Taniguchi Lectures on Principal Bundles on Elliptic Fibrations

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Abstract

In this talk we discuss the description of the moduli space of principal G-bundles on an elliptic fibration $X \to S$ in terms of cameral covers and their distinguished Prym varieties. We emphasize the close relationship between this problem and the integrability of Hitchin's system and its generalizations. The discussion roughly parallels that of [D2], but several examples are included and some important steps of the argument are worked out.

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1 Introduction

1.1 The Question

Consider an elliptic fibration $\pi : X \to S$ with section $\sigma : S \to X$. We are interested in describing $\mathcal{M}_X^G$, the moduli space of principal G-bundles on X.

Here G could be any complex reductive group. The case relevant to string theory is when X is a Calabi-Yau manifold of dimension $= 2, 3, 4$, and $G = E8 \times E8$, or $\mathrm{Spin}(32)/Z_2$, or a subgroup of either. We will divide the answer into three steps.

1.2 Steps:

- Describe $\mathcal{M}_E^G$, E = elliptic curve. This is the special case when the base S is a point. We will discuss this case in section 2.

- Put the individual moduli spaces into a family $\mathcal{M}_{X/S}^G \to S$ with fiber $\mathcal{M}_{E_s}^G$ over the elliptic curve E_S, $s \in S$. Describe the space of sections $\Gamma(S, \mathcal{M}_{X/S}^G)$.

- By sending a bundle to the family of its restrictions to fibers, we get a fibration:

  $$\mathcal{M}_X^G \to \Gamma\left(S, \mathcal{M}_{X/S}^G\right)$$

  sending

  $$P \mapsto (s \mapsto P_s := P|_{E_s}).$$

  The remaining issue then is to describe the fibers. This will be done in section 3.

1.3 The Answer

- $\mathcal{M}_E^G$ can be identified with the quotient $\mathcal{M}_E^T/W$, where T is the maximal torus in G and W is its Weyl group. $\mathcal{M}_E^T$, the moduli space of T-bundles of degree 0 on E, is an abelian variety, isomorphic to E^r (r is the rank of G), or more canonically to $Hom(\Lambda, E)$, where $\Lambda := Hom(T, \mathbf{C}^*)$ is the character lattice of G. $\mathcal{M}_E^G$ itself is a weighted projective space. This was proved originally by

Looijenga [L1,L2] and also by Bernstein and Shvartsman [BS]. It was reproved and interpreted recently by Friedman, Morgan and Witten [FMW].

- $\Gamma\left(S, \mathcal{M}^G_{X/S}\right)$ is another weighted projective space, since $\mathcal{M}^G_{X/S} \to S$ is a weighted projectivization of a vector bundle. For groups other than $E8$, this was proved by Wirthmuller [W], and closely related results were obtained by K. Saito [S]. The situation for $E8$ is not known. The crucial observation for us is that, for any G, the base $\Gamma\left(S, \mathcal{M}^G_{X/S}\right)$ parametrizes a family of W-Galois covers $\widetilde{S} \to S$ which we call *cameral covers*: The cover corresponding to a section $S \to \mathcal{M}^G_{X/S}$ is the pullback of the cover $\mathcal{M}^T_{X/S} \to \mathcal{M}^G_{X/S}$.

- The main result of [D3] is that the fiber of $\mathcal{M}^G_X$ over the point of $\Gamma\left(S, \mathcal{M}^G_{X/S}\right)$ corresponding to a cover $\widetilde{S} \to S$ can be identified with the *distinguished Prym* variety $Prym_\Lambda(\widetilde{S})$ introduced in [D1] . When $\widetilde{S}$ is non singular, this is a product of an abelian variety and a finite group. It can be described as the kernel of a homomorphism from $Hom_W(\Lambda, Pic(\widetilde{S}))$ to the finite group $H^2(W, \Lambda)$. The identification of the fiber with this group is non-canonical; the fiber is really a non-trivial torser over it.

