Lecture 5: Theories with More Supersymmetry

The title might suggest that we discuss all possible theories in $n \leq 4$ dimensions with $s = 4$ supersymmetries, in $n \leq 6$ dimensions with $s = 8$ supersymmetries, and in $n \leq 10$ dimensions with $s = 16$ supersymmetries. And perhaps it also suggests supergravity. Sorry! Our more modest goal is first of all to introduce the superspacetime $M^{4|4}$ and the basic theories with $P^{4|4}$ symmetry. They and their dimensional reductions are also useful for discussing theories with more than 4 supersymmetries. Secondly, we want to summarize some of the geometry which enters into theories with 8 or 16 supersymmetries in the maximal dimension. As far as the dimensional reductions, each exhibits its own peculiarities and we select a few representative cases to illustrate some of the phenomena which occur. We will not discuss theories with *chiral* supersymmetry beyond the minimal amount.

We begin with bosonic theories again. First, we add some perspective to our general discussion at the beginning of Lecture 4 about classical bosonic lagrangians. Namely, in the application to *quantum* field theory we must distinguish between lagrangians which can be used to define *fundamental, microscopic* theories valid at arbitrarily small distance (large energy) and lagrangians used to describe an *effective, macroscopic* theory at large distance (small energy). Second, we show how dimensional reduction of a gauge field produces additional scalar fields.

Then we turn to our main task, which is the description of theories with symmetry group $P^{4|4}$. We write these theories in the superspacetime $M^{4|4}$. The description of the σ-model is similar to that of the σ-model in $M^{3|2}$ treated in Lecture 4, only now the superfield is constrained to be *chiral* and the target manifold is *Kähler*. Also, the potential energy is built out of a *holomorphic* function called the *superpotential*. Gauge theory on $M^{4|4}$ has a similar description to that on $M^{3|2}$, but now the coupled model with scalars in a Kähler manifold X exhibits a new feature: a term in the potential energy coming from the *moment map* of the G action on X.

The target manifold of a σ-model with 8 supersymmetries is *hyperkähler*. In the coupled model we meet the hyperkähler moment map analogously to the Kähler case. As indicated in Table 5 there is no σ-model with 16 supersymmetries. There is, however, a pure supersymmetric gauge theory in $n \leq 10$ dimensions. Neither the σ-model with 8 supersymmetries nor the gauge theory with 16 supersymmetries has a superspacetime description (off-shell). At the end of the lecture we demonstrate using components that the minimal supersymmetric Yang-Mills lagrangian is indeed supersymmetric in $n = 10$ dimensions.

We meet more interesting geometry when we consider dimensional reductions of these models, especially in the possible effective lagrangians. This is a large topic and we include only a few examples to illustrate some of the ideas involved. One notable feature which has no analog in ordinary Minkowski space: There exist nontrivial superPoincaré invariant connections in superspacetimes. In some cases with extended supersymmetry we can use these as background connections to obtain special mass terms in lagrangians.

Fundamental vs. effective lagrangians

The bosonic lagrangian (4.10) we considered in Lecture 4 covers many cases of interest, but is hardly the most general classical lagrangian with a connection A and a scalar field ϕ. For example, we might write terms which involve higher powers of first derivatives—say, $|F_A|^4$ or $|d_A\phi|^4$—or terms involving higher derivatives—say $|\nabla_A F_A|^2$ or $|\nabla_A d_A\phi|^6$. Even if we stick to terms which are at most quadratic in the first derivatives, there are more general terms which we can write. For example, we might allow the coefficient in front of the Yang-Mills action to depend on the scalar field:

$$-\frac{1}{2g^2(\phi)}|F_A|^2\ |d^nx|, \tag{5.1}$$

where $g^2\colon X\to\mathbb{R}^{>0}$ is a G-invariant function. Also, there are special terms we can write in specific dimensions. In $n=2$ dimensions if $\langle\cdot\rangle\colon\mathfrak{g}\to\mathbb{R}$ is an infinitesimal character of G—a trace on $\mathfrak{g}$—then we have the *topological term*

$$\theta\langle F_A\rangle. \tag{5.2}$$

Here θ is a real coupling constant. The term is called topological since it is locally constant on the space of fields with finite action. Notice that it is a 2-form, and to turn it into a density we need to fix an orientation. Said differently, theories with nonzero θ are not invariant under orientation-reversing transformations in the Lorentz group. Such terms are said to be *CP noninvariant*. There is another 2-form term possible in 2-dimensional theories with a $\mathfrak{g}$-valued scalar field σ:

$$\langle F_A,\sigma\rangle. \tag{5.3}$$

Again it is a 2-form and it breaks CP invariance. (For $G=\mathbb{T}$ the circle group, (5.3) is obtained from (5.2) by promoting the real constant θ to a variable $\mathbb{T}$-valued field σ.) In $n=4$ dimensions we use the inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$ to write a topological term

$$\theta\langle F_A\wedge F_A\rangle \tag{5.4}$$

for θ a real coupling constant. In $n=3$ dimensional gauge theories we can also include a Chern-Simons term.

In geometry we study many of these classical lagrangians on curved Riemannian manifolds M, depending on the geometric problem of interest. In *quantum* field theory lagrangians are used both to describe *fundamental* (*microscopic*) theories and *effective* (*macroscopic*) theories. It is often said that the microscopic theory is the *question* (*problem*) and the macroscopic theory the *answer* (*solution*). Given a well-posed question there is in principle a method of computing the answer, but it is intractable. One tries instead to use physical insight and symmetry principles to determine the form of the effective lagrangian.

A fundamental theory should make sense at arbitrarily high energies, and so microscopic lagrangians are constrained by *renormalizability* and *asymptotic freedom*. Both concepts refer to perturbation theory—we compute everything as a perturbation of a free theory. And both put constraints on the data (4.8) used to define the theory. As we might expect, the constraints depend strongly on the dimension. In $n = 2$ dimensions a σ-model is renormalizable for any target manifold X, but in higher dimensions the theory is renormalizable only if X is a vector space. Then for renormalizability in $n = 3$ dimensions the potential V must be a polynomial of degree at most 6, and in $n = 4$ dimensions it must be a polynomial of degree at most 4. Gauge theories are renormalizable in $n \leq 4$ dimensions. Renormalizability excludes higher derivative terms and a nontrivial coupling (5.1), though the terms (5.2)–(5.4) described above are possible. There are no renormalizable (nonfree) theories above $n = 4$ dimensions. Asymptotic freedom is determined by computing the 1-loop β function. It also depends on the spinor fields in the theory, and since we have not included those we will not try to describe the precise statements. Suffice it to say that there are asymptotically free theories and that in $n = 4$ dimensions they involve nonabelian gauge fields.

An *effective lagrangian* is used to describe the behavior of a quantum field theory at large distances (low energies). Here the physical input goes beyond the purely formal. The first step is to determine what the proper degrees of freedom are to describe the long-range behavior of the system. These are massless fields, but they may be quite different than the fields which are used in the fundamental lagrangian to define the theory. For example, we may start with a gauge theory with gauge group G but at long distance only see a subgroup of G because of the Higgs mechanism. Or, say in $n = 3$ dimensions, we may start with a nonabelian gauge theory and then discover that at long distances we have a massless abelian gauge field. In that case we should write the effective lagrangian in terms of a dual scalar field; recall that we mentioned this duality at the end of Lecture 3. Sometimes in the effective action we use a scalar field which is made as the product of spinor fields. In other cases we find fields whose expression in terms of the fields in the fundamental lagrangian is highly nonlocal and nontrivial (and nonexplicit). In $n = 4$ dimensions we might find composite massless gauge fields with gauge group G' which is not a subgroup of the gauge group G in the fundamental theory. As should be clear, identifying the proper degrees of freedom at long distances requires physical insight into the *dynamics* of the theory.

Once the fields are determined, the effective action is usually said to include all possible terms compatible with the symmetries of the theory, but with coefficients to be determined. When the symmetries of the theory include supersymmetries, these coefficients are often highly constrained and more can be said. The terms are organized into a *derivative expansion*, where the leading order contributions come from terms with a small number of derivatives. This reflects the fact that we are studying low energy field configurations which are essentially constant. As in (4.10) we will restrict ourselves to terms at most quadratic in first derivatives, but now we are not constrained by renormalizability. Nor is asymptotic freedom—a property which refers to the behavior at short distances—relevant. The corresponding property at large distances is *infrared freedom*, and this is necessary if we want to describe the long-range behavior as perturbation of a free theory. (It is not necessarily true that a theory is infrared free. It may instead be described by a nontrivial conformal field theory at long distance, often described by saying that the theory "flows to a nontrivial fixed point". There are many nontrivial conformal theories in $n = 2$ dimensions and they exist in higher dimensions as well.) Now if the effective action is written in terms of massless scalars only, and if there is no potential function V, then the manifold X is called the *quantum moduli space*. It often differs from the classical moduli space and is one of the key objects of interest. The quantum moduli space is a *complete* Riemannian "manifold" possibly with singularities. At a singularity there is new physics—more fields are massless. We can include massive scalars and a potential, in which case the quantum moduli space is determined as in (4.11). For example, near a singularity there is a description in terms of more fields and a potential V with $V^{-1}(0)/G$ singular. This illustrates that there may be no single global description of the quantum moduli space—for example, different fields may be used to describe different local regions. As for the massless gauge fields, in general the coupling constants in the effective lagrangian depend on the scalar fields as in (5.1). (In fact, there is a coupling constant for each simple or abelian factor of the gauge group G. Put differently, the invariant inner product $\langle\cdot,\cdot\rangle$ is a function of the scalar fields.)

Although we have only discussed bosonic lagrangians in this subsection, we can and should also include fermions in both microscopic and macroscopic lagrangians. Of course, they are required in supersymmetric lagrangians.

It should be said that without supersymmetry the massless scalars one encounters are usually Goldstone bosons. That is, they are due to spontaneous global symmetry breaking. In supersymmetric theories there are nontrivial quantum moduli spaces which are not homogeneous.

