ICMP lecture on Heterotic/F-theory duality

Ron Donagi [1]

Department of Mathematics, University of Pennsylvania, Philadelphia, PA 19104-6395

ABSTRACT

The heterotic string compactified on an $n-1$-dimensional elliptically fibered Calabi-Yau $\pi_H : Z \to B$ is conjectured to be dual to F-theory compactified on an n-dimensional Calabi-Yau $\pi_F : X \to B$ fibered over the same base with elliptic K3 fibers. In particular, the moduli of the two theories should be isomorphic. The cases most relevant to the physics are $n = 2, 3, 4$, i.e. the compactification is to dimensions $d = 8, 6$ or 4 respectively. Mathematically, the richest picture seems to emerge for $n = 3$, where the moduli space involves an analytically integrable system whose fibers admit rather different descriptions in the two theories. The purpose of this talk is to review some of what is known and what is not yet known about this conjectural isomorphism.

[1] email: donagi@math.upenn.edu

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1 Introduction

The heterotic string compactified on an $n-1$-dimensional elliptically fibered Calabi-Yau $\pi_H : Z \to B$ is conjectured to be dual to F-theory compactified on an n-dimensional Calabi-Yau $\pi_F : X \to B$ fibered over the same base with elliptic K3 fibers. In particular, the moduli of the two theories should be isomorphic. The cases most relevant to the physics are $n = 2, 3, 4$, i.e. the compactification is to dimensions $d = 8, 6$ or 4 respectively. Mathematically, the richest picture seems to emerge for $n = 3$, where the moduli space involves an analytically integrable system whose fibers admit rather different descriptions in the two theories. The purpose of this talk is to review some of what is known and what is not yet known about this conjectural isomorphism.

Various aspects of heterotic compactifications on elliptic Calabi-Yaus were explored in [1], [2], [3]. We emphasize the role of cameral (and spectral) covers, which were introduced in [4], and applied to integrable systems in [5] and to the heterotic moduli in [2]. F-theory was created in [6]; some of the ideas we need here were developed in [7]. Results about the comparison of the "base" moduli in the two theories appear in [1], [3] as well as the earlier [8], [9], and elsewhere. The comparison of the "fiber" moduli appears in [10].

2 Eight-dimensional compactifications

In a compactification to any dimenesion, the heterotic moduli include the $(n-1)$-dimensional elliptically fibered Calabi-Yau $Z \to B$ with a section $\sigma : B \to Z$, together with a principal G-bundle on Z, where $G = E8 \times E8$ or $Spin(32)/\mathbf{Z}_2$ (or one of their subgroups), and additional fields depending on the dimension. The F-theory moduli include the double fibration $X \to B_F \to B$, where $B_F \to B$ is a $\mathbf{P}^1$-bundle (we will refer to B, B_F respectively as the heterotic and F-theoretic bases) and $X \to B_F$ is an elliptic fibration with section $\sigma_F : B_F \to X$, and additional fields depending on the dimension.

In an eight-dimensional compactification, the base B is a point, so the heterotic CY is an elliptic curve $Z = E$ equipped with a pair of E_8 bundles, while the F-theoretic CY is an elliptic K3 surface X. It is well-known that the moduli spaces $\mathcal{M}_H, \mathcal{M}_F$ for these two types of data are both given by the Narain space [11]

$$O(2,18;\mathbf{Z})\backslash O(2,18;\mathbf{R})/(O(2;\mathbf{R}) \times O(18;\mathbf{R}))$$

In a sense, this is the basic form of the duality. When compactified further to six or four dimensions, the base of the integrable system for each theory will parametrize families (one or two complex-dimensional) of the eight dimensional data. (The fiber data is new to these further compactificatiuons, and will be handled differently.)

