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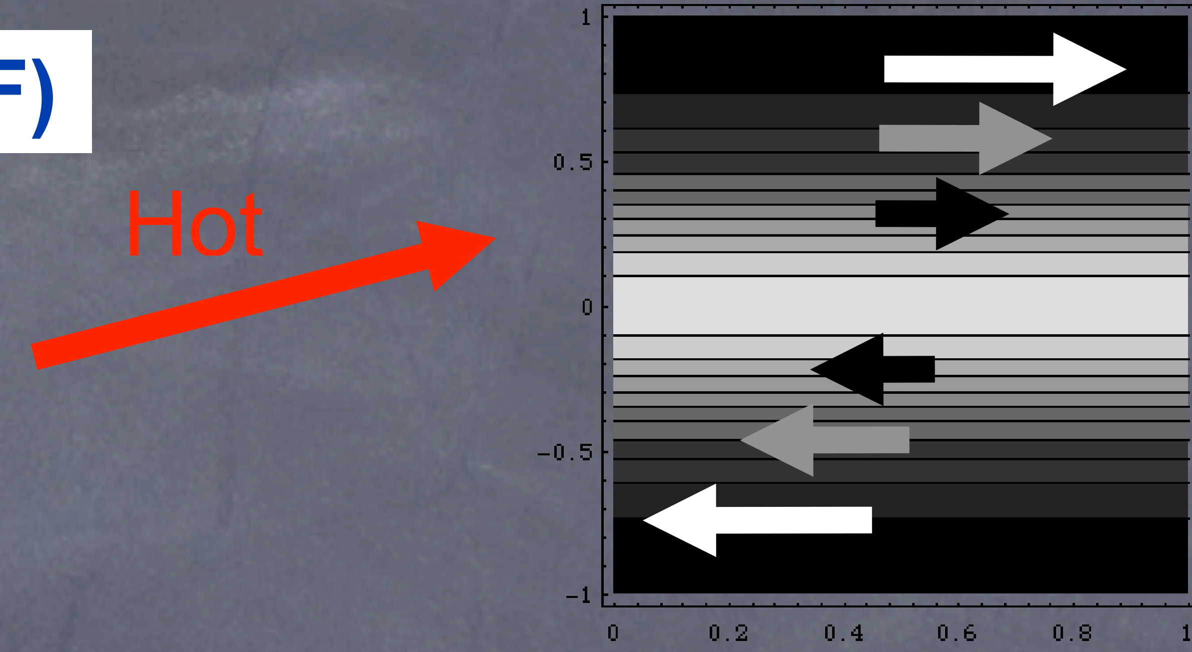

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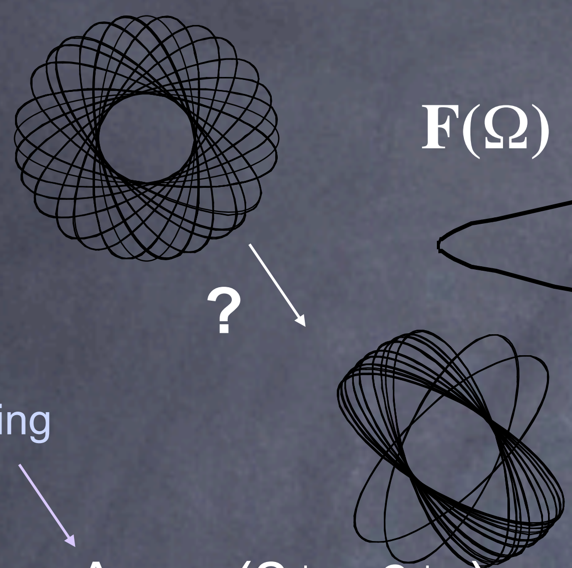

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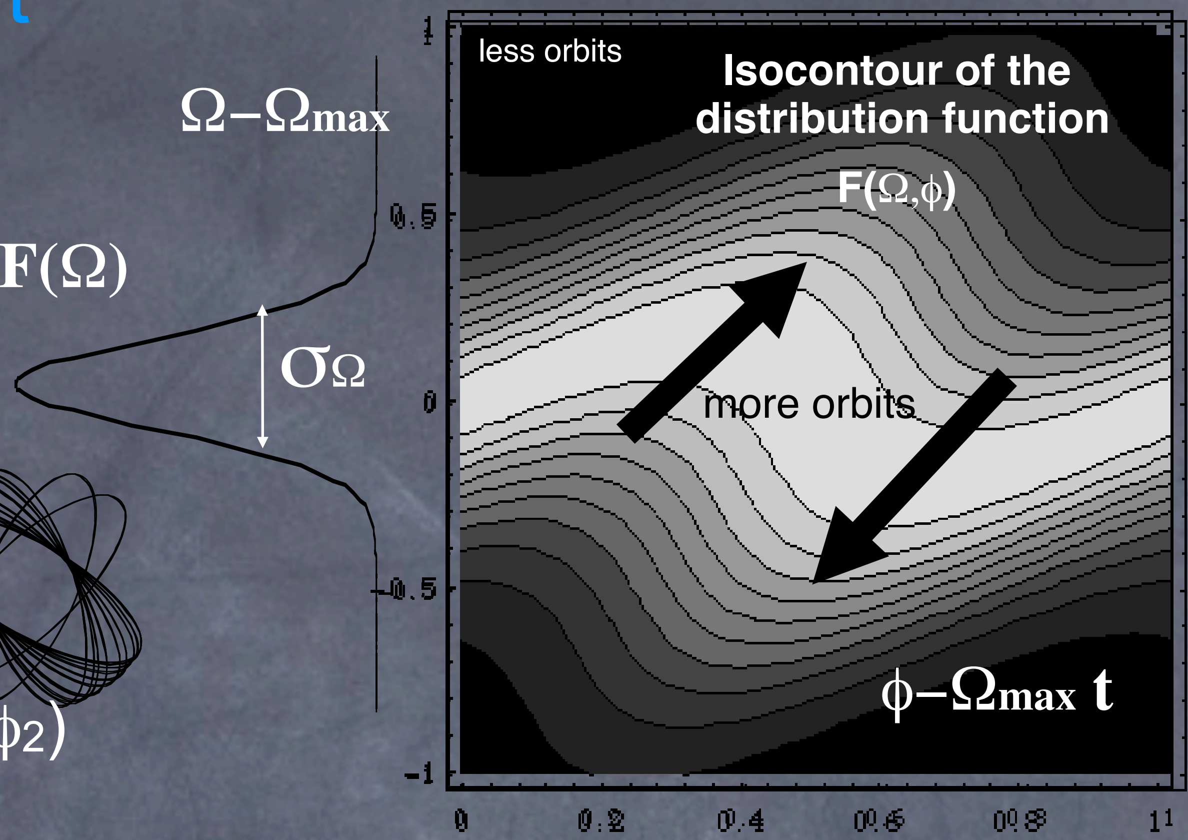
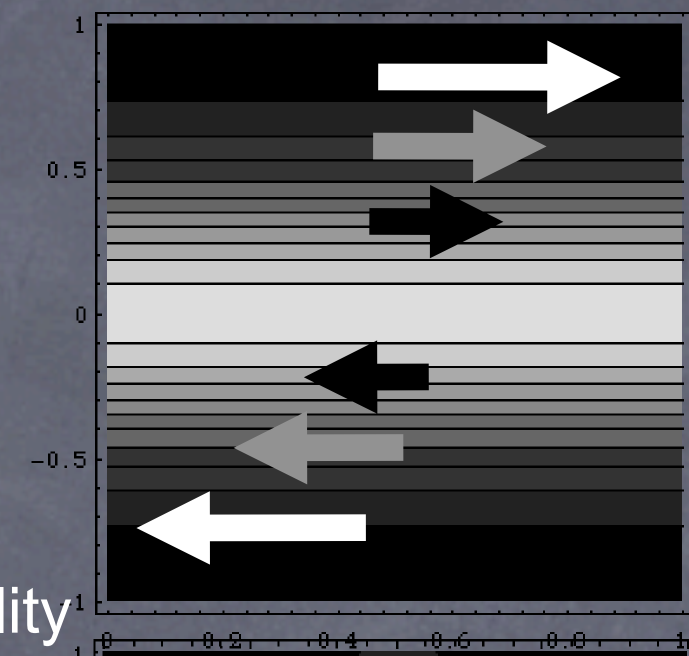