Finally, in applications to geometry we often analytically continue a lagrangian to Euclidean space and then try to formulate a version on curved Riemannian manifolds. Thus, for example, geometers and topologists get great mileage by studying the Yang-Mills equations on Riemannian 4-manifolds. As for supersymmetric lagrangians, they can be formulated on curved Riemannian manifolds in many cases using a device called *topological twisting*, but we will not consider them here.

Dimensional reduction of gauge fields

Suppose $\mathbf{A}$ is gauge field on Minkowski spacetime $\check{M}^{n+r}$ invariant under the spatial translations $\partial/\partial x^j$, $j = n, \dots, n+r-1$. (The indices $j, k, \dots$ in this subsection always run over the last r spatial directions. We use standard coordinates on Minkowski spacetime as in previous lectures.)

This means that the indicated spatial translations have been lifted to commuting vector fields on the total space of the principal bundle $P \to \check{M}^{n+r}$ which hosts the connection $\mathbf{A}$, and that the Lie derivative of the connection form along these vector fields vanishes. Then we write

$$\mathbf{A} = A + \sigma_j dx^j,$$

where A does not involve any dx^j. We view $\check{M}^n$ as the quotient of $\check{M}^{n+r}$ by the last r spatial translations. Then A is a connection on $\check{M}^n$ on the quotient bundle $\overline{P} \to \check{M}^n$ and each σ_j is a scalar field with values in the adjoint bundle $\operatorname{ad}\overline{P}$. Now the curvature of $\mathbf{A}$ is

$$F_{\mathbf{A}} = F_A \ + \ d_A\sigma_j \wedge dx^j \ + \ \frac{1}{2}[\sigma_j, \sigma_k]\, dx^j \wedge dx^k,$$

which is a sum of orthogonal terms, and so

$$-\frac{1}{2}|F_{\mathbf{A}}|^2 = -\frac{1}{2}|F_A|^2 \ + \ \sum_j \frac{1}{2}|d_A\sigma_j|^2 \ - \ \frac{1}{2}\sum_{j<k} |\,[\sigma_j, \sigma_k]\,|^2.$$

Thus the dimensional reduction of the Yang-Mills lagrangian (4.5) has the canonical kinetic terms for the connection A and the scalar fields σ_j and a potential energy term

$$V = \frac{1}{2}\sum_{j<k} |\,[\sigma_j, \sigma_k]\,|^2. \tag{5.5}$$

Sometimes the real scalar fields σ_j are written in terms of complex scalar fields. For example, if $r = 2$ we set

$$\sigma = \frac{\sigma_1 + \sqrt{-1}\sigma_2}{\sqrt{2}};$$

it is a scalar field with values in the complexified adjoint bundle $\operatorname{ad}_{\mathbb{C}} P$. Then the potential energy (5.5) is

$$V = \frac{1}{2}\,\|\,[\sigma, \bar{\sigma}]\,\|^2. \tag{5.6}$$

Our convention is that the double bar notation '$\|\cdot\|$' denotes the norm built from the inner product space on a hermitian vector space. We always use single bars for quadratic forms built from bilinear forms.

Quantum particles in $n = 4$ dimensions

As we indicated in Lecture 3, the quantization of free classical field theories leads to a one particle Hilbert space which is a representation of an appropriate (super)symmetry group. Therefore, the representation theory of supersymmetry groups constrains the possible free classical field theories, and since nonfree theories can be regarded as perturbations of free theories, it constrains nonfree theories as well. Before launching into the construction of supersymmetric theories in $n = 4$ dimensions with $s = 4$ supersymmetries, we carry out the easy analysis of the representations of $P^{4|4}$.

As usual, we restrict our analysis to representations where only scalars (spin 0), spinors (spin 1/2), and vectors (spin 1) occur. First, without supersymmetry we seek representations of the bosonic symmetry group P^4, as discussed in Lecture 1. In the massless case the little group is Spin(2). The irreducible complex representations are one dimensional and indexed by the *helicity* $j \in \frac{1}{2}\mathbb{Z}$. (The helicity is the spin in the direction of the motion.) But recall that physical representations are constrained by CPT invariance, which in even dimensions means that the representation of the little group is real or quaternionic. Here we see that if helicity $j \neq 0$ occurs, then so does helicity $-j$ with the same multiplicity. In other words, the helicities are symmetric about 0. In the massive case the little group is Spin(3). Now all representations are real or quaternionic, so CPT gives no restriction. We decompose (complex) representations under a maximal torus $\mathrm{Spin}(2) \subset \mathrm{Spin}(3)$ into one dimensional representations indexed by the *spin* $j \in \frac{1}{2}\mathbb{Z}$.

	-1	$-1/2$	0	$1/2$	1
massless scalar			1		
massive scalar			1		
massless spinor		1		1	
massive spinor		1		1	
massless vector	1				1
massive vector	1		1		1

Table 6: Particle representations of P^4

The various possibilities are summarized in Table 6. In the columns we indicate the dimension of the helicity or spin j representation of Spin(2). CPT requires that the table be symmetric about the vertical line $j = 0$. The most noteworthy feature is the difference between the massive and massless vector: a massive vector has 3 degrees of freedom whereas a massless vector has 2 degrees of freedom. The massless vector representation does not deform to the massive vector representation; rather, the sum of the massless vector and massless scalar deforms to a massive vector. In field theory this deformation is realized via the Higgs mechanism.

As discussed in Lecture 3, in the supersymmetric case we must represent a Clifford algebra as well. For $P^{4|4}$ this algebra has 4 generators in the massive case and 2 generators in the massless case. (The quadratic form is degenerate in the massless case leading to a Clifford algebra with 2 generators.) The minimal complex Clifford module $W = W^0 \oplus W^1$ has $\dim W^0 = \dim W^1 = 2$ in the massive case and $\dim W^0 = \dim W^1 = 1$ in the massless case. There must be an intertwining representation of the little group, and from the bracketing relations in the algebra $\mathfrak{p}^{4|4}$ it is easy to see that the spins which occur in the massive case are j, $j+1/2$, $j+1/2$, and $j+1$. In the massless case the helicities are j and $j+1/2$. Finally, we impose spin-statistics and require that W^0 consist of representations with integer spin or helicity and that W^1 consist of representations with half-integer spin or helicity. The possibilities are summarized in Table 7. There is no multiplet with a single real scalar; the minimal multiplet with scalar particles has 2 scalars. Its name 'chiral multiplet' derives from the description in terms of fields, which we discuss presently. We learn that this description must involve 2 real scalar fields, or equivalently one complex scalar field. The chiral multiplets and the massless vector multiplet each include a spinor with 2 physical degrees of freedom. So we expect the chiral and massless vector multiplets to include a single spinor field in the field description. We will encounter the massive vector multiplet through a supersymmetric version of the Higgs mechanism.

	-1	$-1/2$	0	$1/2$	1
massless chiral multiplet		1	2	1	
massive chiral multiplet		1	2	1	
massless vector multiplet	1	1		1	1
massive vector multiplet	1	2	2	2	1

Table 7: Multiplets for $P^{4|4}$

The superspacetime $M^{4|4}$

As indicated in Table 1, the 4 dimensional real spin representation S of $\mathrm{Spin}(1,3)$ decomposes under complexification as

$$S_{\mathbb{C}} \cong S' \oplus S'', \qquad S'' \cong \overline{S'}.$$

Each of S', S'' has complex dimension 2 and each carries an invariant complex volume form ϵ. We build superspacetime using ΠS^*, and so fix complex odd linear coordinates θ^1, θ^2 on $\Pi S'^*$. We use the conjugate coordinates $\bar\theta^{\dot 1}, \bar\theta^{\dot 2}$ on $\Pi S''^*$. We use the notation '$\bar\theta^{\dot\alpha}$' for the conjugate coordinates; the use of dotted indices is analogous to the use of barred indices on complex manifolds. Recall that the vector space V underlying Minkowski spacetime has complexification

$$V_{\mathbb{C}} \cong S'^* \otimes S''^*, \tag{5.7}$$

and so it is natural to use complex coordinates $y^{\alpha\dot\beta}$ related to the odd complex coordinates $\theta^\alpha, \bar\theta^{\dot\beta}$ by (5.7). They are related to the standard x coordinates by the formulas

$$\begin{aligned} y^{1\dot1} &= \frac{x^0 + x^1}{2}, \\ y^{2\dot2} &= \frac{x^0 - x^1}{2}, \\ y^{1\dot2} &= \frac{x^2 + \sqrt{-1}\,x^3}{2}, \\ y^{2\dot1} &= \frac{x^2 - \sqrt{-1}\,x^3}{2}. \end{aligned}$$

There are nice formulas for the metric, wave operator, Dirac operator, star operator, etc. in this coordinate system.

As always we work in terms of a global framing rather than global coordinates, but here we use *complex* vector fields. The even elements of the framing are the bi-invariant vector fields

$$\partial_{\alpha\dot\beta} = \frac{\partial}{\partial y^{\alpha\dot\beta}}$$

and the odd elements are the left-invariant vector fields

$$(5.8) \qquad \begin{aligned} D_\alpha &= \frac{\partial}{\partial\theta^\alpha} - \bar\theta^{\dot\beta}\partial_{\alpha\dot\beta}, \\ \overline{D}_{\dot\alpha} &= \frac{\partial}{\partial\bar\theta^{\dot\alpha}} - \theta^\beta\partial_{\beta\dot\alpha}. \end{aligned}$$

The nontrivial bracket relation is

$$[D_\alpha, \overline{D}_{\dot\beta}] = -2\partial_{\alpha\dot\beta}.$$

The superPoincaré symmetries are generated by right invariant vector fields as before. We have the inclusion

$$i\colon \check M^4 \hookrightarrow M^{4|4}.$$

We assume that our coordinates are chosen so that the volume forms ϵ satisfy (4.15).

Note that our conjugation convention (1.???) makes the two equations in (5.8) really look conjugate. The conventions in many physics references lead to different looking formulas.