Because of the T-dualities, this correspondence is not (algebro-) geometric; it involves the period map, which is only analytic. A simple geometric recipe for recovering Z from X for $G = E_8 \times E_8$ was nevertheless given by Morrison and Vafa [7]: write X in Weierstrass form

$$X : \quad y^2 = x^3 + xf + g$$

where f, g are sections of $\mathcal{O}(8), \mathcal{O}(12)$ on the base $\mathbf{P}^1$. In terms of a coordinate t on $\mathbf{P}^1$, write

$$f=\sum_{i=1}^{8} f_i t^i, \quad g=\sum_{j=0}^{12} f_j t^j.$$

Then the equation of the elliptic curve is given by the middle terms

$$Z: \quad y^2=x^3+xf_4+g_6.$$

Further, it is pointed out in [1] that the data of an E_8 bundle on Z is equivalent to embedding Z in a rational elliptic surface dP_9 with a section. We will recall this equivalence below. Such a surface can be obtained by blowing up 9 points in $\mathbf{P}^2$. (Blowing down the section yields a del Pezzo surface dP_8, the blow up of $\mathbf{P}^2$ at 8 points.) It can also be given in Weierstrass form $X': \quad y^2=x^3+xf'+g'$, where f', g' are sections of $\mathcal{O}(4), \mathcal{O}(6)$ on the base $\mathbf{P}^1$. The recipe of [7] thus extends naturally: given X we recover Z as above, while the two E_8 bundles are those which correspond to the two dP_9's:

$$X': \quad y^2=x^3+xf'+g', \qquad X'': \quad y^2=x^3+xf''+g'',$$

where

$$f'=\sum_{i=0}^{4} f_i t^i, \; g'=\sum_{j=0}^{6} f_j t^j,$$

and

$$f''=\sum_{i=4}^{8} f_i t^i, \quad g''=\sum_{j=6}^{12} f_j t^j,$$

which has the same general form, as seen by exchanging t and t^{-1}.

Appealing as it is, this description is valid only in a certain limit. The problem is with the coordinate t on $\mathbf{P}^1$: it is defined only up to $\mathbf{P}GL(2,\mathbf{C})$. Actually, slightly less is required: the three parameters in t can be identified as the 2 points $t=0, t=\infty$, plus a scale. Only the two points are required in order to specify the upper and lower parts X', X"", while the rescaling acts on each by isomorphisms. So let $\widetilde{\mathcal{M}}_F$ denote the $\mathbf{P}^1$ bundle over $\mathcal{M}_F$ whose fiber over $X \in \mathcal{M}_F$ is the $\mathbf{P}^1$ which is the base of the elliptic fibration on X. Then $\widetilde{\mathcal{M}}_F \times_{\mathcal{M}_F} \widetilde{\mathcal{M}}_F$ is a $\mathbf{P}^1 \times \mathbf{P}^1$ bundle over $\mathcal{M}_F$, and the space we need is the open subset $\widetilde{\widetilde{\mathcal{M}}}_F \subset \widetilde{\mathcal{M}}_F \times_{\mathcal{M}_F} \widetilde{\mathcal{M}}_F$ obtained by removing the diagonal in $\mathbf{P}^1 \times \mathbf{P}^1$. The above Morrison-Vafa-Friedman-Morgan-Witten recipe gives a global map $\widetilde{\widetilde{\mathcal{M}}}_F \to \mathcal{M}_H$. In order to map $\mathcal{M}_F$ to $\mathcal{M}_H$ we need a section $\mathcal{M}_F \to \widetilde{\widetilde{\mathcal{M}}}_F$. This does *not* exist globally, but it can be given on a codimension-1 boundary stratum. In $\mathcal{M}_F$ this stratum corresponds to degenerations of X to the union $X' \cup X''$ of two dP_9's intersecting along an elliptic fiber. the picture is that the base $\mathbf{P}^1$ degenerates into the union of two $\mathbf{P}^1$'s intersecting in a point, the bundles $\mathcal{O}(8), \mathcal{O}(12)$ go into $(\mathcal{O}(4), \mathcal{O}(4))$ and $(\mathcal{O}(6), \mathcal{O}(6))$, and X goes to an elliptic fibration over this singular base. In this limit both the points $t=0,\infty$ go towards the singular point. The corresponding locus in $\mathcal{M}_H$ is where the size of the elliptic curve Z goes to ∞. This gets rid of the only non-algebraic parameter, thus allowing for an algebraic interpretation of the duality. Explicitly, consider a two-parameter family of $K3$ surfaces $X_{a,b}$ where f_i is multiplied by a^{4-i} for $i \leq 4$ and by b^{i-4} for $i \geq 4$, and similarly for the g_j. Then, because of the above-mentioned scale parameter, $X_{a,b}$ depends up to isomorphism only on the product ab. The degeneration is obtained by letting $a \to 0$ with $b=1$, or equivalently by letting $b \to 0$ with $a=1$. From the point of view of either of the two limiting $\mathbf{P}^1$'s, "half the $K3$" (e.g. 12 of the 24 singular fibers) has moved to ∞.