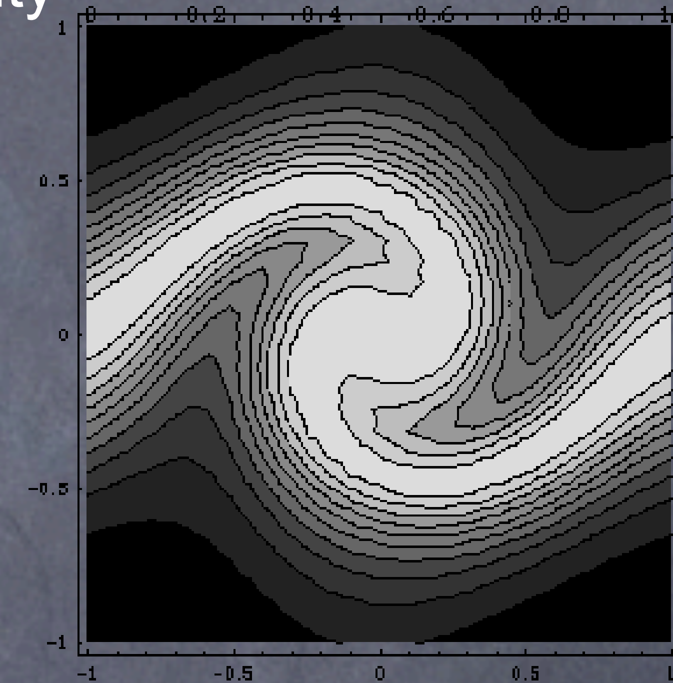
← precession rate

Hot



Heat quenches the instability

Cold



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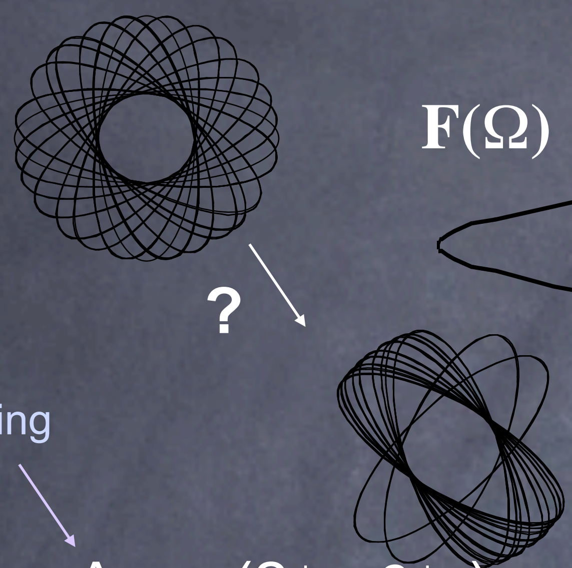

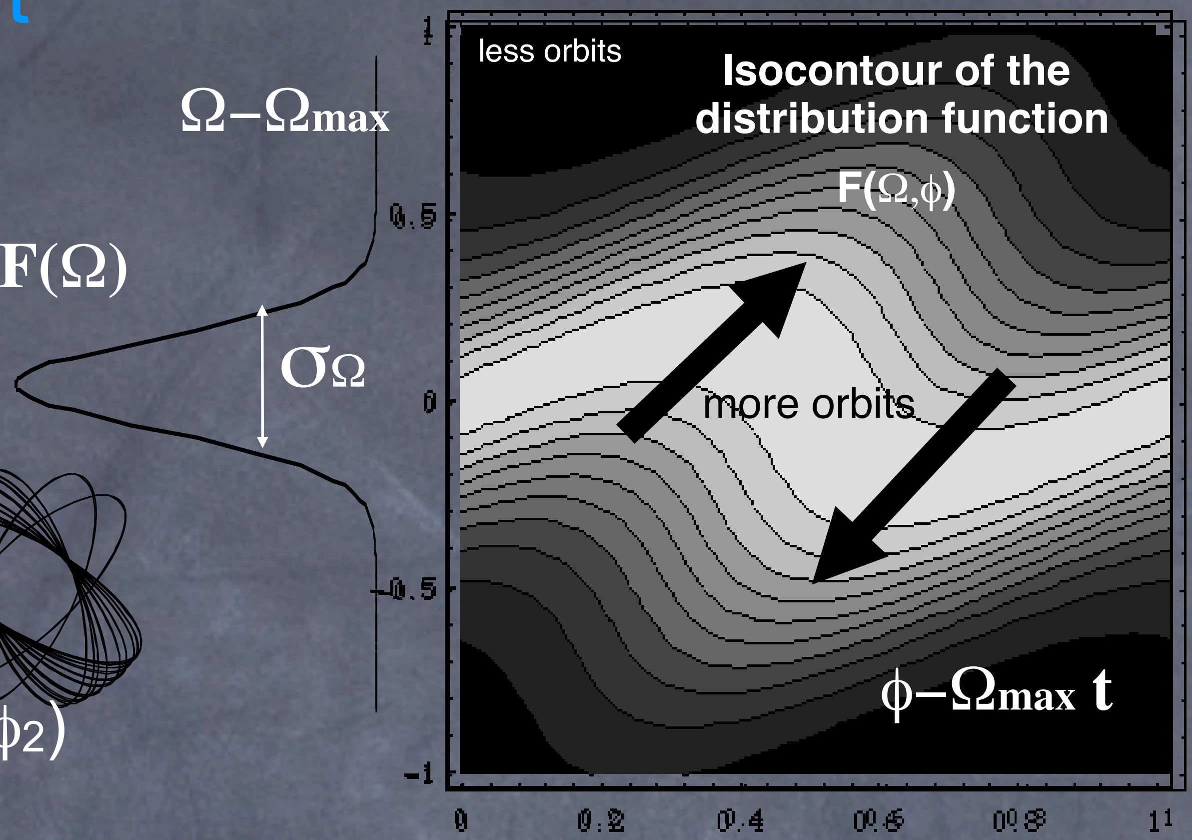

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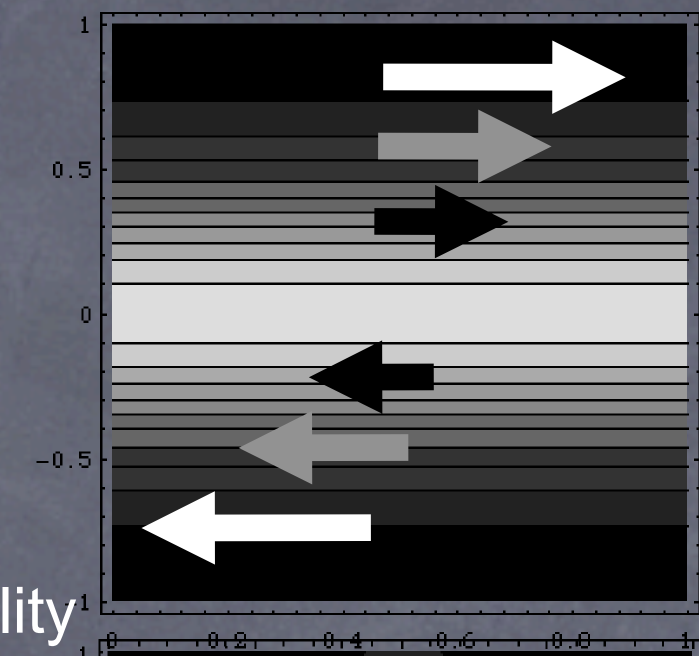

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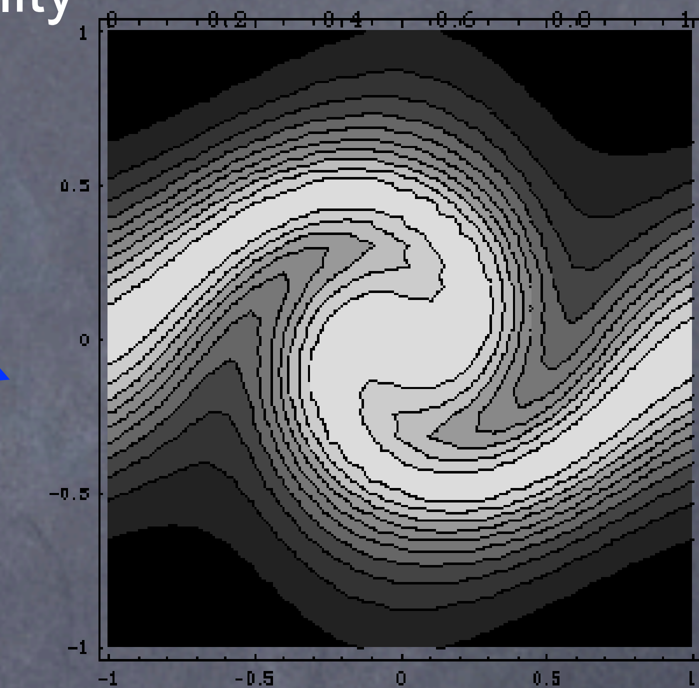