  A different description of the fiber is available in case G is of type E_n. In this case, as proposed in [K] and in [FMW], the cameral cover can be replaced by a fibration $U \to S$ whose fibers are del Pezzo surfaces. (For more information on del Pezzo surfaces, compare [De].) In [CD] it is shown that the fiber can then be reinterpreted as the relative Deligne cohomology group $\mathcal{D}(U/S)$, whose connected component is the relative intermediate Jacobian $J_3(U/S)$. (A similar result had been proved earlier by Kanev [K] for the case that the base is $\mathbf{P}^1$ and the structure group is $E_n, \quad n \leq 7$. A similar result over $\mathbf{P}^1$ but allowing $G = E_8$ is announced in [FMW2].)

1.4 An Analogy

Before proceeding, we sketch an analogous problem whose solution [D2] provides the motivation for our approach to the present question, as well as being one of the key ingredients in its solution. In brief, we want to think of a principal G-bundle on the elliptic fibration $\pi : X \to S$ as a kind of G-Higgs bundle on S "taking its values in the fibers".

The problem is to describe the moduli space of "K-valued principal G-Higgs bundles" on our base S:

$$\mathrm{Higgs}^G_{S,K} := \{(P,\phi) \mid \begin{array}{l} P : \text{a principal G} - \text{bundle on } S \\ \phi \in \Gamma(S, adP \otimes K) \end{array} \}$$

Here K is any line bundle on S. In case S is a curve and K is the canonical bundle, this is the total space of Hitchin's integrable system. In general there is a natural map which sends a Higgs bundle to the collection of values of the basic G-invariant polynomials $f_i,\ i = 1,\ \ldots,\ r := \mathrm{rank}(g)$:

$$\begin{array}{rl} \mathrm{Higgs}^G_{S,K} \to \Gamma(\oplus K^{d_i}) & \\ (P,\phi) \mapsto \ (f_i(\phi))_{i=1}^r & \in \Gamma\left(S, \oplus_{i=1}^r K^{\otimes d_i}\right) \end{array}$$

Here d_i is the degree of f_i. Let $\mathbf{t}$ be the Lie algebra of T. The base $\Gamma\left(\oplus K^{d_i}\right)$ parametrizes sections of the bundle $\mathbf{t} \otimes K \to s$. Now just as in the elliptic case, pulling back via a section induces a W-Galois cameral cover $\widetilde{S} \to S$. So the base $\Gamma\left(\oplus K^{d_i}\right)$ parametrizes cameral covers $\widetilde{S} \to S$. The main result of [D2] is that the fiber over a general $[\widetilde{S}]$ is the distinguished ed Prym, i.e. the same subgroup $Prym_\Lambda(\widetilde{S})$ of $Hom_W(\Lambda, Pic\widetilde{S})$ encountered in the elliptic story.

Here is a heuristic explanation of this result. Giving a (diagonalizable) endomorphism ϕ of a vector space V is equivalent to giving a decomposition of V into eigenspaces, plus an assignment of an eigenvalue to each. Next, do this in families, i.e. start with a K-valued $GL(n)$-Higgs bundle (P,ϕ) over S. If we make the unrealistic assumption that ϕ is diagonalizable with distinct eigenvalues (=regular semisimple) above each point $s \in S$, then these eigenvalues fit together to form an n-sheeted *spectral cover* $\overline{S} \to S$, while the (one-dimensional) eigenspaces form a line bundle over $\overline{S}$. A more realistic assumption might be that ϕ may have repeated eigenvalues, but has a unique Jordan block per eigenvalue. (Such ϕ are called *regular*.) In this case, the cover $\overline{S} \to S$ swept out by the eigenvalues is ramified, but it still carries a bundle of eigenlines. Note that $\overline{S}$ comes with an embedding in the total space of K. The cameral cover $\widetilde{S}$ is the $n!$-sheeted cover which is the Galois closure of $\overline{S}$, i.e. a point of $\widetilde{S}$ is an ordered n-tuple of points in a fiber of $\overline{S} \to S$. This looks "too big"; but it turns out to provide a natural and uniform way to extend the vector bundle description to other structure groups, without having to fix a representation. If we do pick an irreducible representation ρ_λ of G, with highest weight λ, we recover the spectral cover $\overline{S}_\lambda$ of the associated $GL(n)$-Higgs bundle as the quotient of $\widetilde{S}$ by the Weyl subgroup fixing λ. (More precisely, $\overline{S}_\lambda$ has various components corresponding to the Weyl orbits of weights of ρ_λ, and each of these is a quotient of $\widetilde{S}$ by a Weyl subgroup.) The line bundles on $\overline{S}$ are replaced by their pullbacks to $\widetilde{S}$, where they can be characterized in terms of their behavior under the action of W.