3 Heterotic moduli in six dimensional compactifications

In six dimensional compactifications, we see integrable systems come up naturally on both sides. We will explore these integrable systems in the next two sections.

On the heterotic side, the moduli space is fibered over the moduli of K3 surfaces Z. The fiber over a given surface Z is the moduli space $\mathcal{M}_Z^G$ of semistable G-bundles on Z. Now it was shown by Mukai [12] that $\mathcal{M}_Z^G$ is a holomorphic symplectic manifold, and in fact it is an *algebraically integrable system.* This means that it admits a natural fibration $p : \mathcal{M}_Z^G \to S$ whose fibers are Lagrangian (=maximal isotropic) subvarieties $P_s := p^{-1}(s)$, and for generic $s \in S$, each connected component of the fiber P_s is an abelian variety. We will describe this structure in a way which extends to other dimensions. The heterotic fibration p sends a G-bundle to *the family of its restrictions to elliptic fibers of* π. To make sense of this, consider for each elliptic curve E the moduli space $\mathcal{M}_E^G$ of semistable G-bundles on E. Given the family of elliptic curves $\pi : Z \to B$, we can put the family of moduli spaces together to obtain a fibration $\mathcal{M}_{Z/B}^G \to B$ whose fiber over $b \in B$ is $\mathcal{M}_{E_b}^G$. Now the *base* of the heterotic fibration p is

$$S := \{\text{sections} : B \to \mathcal{M}_{Z/B}^G\}.$$

In [1] it is observed that this base S is a weighted projective space. This follows immediately from the fact that, by a theorem of Looijenga [13], $\mathcal{M}_E^G$ is itself a weighted projective space. There are several geometric interpretations for a point $s \in S$; depending on your taste, you can think of it as parametrizing a cameral cover $\widetilde{\pi} : \widetilde{B} \to B$, a spectral cover $\overline{\pi} : \overline{B} \to B$, or a del Pezzo fibration $\pi : U \to B$.

Cameral covers were introduced in [4], and were related to Higgs-type integrable systems in [5] and to principal bundles on elliptic fibrations in [2]. A cameral cover is a Galois cover $\widetilde{\pi} : \widetilde{B} \to B$ with group W, the Weyl group of G. There are some restrictions on the local structure: it is locally pulled back from a certain standard W-cover. One way to describe this standard cover is as the map $\mathcal{M}_E^T \to \mathcal{M}_E^G$, where $\mathcal{M}_E^T$ is the moduli space of T-bundles on E, with T the maximal torus of G. Specifying a T-bundle, on any variety Y, is the same as giving a homomorphism $\Lambda \to Pic(Y)$, where $\Lambda \approx Hom(T, \mathbf{C}^*)$ is the lattice of *characters* of G. So $\mathcal{M}_E^T = Hom(\Lambda, E)$ and two points here map to the same point in $\mathcal{M}_E^G$ if and only if they differ by the action of W on Λ. A point $s \in S$ determines both a cameral cover $\widetilde{\pi} : \widetilde{B} \to B$ and a W-equivariant family of maps $v_\lambda : \widetilde{B} \to Z$, depending linearly on $\lambda \in \Lambda$ and commuting with the projection to B. In fact, this data suffices to determine $s \in S$ uniquely. (Actually, some care is needed in order to handle the possibility of a semistable bundle on Z whose restrictions to some, or even all, the elliptic fibers may be unstable.)

Spectral covers are quotients of the cameral cover by the action of a Weyl subgroup $W_0 \subset W$. They can be identified with the image of v_λ for any λ which is fixed precisely by W_0. It is often easier to write down explicit families of spectral covers, for example as linear systems of divisors in Z. On the other hand, the cameral covers are somewhat more natural, and consequently are easier to use to obtain a precise desccription of the fibers. We will discuss del Pezzo fibrations and see some concrete examples of spectral covers below.

