From kinetic theory to 'jamming': Dynamical arrest due to correlations in a sheared hard-particle fluid.

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Hard particle fluids:

- Interaction potential 0 or ∞ — no intrinsic energy scale.
- Temperature just sets velocity scale.
- Zero interaction time — collisional interactions.
- Athermal — no random noise.
- Static state: Isostatic packing. (D+1) contacts per perticle (apart from rattlers).
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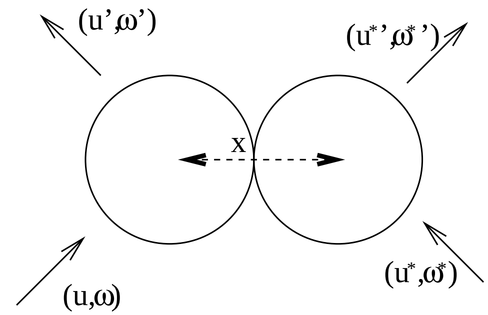
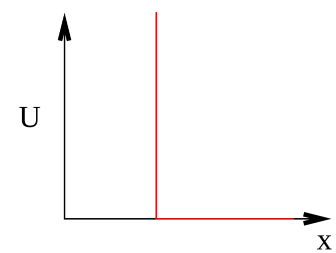
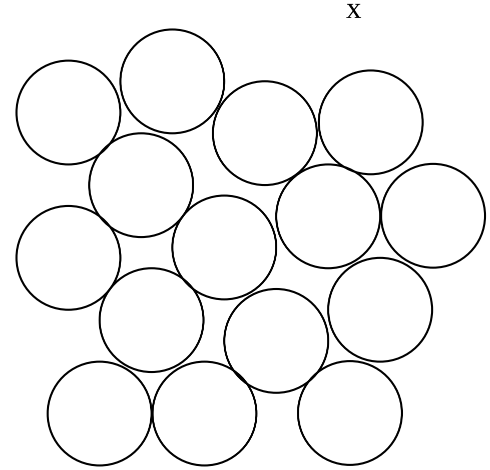
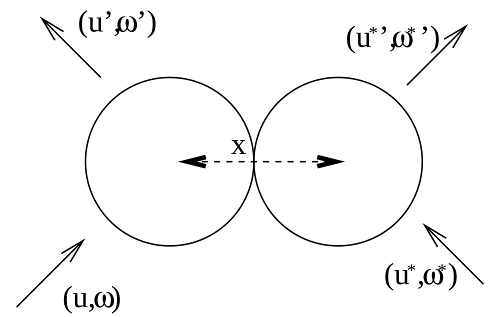


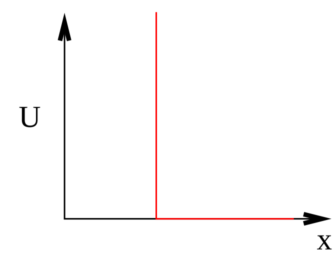


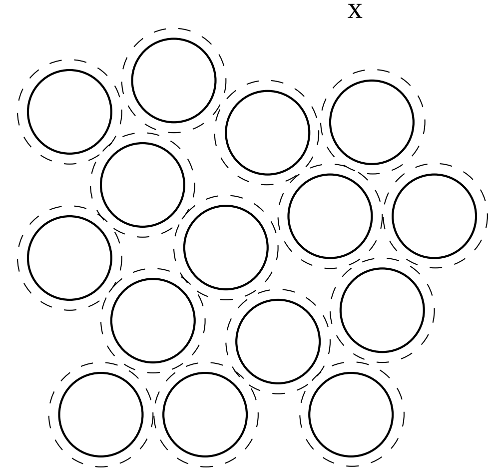


Hard particle fluids:

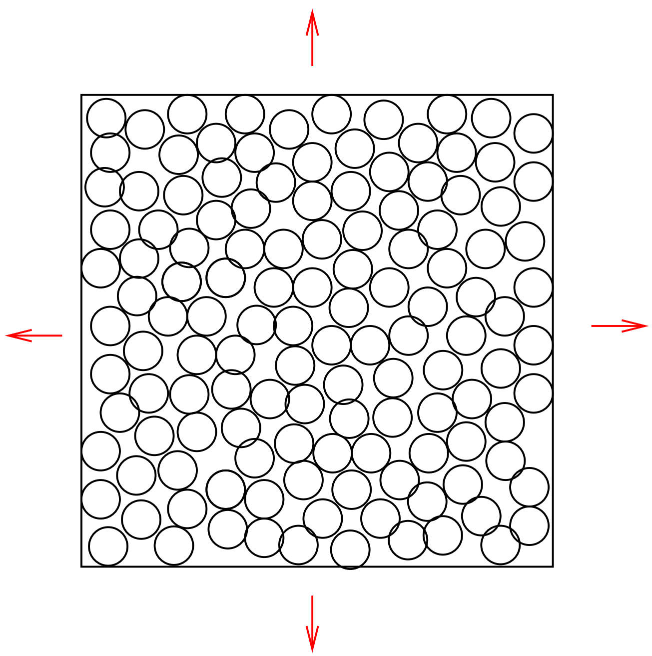
- Interaction potential 0 or ∞ — no intrinsic energy scale.
- Temperature just sets velocity scale.
- Zero interaction time — collisional interactions.
- Athermal — no random noise.
- Static state: Isostatic packing. (D+1) contacts per perticle (apart from rattlers).
- Slight decrease in volume fraction — binary contact regime.