The roles of the various ingredients become clearer when we introduce the notion of an abstract *principal G-Higgs bundle* on S. This is a pair $(P_S, \mathcal{C})$ where P_S is a principal G-bundle on S, and $\mathcal{C} \subset ad(P_S)$ is a vector subbundle whose fibers are centralizers of regular elements. We then think of a K-valued Higgs bundle as an abstract Higgs bundle $(P_S, \mathcal{C})$ plus a section $\phi \in \Gamma(\mathcal{C} \otimes K)$. An abstract Higgs bundle corresponds to an (abstract) cameral cover $\widetilde{S} \to S$ together with a point in its distinguished Prym. The additional data needed for a K-valued Higgs bundle then amounts to a collection of "value maps" $v(\lambda) : \widetilde{S} \to K$, one for each character λ of our group G; the image of $v(\lambda)$ is the λ-spectral cover. These value maps are equivariant under the W action on the λ and on $\widetilde{S}$.

The point of our analogy is the following. According to Atiyah [A], a semistable rank n vector bundle V_E on an elliptic curve E decomposes as a sum of simple pieces which generically (in the "regular semisimple case") are distinct line bundles. We may therefore think of the vector bundle as consisting of a decomposition (into line subbundles) plus an assignment of a "value" to each; but this value now lives in $Pic^0(E)$, which we canonically identify with E itself. (The corresponding statement for a generic *principal* bundle P_E is simply that its structure group can be reduced to T.) A bundle P_X on the elliptic fibration $\pi : X \to S$ can therefore be interpreted as a cameral cover $\widetilde{S}$, plus a point in its distinguished Prym, plus a value map $v : \widetilde{S} \times \Lambda \to X$. The first two ingredients specify an abstract Higgs bundle $(P_S, \mathcal{C})$ on S, the one obtained by restricting our principal bundle P_X as well as its Atiyah decomposition to S (which is identified, via σ, with the 0-section of X); the value map tells us how to lift $(P_S, \mathcal{C})$ to a bundle on all of X, just as the value map $v : \widetilde{S} \times \Lambda \to K$ tells us how to lift $(P_S, \mathcal{C})$ to a K-valued Higgs bundle.

2 Bundles on an elliptic curve

Our setup is as follows. $(E, 0)$ is an elliptic curve, which we identify with its dual $Pic^0\ (E)$. G is a simply connected, complex semisimple group with Lie algebra $\mathbf{g}$. (Most of what we say extends to reductive G.) B is a Borel subgroup, T a maximal torus, $N := N_G(T)$ the normalizer of T in G, $W := N/T$ the Weyl group, $\Lambda := Hom(T, C^*)$ the lattice of characters of G (i.e. of T), and $r = \text{rank } G = dim\ T$.

We will work with $\mathcal{M}_E^G$, the moduli of semistable G-bundles on E, and its abelian analogue $\mathcal{M}_E^T = Hom(\Lambda, E) \approx E^r$. Their relationship is:

$$\mathcal{M}_E^G = \mathcal{M}_E^T / W.$$

Looijenga [L1,L2] showed that this quotient is in fact a weighted projective space. This result has recently been reproved and interpreted in [FMW]. Wirthmuller [W] showed that the identification is canonical in families, so we get a weighted projective bundle over the parameter space $\Gamma(S, \oplus K^{d_i})$ of cameral covers. Wirthmuller's result holds for all simple groups except, unfortunately, $E8$. In case $G = E8$ it is not known whether this holds. Friedman, Morgan, and Witten get around this difficulty by restricting attention to fibrations $\pi : X \to S$ whose singular fibers have nodes (but no cusps). Finer information about the naturality of the trivialization and its modular properties has been obtained by K. Saito [S].