Supersymmetric σ-model on $M^{4|4}$

Our discussion of the representation theory of $P^{4|4}$ makes clear that we cannot write a sensible (free) theory using a real superfield $M^{4|4} \to \mathbb{R}$. Rather, we expect to use a complex superfield

$$(5.9) \qquad \Phi\colon M^{4|4} \longrightarrow \mathbb{C}.$$

An unconstrained complex superfield would lead to too many component fields. For example, the restriction of first derivatives to $\check{M}^4$ would give both a complex spinor field in S' and an independent complex spinor field in S''. Upon quantization of a canonical free lagrangian we would obtain two spinor particles, whereas Table 7 indicates that we should only find one spinor particle in a multiplet with scalars. Therefore, we expect to put a constraint on the superfield (5.9). It turns out that the appropriate constraint is that we require Φ to be *chiral*, which means that its derivatives along the left-invariant distribution spanned by the $\overline{D}_{\dot\alpha}$ vanish:

$$\overline{D}_{\dot\alpha}\Phi = 0. \tag{5.10}$$

In this case $\overline{\Phi}$ is called *antichiral*. Chiral superfields on $M^{4|4}$ are analogous to holomorphic functions on a complex manifold; (5.10) is analogous to the $\bar\partial$ equation. Sums and products of chiral superfields are chiral. Let $\mathcal{C}^*$ denote the sheaf of chiral superfields. Then

$$H^1(M^{4|4}, \mathcal{C}^*) = 0. \tag{5.11}$$

This is easy to see in components, for example.

The component fields of a chiral superfield are defined analogously to (4.16):

$$\begin{aligned} \phi &= i^*\Phi \\ \psi_\alpha &= \frac{1}{\sqrt{2}} i^* D_\alpha \Phi \\ F &= i^*\big(-\frac{1}{2}\epsilon^{\alpha\beta} D_\alpha D_\beta\big)\Phi \end{aligned}$$

Whereas in $M^{3|2}$ the scalar multiplet consists of real fields, here the *chiral multiplet* $\{\phi, \psi, F\}$ consists of complex fields. In particular, there is a complex auxiliary field F. The free lagrangian in superspacetime is

$$\mathcal{L} = \frac{1}{4}\Phi\overline{\Phi}\,|d^4x|\,d^4\theta. \tag{5.12}$$

When we expand in components we obtain

$$L = \Big\{ \|d\phi\|^2 + \bar\psi \not{D}\psi + \|F\|^2 \Big\}\,|d^4x|. \tag{5.13}$$

- As noted, F is an auxiliary field and in this simple model its (algebraic) equation of motion is $F = 0$. In the future we eliminate auxiliary fields from component lagrangians. Also, we will usually only write the bosonic part of the component lagrangian.

- Our definition of the component lagrangian (4.20) in this case differs from integration over $d^4\theta$, as anticipated in (4.???). We use the string of derivatives on the left hand side of the equation
$$\frac{1}{2}i^*(D_1\overline{D}_{\dot{1}}\overline{D}_{\dot{2}}D_2 + D_2\overline{D}_{\dot{2}}\overline{D}_{\dot{1}}D_1) = \int d^4\theta \;+\; \square\, i^*$$
to define L, and it differs from integration by the exact term indicated on the right hand side.

The *nonlinear* σ-model is more interesting. We first replace X by a complex manifold. It is easy to define a chiral map
$$\Phi\colon M^{4|4} \longrightarrow X;$$
Φ is chiral if $D_\alpha\Phi$ is a vector field of type $(1,0)$ on X. Equivalently, Φ is chiral if the pullback of every local holomorphic function on X is chiral in the sense previously defined. Now to write the σ-model lagrangian we need to impose a Riemannian structure on X. In fact, the model is defined for X a *Kähler* manifold. To see why X has to be Kähler, we might try to relax some part of the Kähler structure (e.g. take a hermitian manifold whose $(1,1)$-form is not closed) and then rediscover the Kähler structure by imposing supersymmetry in the component lagrangian. Or, we could dimensionally reduce to $n = 3$ dimensions and ask that the σ-model, which always has $s = 2$ supersymmetries, carry $s = 4$ supersymmetries. We content ourselves here with simply defining the model assuming X is Kähler to begin. In fact, we make the simplifying assumption that X is *Hodge*. This means that X carries a holomorphic hermitian line bundle $L \to X$ with curvature
$$\operatorname{curv}(L) = \sqrt{-1}\,\omega,$$
where ω is the Kähler form. Now if $\Phi\colon M^{4|4} \to X$ is chiral, then there exist nonzero chiral sections $\tilde{\Phi}$ of $\Phi^*L \to M^{4|4}$. This follows from (5.11). The superspacetime lagrangian is written in terms of such a lift of Φ:

$$\mathcal{L} = \frac{1}{2}\log\|\tilde{\Phi}\|^2\,|d^4x|\,d^4\theta. \tag{5.14}$$

- The *action*—the integral of the lagrangian density—only depends on Φ, not on the lift $\tilde{\Phi}$. This is familiar from other examples, e.g. a Wess-Zumino-Witten term in a σ-model. Our situation is better since the component lagrangian, which is a partial integral of the lagrangian density, is also independent of the lift.
- Usually one writes instead of (5.14) a local formula in terms of a Kähler potential K:

$$\mathcal{L} = \frac{1}{2}K(\Phi,\overline{\Phi})\,|d^4x|\,d^4\theta. \tag{5.15}$$

Then a simple argument shows that the component lagrangian is independent of the choice of K. This expression has the disadvantage that it is only local, but the advantage that it is defined for Kähler manifolds which are not Hodge.

- The component lagrangian is similar to (5.13), but now there is a curvature term as in (4.23). Since we will not need the explicit formula, we omit it. The bosonic part is the standard σ-model with target X.

The superpotential

Suppose

$$W : X \longrightarrow \mathbb{C}$$

is a global holomorphic function. Then $\Phi^*(W)$ is a chiral superfield on $M^{4|4}$. It turns out that

$$\Phi^*(W)\,|d^4x|\,d^2\theta \tag{5.16}$$

is a superPoincaré-invariant density, but on a superspace $M^{4|2_{\mathbb{C}}}$ whose odd directions are *complex* and modeled on $\Pi S'^*$. Furthermore, we can integrate out $d^2\theta$ using the expression

$$i^*(-\frac{1}{2}\epsilon^{\alpha\beta}D_\alpha D_\beta) = \int d^2\theta,$$

and the result is supersymmetric.

The total superspacetime lagrangian of the σ-model with superpotential W is

$$\mathcal{L} = \Big\{\frac{1}{2}\log\|\overline{\Phi}\|^2\,d^4\theta \;+\; \frac{1}{2}\Phi^*(W)\,d^2\theta \;+\; \frac{1}{2}\overline{\Phi^*(W)}\,d^2\bar\theta\Big\}\;|d^4x|.$$

- We include the complex conjugate of (5.16) so that the lagrangian is real, as it must be in Minkowski spacetime.
- There are 3 different superspacetimes at work in this super lagrangian.
- The auxiliary field has a nontrivial field equation

$$F = -\phi^*\operatorname{grad} W.$$

- As in the σ-model on $M^{3|2}$ with potential, in this model with superpotential there is a scalar potential energy term

$$V = \|\operatorname{grad} W\|^2 \tag{5.17}$$

  and also a fermion mass term built from the Hessian of W.
- The analysis of the theory—its vacuua, dimensional reductions, etc.—is entirely parallel to the story in Lecture 4.
- The R-symmetry of $M^{4|4}$ is broken by a generic superpotential W, but in certain cases one can assign a transformation law to Φ so that the theory with superpotential has R-symmetry.

Gauge theory on $M^{4|4}$

Fix a compact Lie group G and a bi-invariant inner product $\langle\cdot,\cdot\rangle$ on its Lie algebra $\mathfrak{g}$. As asserted in Lecture 4, a gauge field in a minimal supersymmetric gauge theory is a connection $\mathcal{A}$ constrained to be flat along the left-invariant odd distribution (see (4.38)). We remark that on $M^{4|4}$ the description of the unconstrained field is a bit more involved than on $M^{3|2}$, and we omit it. Again the basic invariant of a constrained connection is the even-odd curvature

$$W_\alpha = -\frac{1}{4}\epsilon^{\dot\beta\dot\gamma}[\nabla_{\alpha\dot\beta}, \overline{\mathcal{D}}_{\dot\gamma}],$$

which is a section of the complexified adjoint bundle. The Bianchi identity implies that W_α is chiral. There are further ramifications of the Bianchi identities for the superspacetime curvature; in particular, they express the entire curvature in terms of W_α. The component fields are

$$\begin{aligned} A &= i^*\mathcal{A}, && \text{connection on a bundle } P \to \check{M}^4;\\ \lambda_\alpha &= i^*W_\alpha, && \lambda \in C^\infty\big(\check{M}^4, \Pi(\mathrm{ad}_{\mathbb{C}} P \otimes S')\big);\\ \sqrt{-1}\,E &= i^*\Big(-\frac{1}{2}\epsilon^{\alpha\beta}\mathcal{D}_\alpha W_\beta\Big), && E \in C^\infty(\check{M}^4, \mathrm{ad}\, P). \end{aligned}$$

- The set $\{A, \lambda, E\}$ of fields is called the *vector multiplet*. Here E is a *real* auxiliary field. ('Real' means that E takes values in the Lie algebra of the compact gauge group G, or rather the real adjoint bundle.)
- Usually 'D' is used in place of 'E', but we prefer not to have yet another dee.
- The symmetric part of $i^*\mathcal{D}_\alpha W_\beta$ is the self-dual part of the curvature of A.
- The supersymmetry transformation laws for the component fields are easily derived by applying (4.17).