- Linear (marginal) instability

mode (m=2)

Azimuthal Instability

$$\int \frac{\partial F/\partial \Omega}{\Omega - \Omega_p}\, d\Omega = \frac{-2^2}{4\pi G A^2}$$

precession rate

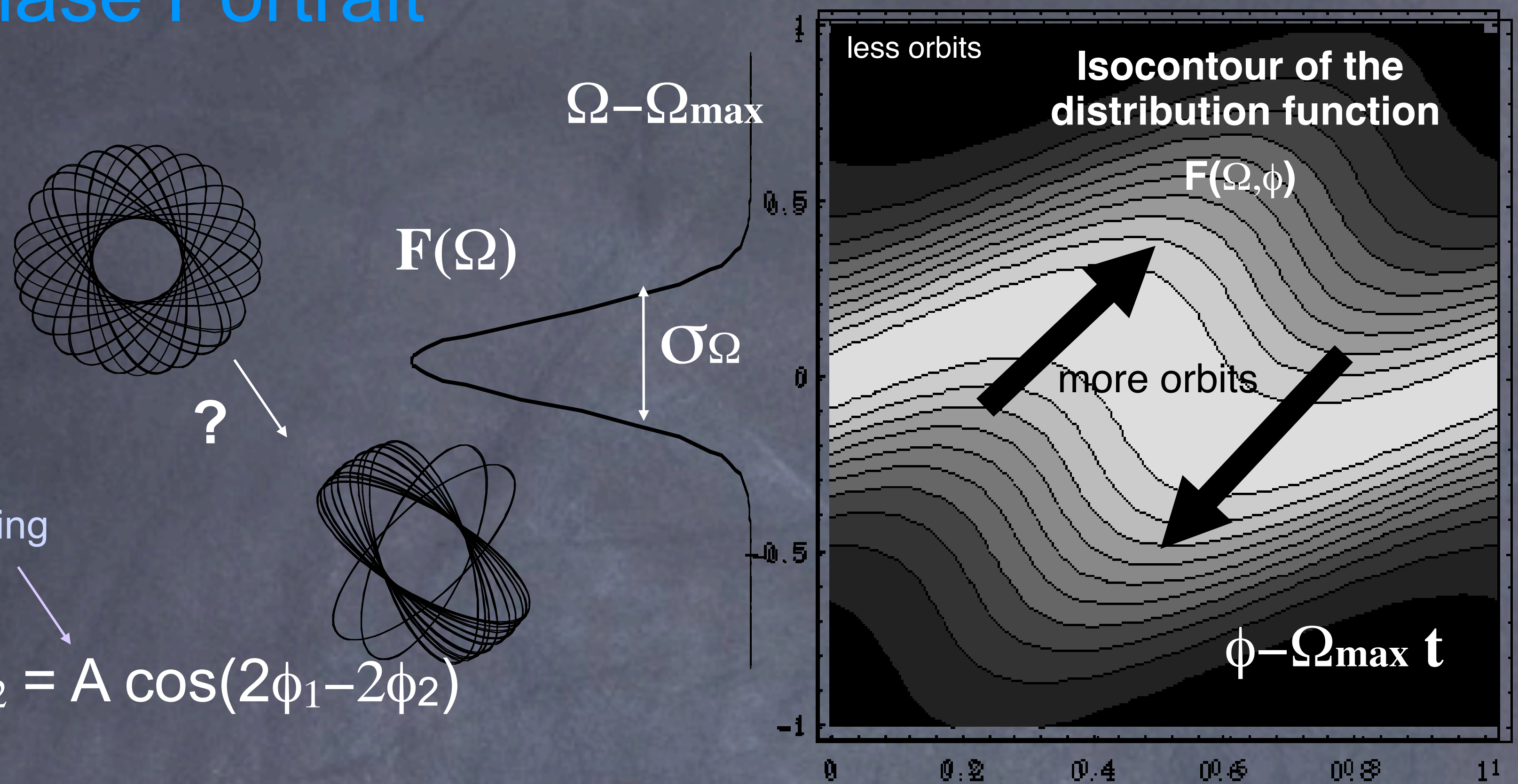
gravity

$$\left\langle \partial^2 F/\partial \Omega^2 \right\rangle \approx \frac{M}{\sigma_\Omega^2}$$

Maxwellian

Tumbling orbit instability (a.k.a. HMF)

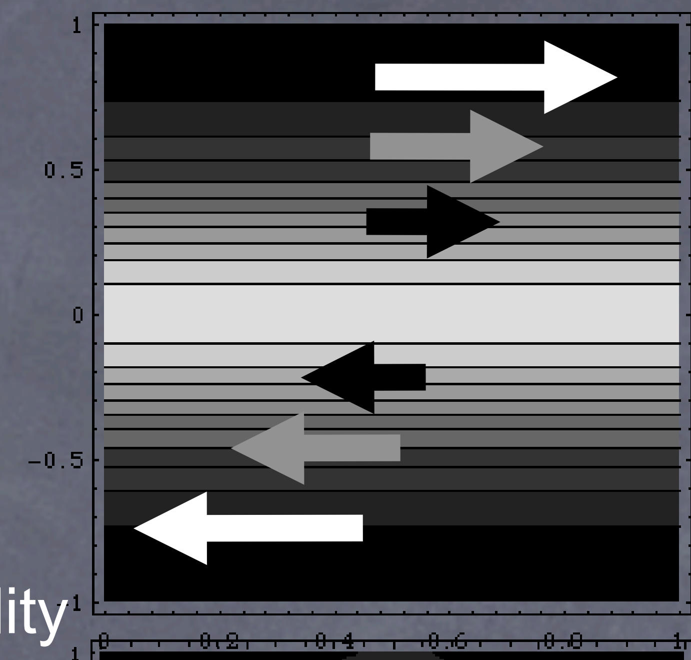
•Phase Portrait



Coupling

$\Psi_{12} = A\cos(2\phi_1 - 2\phi_2)$

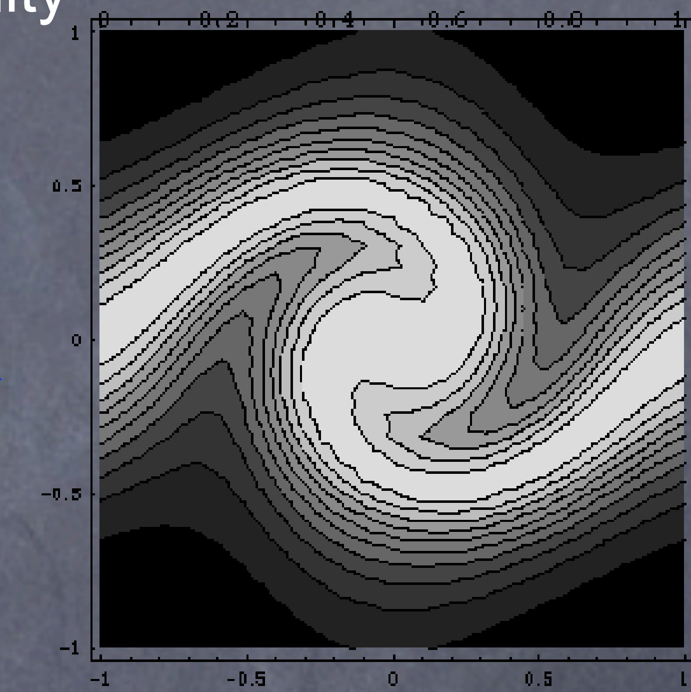
Hot



Heat quenches the instability

precession rate

Cold



• Linear (marginal) instability

mode (m=2)

Azimuthal Instability

$$\int \frac{\partial F/\partial\Omega}{\Omega - \Omega_p} d\Omega = \frac{-2^2}{4\pi G A^2}$$

precession rate

gravity

$$\left\langle \partial^2 F/\partial\Omega^2 \right\rangle \approx \frac{M}{\sigma_\Omega^2}$$