2.1 Regular vs. Semisimple

A G-Bundle $P \to E$ is:

- semisimple if $P \approx P_T \overset{T}{\times} G$, for some T-bundle P_T.
- regular if $h^0(E, ad\ P) = r$.

Points of $\mathcal{M}_E^G$ correspond to S-equivalence classes of bundles. The generic class has a unique representative, which is both regular and semisimple. Each class has a unique semisimple representative *and* a unique regular representative.

2.2 More of the Analogy

An element $g \in G$ is:

- semisimple if it is in some conjugate of T,
- regular if dim $Z_G(g) = r$.

We illustrate these notions in case $G = SL(2)$:

- $\pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ are the elements which are semisimple but not regular.
- $\pm \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ are regular but not semisimple ($* \neq 0$).
- Regular semisimple bundles: $\mathcal{O}(p) \ \oplus \ \mathcal{O}(-p), \ p \neq -p \in E^u$.
- Semisimple irregular: $\mathcal{O}(p) \ \oplus \ \mathcal{O}(p), \quad p = -p$.
- Regular non semisimple: $\mathcal{O}(p) \ \otimes \ A, \quad p = -p$, where A is Atiyah's bundle, given as a non-trivial extension:

$$0 \ \to \mathcal{O} \ \to A \ \to \mathcal{O} \ \to 0. \tag{1}$$

2.3 All Bundles

In order to work in families, we need to understand not only the regular or the semisimple bundles. How can we describe *all* the G-bundles on E?

The semisimple bundles come from T-bundles. These are parametrized by $H^1(E,T)$, over which there is a universal T-bundle, hence a universal "Poincare" semisimple G-bundle. (The "T" in $H^1(E,T)$ is really the sheaf of holomorphic sections of T. A more accurate notation would be $T(\mathcal{O}_E)$.)

The other bundles come from C-bundles, where $C = Z_G(g)$ is a regular centralizer, i.e. the centralizer of a regular element $g \in G$. Since C is still abelian these are parametrized by $H^1(E,C)$, and again there is a universal family.

As an example, let g be one of the regular but non-semisimple elements in $G = SL(2)$. Then $C = Z_G(g)$ is isomorphic to $\mathbf{C} \times \mathbf{Z}_2$, so $H^1(E,C) \approx \mathbf{C} \times \mathbf{Z}_2 \times \mathbf{Z}_2$.

When we go to the associated G-bundles, we get only the four distinct objects of (1): the regular non-semisimple bundles $\mathcal{O}(p) \otimes A, \quad p = -p$. Each of these carries a one-dimensional family of inequivalent C-structures, parametrized by the extension class in (1).

So in order to describe all bundles uniformly, we need to fit the maximal tori and the other regular centralizers into a single family. We know that the family of maximal tori is parametrized by the quotient G/N. It has a natural W-Galois cover, G/T, parametrizing pairs consisting of a maximal torus and a Borel subgroup containing it:

$$\begin{array}{ccc} \{T \subset B\} & = & G/T \\ \downarrow & & \downarrow \\ \{T\} & = & G/N \end{array} \tag{2}$$

To obtain the parameter space of regular centralizers, we embed G/N in the Grassmannian of r-dimensional subspaces of the lie algebra $\mathbf{g}$, and take an appropriate open subset of the closure:

$$\begin{array}{ccc} \{C \subset B\} & = & \overline{G/T} \\ \downarrow & & \downarrow \\ \{C\} & = & \overline{G/N} \end{array} \tag{3}$$