The fundamental lagrangian has a *complex* coupling constant τ, which is usually written in terms of real coupling constants g^2, θ:

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi\sqrt{-1}}{g^2}.$$

We require that $g^2 > 0$, or equivalently that τ live in the upper half plane. The lagrangian in superspacetime is

$$\mathcal{L} = -\frac{1}{16\pi}\,\mathrm{Im}(\tau\epsilon^{\alpha\beta}\langle W_\alpha, W_\beta\rangle\, d^2\theta)\,|d^4x|. \tag{5.18}$$

The expansion into a component lagrangian is

$$L = \frac{1}{g^2}\Big\{-\frac{1}{2}|F_A|^2 + \langle\bar\lambda \not{D}_A\lambda\rangle + \frac{1}{2}|E|^2\Big\}\,|d^4x| + \frac{\theta}{16\pi^2}\langle F_A\wedge F_A\rangle.$$

- Notice that the superspacetime lagrangian (5.18) involves only $d^2\theta$ terms, no $d^4\theta$ terms. In fact, we only encounter $d^4\theta$ terms in the kinetic term for the chiral superfield.
- We eliminate the auxiliary field E by its equation of motion $E = 0$.
- For $\theta = 0$ the component lagrangian is a density and does not require a choice of orientation. Thus it is invariant under orientation-reversing Poincaré transformations. But for $\theta \neq 0$ we need an orientation, and so the θ-term (which we anticipated in (5.4)) breaks CP invariance.
- In the quantum theory the integrand of the functional integral is $e^{iL/\hbar}$, and then the integrality of characteristic classes implies that θ is a periodic variable.
- We wrote the theory assuming the gauge group G is simple. Otherwise, we have coupling constants for each simple or abelian factor. We previously absorbed the coupling constant into the inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$, but here it is more natural to normalize the inner product so that θ is periodic with period 2π.

The general theory on $M^{4|4}$

The most general supersymmetric theory (without gravity) includes both a gauge superfield and a chiral superfield. We assume the chiral superfield takes values in a Hodge manifold to simplify the presentation; as before there is also a formulation for Kähler targets.

The data which defines the theory is

$$\begin{array}{ll} G & \text{compact Lie group} \\ \langle\cdot,\cdot\rangle & \text{bi-invariant scalar product on } G \\ L \to X & \text{Hodge manifold on which } G \text{ acts} \\ W\colon X \longrightarrow \mathbb{C} & \text{superpotential invariant under } G \end{array} \tag{5.19}$$

The action of G on X is by automorphisms of the Kähler structure, and we assume we are given a lift to an action on L. The reader should compare with the data (4.8) required in the most general bosonic theory. The fields in the model are

$$\begin{array}{ll} \mathcal{A} & \text{connection on a principal } G \text{ bundle } \mathcal{P} \longrightarrow M^{4|4} \\ \Phi\colon \mathcal{P} \longrightarrow X & \text{chiral equivariant map} \end{array}$$

Of course, Φ can be viewed as a chiral section of a bundle over $M^{4|4}$ with typical fiber X. To write the lagrangian we lift to a nonzero equivariant chiral map $\tilde{\Phi}\colon \mathcal{P} \to L$ as before. Now the general lagrangian we want to consider is

$$\mathcal{L} = \left\{ \frac{1}{2}\log\|\tilde{\Phi}\|^2\, d^4\theta + \operatorname{Re}\Big[\big(\frac{\sqrt{-1}\,\tau}{16\pi}\epsilon^{\alpha\beta}\langle W_\alpha, W_\beta\rangle + \Phi^* W\big)\, d^2\theta\Big] \right\} |d^4x|. \tag{5.20}$$

A new feature enters into the component lagrangian: the *moment map*

$$\mu\colon X \longrightarrow \mathfrak{g}^*$$

of the action of G on X. It is defined by the lift of the action to L:

$$\mu(\zeta) = -\sqrt{-1}\,\mathrm{vert}_1(\hat{\zeta}), \qquad \zeta \in \mathfrak{g},$$

where $\hat{\zeta}$ is the vector field on L induced by ζ and we take the vertical part of $\hat{\zeta}$ at points of unit norm. Then after eliminating auxiliary fields the bosonic part of the component lagrangian is

$$L_{\mathrm{bos}} = \Big\{\big(-\frac{1}{2g^2}|F_A|^2 + \|d_A\phi\|^2\big) - \phi^*\big(\|\operatorname{grad} W\|^2 + 2|\mu|^2\big)\Big\}\,|d^4x| \;+\; \frac{\theta}{16\pi^2}\langle F_A \wedge F_A\rangle. \tag{5.21}$$

- The potential function

$$V = \|\operatorname{grad} W\|^2 + 2|\mu|^2 \tag{5.22}$$

  has contributions from both the superpotential and the moment map. The first term comes from the complex auxiliary field F, the second from the real auxiliary field E. As expected in a supersymmetric theory, V is nonnegative.
- For a *fundamental* lagrangian we require renormalizability. This means that X is a complex inner product space and W a polynomial of degree at most 3.
- For an *effective* lagrangian we replace the constant τ in (5.20) by a holomorphic function

$$\tau\colon X \longrightarrow \mathbb{C}. \tag{5.23}$$

  Recall that τ must take values in the upper half plane. In the bosonic lagrangian we find that both g^2 and θ are functions of ϕ.
- The moduli space of vacuua is

$$\mathcal{M}_{\mathrm{vac}} = \mu^{-1}(0) \cap \mathrm{Crit}(W)\;/\;G.$$

  When $W = 0$ it is the Kähler (symplectic) quotient $X//G$. In general, W drops to a holomorphic function $\overline{W}$ on the Kähler quotient and the moduli space of vacuua is the set of critical points of $\overline{W}$ on this quotient. We can often identify it with the set of critical points of W on the quotient of X by the complexification of G.
- We can always shift the moment map by a character $G \to \mathbb{T}$. This holds even if there is no chiral superfield—X is a point. In that case the basic data (5.19) includes a lift of the G action to the Hodge line $\mathbb{C} \to \mathrm{pt}$, which is precisely a character of G. The corresponding term in the action, a constant which is twice the norm square of the infinitesimal character, is called a *Fayet-Iliopoulos* term.

Example (super Higgs mechanism). As a simple example to illustrate the Fayet-Iliopoulos term, take $G = \mathbb{T}$ to be the circle group and $X = \mathbb{C}$ with its standard Kähler structure and the circle group acting by multiplication. Then one possible moment map is $\mu(z) = \frac{1}{2}|z|^2$. We take $W = 0$. So the potential energy is $V = \frac{1}{2}|z|^4$, which has a *degenerate* zero at the origin. The moduli space of vacuua consists of a single point at which there is a massless gauge field and a massless complex scalar. Now we shift the moment map by a character:

$$\mu(z) = \frac{1}{2}|z|^2 - \frac{a^2}{2}, \qquad a \in \mathbb{R}.$$

Then the new potential is the quartic

$$\frac{1}{2}(|z|^2 - a^2)^2.$$

The moduli space of vacuua is the set of zeros of V divided by the $\mathbb{T}$ action, so again a single point. But now the Higgs mechanism is in play and there are no massless scalar fields or massless gauge fields.

Dimensional reductions of theories with $s = 3$ and $s = 4$ supersymmetries

In Lecture 4 we considered in detail the dimensional reduction of the supersymmetric σ-model with symmetry group $P^{3|2}$. The scalar superfield $\Phi\colon M^{3|2} \to X$ reduces to a scalar superfield on the superspacetime $M^{n|2}$ with $n = 2$ or $n = 1$. The supersymmetric σ-model on $M^{4|4}$ reduces similarly. There are superspacetimes $M^{n|4}$ with $n < 4$ and a chiral superfield $\Phi\colon M^{4|4} \to X$ reduces to a chiral superfield on the reduced superspacetime. For nontrivial target manifolds the quantum theory is only well-defined if $n = 2$ or $n = 1$. The theory on $M^{2|(2,2)}$ has been much studied; in the classical version the BPS story leads to the equations of holomorphic Morse theory. The Hilbert space of supersymmetric quantum mechanics on $M^{1|4}$ is the space of complex differential forms on the Kähler manifold X, and the 4 supercharges are written in terms of the $\bar\partial$ operator and its conjugate and adjoint. The supersymmetry group $P^{1|4}$ and its R-symmetry group $SU(2)$ are represented on the complexified forms. In particular, the action of the R-symmetry group incorporates the Kähler identities. Notice that the group of R-symmetries is the double cover of the group of spatial rotations in $P^{4|4}$ acting by outer automorphisms on the subgroup $P^{1|4}$. Thus we have a geometric explanation of the Kähler identites using supersymmetry.

To this point we have not considered the dimensional reduction of a supersymmetric *gauge* theory. We explained earlier that in the bosonic case the reduced theory has scalar fields coming from spatial components of the connection in higher dimensions. So in a supersymmetric gauge theory we learn that a vector multiplet with extended supersymmetry contains scalar fields. Since supersymmetry links these scalar fields to a connection, we expect that at least in some cases the geometry of these scalars in an effective theory is constrained in a different manner than that of scalar fields which are in scalar multiplets and not linked to connections. The geometry of the scalars in a vector multiplet with extended supersymmetry goes by the generic name *special geometry*. We mention in passing that scalar fields in theories with *local* supersymmetry (supergravity) are also constrained, and the constraints lead to different geometric structures. In both local and global supersymmetry the geometry depends strongly on the dimension of spacetime and the amount and type of supersymmetry.

Supersymmetric lagrangians with both vector superfields and scalar superfields include a wide variety of terms. The few examples we discuss serve only as an introduction, not as an exhaustive catalog.