To complete our description of the heterotic integrable system, we need to identify the fibers of the integrable system $p : \mathcal{M}_Z^G \to S$. This was done in [2], based on [5]: the fiber $p^{-1}(s)$ corresponding to a cameral cover $\tilde{\pi} : \tilde{B} \to B$ is given by the *distinguished Prym*:

$$Prym_\Lambda(\tilde{B}') := Hom_W(\Lambda, Pic\tilde{B}).$$

When $\tilde{B}$ is a non-singular projective variety (a curve, in our current dimension), $Prym_\Lambda(\tilde{B}')$ will be the product of its connected component $Prym^0_\Lambda(\tilde{B}')$, an abelian variety which up to a finite cover is an abelian subvariety of $Pic^0(\tilde{B})$, with a discrete abelian group of connected components. (There is a fine point here, which is that $p^{-1}(s)$ is naturally identified with a *torser* over $Prym_\Lambda(\tilde{B}')$; this means that (if nonempty) they are isomorphic, but not naturally; so the family of all $Prym_\Lambda(\tilde{B}')$ will be different than the total space $\mathcal{M}_Z^G$.)

4 F-theoretic moduli in six-dimensional compactifications

The continuous moduli which occur in six-dimensional compactifications of F-theory include the complex structure of the Calabi-Yau threefold X, the non-zero holomorphic volume form $\omega_X \in H^{3,0} = H^0(K_X)$, and the Ramond-Ramond fields which specify a point of $H^3(X, \mathbf{R})$ modulo $H^3(X, \mathbf{Z})$. This torus, called the intermediate Jacobian of X, has a natural complex structure,

$$J^3(X) := H^3(X, \mathbf{Z})\backslash H^3(X, \mathbf{C})/(H^{30} \oplus H^{21}).$$

According to [14],[15], the full moduli space for F-theory compactifications to 4 dimensions includes some discrete parameters as well; the intermediate Jacobian should be replaced by the *Deligne cohomology group* :

$$0 \to J^3(X) \to \mathcal{D} \to H^{2,2}(X, \mathbf{Z}) \to 0,$$

or rather by the part of the Deligne cohomology mapping to the primitive cohomology $H_0^{2,2}(X, \mathbf{Z})$. In six dimensional compactifications we may expect this primitive group, in generic situations, to be finite; nevertheless, it is natural to include it in our moduli.

It was proved in [16] that this full moduli space - including the complex structure parameters, the holomorphic volume form, and the Deligne cohomology - is a holomorphic integrable system. The origin and nature of this integrable system seem very different than those of the heterotic system. One obvious distinction is that the heterotic fibers are abelian varieties, that is they come with definite polarizations; but the intermediate Jacobians obtained on the F-theory side are non algebraic varieties: their polarizations are indefinite, of signature $(1, h^{21})$.

We sketch one way to see that this is indeed an integrable system. Let us start with a very general question: given a family $\mathcal{X} \to S$ of g-dimensional, polarized complex tori over a g-complex-dimensional base S, is there a holomorphic symplectic form Σ on $\mathcal{X}$ such that the fibers are Lagrangian?

The answer is that such a Σ corresponds to a field of symmetric cubic tensors on the base S satisfying an integrability condition which ensures that it can be given locally as third partials of a single function, the prepotential.

Indeed, given Σ we get an isomorphism of the holomorphic tangent bundle $T(S)$ of the base with the relative cotangent bundle $T^*(\mathcal{X}/S)$ along the fibers $X_s, s \in S$. By partitioning a basis of $H^1(X_s, \mathbf{Z})$ into "α" and "β" isotropic subspaces (for some $s_0 \in S$ and hence via the Gauss-Manin connection for all nearby $s \in S$) we get a local holomorphic trivialization of $T^*(\mathcal{X}/S)$, and hence of $T(S)$. this allows us to identify S locally as an open subset of a vector space V. The period matrix of the fiber X_s then lives naturally in Sym^2V^*. The holomorphic structure of the fibration $\mathcal{X} \to S$ together with such a trivialization is given by its period map

$$p : S \to Sym^2V^*.$$

At each point of S, the differential of the period map, $dp : V \to Sym^2V^*$, can be identified as an element of $V^* \otimes Sym^2V^*$. The result proved in [16] is that this $\mathcal{X}$ admits a symplectic form Σ inducing the given trivialization and for which the fibration is Lagrangian if and only if dp is in the subspace Sym^3V^* of $V^* \otimes Sym^2V^*$. In this case there exists locally a prepotential function whose second partials give the period matrix p, so the third partials give dp, the cubic.