The key point is that over this base there exists a universal object $\mathcal{U}_E$ parametrizing the data:

$$\{\text{bundles } P \to E \text{ trivialized at } 0 \in E + \text{reduction to a regular centralizer}\}.$$

Example: $G = SL(2)$

$$\begin{array}{ccccccc} G/T & \hookrightarrow & \overline{G/T} & & \mathbf{P}^1 \times \mathbf{P}^1 \setminus \mathit{diagonal} & \subset & \mathbf{P}^1 \times \mathbf{P}^1 \\ \downarrow & & \downarrow & = & \downarrow & & \downarrow \\ G/N & \hookrightarrow & \overline{G/N} & & \mathbf{P}^2 \setminus \mathit{conic} & \subset & \mathbf{P}^2 \end{array}$$

In this case, $\mathcal{U}_E$ is obtained from $\overline{\mathcal{U}_E} := (\mathbf{P}^1 \times \mathbf{P}^1 \times E)/\mathbf{Z}_2$. First we resolve the singularities. This yields degenerate fibers of type I_0^*. We then discard multiple components of fibers. This gives us cu_E which maps onto $\mathbf{P}^2 = \overline{G/N}$. The off-diagonal fibers are isomorphic to $E \approx Pic^0(E)$, which parametrizes the $T = \mathbf{C}^*$-bundles on E. The fibers over points of the diagonal are isomorphic to type I_0^* curves with the central, multiplicity-2 component removed; this leaves four disjoint copies of $\mathbf{P}^1 \setminus \infty \approx \mathbf{C}$, which exactly matches our previous description of $H^1(E, C)$ in this case.

To do this for an arbitrary group, we start with $\overline{\mathcal{U}}_E = (\overline{G/T} \times \mathcal{M}_E^T)/W$ and resolve its singularities to obtain $\mathcal{U}_E^+$. We let $\mathcal{U}_E \subset \mathcal{U}_E^+$ be the open subset obtained by removing the proper transform of the components with multiplicity ≥ 1, i.e. the components where some element of W stabilizes the centralizer C but does not stabilize the T-bundle L:

$$\begin{array}{ccc} \overline{\mathcal{U}}_E & = & (\overline{G/T} \times \mathcal{M}_E^T)/W \\ \cup & & \cup \\ \overline{\mathcal{U}}'_E & = & \{(C, L) \mid Stab_W^C \subset Stab_W L\} \\ \uparrow & & \\ \mathcal{U}_E & & \end{array} \tag{4}$$

$$\begin{array}{ccc} H^1(E, C) & \hookrightarrow & \mathcal{U}_E \\ \downarrow & & \downarrow \\ \{C\} & \in & \overline{G/N} \end{array} \tag{5}$$

3 Fibrations

3.1 Regularized Bundles

We now move on to the general case: $\pi : X \to S$ is an elliptic fibration with a section $\sigma : S \to X$. The restriction of our principal bundle $P \to X$ gives a principal bundle $P_S \to S$ on the base. The sheaf of automorphisms along the fibers, $Aut_S(P) := \pi_* Ad(P) \subset Ad(P_S)$, is a subsheaf of $Ad(P_S)$.

The bundle P_S determines the bundle P_S/N of maximal tori in the fibers, as well as the family $\overline{P_S/N}$ of regular centralizers in P_S, etc.

A section $c : S \longrightarrow \overline{P_{S/N}}$ determines an abelian group scheme $\mathcal{C} \to S$, which is a subgroup scheme of $Ad(P_S)$:

$$\begin{array}{ccc} \mathcal{C} & \hookrightarrow & Ad(P_S) \\ & \searrow \quad \swarrow & \\ & S & \end{array} \tag{6}$$

By a *regularized G-bundle* on X we mean a triple $(P,\ c : S \to \overline{P_S/N},\ P^{\mathcal{C}})$, where $\mathcal{C} \subset Ad\ P_S$ is the group scheme determined by c as above, $P^{\mathcal{C}}$ is a $\pi^*\mathcal{C}$-torser on X, and $P = P^{\mathcal{C}} \times_{\mathcal{C}} Ad\ P_S$. This amounts to a reduction of the structure group of P to a group scheme $\mathcal{C}$ of regular centralizers. We note that if P is everywhere regular then it has a unique regularization.