As a first example, consider pure supersymmetric gauge theory on $M^{3|2}$ reduced to $M^{2|(1,1)}$. From the component description (4.???) we see that the reduced vector multiplet has a real scalar field, a spinor field of each chirality, and a connection. Since there is a superspacetime description in $n = 3$ dimensions, we obtain such a description in $n = 2$ dimensions. The field is a connection $\mathcal{A}$ on $M^{2|(1,1)}$, but the constraint is no longer simply (4.38)—we allow one component of the curvature along the odd distribution to be nonzero. Namely, in the superspacetime $M^{2|(1,1)}$ we use a framing by real left invariant vector fields—two odd vector fields D_+, D_- and two even vector fields ∂_+, ∂_-—with brackets (see (3.13)):

$$\begin{aligned} [D_+, D_+] &= -2\partial_+ \\ [D_-, D_-] &= -2\partial_- \\ [D_+, D_-] &= 0, \end{aligned} \tag{5.24}$$

We still demand zero curvature in the D_+, D_+ and D_-, D_- directions, but allow a nonzero bracket for the covariant derivatives in the D_+, D_- directions:

$$[\mathcal{D}_+, \mathcal{D}_-] = 2\Sigma \tag{5.25}$$

for some adjoint-valued scalar superfield Σ. The restriction of Σ to $\check{M}^2$ is the adjoint-valued scalar field of the vector multiplet. The Bianchi identities allow us to write all components of the curvature in terms of Σ. We find that the reduction of the superfield lagrangian (4.???) to $M^{2|(1,1)}$ is

$$\mathcal{L} = \frac{1}{2}\langle \mathcal{D}_+\Sigma, \mathcal{D}_-\Sigma\rangle \, |d^2x| \, d^2\theta.$$

This is precisely the σ-model lagrangian for a scalar superfield Σ; it is the dimensional reduction of (4.19) to $M^{2|(1,1)}$. Where, then, is the connection A in the component formulation? The auxiliary component field F of the superfield Σ (see (4.16)) is (up to a sign) the Hodge star of the curvature of A. The bosonic lagrangian (4.23) reduces to the correct lagrangian in $\check{M}^2$ for the adjoint-valued scalar field.

Note the following:

- At least formally, the scalar superfield Σ is related to the central extension of the supersymmetry algebra: Compare (3.13) with (5.24) and (5.25).
- This example illustrates a general feature of vector superfields in superspacetimes with more than the minimal amount of supersymmetry: The curvature constraint allows nonzero components of the curvature along the odd distribution.

An effective lagrangian for a scalar superfield Σ on $M^{2|(1,1)}$ which is at most quadratic in first derivatives can be written for a general metric on the manifold X where Σ takes its values. But if we write such an effective lagrangian for the vector superfield on $M^{2|(1,1)}$ described here, then we cannot allow a general G-invariant metric on the Lie algebra $X = \mathfrak{g}$ where Σ takes its values. Otherwise, the term in the component lagrangian with the auxiliary field, which here is the kinetic term for the connection A, might not be quadratic in the curvature F_A. Rather, the most general metric we can allow on $\mathfrak{g}$ is a metric conformally equivalent to the constant metric. We can include a potential term as well. So the most general effective superspacetime lagrangian to the order in the derivative expansion we are considering is

$$\mathcal{L}_{\mathrm{eff}} = \left\{ \frac{1}{2g^2(\Sigma)} \langle \mathcal{D}_+\Sigma, \mathcal{D}_-\Sigma\rangle + \Sigma^*(h) \right\} |d^2x|\, d^2\theta,$$

where $g^2 \colon \mathfrak{g} \to \mathbb{R}^{>0}$ and $h \colon \mathfrak{g} \to \mathbb{R}$ are arbitrary functions. Notice that the quadratic function $h = \langle \Sigma, \Sigma\rangle$ gives in the component lagrangian a term proportional to (5.3). So (5.3) does appear in supersymmetric lagrangians.

In $n = 2$ dimensions a gauge field has no physical degrees of freedom, and so we may as well write an effective lagrangian simply in terms of the scalar superfield Σ, forgetting the vector superfield $\mathcal{A}$. Then we can have as target an arbitrary Riemannian manifold X and an arbitrary potential function. This simple example shows that the identification of the correct physical degrees of freedom dramatically affects the set of possible effective lagrangians. In one case the manifold of scalars is the Lie algebra $\mathfrak{g}$ with a conformally trivial metric; in the other it is an arbitrary Riemannian manifold.

It is instructive to consider the effective lagrangian (to the order we are considering) for a model with a vector superfield $\mathcal{A}$ and a scalar superfield Φ with values in a manifold Y. Then the manifold of scalars is $X = Y \times \mathfrak{g}$, and the Riemannian metric is constrained to be of the form

$$ds^2_X(\phi, \sigma) = ds^2_Y(\phi) + \frac{1}{g^2(\phi,\sigma)} \langle \cdot, \cdot \rangle.$$

Next, we ask a general question: Are there nontrivial connections on a superspacetime $M^{n|s}$ invariant under the supersymmetry group $P^{n|s}$? The answer to the analogous question on $\check{M}^n$

is no—by Lorentz invariance all components of the curvature vanish. This Lorentz invariance argument also shows that for a $P^{n|s}$-invariant connection on $M^{n|s}$ the even-even and even-odd components of the curvature vanish, but it says nothing about the odd-odd components. The translational invariance says that the odd-odd components are covariant constant. Hence a $P^{n|s}$-invariant connection has odd-odd curvature $\{\mathcal{D}_a, \mathcal{D}_b\} = 2\Sigma_{\alpha\beta}$ for $\{\Sigma_{\alpha\beta}\}$ a collection of *commuting* elements of the gauge algebra. (The covariant constancy implies that the $\Sigma_{\alpha\beta}$ pairwise commute.)

In supersymmetric theories we use *constrained* connections. With minimal supersymmetry ($M^{2|(1,0)}$, $M^{3|2}$, $M^{4|4}$, $M^{6|(8,0)}$) the constraint says that all odd-odd components of the curvature vanish, and so every superPoincaré invariant constrained connection is trivial. But as we have seen, with extended supersymmetry some odd-odd components can be nonzero and in that case there are nontrivial superPoincaré invariant connections. If we treat such a connection as fixed and couple a variable scalar superfield to it, then we can write supersymmetric terms for the scalar superfield which end up involving additional masses related to the $\Sigma_{\alpha\beta}$. Such terms are not dimensional reductions.

As a concrete example, consider a theory on the superspacetime $M^{3|4}$. We use the odd left-invariant vector fields (5.8), only now we have $\partial_{1\dot{2}} = \partial_{2\dot{1}}$. A constrained connection has all odd-odd components of curvature vanishing except

$$\operatorname{Im}\,[\mathcal{D}_1, \mathcal{D}_{\dot{2}}] = 2\Sigma$$

for some *real* adjoint-valued scalar superfield Σ. The Bianchi identity gives constraints on Σ. A connection with Σ constant and all other components of curvature vanishing is superPoincaré invariant. Now let a chiral superfield Φ take values in an associated hermitian vector bundle. Then the supersymmetric kinetic term (see (5.12))

$$\mathcal{L} = \frac{1}{4}\|\Phi\|^2 \; |d^3x|\, d^4\theta$$

gives in components

$$L = \|d\phi\|^2 + \langle\bar{\psi}\not{D}\psi\rangle + \|\sigma\cdot\phi\|^2 + \langle\bar{\psi}, \sigma\cdot\psi\rangle,$$

where σ is the constant value of Σ. So we have *real* mass terms. (Note that complex mass terms for Φ are easily obtained from a superpotential (5.16).) Such terms are not dimensional reductions from 4 dimensions, nor can they be written in superspacetime just in terms of Σ, Φ (without the background connection).

This construction works in other dimensions and always gives mass terms. However, the $M^{3|4}$ example in the previous paragraph is the only one I know where there is no alternative simpler superspacetime expression.

Finally, consider the reduction of the supersymmetric gauge theory on $M^{4|4}$ reduced to $M^{2|(2,2)}$. Now we obtain two real adjoint-valued scalars, or equivalently one complex adjoint-valued scalar. These have a superfield description analogous to (5.25). Namely, in the superspace $M^{2|(2,2)}$ we use a framing by complex vector fields $D_+, \overline{D}_+, D_-, \overline{D}_-, \partial_+, \partial_-$, and then define the $\mathfrak{g}_{\mathbb{C}}$-valued superfield Σ by

$$[\mathcal{D}_-, \overline{\mathcal{D}}_+] = \sqrt{2}\,\Sigma.$$

The Bianchi identities imply that Σ is a *twisted chiral superfield*: $\overline{\mathcal{D}}_+\Sigma = \mathcal{D}_-\Sigma = 0$. It turns out that the reduced lagrangian is an integral over $d^4\theta$ of a constant times the norm square of Σ. Now the complex auxiliary field of Σ includes both the Hodge star of the curvature of the reduced connection A and a real auxiliary field. When we solve for the auxiliary field we obtain the potential energy

$$V = \frac{1}{2}\|\,[\sigma, \bar{\sigma}]\,\|^2$$

as expected from (5.6). As before, we have two different stories for a possible effective lagrangian depending on whether we write it in terms of an arbitrary twisted chiral superfield or a vector superfield. (In the former case, we may as well take the field to be chiral and then the target is an arbitrary Kähler manifold with superpotential.)

There are more interesting possibilities in effective theories which also include a chiral superfield.

In these examples we do not meet any nontrivial special geometry for the scalars in the vector multiplet. For that we need models with more supersymmetry.

Theories with $s = 8$ supersymmetries

Here we simply survey the results and do not give many details. Recall from Table 5 that for the σ-model there is no known superspacetime model in general, so we cannot present the theory in a manifestly supersymmetric way. This means that the model in components has no auxiliary fields and that the supersymmetry algebra does not close off-shell. For the gauge theory there is a superspacetime model, but we do not formulate that here. The model in components has 3 real adjoint-valued auxiliary fields.

We begin our discussion in the maximal dimension $n = 6$ where $s = (8, 0)$ supersymmetry is the minimal amount. Consider briefly the representations of $P^{6|(8,0)}$. The massless little group is Spin(4) and the Clifford algebra we must represent has 4 generators, so the minimal Clifford module is $W = W^0 \oplus W^1$ with $\dim W^0 = \dim W^1 = 2$. All representations of Spin(4) are real or quaternionic, so CPT imposes no constraint. The minimal solution has W^0 consisting of 2 scalars and W^1 consisting of a chiral spinor. This multiplet is called a *half-hypermultiplet*. It is the first example we meet of a representation of a superPoincaré group which has no realization in terms of fields. The *hypermultiplet* is, not surprisingly, two half-hypermultiplets; it contains 4 scalars. It has a realization as a multiplet of fields which contains a quaternionic-valued scalar field. The little group representation for the *massless vector multiplet* has even part the 4 dimensional spin 1 representation of Spin(4) and odd part two copies of a chiral spin representation. A massive representation has little group Spin(5) $\cong Sp(2)$ and the restriction of any spin representation to Spin(4) $\cong Sp(1)\times Sp(1)$ contains spinors of both chirality. It follows that the hypermultiplet, whose little group representation consists of chiral spinors, does not deform to a massive representation.