Jeans Instability

projected velocity

$$\int \frac{\partial F/\partial V_{//}}{V_{//} - W} dV_{//} = \frac{-k^2}{4\pi G}$$

phase velocity

Maxwellian $\frac{\rho}{c^2} = \frac{k^2}{4\pi G}$

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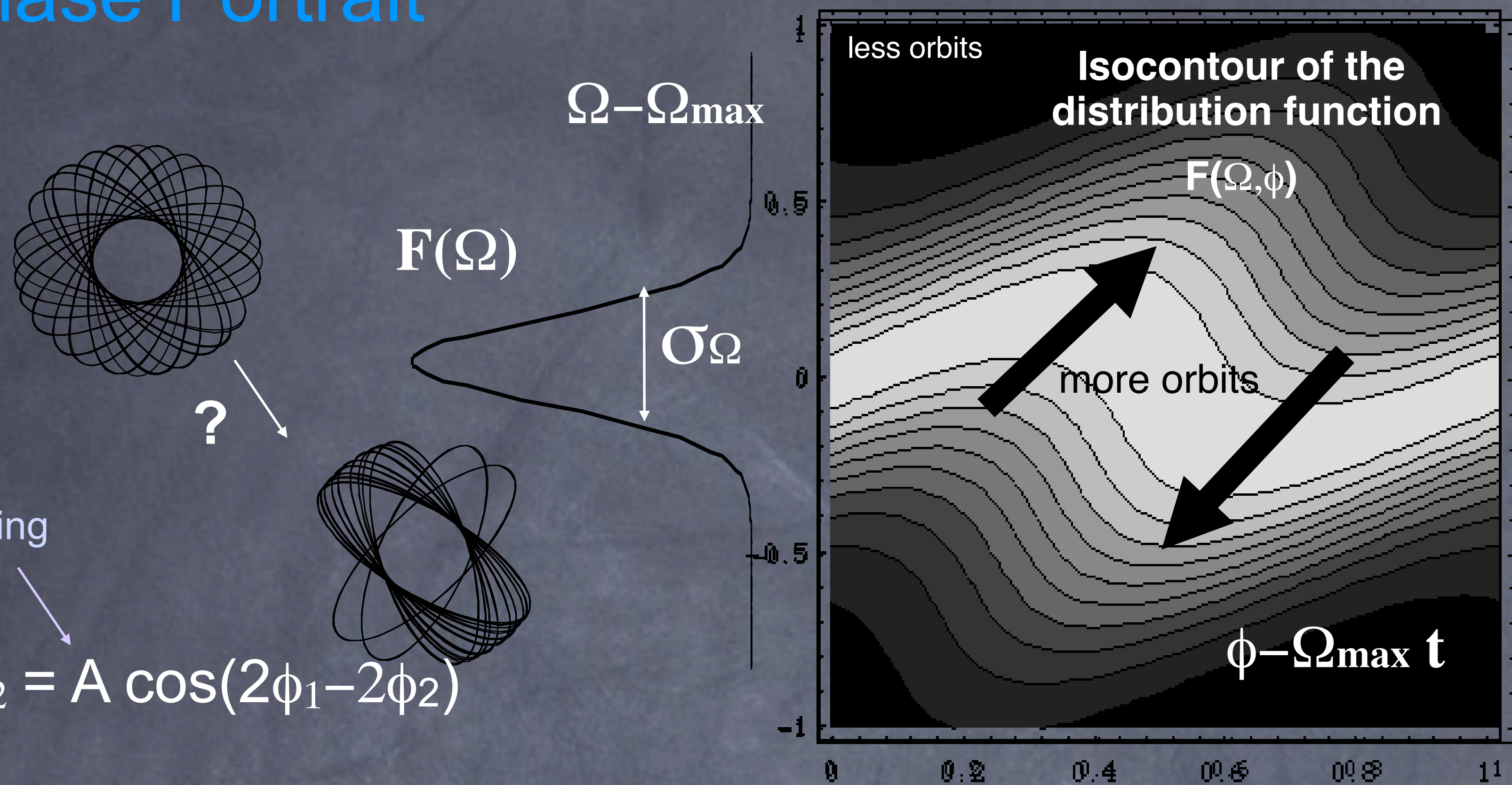

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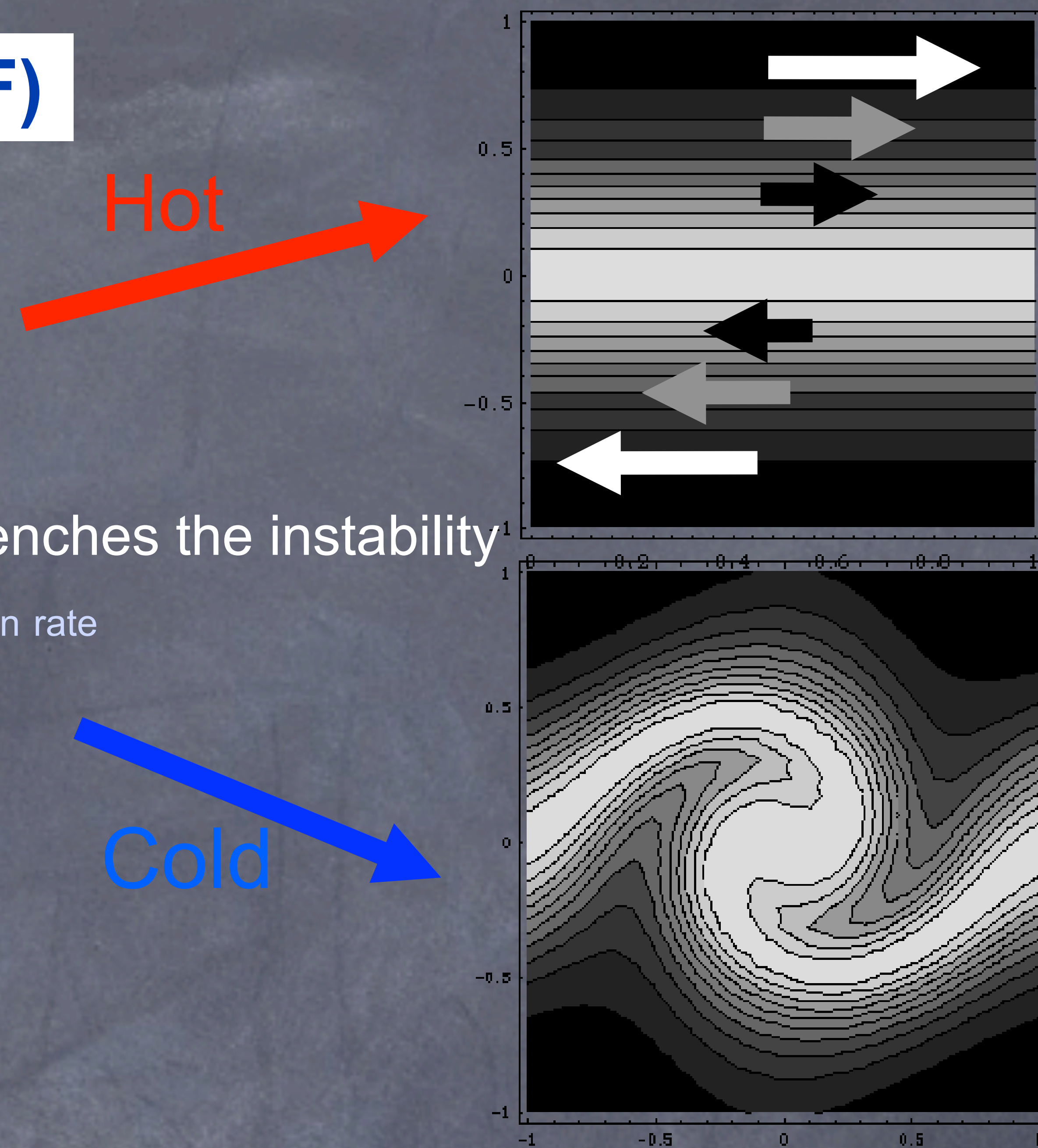

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Contest: gravity vs inertia : spontaneous evolution ?