3.2 Cameral Covers and Spectral Data

A Cameral Cover is a W-Galois cover $\widetilde{S} \to S$ modelled on $\overline{G/T} \to \overline{G/N}$. (This means that the restriction $\widetilde{S}_0 \to S_0$ of $\widetilde{S} \to S$ to a sufficiently small open neighborhood S_0 of each point $s \in S$ is the pullback of $\overline{G/T} \to \overline{G/N}$ via a map of S_0 to $\overline{G/N}$.) An equivalent notion is obtained if we model our covers instead on $\mathbf{t} \to \mathbf{t}/W$.

Our **Spectral Data** consists of:

1. a cameral cover $\widetilde{S} \to S$
2. a W-equivariant morphism $v : \widetilde{S} \to \mathcal{M}^T_{X/S}$ (or equivalently: $v' : \widetilde{S} \times \Lambda \to X$).
3. a point of the distinguished Prym variety $Prym_\Lambda(\widetilde{S})$, i.e. a homomorphism $\mathcal{L} : \Lambda \to Pic(\widetilde{S})$ (equivalently, a T-bundle on $\widetilde{S}$), which satisfies a certain twisted W-equivariance condition.

Remarks

(1) Following the analogy of section 1.4, the map v is called the *value map*. For each $\lambda \in \Lambda$, the image $v'(\widetilde{S} \times \lambda)$ is the *spectral cover* for the representation of G of highest weight λ. Some typical examples: For $G = SL(n)$, there is an n-sheeted spectral cover $\overline{S} \to S$ corresponding to the first fundamental weight. A point of $\widetilde{S}$ is an ordering of the n points in a fiber of $\overline{S}$. A point of the spectral cover corresponding to the k-th fundamental weight amounts to a choice of k unordered points in a fiber. For $G = E_n$, the root system can be identified with the collection of lines on a del Pezzo surface dP_n obtained by blowing up n points in $\mathbf{P}^2$. A point of the smallest spectral cover then corresponds to one of the lines on the surface, while a point of the cameral cover is specified by the choice of an ordered n-tuple of disjoint lines.

(2) Since W acts on both Λ and $Pic(\widetilde{S})$, we can consider the group $Hom_W(\Lambda, Pic(\widetilde{S}))$ of all W-equivariant homomorphisms. There is a natural map (cf. [D2])

$$Hom_W(\Lambda, Pic(\widetilde{S})) \to H^2(W, T),$$

and the homomorphisms $\mathcal{L}$ which we allow form one coset of its kernel. Elements of the finite group $H^2(W,T)$ are given by classes of extensions of W by T; our coset is the inverse image of the class $[N] \in H^2(W,T)$ of the normalizer N. Actually, there is a further shift depending on the ramification divisor of $\widetilde{S}$ over S, but we will ignore this here. More details can be found in [D2]. For the purpose of this talk, we can consider the distinguished Prym as a black box: all we need to know is that it (together with the cameral cover) uniquely determines a principal G-Higgs bundle. We recall this notion which appeared in the introduction:

A Principal G-Higgs Bundle on S is a pair $(P_S, \mathcal{C})$ where P_S is a principal G-bundle on S, and $\mathcal{C} \subset ad(P_S)$ is a family of regular centralizers.

3.3 The Main Result

Theorem: There is a natural equivalence between regularized G-bundles $(P,\ c : S \to \overline{P_S/N},\ P^{\mathcal{C}})$ on an elliptic fibration $\pi : X \to S$, and spectral data $(\widetilde{S} \to S, v, \mathcal{L})$.