In the field representation, the *linear* hypermultiplet has a quaternionic-valued scalar field, as mentioned above. Just as there is a constraint on the target of a *nonlinear* σ-model with 4 supersymmetries—it must be Kähler—here we find that the target of a σ-model with 8 supersymmetries must be *hyperkähler*. In $n = 6$ dimensions there is no supersymmetric potential term possible. This is consistent with the representation theory discussed above where we found that no mass term is possible. There is a supersymmetric gauge theory, as we have mentioned several times. The vector multiplet consists of a connection, two chiral spinor fields with values in the adjoint bundle, and three real auxiliary fields with values in the adjoint bundle. To write the coupled model we need:

G	compact Lie group
$\langle\cdot,\cdot\rangle$	bi-invariant scalar product on G
X	hyperkähler manifold on which G acts
$\mu_{\mathbb{H}}\colon X \longrightarrow \mathfrak{g}\otimes \operatorname{Im}\mathbb{H}$	hyperkähler moment map

The appearance of the moment map is not surprising given (5.21). The bosonic lagrangian is then

$$L_{\text{bos}} = -\frac{1}{2}|F_A|^2 + \frac{1}{2}|d_A\phi|^2 - \phi^*\big(2|\mu_{\mathbb{H}}|^2\big),$$

with nonnegative potential energy

$$V = 2|\mu_{\mathbb{H}}|^2. \tag{5.26}$$

The moduli space of classical vacuua is the *hyperkähler quotient*

$$\mathcal{M}_{\text{vac}} = \mu_{\mathbb{H}}^{-1}(0)\,/\,G.$$

The ambiguity in choosing the moment map $\mu_{\mathbb{H}}$ leads to Fayet-Iliopoulos terms as in theories with 4 supersymmetries.

We could consider the dimensional reduction in all dimensions ≤ 5, but for lack of time, space, and energy we consider only the reduction to $n = 4$ dimensions. These are theories invariant under $P^{4|8}$, the so-called $N = 2$ supersymmetry group in $n = 4$ dimensions. In this case a supersymmetric potential function does exist, and it is specified by a vector field ζ on the hyperkähler manifold X which is an infinitesimal automorphism of the hyperkähler structure. In the coupled model with gauge group G, we demand that G fix ζ. Then the supersymmetric lagrangian of the general form (5.6) (with fermions added) is determined by the data:

$$\begin{array}{ll} G & \text{compact Lie group} \\ \langle\cdot,\cdot\rangle & \text{bi-invariant scalar product on } G \\ X & \text{hyperkähler manifold on which } G \text{ acts} \\ \mu_{\mathbb{H}}\colon X \longrightarrow \mathfrak{g}\otimes \operatorname{Im}\mathbb{H} & \text{hyperkähler moment map} \\ \zeta & \text{infinitesimal hyperkähler automorphism commuting with } G. \end{array} \tag{5.27}$$

Now the scalar fields in the model are

$$\begin{array}{ll} \phi & \text{values in } X \\ \sigma & \text{values in } \mathfrak{g}_{\mathbb{C}}. \end{array}$$

Of course, both scalar fields take values in a twisted bundle determined by the connection A. Both scalar fields are complex. The bosonic lagrangian has the canonical kinetic terms and the potential energy is

$$V = \frac{1}{2}\|\,[\sigma,\bar\sigma]\,\|^2 \;+\; \|\,\sqrt{2}\,\hat\sigma_\phi + \zeta_\phi\,\|^2 \;+\; 2|\mu_{\mathbb{H}}|^2. \tag{5.28}$$

Here $\hat\sigma$ is the complex vector field on X induced by the action of $\sigma \in \mathfrak{g}_{\mathbb{C}}$.

- We met the first term of V in (5.6) and the last term in (5.26). The second term is new. Without $\sqrt{2}\,\hat\sigma_\phi$ it is analogous to (5.17); the appearance of $\sqrt{2}\,\hat\sigma_\phi$ becomes clear in the formulation on $M^{4|4}$ which we consider below.
- In a fundamental lagrangian the manifold X is a quaternionic vector space and the vector field ζ is a quaternionic linear transformation.
- We discuss effective lagrangians below.
- In general, the moduli space $\mathcal{M}_{\text{vac}}$ is singular with different branches and different long-range physics (e.g. massless spectrum) on each of the three branches. To find $\mathcal{M}_{\text{vac}}$ we set each of the terms in (5.28) to zero and divide by the action of G.

It is convenient to formulate the $P^{4|8}$-invariant theories in terms of the superspace $M^{4|4}$. This makes the $P^{4|4}$ supersymmetry manifest. We use the terminology '$N = 2$' to refer to multiplets for $P^{4|8}$ and '$N = 1$' to refer to multiplets for $P^{4|4}$.

The $M^{4|4}$ superfield representation of an $N = 2$ $\mathbb{H}$-valued hypermultiplet is a pair (Φ_1, Φ_2) of $N = 1$ $\mathbb{C}$-valued chiral multiplets. Here we need to write $\mathbb{H} \cong \mathbb{C} \oplus \mathbb{C}$. Similarly, to write a hypermultiplet with values in a hyperkähler manifold X in terms of $N = 1$ chiral multiplets, we need to fix a distinguished Kähler structure on X. An $N = 2$ vector multiplet with gauge group G is represented by a pair $(\mathcal{A}, \Sigma)$ consisting of an $N = 1$ vector multiplet $\mathcal{A}$ with gauge group G and an $N = 1$ chiral multiplet Σ with values in $\mathfrak{g}_{\mathbb{C}}$. To formulate the general theory we need to modify the data (5.27) to fit the form (5.19). Thus we fix a distinguished Kähler structure, and decompose the moment map as

$$\mu_{\mathbb{H}} = (\mu_{\mathbb{R}}, \mu_{\text{hol}}).$$

Here μ_{hol} can be thought of as the moment map for the complexified $G^{\mathbb{C}}$ action, where X has the holomorphic symplectic form η given from the hyperkähler structure and the distinguished Kähler structure. The manifold of scalars is

$$\tilde{X} = X \times \mathfrak{g}_{\mathbb{C}}$$

and we have the Kähler moment map

$$\mu(\phi, \sigma) = \mu_{\mathbb{R}}(\phi) + \frac{\sqrt{-1}}{2}[\sigma, \bar{\sigma}].$$

The superpotential W we use is built out of the holomorphic moment map and the vector field ζ on X:

$$W = \mu_{\zeta} + \sqrt{2}\,\tilde{\mu}_{\text{hol}}, \tag{5.29}$$

where μ_{ζ} is the holomorphic moment map for the vector field ζ (in other words, $\iota(\zeta)\eta = -\partial W$) and

$$\mu_{\text{hol}} \colon X \longrightarrow \mathfrak{g}_{\mathbb{C}}^{*}$$

is transposed to

$$\tilde{\mu}_{\text{hol}} \colon X \times \mathfrak{g}_{\mathbb{C}} \longrightarrow \mathbb{C}.$$

The formula for the scalar potential function (5.28) then follows from (5.22).

We encounter an interesting example of special geometry when we consider the effective lagrangian for an $N = 2$ vector multiplet. Let's take the gauge group to be the abelian group $G = \mathbb{T}^d$. Then the complexified Lie algebra is $\mathfrak{g}_{\mathbb{C}} \cong \mathbb{C}^d$, and this is the manifold on which the scalars take their values. In fact, we will find that the description we give is local and only holds on an open set $U \subset \mathbb{C}^d$. The quantum moduli space of a theory is patched together from such local descriptions (and typically also has singularities). We write the effective lagrangian in terms of $N = 1$ superfields on $M^{4|4}$, as in the previous paragraph. Choosing the standard basis of $\mathbb{C}^d$ we decompose the

$N=1$ vector multiplet $\mathcal{A}$ into components $\mathcal{A}^i$. Then the most general effective lagrangian to the order we consider which is $P^{4|4}$ invariant was already written in (5.20) and (5.23). It is:

$$\mathcal{L}_{\text{eff}} = \Big\{\frac{1}{2}K(\Sigma,\overline{\Sigma})\,d^4\theta \;-\; \operatorname{Im}\big(\frac{\tau_{ij}(\Sigma)}{16\pi}\epsilon^{\alpha\beta}\langle W^i_\alpha, W^j_\beta\rangle\,d^2\theta\big)\Big\}\;|d^4x|.$$

Here K is a Kähler potential on U and τ is a holomorphic function from U to the upper half plane.

- K is constrained by the fact that the Kähler form is positive definite, and τ is constrained by the fact that it takes values in the upper half plane.
- We used the local form (5.15) for the first term.
- The most general $P^{4|4}$-invariant lagrangian includes a superpotential, but from (5.29) we see that there is no superpotential in a $P^{4|8}$-invariant lagrangian.

When we impose $N=2$ supersymmetry (or $P^{4|8}$-invariance) we find that K and τ are coupled; there is a *holomorphic prepotential*

$$\mathcal{F}\colon U\longrightarrow\mathbb{C}$$

such that

$$K=\frac{1}{2}\operatorname{Im}\big(\frac{\partial\mathcal{F}}{\partial z^i}\bar z^i\big)$$

and

$$\tau_{ij}=\frac{\partial^2\mathcal{F}}{\partial z^i\partial z^j}$$

in standard coordinates on $U\subset\mathbb{C}^d$. These are the defining equations of *special Kähler geometry*. The most common examples of special Kähler manifolds occur as the base space of an algebraic integrable system. Seiberg and Witten computed the low energy effective lagrangian of the model with $G=SU(2)$ (with and without hypermultiplets in the fundamental representation and adjoint representation). Subsequent work treated more general gauge groups and matter representations, and the connection with integrable systems emerged.