In our case, let $\mathcal{M}$ be the moduli space of complex structures on X, and let $S := \widetilde{\mathcal{M}}$ be the moduli space of complex structures on X together with a nonzero holomorphic volume form ω, so S is a C^*-bundle over $\mathcal{M}$. The cubic is, of course, Yukawa's:

$$Sym^3T_s\widetilde{\mathcal{M}} \to Sym^3T_s\mathcal{M} = Sym^3H^1(X_s, TX_s) \to H^3(X_s, \wedge^3TX_s) \to H^3(X_s, \mathcal{O}_{X_s}) \to \mathbf{C}$$

where each of the last two maps is given by contraction with the volume form ω. This then gives rise to the holomorphically integrable system of intermediate Jacobians. For extension to the family of Deligne cohomology groups, an additional fact is needed, namely that the Abel-Jacobi image of any family of null-homologous algebraic cycles is an isotropic subspace with respect to the symplectic structure just constructed on the family of intermediate Jacobians.

5 Duality correspondence along the base

As explained earlier, a point in the base of either integrable system corresponds to a family, parametrized by the heterotic base B, of the eight-dimensional data. The correspondence along the base is thus obtained by putting the 8-dimensional correspondence into families. On the F side, the base is just the moduli space of Calabi-Yaus (with a holomorphic volume form). On the heterotic side there are three useful descriptions for a point of the base: it represents either a cameral cover (with the additional data of the evaluation map $v : \Lambda \times \widetilde{B} \to Z$), or a spectral cover, or a del Pezzo fibration.

We have already discussed cameral and spectral covers. When the structure group is of type A_{n-1}, the cameral cover is an $n!$-sheeted cover. The basic spectral cover $\overline{B}$, corresponding to the first fundamental weight, is n-sheeted, and the one corresponding to the k-th weight is $\binom{n}{k}$-sheeted (one of its points is given by an unordered k-tuple of points in a fiber of $\overline{B}$). For E_8, the cameral cover has degree $\#W = 696,729,600 = 2^{14}3^5 5^2 7$, while the smallest spectral cover has degree 240.

Let D be a del Pezzo surface dP_8. The blowup picture identifies $H^2(D, \mathbf{Z})$ with the indefinite lattice $\mathbf{Z}^{(8,1)}$, so the primitive cohomology $H_0^2(D, \mathbf{Z})$ is identified with the E_8 lattice $\Lambda \subset \mathbf{Z}^{(8,1)}$; this identification is determined only up to the action of W. An embedding of an elliptic curve E as an anticanonical divisor in D then induces a homomorphism $\Lambda \to Pic^0(E) = E$, up to the W action on Λ. But as we saw, a homomorphism gives a point of $Hom(\Lambda, E) = \mathcal{M}_E^T$, and two points here map to the same point in $\mathcal{M}_E^G$ if and only if they differ by the action of W on Λ. So the embedding of E in D is exactly equivalent to specifying a point of $\mathcal{M}_E^G$. Explicitly, an E_8 bundle on E is given by 8 line bundles of degree 0, hence 8 points of E; the embedding of E into D is recovered by first embedding E into $\mathbf{P}^2$ by the linear system $3o$, where $o \in E$ is the origin, and then blowing up $\mathbf{P}^2$ at the images of the 8 points of E. This construction works well in families [10], giving the interpretation of a point of the base S as a fibration $U \to B$ whose fibers are dP_8 surfaces together with a (relatively anticanonical) embedding of the heterotic Calabi-Yau Z into U. Given this description, we recover the spectral cover as parametrizing the 240 lines in each del Pezzo fiber, while the cameral cover parametrizes 8-tuples of disjoint lines.