Sketch of the proof

The regularized bundle $(P,\ c : S \to \overline{P_S/N},\ P^{\mathcal{C}})$ determines by restriction to the base a principal G-Higgs bundle $(P_S, \mathcal{C})$. The latter is equivalent, by [D2] ("integrability of the Hitchin system"), to items **(1,3)** of the spectral data. In the next subsection we will give a direct construction for recovering the value map (=data **(2)**) from a regularized bundle. In the last subsection we illustrate, in case $G = SL(3)$, how data **(2)** allows us to lift a principal G-Higgs bundle on S to a regularized G-bundle on X. The idea is to use the "universal family" $\mathcal{U}_{X/S} \to S$ with fibers $\mathcal{U}_E$; this includes enough data to rigidify the problem, resulting in the unique extension.

$$\begin{array}{ccc} \text{Regularized G} - \text{bundle on X} & \overset{\text{direct construction}}{\Longrightarrow} & (\mathbf{1}, \mathbf{2}, \mathbf{3}) \\ \text{forget} \downarrow & & \\ \text{Principal G-Higgs bundle on S} & \overset{\text{integrability of Hitchin's}}{\Longleftrightarrow} & (\mathbf{1}, \mathbf{3}) \end{array} \tag{7}$$

$$\begin{array}{ccccc} \{\text{Embedded cameral covers}\} & = & \Gamma(S, \mathcal{M}^G_{X/S}) & \Leftrightarrow & (\mathbf{1},\mathbf{2}) \\ Prym_\Lambda(\widetilde{S}) & \approx & Hom_W(\Lambda, Pic\ \widetilde{S} & \Leftrightarrow & (\mathbf{3}) \end{array}$$

3.4 From a regularized G-Bundle to a value map.

The fiber of the downward arrow in (7) is the space $\mathcal{M}^{\mathcal{C}}_E$ of all $\mathcal{C}$-torsers $P^{\mathcal{C}}$ on X with given $\mathcal{C} \subset Ad\ P_S$. We need to show that this is isomorphic to the space of data **(2)**, i.e. of W-equivariant maps $v : \widetilde{S} \to \mathcal{M}^T_{X/S}$. This can be checked point-by-point in the base S, but we have to pay close attention to the scheme structure of $\widetilde{S}$. Take S to be a point, so $X = E$. Let $(G/B)^C$ be the *subscheme* of G/B parametrizing Borel subgroups which contain the given regular centralizer C. The claim is:

$$\mathcal{M}^C_E \approx Maps_W((G/B)^C, \mathcal{M}^T_E). \tag{8}$$

First, we construct the morphism

$$\mathcal{M}^C_E \times (G/B)^C \to \mathcal{M}^T_E. \tag{9}$$

Set-theoretically, this sends:

$$\left(P^C, B\right) \mapsto \left(P^C \times^C B\right) \times^B T. \tag{10}$$

To do this scheme theoretically, note that over $(G/B)^C$ there are bundles
$\mathcal{C} \hookrightarrow \mathcal{B} \to \mathcal{T}$:

- $\mathcal{B}$ = restriction of the universal B-bundle on G/B
- $\mathcal{C}$ = the trivial C-bundle
- $\mathcal{T} := \mathcal{B}/[\mathcal{B}, \mathcal{B}]$, the trivial T-bundle

We see that a C-bundle P^C on E extends to a $\mathcal{C}$-bundle $P^{\mathcal{C}}$ on $E \times (G/B)^C$. This in turn induces bundles $P^{\mathcal{B}}$ and $P^{\mathcal{T}}$. The latter is a T-bundle on $E \times (G/B)^C$, so it is given by a classifying morphism $(G/B)^C \to \mathcal{M}_E^T$, which is what we wanted for a fixed C-bundle P^C. The existence of the Poincare bundle on $\mathcal{M}_E^C$ implies that this globalizes to a morphism $\mathcal{M}_E^C \times (G/B)^C \to \mathcal{M}_E^T$, as claimed.