Finally, we mention that if we reduce the σ-model to $n=1$ dimensions, then in the quantum theory we obtain a representation of the supersymmetry algebra and R-symmetries on the space of complex differential forms on a hyperkähler manifold. In particular, the R-symmetry group is Spin(5)—the double cover of the group of rotations in the 5 spatial directions in $\check{M}^6$.

Theories with $s=16$ supersymmetries

We begin in the maximal dimension $n=10$ with $P^{10|(16,0)}$-invariant theories. As for the representation theory, the massless little group is Spin(8), and the relevant Clifford algebra has 8 generators. So a minimal complex Clifford module $W\cong W^0\oplus W^1$ has $\dim W^0=\dim W^1=8$. The only solution with spins ≤ 1 is to take W^0 to be the vector representation of Spin(8) and W^1 a half-spinor representation of Spin(8). This is the vector multiplet and the field content is that of minimal supersymmetric Yang-Mills theory. So we can hope that the lagrangian

$$L = \{-\frac{1}{2}|F_A|^2 + \frac{1}{2}\langle\lambda \not{D}\lambda\rangle\}\,|d^{10}x| \tag{5.30}$$

for supersymmetric *nonabelian* Yang-Mills is supersymmetric, and indeed that is the case as we illustrate below. Here λ is a *dual* spinor field with values in the adjoint bundle.

- There is no σ-model as the representation theory indicates. That is the argument in Nahm's theorem which lies behind the corresponding blank entry Table 5.
- There are also theories of supergravity in $n = 10$ dimensions (and $n = 11$ dimensions). Their field content can easily be determined by these representation theory arguments.
- The reduction of the $n = 10$ theory to any dimension $n \leq 10$ is the gauge theory with maximal supersymmetry. The $P^{4|16}$-invariant theory in $n = 4$ dimensions, called '$N = 4$ Yang-Mills', is predicted to be a conformally invariant quantum field theory which exhibits exact $SL(2,\mathbb{Z})$ electromagnetic duality. Notice that there are 6 $\mathfrak{g}$-valued scalar fields, and we can organize them into 3 $\mathfrak{g}_{\mathbb{C}}$-valued scalar fields. If we write the classical theory in $M^{4|4}$, then there is a quartet of superfields $\{\mathcal{A}, \Sigma_1, \Sigma_2, \Sigma_3\}$ consisting of a vector superfield and 3 chiral superfields. The fundamental theory has the canonical lagrangian (5.20) with superpotential
$$W = \sqrt{2}\,\langle\Sigma_1, [\Sigma_2, \Sigma_3]\rangle.$$

We now show that (5.30) is supersymmetric. It is useful to consider the problem in general, meaning in n dimensions with s supersymmetries. Consider first the abelian theory, which is free. Upon quantization, according to the discussion at the end of Lecture 3, the gauge field A has $n - 2$ physical degrees of freedom and the spinor field has $s/2$ physical degrees of freedom. With one exception the number of bosonic and fermionic degrees of freedom in a supersymmetric theory must be equal, from which $n = 2 + s/2$. This is satisfied only for $n = 3, 4, 6, 10$ with minimal supersymmetry $s = 2, 4, 8, 16$. Our analysis below applies to these cases; in all other cases the representation theory shows that (5.30) cannot be supersymmetric (in either the abelian or nonabelian case). This argument is in accord with Table 5, but there is an additional theory listed there in $n = 2$ dimensions with $s = (1, 0)$ supersymmetry. This falls under the exception referred to above. Namely, with any chiral supersymmetry $s = (s^+, 0)$ in $n = 2$ dimensions, the left-handed supersymmetries act trivially on right-handed fields (which are annihilated by ∂_+). So there is an $s = (1, 0)$ supersymmetric multiplet with a single right-handed spinor field.

Now we write (5.30) in more detail. We work in n dimensional Minkowski space and fix a real spin representation S. Thus we have pairings (3.1), (3.2) which satisfy the Clifford relation (3.3).

The spinor field λ takes values in the *dual* space S^*. The lagrangian is[1]

$$L = \big(-\frac{1}{4}g^{\mu\sigma}g^{\nu\rho}\langle F_{\mu\nu}, F_{\sigma\rho}\rangle + \frac{1}{2}\Gamma^{\sigma}_{\alpha\beta}\langle\lambda^\alpha, D_\sigma\lambda^\beta\rangle\big)\ |d^nx|, \tag{5.31}$$

where $\langle\cdot,\cdot\rangle$ is the invariant inner product on the gauge algebra, the covariant derivative is $D_\sigma = \partial_\sigma + A_\sigma$, and the curvature is $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \frac{1}{2}[A_\mu, A_\nu]$. We want a theory with supersymmetry algebra (3.10), where the bracket of the odd elements is (3.9). Extrapolating from the three dimensional case (4.40), we postulate the action of the even vector field $\eta^\alpha Q_\alpha$ on the fields to be

$$\begin{aligned}(\hat\zeta A)_\mu &= \eta^\alpha g_{\mu\nu}\Gamma^{\nu}_{\alpha\beta}\lambda^\beta \\ \hat\zeta\lambda^\alpha &= \frac{1}{2}\eta^\beta\tilde\Gamma^{\mu\alpha\gamma}\Gamma^{\nu}_{\gamma\beta}F_{\mu\nu}.\end{aligned} \tag{5.32}$$

Each side of the first equation is a (component of a) 1-form with values in the adjoint bundle.

There are two things we need to check: (i) $\hat\zeta$ is a (nonmanifest) symmetry of L; and (ii) the Lie bracket of transformations (5.32) is consistent with (3.9). We will find the conditions under which (i) and (ii) hold.

First, we investigate $\mathrm{Lie}(\hat\zeta)L$. The variation of the curvature is

$$\hat\zeta F_{\mu\nu} = \eta^\alpha g_{\nu\sigma}\Gamma^{\sigma}_{\alpha\beta}D_\mu\lambda^\beta - \eta^\alpha g_{\mu\sigma}\Gamma^{\sigma}_{\alpha\beta}D_\nu\lambda^\beta,$$

from which we easily compute

$$\hat\zeta\big(-\frac{1}{4}g^{\mu\sigma}g^{\nu\rho}\langle F_{\mu\nu}, F_{\sigma\rho}\rangle\big) = -\eta^\alpha g^{\mu\sigma}\Gamma^{\nu}_{\alpha\beta}\langle F_{\mu\nu}, D_\sigma\lambda^\beta\rangle. \tag{5.33}$$

In the variation of the second term in (5.31) we must remember to include the variation of the covariant derivative $D_\sigma = \partial_\sigma + A_\sigma$. Also, since we only want to check whether $\hat\zeta$ is a *nonmanifest* symmetry, we are free to compute up to exact terms—that is, to integrate by parts. (To compute the Noether current we would keep track of the exact terms.) We write '$\sim$' to mean 'equal up to exact terms'. Then

$$\begin{aligned}\hat\zeta\big(\frac{1}{2}\Gamma^{\sigma}_{\alpha\beta}\langle\lambda^\alpha, D_\sigma\lambda^\beta\rangle\big) &\sim \frac{1}{2}\Gamma^{\sigma}_{\alpha\beta}\langle\eta^\gamma\tilde\Gamma^{\mu\alpha\delta}\Gamma^{\nu}_{\delta\gamma}F_{\mu\nu}, D_\sigma\lambda^\beta\rangle + \frac{1}{2}\Gamma^{\sigma}_{\alpha\beta}\langle\lambda^\alpha, \eta^\gamma g_{\sigma\nu}\Gamma^{\nu}_{\gamma\delta}[\lambda^\delta,\lambda^\beta]\rangle \\ &\sim \frac{1}{2}\eta^\alpha\Gamma^{\sigma}_{\beta\delta}\tilde\Gamma^{\mu\gamma\delta}\Gamma^{\nu}_{\alpha\gamma}\langle F_{\mu\nu}, D_\sigma\lambda^\beta\rangle - \frac{1}{2}\eta^\gamma g_{\sigma\nu}\Gamma^{\sigma}_{\alpha\beta}\Gamma^{\nu}_{\gamma\delta}\langle\lambda^\alpha, [\lambda^\delta,\lambda^\beta]\rangle \\ &\sim \eta^\alpha g^{\mu\sigma}\Gamma^{\nu}_{\alpha\beta}\langle F_{\mu\nu}, D_\sigma\lambda^\beta\rangle - \frac{1}{2}\eta^\gamma g_{\sigma\nu}\Gamma^{\sigma}_{\alpha\beta}\Gamma^{\nu}_{\gamma\delta}\langle\lambda^\alpha, [\lambda^\delta,\lambda^\beta]\rangle.\end{aligned}$$

[1] Unfortunately, in these notes we have used Greek indices $(\alpha,\beta,\gamma,\delta,\epsilon)$ for spinors and also Greek indices (μ,ν,σ,ρ) for vectors. We'll fix that confusion (and many other points as well) in the published version.

In the last step we use the Clifford relation $\Gamma^{\mu}_{\beta\delta}\tilde{\Gamma}^{\sigma\gamma\delta} + \Gamma^{\sigma}_{\beta\delta}\tilde{\Gamma}^{\mu\gamma\delta} = 2g^{\mu\sigma}\delta^{\gamma}_{\beta}$ and the Bianchi identity $D_\sigma F_{\mu\nu} + D_\nu F_{\sigma\mu} + D_\mu F_{\nu\sigma} = 0$. Now the first term in the last line cancels against (5.33), and the second term vanishes if and only if the quartic form

$$Q_{\alpha\beta\gamma\delta} = g_{\sigma\nu}\big(\Gamma^{\sigma}_{\alpha\beta}\Gamma^{\nu}_{\gamma\delta} + \Gamma^{\sigma}_{\alpha\delta}\Gamma^{\nu}_{\beta\gamma} + \Gamma^{\sigma}_{\alpha\gamma}\Gamma^{\nu}_{\delta\beta}\big)$$

vanishes. Hence:

- If the gauge algebra is abelian, then $\hat{\zeta}$ is a symmetry of L in any dimension and with any spin representation S.
- If the gauge algebra is nonabelian, then $\hat{\zeta}$ is a symmetry of L if and only if the quartic form

$$s^* \longmapsto |\,\Gamma(s^*, s^*)\,|^2, \qquad s^* \in S^*, \tag{5.34}$$

  vanishes identically.