In case the structure group of the bundle is the full E_8, the generic del Pezzo fiber will be non-singular. A singularity of type ADE on the generic del Pezzo implies that the structure group of the equivalent cameral cover is reduced to the Weyl group of the complementary subgroup in $E_8 \times E_8$. (This can also be expressed in terms of the spectral cover, which will then have the zero section as a component with appropriate multiplicity.) This results in a reduction of the structure group of the heterotic bundle to the complementary subgroup in $E_8 \times E_8$. This reduction shows up as enhanced symmetry of the compactified theory, so it must be visible also in a family of ADE singularities of the corresponding type in the F-theoretic Calabi-Yau. We conclude that the two n-dimensional objects: the F-theoretic Calabi-Yau X and the heterotic pair-of-delPezzo-fibrations $U' \cup U''$ must have the *same* types of singularities. (Except that the heterotic object is also singular along the divisor Z.) In the geometric limit, this matching is evident, as the two varieties are actually isomorphic here. As we move into the interior of moduli space by returning to finite volume, the heterotic object does not change. The singularity of the F-theoretic X along Z disappears, but the other singularities are supposed to remain. Much evidence for this is presented in [8], [9], and [1].

6 Duality correspondence along the fibers

Does duality preserve the integrable system structures? Not quite. One obvious difference is in the dimensions: the F-theoretic integrable system involves all the moduli, while on the heterotic side the system itself is the fiber of a map of the heterotic moduli to the moduli of $K3$. A second important difference was already noted above: the heterotic fibers are algebraic (abelian) varieties, but the intermediate Jacobians on the F-theory side are non algebraic complex tori, with "Lorentzian" polarizations of signature $(1, h^{21})$.

Going to the "geometric" limit which was described in the eight-dimensional context resolves both of these difficulties. On the heterotic side, we take the Kahler metric on

B as well as the size of the elliptic fibers to be large. On the F side The Calabi-Yau X splits into $X' \cup X''$ which is an elliptic fibration over $B'_F \cup B''_F$, while the latter becomes a $\mathbf{P}^1 \cup \mathbf{P}^1$ bundle over B. The intersection $X' \cap X''$ is again the heterotic Z. This degeneration kills the H^{30} term, so the surviving part of the intermediate Jacobian,

$$J^3(X) \to J^3(X') \times J^3(X'')$$

is now indeed an abelian variety. The result, proved in [10], is that the continuous part $J^3(X') \times J^3(X'')$ of the F-theory moduli is identified with the continuous part $Prym^0_\Lambda(\widetilde{B}') \times Prym^0_\Lambda(\widetilde{B}'')$ of the heterotic moduli, while the F-theory discrete part embeds as a subgroup of finite index in the corresponding heterotic discrete group. This possible finite index may well vanish in general, yielding an actual isomorphism; in our approach this depends on the vanishing of certain (finite) group cohomologies which have not yet been computed. In the physically important case that the base B is $\mathbf{P}^1$, precisely such an isomorphism is announced in [17].

The result proved in [10] can actually be stated *without* going to the limit: it identifies the Deligne group $\mathcal{D}(U')$ of the heterotic del Pezzo fibration as a subgroup of finite index in the heterotic fiber moduli, which are given by $Prym^0_\Lambda(\widetilde{B}')$. The additional feature present in the limit is that the heterotic and F-theoretic del Pezzo fibrations can be identified, so the general algebro-geometric result becomes relevant to the matching of the fibers between the two theories.

7 Four dimensional compactifications

The analysis of cameral and spectral covers and del Pezzo fibrations works in any dimension. It allows parametrization of bundles on Z whose restriction to each elliptic fiber is semistable, together with a bit of extra data, a regularization [2]. In particular, the moduli space of these bundles is still fibered over the space of covers, and the fiber is still given by the distinguished Prym, $Prym_\Lambda(\widetilde{B})$. The result of [10] is also still valid: the Deligne group of the del pezzo fibration a subgroup of finite index in $Prym_\Lambda(\widetilde{B})$. For a bundle whose restriction to a particular elliptic fiber E is not semistable, the corresponding spectral or cameral "cover" will contain the entire fiber E, so it is no longer an (everywhere finite) cover. It seems that the spectral description of the heterotic moduli can be extended, with modifications, to include this case, but further work needs to be done to substantiate this. Various results on four dimensional compactifications can be found in [1],[3] and elsewhere.

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