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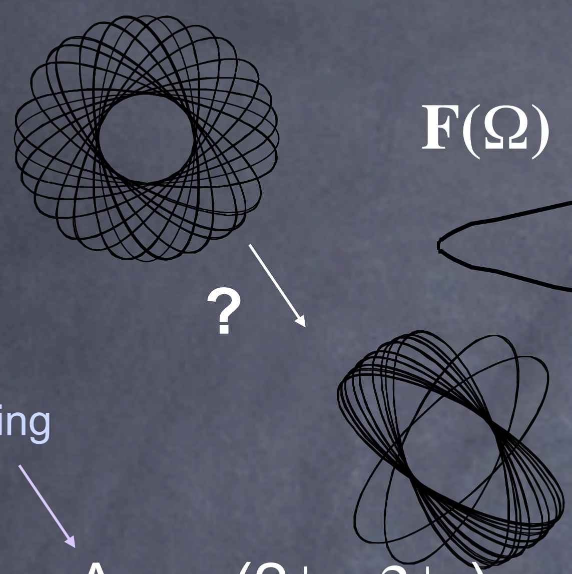
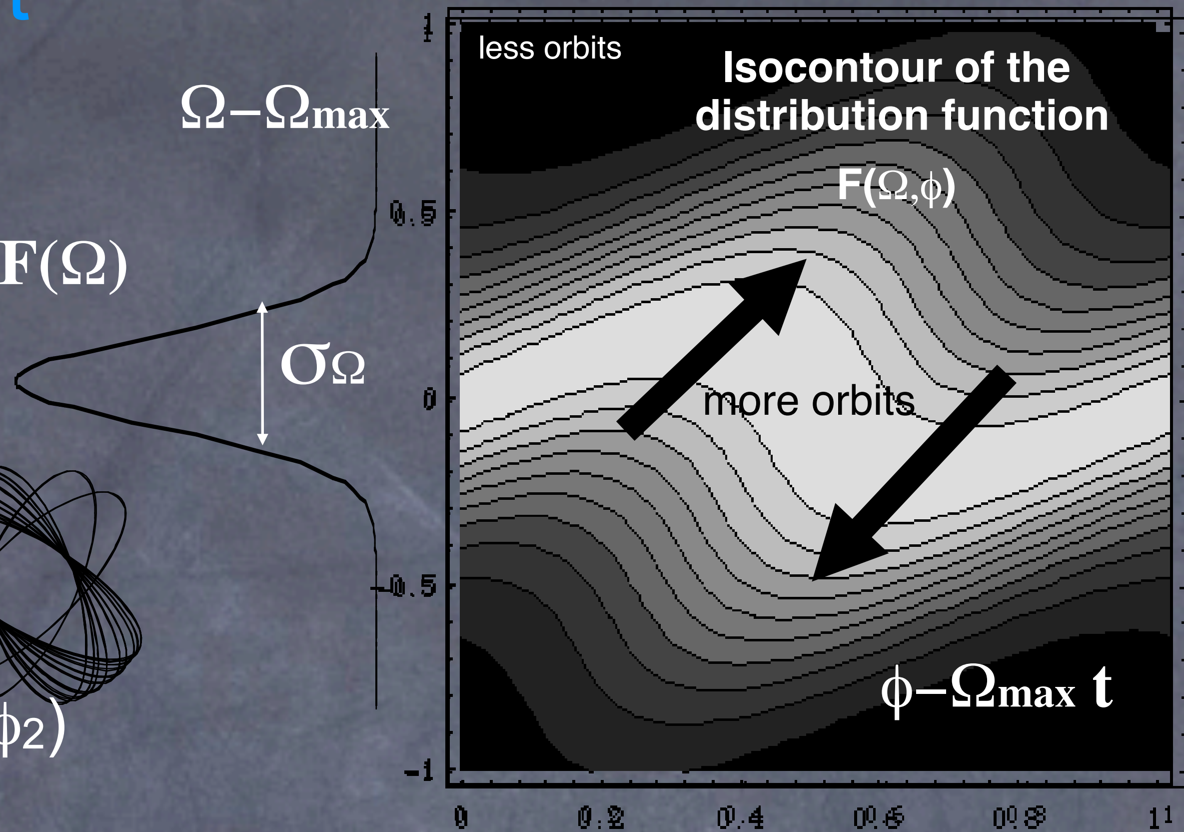


Coupling

Ψ_{12} = A cos($2\phi_1 - 2\phi_2$)



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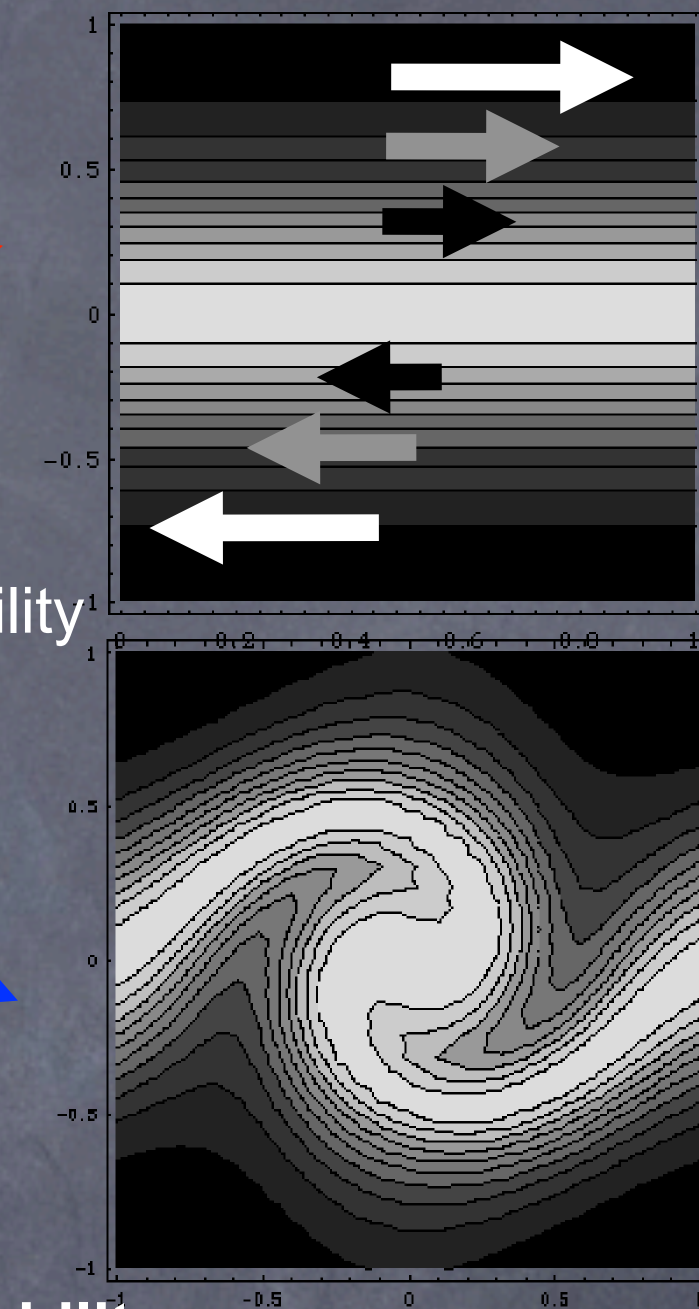

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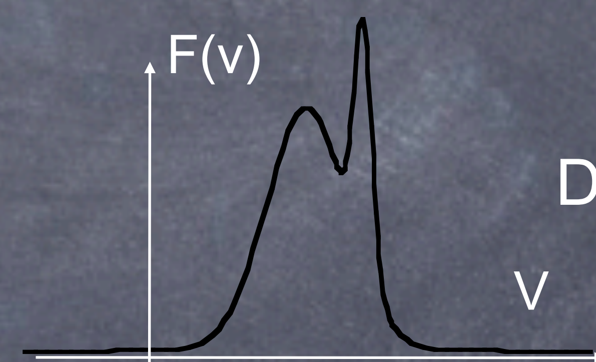
Two stream instability



Donkey streams ?