Now W acts on $(G/B)^C$ and on $\mathcal{M}_E^T$, and we need to show that the map (9) is equivariant for these actions. This is actually easier to see in a global setup: Letting C vary over $\overline{G/N}$, the two sides of (9) become:

$$\begin{array}{c} \widetilde{\mathcal{U}_E} := \mathcal{U}_E \times_{\overline{G/N}} \overline{G/T}. \to \mathcal{M}_E^T \times \overline{G/N} \\ \nwarrow \ \nearrow \\ (W \text{ acts here}) \end{array} \tag{11}$$

We put together the following facts:

- W equivariance is clear over G/T, since $(G/B)^C$ is a W-torser there, so both sides of (8) equal cm_E^T;
- $\mathcal{U}_E$ is smooth (of dimension r) over $\overline{G/N}$, which implies that:
- $\widetilde{\mathcal{U}_E}$ is smooth (of dimension r) over $\overline{G/T}$. Therefore:
- the inverse image of G/T is dense,

so we conclude that W-equivariance holds everywhere.

3.5 Inversion

Now that we have constructed a map from the left hand side of (8) to the right hand side, we need to show that it is an isomorphism. (Our discussion here is based on the forthcoming manuscript [DG].) To that end, we decompose our regular centralizer C:

$$C = C_{ss} \times C_{unip} \tag{12}$$

into its semisimple part (a torus) and unipotent part (everything else: a vector group and a finite abelian group of components). The map in (8) decomposes:

$$\begin{array}{ccccc} \mathcal{M}_E^C & = & \mathcal{M}_E^{C_{ss}} & \times & \mathcal{M}_E^{C_{unip}} \\ \downarrow & & \downarrow & & \downarrow \\ Maps_W\left((G/B)^C, \mathcal{M}_E^T\right) & = & Maps_W\left((G/B)^C_{red}, \mathcal{M}_E^T\right) & \times & Maps_W\left((G/B)^C_{conn}, \mathcal{M}_E^T\right) \end{array}.$$

Here $(G/B)^C_{red}$ is the reduced structure underlying the finite scheme $(G/B)^C$, while $(G/B)^C_{conn}$ is one of its connected components, a one-pointed scheme. We can analyze the two pieces separately:

- The semisimple part: both sides are isomorphic to the invariant locus $(\mathcal{M}_E^T)^{W_0}$, where W_0 is the subgroup of W stabilizing a point of $(G/B)^C_{red}$.
- The unipotent part: both terms are $\mathcal{M}_E^{C_{unip}} = C_{unip}$. In particular, this part of the answer is independent of the elliptic curve E.

Combining these, we see that (8) is indeed an isomorphism as we claimed.

The following example illustrates in some detail the contributions of the toral, vector and finite parts of C for the three types of regular centralizers in $G = SL(3)$. We describe each regular centralizer, then the moduli space $\mathcal{M}_E^C$, the coordinate ring $\mathcal{O}((G/B)^C)$ of $(G/B)^C$, and finally $Maps_W((G/B)^C, \mathcal{M}_E^T)$.

Example: $G = SL(3)$

C	$\mathcal{M}_E^C$	$\mathcal{O}((G/B)^C)$	$Maps_W((G/B)^C, \mathcal{M}_E^T)$
$\begin{pmatrix} * & & \\ & * & \\ & & * \end{pmatrix}$	$\Lambda \otimes E \approx E^2$	$\mathbf{C}[W]$	$\Lambda \otimes E \approx E^2$
$\begin{pmatrix} a & b & \\ & a & \\ & & a^{-2} \end{pmatrix}$	$\mathbf{C} \times E$	$\mathbf{C}[\epsilon]/\epsilon^2 \times \mathbf{C}[W/Z_2]$	$\mathbf{C} \times E$
$\begin{pmatrix} \omega & a & b \\ & \omega & a \\ & & \omega \end{pmatrix}$ $\omega^3 = 1$	$\mathbf{C}^2 \times E[3]$	$\mathbf{C}[x,y,z]/(\sigma_i = 0)$ $\sigma_i = \sigma_i(x,y,z)$ $i = 1,2,3$	$C^2 \times E[3]$

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