We analyze this last condition below.

Next, we investigate the Lie algebra generated by the symmetry transformation (5.32). Let $\eta_1^\alpha, \eta_2^\beta$ be odd parameters and $\hat{\zeta}_i$ the action (5.32) of $\eta_i^\alpha Q_\alpha$. In the abstract Lie algebra (3.9) we have

$$[\eta_1^\alpha Q_\alpha, \eta_2^\beta Q_\beta] = -\eta_1^\alpha\eta_2^\beta[Q_\alpha, Q_\beta] = 2\eta_1^\alpha\eta_2^\beta\Gamma^{\sigma}_{\alpha\beta}\partial_\sigma.$$

When acting on the spinor field λ, we replace ∂_σ with the covariant derivative D_σ. When acting on the connection A, we use Lie derivative by the horizontal lift, and it is easy to compute that the action of ∂_σ on A has μ^{th} component equal to $F_{\sigma\mu}$. Recall from Lecture 2 that since we use *left* actions of symmetries, the bracket on the space of fields is opposite to the bracket in the abstract supersymmetry algebra. But recall too that the element ∂_σ in the abstract supersymmetry algebra acts by *minus* the derivative on fields. These two minus signs cancel in the sense that we expect

$$[\hat{\zeta}_1, \hat{\zeta}_2]f = 2\eta_1^\alpha\eta_2^\beta\Gamma^{\sigma}_{\alpha\beta}\partial_\sigma f \tag{5.35}$$

for any field f.

We first verify (5.35) for the connection field. Using (5.32) we find:

$$\begin{aligned}\big([\hat{\zeta}_1, \hat{\zeta}_2]A\big)_\mu &= -\frac{1}{2}\eta_1^\alpha\eta_2^\beta g_{\mu\nu}(\tilde{\Gamma}^{\rho\gamma\delta}\Gamma^{\nu}_{\beta\gamma}\Gamma^{\sigma}_{\alpha\delta} + \tilde{\Gamma}^{\rho\gamma\delta}\Gamma^{\nu}_{\alpha\gamma}\Gamma^{\sigma}_{\beta\delta})F_{\rho\sigma} \\ &= -\frac{1}{2}\eta_1^\alpha\eta_2^\beta g_{\mu\nu}(4g^{\rho\nu}\Gamma^{\sigma}_{\alpha\beta})F_{\rho\sigma} \\ &= 2\eta_1^\alpha\eta_2^\beta\Gamma^{\sigma}_{\alpha\beta}F_{\sigma\mu}.\end{aligned}$$

To pass to the second line we added to the first line terms in parentheses with ρ and ν exchanged; then we used the Clifford identity (3.3). This is the desired result. To check the commutation relation (5.35) on the spinor field we need to also impose the equation of motion

$$\Gamma^{\sigma}_{\alpha\beta} D_\sigma \lambda^\alpha = 0, \qquad \text{for all } \beta. \tag{5.36}$$

(Recall Example 3 in Lecture 2 where the $U(n)$ symmetry of n coupled harmonic oscillators only "closes" on-shell.) Then we compute from (5.32)

(5.37)
$$\begin{aligned}
[\hat{\zeta}_a, \hat{\zeta}_2]\lambda^\gamma &= -\frac{1}{2}\eta_1^\alpha \eta_2^\beta g_{\mu\nu}(\tilde{\Gamma}^{\sigma\gamma\delta}\Gamma^{\nu}_{\delta\beta}\Gamma^{\mu}_{\alpha\epsilon} + \tilde{\Gamma}^{\sigma\gamma\delta}\Gamma^{\nu}_{\delta\alpha}\Gamma^{\mu}_{\beta\epsilon} - \tilde{\Gamma}^{\nu\gamma\delta}\Gamma^{\sigma}_{\delta\beta}\Gamma^{\mu}_{\alpha\epsilon} - \tilde{\Gamma}^{\nu\gamma\delta}\Gamma^{\sigma}_{\delta\alpha}\Gamma^{\mu}_{\beta\epsilon}) D_\sigma \lambda^\epsilon \\
&= \frac{1}{2}\eta_1^\alpha \eta_2^\beta g_{\mu\nu}\tilde{\Gamma}^{\sigma\gamma\delta}\Gamma^{\nu}_{\delta\epsilon}\Gamma^{\mu}_{\alpha\beta} D_\sigma\lambda^\epsilon - \frac{1}{2}\eta_1^\alpha\eta_2^\beta\tilde{\Gamma}^{\sigma\gamma\delta}Q_{\alpha\beta\delta\epsilon}D_\sigma\lambda^\epsilon \\
&\qquad + \frac{1}{2}\eta_1^\alpha \eta_2^\beta g_{\mu\nu}(\tilde{\Gamma}^{\nu\gamma\delta}\Gamma^{\sigma}_{\delta\beta}\Gamma^{\mu}_{\alpha\epsilon} + \tilde{\Gamma}^{\nu\gamma\delta}\Gamma^{\sigma}_{\delta\alpha}\Gamma^{\mu}_{\beta\epsilon}) D_\sigma\lambda^\epsilon.
\end{aligned}$$

Let 'LT' denote the last term. Applying the Clifford identity we see that the symmetric part in ν, σ vanishes by the equation of motion, whence

$$\begin{aligned}
LT &= \frac{1}{2}LT - \frac{1}{4}\eta_1^\alpha\eta_2^\beta g_{\mu\nu}(\tilde{\Gamma}^{\sigma\gamma\delta}\Gamma^{\nu}_{\delta\beta}\Gamma^{\mu}_{\alpha\epsilon} + \tilde{\Gamma}^{\sigma\gamma\delta}\Gamma^{\nu}_{\delta\alpha}\Gamma^{\mu}_{\beta\epsilon}) D_\sigma\lambda^\epsilon \\
&= \frac{1}{2}LT + \frac{1}{4}\eta_1^\alpha\eta_2^\beta g_{\mu\nu}\tilde{\Gamma}^{\sigma\gamma\delta}\Gamma^{\nu}_{\delta\epsilon}\Gamma^{\mu}_{\alpha\beta}D_\sigma\lambda^\epsilon - \frac{1}{4}\eta_1^\alpha\eta_2^\beta\tilde{\Gamma}^{\sigma\gamma\delta}Q_{\alpha\beta\delta\epsilon}D_\sigma\lambda^\epsilon.
\end{aligned}$$

Solving for LT and plugging into (5.37) we have

$$\begin{aligned}
[\hat{\zeta}_1, \hat{\zeta}_2]\lambda^\gamma &= \eta_1^\alpha\eta_2^\beta g_{\mu\nu}\tilde{\Gamma}^{\sigma\gamma\delta}\Gamma^{\nu}_{\delta\epsilon}\Gamma^{\mu}_{\alpha\beta}D_\sigma\lambda^\epsilon - \eta_1^\alpha\eta_2^\beta\tilde{\Gamma}^{\sigma\gamma\delta}Q_{\alpha\beta\delta\epsilon}D_\sigma\lambda^\epsilon \\
&= 2\eta_1^\alpha\eta_2^\beta\Gamma^{\sigma}_{\alpha\beta}D_\sigma\lambda^\gamma - \eta_1^\alpha\eta_2^\beta\tilde{\Gamma}^{\sigma\gamma\delta}Q_{\alpha\beta\delta\epsilon}D_\sigma\lambda^\epsilon.
\end{aligned}$$

Here we use the equation of motion (5.36) to see that the last term in the first line vanishes when σ and ν are exchanged. If the second term vanishes, then this is the desired bracket. Hence:

- The supersymmetry algebra is satisfied on-shell if and only if the quartic form (5.34) vanishes. This applies both to the abelian and nonabelian theories.
- Notice that the abelian theory is *always* invariant under $\hat{\zeta}$, but unless (5.34) vanishes the algebra generated by $\hat{\zeta}$ and the infinitesimal Poincaré transformations is *not* the usual supersymmetry algebra. This does not violate the Haag-Lopuszański-Sohnius theorem since the abelian theory is free, whereas that theorem gives a restriction on the symmetry algebra of theories in which the S-matrix is nondegenerate.

Finally, we investigate conditions under which the quartic form (5.34) vanishes. From our initial discussion of representation theory we know that this happens in at most the four cases $n = 3, 4, 6, 10$ with minimal supersymmetry $s = 2, 4, 8, 16$. For if (5.34) vanishes, then the abelian theory does have the usual supersymmetry algebra and so the one particle Hilbert space is a representation of that algebra. We claim that in the four cases mentioned there is in fact no nonzero invariant form of *any* degree on S^*, much less a nonzero invariant quartic form. For $n = 3, 4, 6$ we identify the spin group $\mathrm{Spin}(1, n-1)$ with $SL(2,\mathbb{F})$, $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, and we identify S^* with $\mathbb{F}^2$. Since $SL(2,\mathbb{F})$ has the open dense orbit $\mathbb{F}^2 \setminus \{0\} \subset \mathbb{F}^2$, there are no nonzero invariant polynomials on $\mathbb{F}^2$. For the remaining case $n = 10$, the 16 dimensional real half-spin representation S^* of $\mathrm{Spin}(1,9)$ restricts to the minimal real spin representation of $\mathrm{Spin}(9) \subset \mathrm{Spin}(1,9)$, and since $\mathrm{Spin}(9)$ is compact there is an invariant metric. The special fact we need is that $\mathrm{Spin}(9)$ acts transitively on the unit sphere in S^*. (This fact is part of the story concerning holonomy groups of irreducible Riemannian manifolds.) Thus the only polynomials on S^* invariant under $\mathrm{Spin}(9)$ are polynomials in the metric. But $\mathrm{Spin}(1,9)$ does not preserve the metric, so there are no nonzero invariant polynomials at all under $\mathrm{Spin}(1,9)$. Therefore, in these four cases (5.34) vanishes as claimed.

This completes the proof that (5.30) is supersymmetric.