In plasma $G \to -e^2/m$

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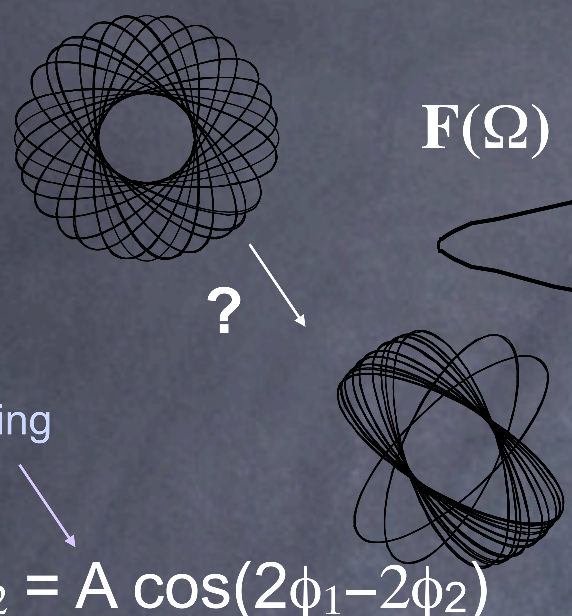

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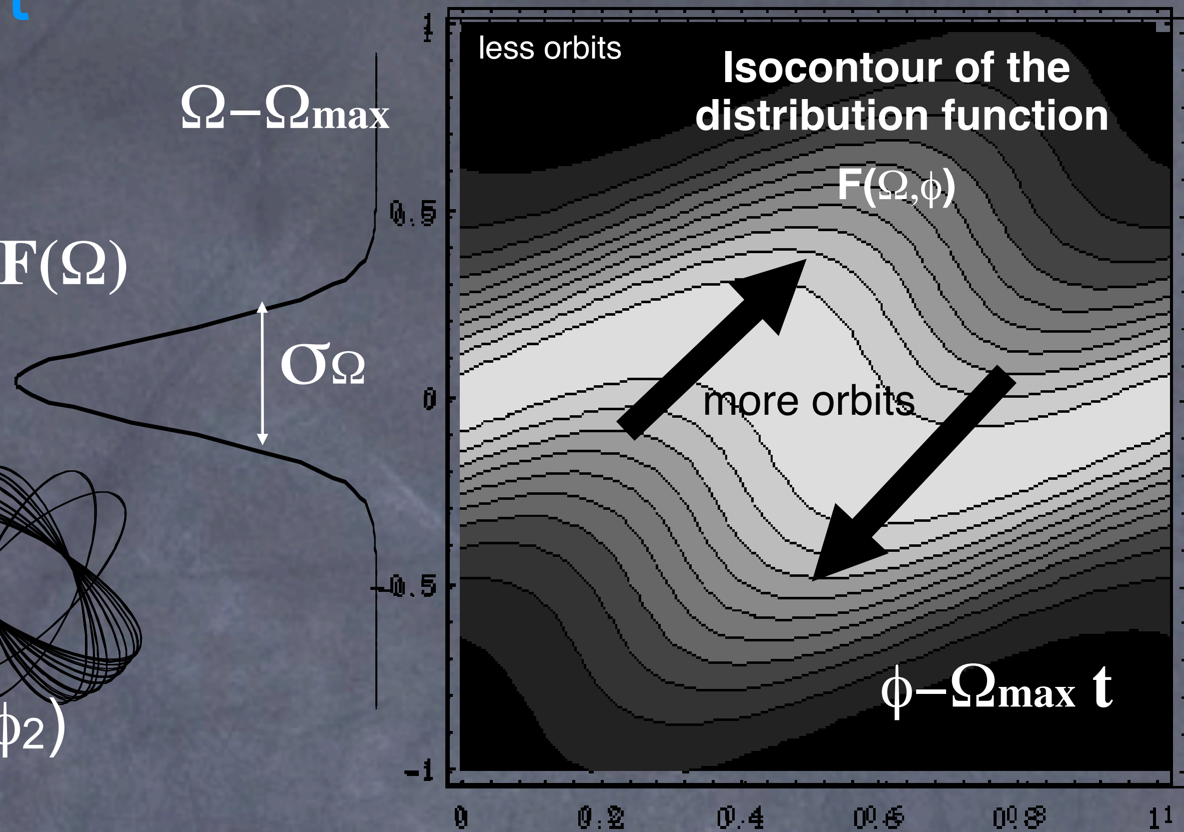

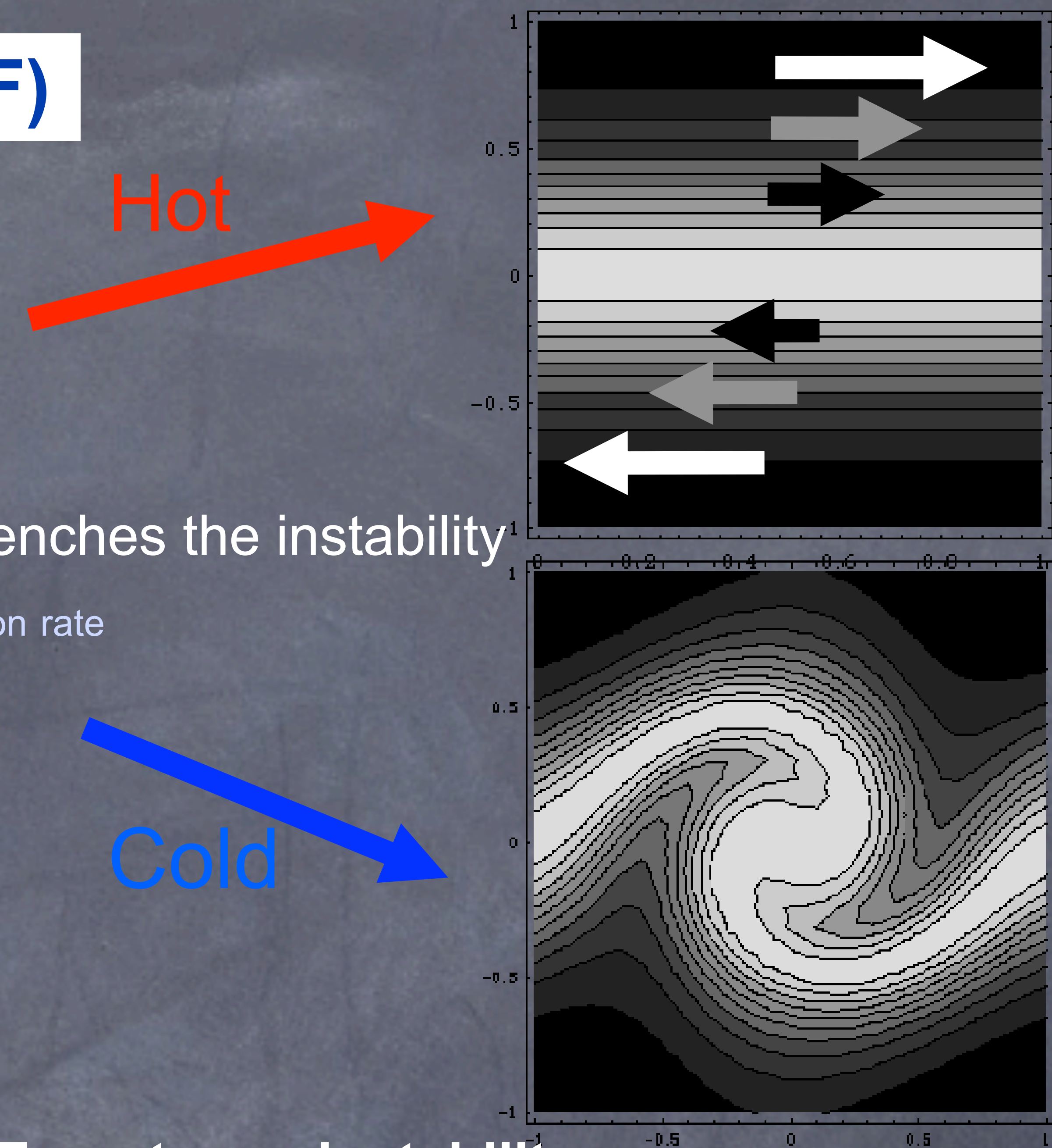

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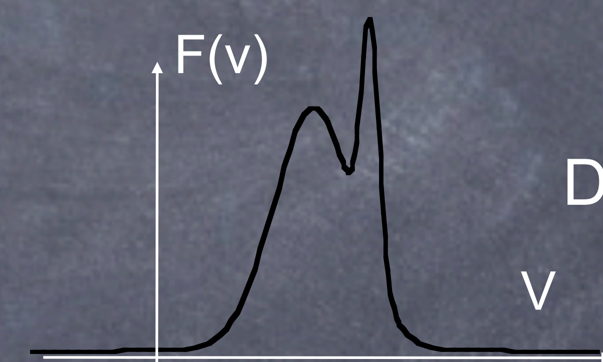

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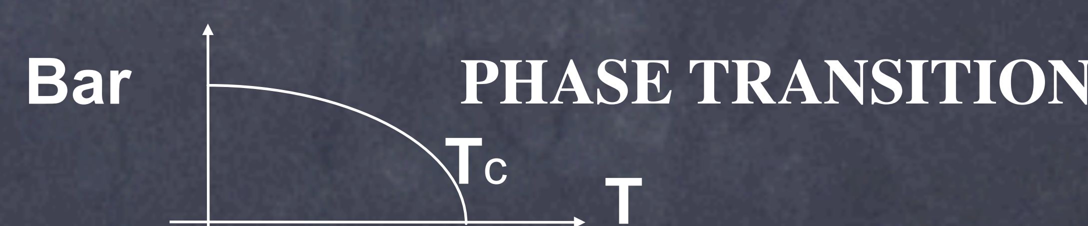
• Thermodynamical criteria

Maximize Entropy at L, E fixed $\frac{A}{k_B T} \geq 2$



Applicable to formation/ dissolution

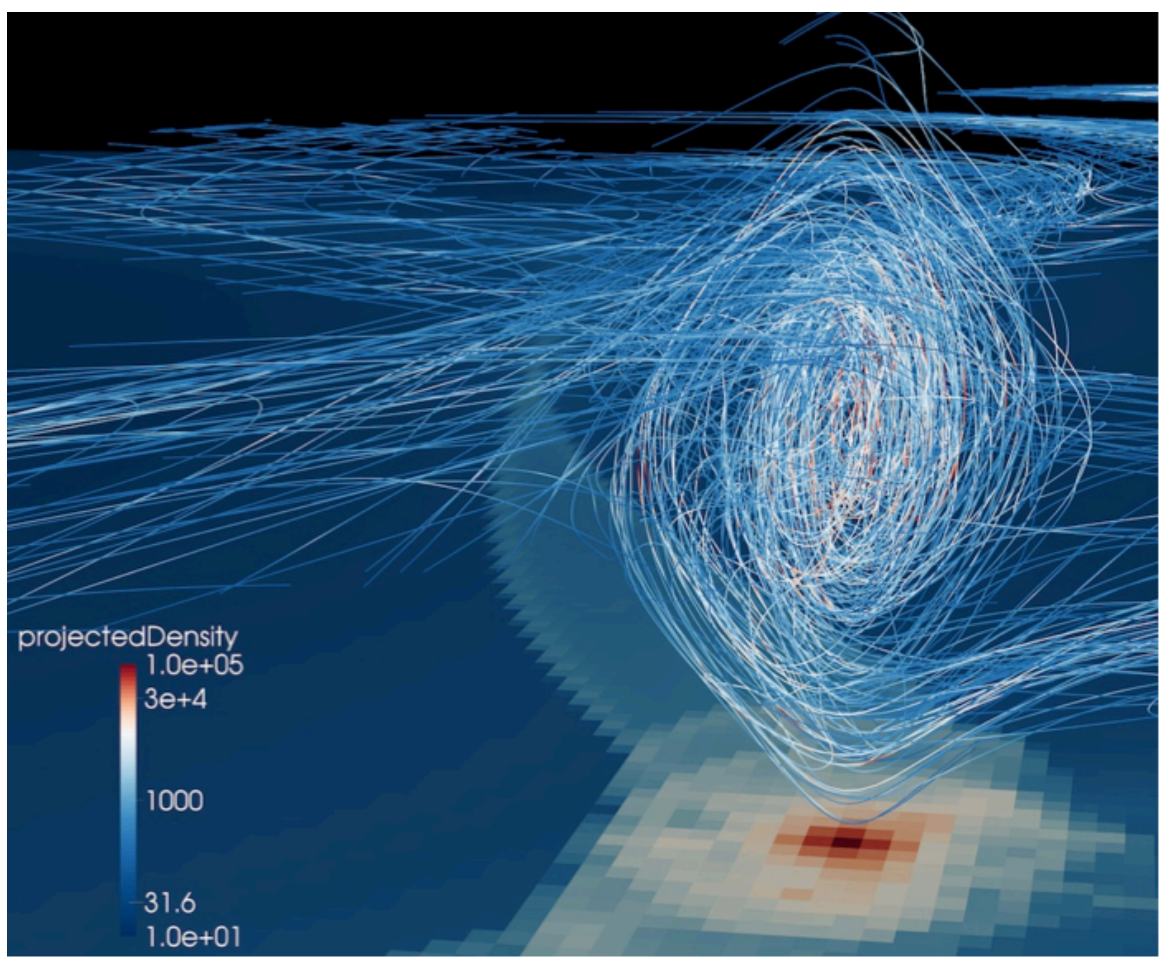
Liouville's Equation

- System of N identical interacting particles, $\boldsymbol{w} = (\boldsymbol{x}, \boldsymbol{v})$.
- Hamiltonian of the system: $H_N = \frac{1}{2}\sum_{i=1}^{N} \boldsymbol{v}_i^2 + \sum_{i<j}^{N} U(\boldsymbol{x}_i - \boldsymbol{x}_j)\,.$
- Individual dynamics governed by Hamilton's equation

$$\frac{\mathrm{d}\boldsymbol{x}_i}{\mathrm{d}t} = \frac{\partial H_N}{\partial \boldsymbol{v}_i} \quad ; \quad \frac{\mathrm{d}\boldsymbol{v}_i}{\mathrm{d}t} = -\frac{\partial H_N}{\partial \boldsymbol{x}_i}\,.$$

- $N-$body DF $f^{(N)}(\boldsymbol{w}_1, ..., \boldsymbol{w}_N, t)$ governed by Liouville's equation

$$\begin{aligned}
0 &= \frac{\partial f^{(N)}}{\partial t} + \mathrm{div}\Big[\dot{\boldsymbol{w}}\, f^{(N)}\Big] \qquad \text{continuity equation} \\
&= \frac{\partial f^{(N)}}{\partial t} + \sum_{i=1}^{N} \bigg\{ \boldsymbol{v}_i \cdot \frac{\partial f^{(N)}}{\partial \boldsymbol{x}_i} + \boldsymbol{F}_i \cdot \frac{\partial f^{(N)}}{\partial \boldsymbol{v}_i} \bigg\} \\
&= \frac{\partial f^{(N)}}{\partial t} + \sum_{i=1}^{N} \bigg\{ \frac{\partial H_N}{\partial \boldsymbol{v}_i} \cdot \frac{\partial f^{(N)}}{\partial \boldsymbol{x}_i} - \frac{\partial H_N}{\partial \boldsymbol{x}_i} \cdot \frac{\partial f^{(N)}}{\partial \boldsymbol{v}_i} \bigg\} \\
&= \frac{\partial f^{(N)}}{\partial t} + \Big[f^{(N)}, H_N\Big]\,.
\end{aligned}$$



- **Exact and reversible equation but in a $6N D$ phase-space.**

BBGKY Hierarchy

- Reduced DF in **$\boldsymbol{6nD}$ phase space**

$$f_n(\boldsymbol{w}_1, ..., \boldsymbol{w}_n, t) = \frac{N!}{(N-n)!} \int \mathrm{d}\boldsymbol{w}_{n+1} \, \mathrm{d}\boldsymbol{w}_N \, f^{(N)}(\boldsymbol{w}_1, ..., \boldsymbol{w}_N, t) \, .$$

- Reduced $n-$body Hamiltonian

$$H_n = \frac{1}{2} \sum_{i=1}^{n} \boldsymbol{v}_i^2 + \sum_{i<j\leq n} U_{i,j} \, .$$

- $n^{\mathrm{th}}-$BBGKY equation for f_n

$$\boxed{\frac{\partial f_n}{\partial t} + [f_n, H_n] = \sum_{i=1}^{n} \int \mathrm{d}\boldsymbol{x}_{n+1} \, \mathrm{d}\boldsymbol{v}_{n+1} \, \frac{\partial U_{i,n+1}}{\partial \boldsymbol{x}_i} \cdot \frac{\partial f_{n+1}}{\partial \boldsymbol{v}_i} \, .}$$

- Content
  - $n-$body dynamics: Liouville's equation and $(n+1)^{\mathrm{th}}$ order *collision term.*
  - Exact hierarchy of equation: **Requires a truncation.**

From BBGKY to Vlasov

- Two-body correlation function

$$\boxed{f_2(\boldsymbol{w}_1,\boldsymbol{w}_2)=f_1(\boldsymbol{w}_1)\,f_1(\boldsymbol{w}_2)+g_2(\boldsymbol{w}_1,\boldsymbol{w}_2)\,.}$$

- **BBGKY** $-\boldsymbol{n}=\mathbf{1}$ **equation**

$$\frac{\partial f_1}{\partial t}+\boldsymbol{v}_1\cdot\frac{\partial f_1}{\partial \boldsymbol{x}_1}-\frac{\partial f_1}{\partial \boldsymbol{v}_1}\cdot\frac{\partial}{\partial \boldsymbol{x}_1}\left[\int\mathrm{d}\boldsymbol{w}_2\,U_{1,2}\,f_1(\boldsymbol{w}_2)\right]=\int\mathrm{d}\boldsymbol{w}_2\,\frac{\partial U_{1,2}}{\partial \boldsymbol{x}_1}\cdot\frac{\partial g_2}{\partial \boldsymbol{v}_1}\,.$$

- Separable system with no particle correlation: $g_2=0$.

$$\begin{cases}\displaystyle\int\mathrm{d}\boldsymbol{v}_2\,f_1(\boldsymbol{x}_2,\boldsymbol{v}_2,t)=\rho(\boldsymbol{x}_2,t)\,,\\ \displaystyle\int\mathrm{d}\boldsymbol{x}_2\,\rho(\boldsymbol{x}_2,t)U(\boldsymbol{x}_1-\boldsymbol{x}_2)=\Phi(\boldsymbol{x}_1,t)\,.\end{cases}\Longrightarrow\boxed{\frac{\partial f_1}{\partial t}+\boldsymbol{v}_1\cdot\frac{\partial f_1}{\partial \boldsymbol{x}_1}-\frac{\partial \Phi}{\partial \boldsymbol{x}_1}\cdot\frac{\partial f_1}{\partial \boldsymbol{v}_1}=0\,.}$$

- We recover **Vlasov equation** for an uncorrelated system of N particles to describe the secular **collisionless** evolution.

From BBGKY to Balescu-Lenard

- Taking into account two-body correlations but $\boxed{\text{truncation at the order } 1/N}$ (*i.e.* $g_3 \equiv 0$).
- **BBGKY–$n=2$ equation**

$$
\begin{aligned}
&\frac{\partial g_2(1,2)}{\partial t} + \left[\boldsymbol{v}_1 \cdot \frac{\partial}{\partial \boldsymbol{x}_1} + \boldsymbol{v}_2 \cdot \frac{\partial}{\partial \boldsymbol{x}_2}\right] g_2(1,2) \\
&- \left[\int \mathrm{d}\boldsymbol{x}_3\, \mathrm{d}\boldsymbol{v}_3\, \frac{\partial U_{1,3}}{\partial \boldsymbol{x}_1} f_1(3) \cdot \frac{\partial}{\partial \boldsymbol{v}_1} + \int \mathrm{d}\boldsymbol{x}_3\, \mathrm{d}\boldsymbol{v}_3\, \frac{\partial U_{2,3}}{\partial \boldsymbol{x}_2} f_1(3) \cdot \frac{\partial}{\partial \boldsymbol{v}_2}\right] g_2(1,2) \\
&- \left[\int \mathrm{d}\boldsymbol{x}_3\, \mathrm{d}\boldsymbol{v}_3\, \frac{\partial U_{1,3}}{\partial \boldsymbol{x}_1} g_2(2,3)\right] \cdot \frac{\partial f_1(1)}{\partial \boldsymbol{v}_1} - \left[\int \mathrm{d}\boldsymbol{x}_3\, \mathrm{d}\boldsymbol{v}_3\, \frac{\partial U_{2,3}}{\partial \boldsymbol{x}_2} g_2(1,3)\right] \cdot \frac{\partial f_1(2)}{\partial \boldsymbol{v}_2} \\
&= \frac{\partial U_{1,2}}{\partial \boldsymbol{x}_1} \cdot \left[\frac{\partial}{\partial \boldsymbol{v}_1} - \frac{\partial}{\partial \boldsymbol{v}_2}\right] f_1(1) f_1(2)
\end{aligned}
$$

- Complex to solve for f_1 and g_2, especially in inhomogeneous systems.
- But VERY symmetric.

Bogoliubov's synchronization hypothesis *(g varies much faster than f)*

From BBGKY to Balescu-Lenard

- Taking into account two-body correlations but $\boxed{\text{truncation at the order } 1/N}$ ($i.e.$ $g_3 \equiv 0$).
- **BBGKY $-n=2$ equation** $\partial_t g(1,2) + V_1(g(1,2)) + V_2(g(1,2)) = S(1,2)$

$$
\begin{aligned}
&\frac{\partial g_2(1,2)}{\partial t} + \left[\boldsymbol{v}_1 \cdot \frac{\partial}{\partial \boldsymbol{x}_1} + \boldsymbol{v}_2 \cdot \frac{\partial}{\partial \boldsymbol{x}_2}\right] g_2(1,2) \\
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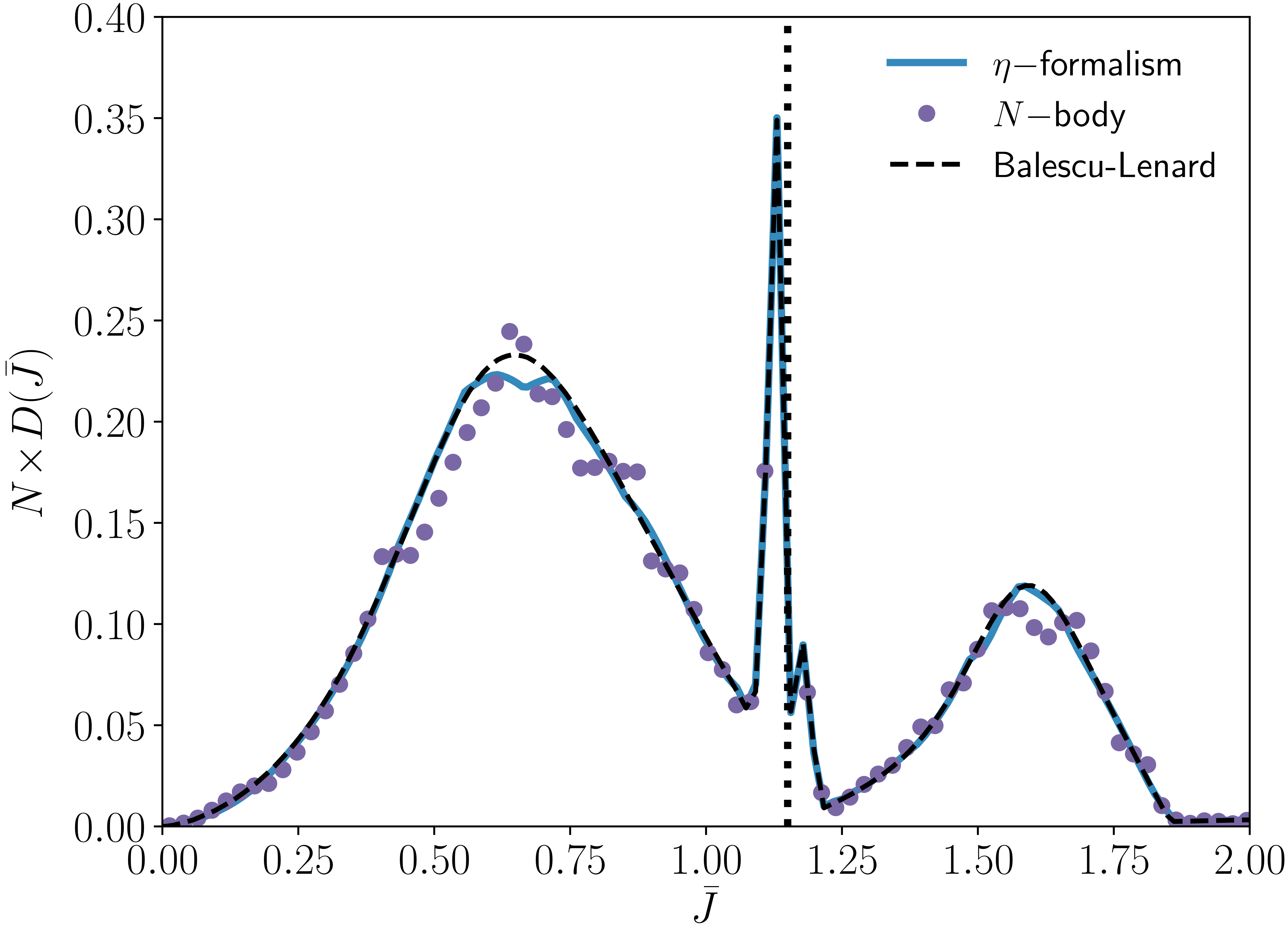
$$g(1,2,t) = \int d1' \int d2' \;\; P_2(1,2|1',2',t)\, S(1',2',0)$$

Structure of BBGKY2 $\Rightarrow$ P_2 factorizes into 2 Vlasov propagators



$\eta-$formalism
$N-$body
Balescu-Lenard
$N \times D(\bar{J})$
$\bar{J}$
0.00
0.05
0.10
0.15
0.20
0.25
0.30
0.35
0.40
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00