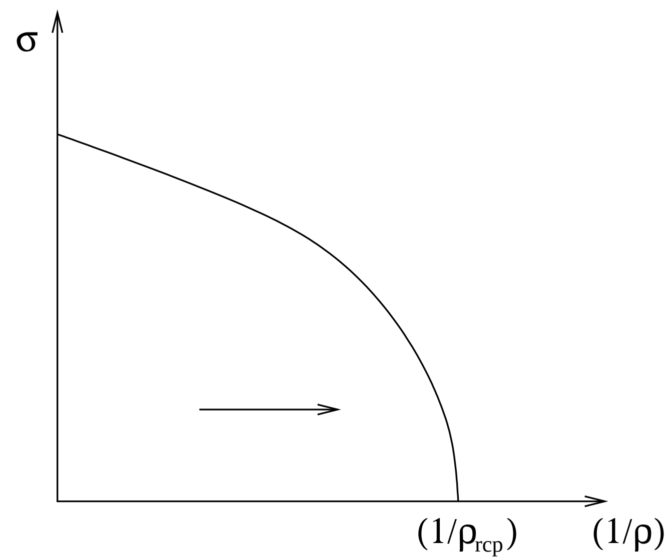


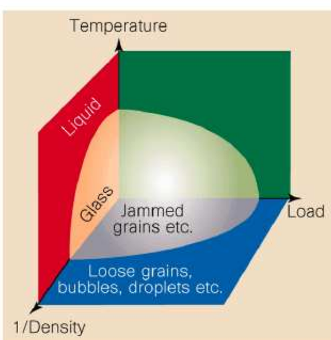


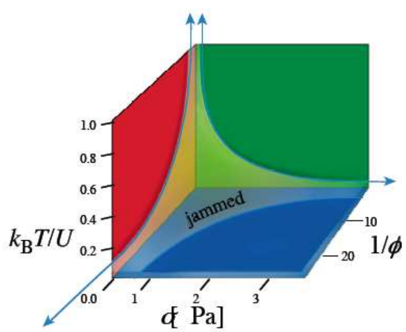


Jamming

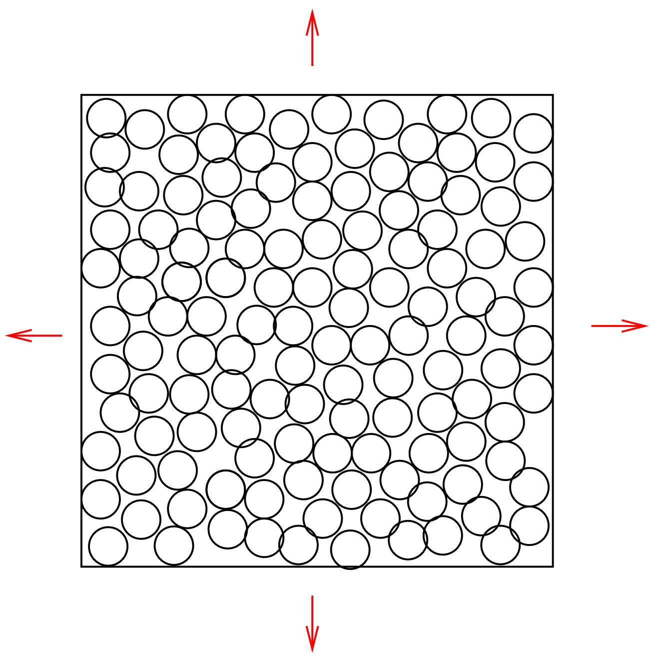


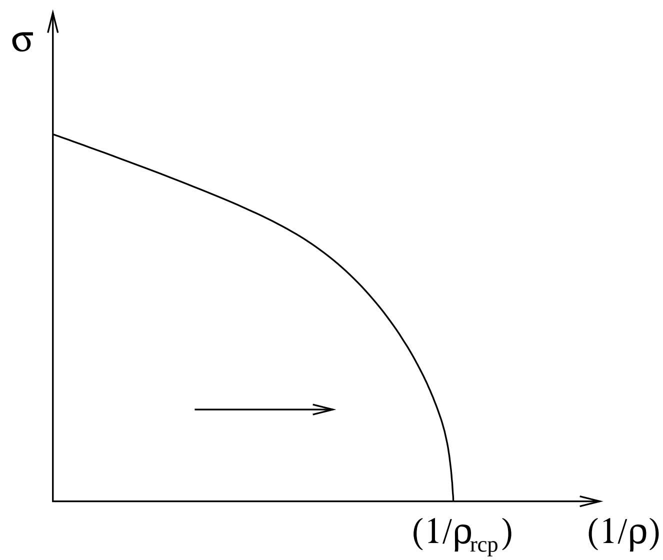

σ
$(1/\rho_{rcp})$
$(1/\rho)$

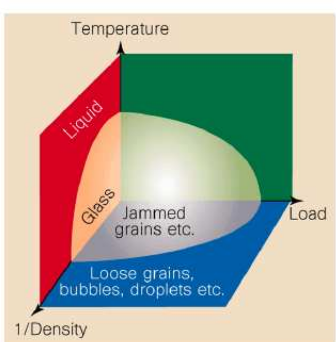

Temperature
Liquid
Glass
Jammed grains etc.
Load
Loose grains, bubbles, droplets etc.
1/Density


1.0
0.8
0.6
0.4
0.2
0.0
$k_B T/U$
jammed
10
20
$1/\phi$
1
2
3
σ [Pa]

UnJamming




σ
(1/ρrcp)
(1/ρ)


Temperature
Liquid
Glass
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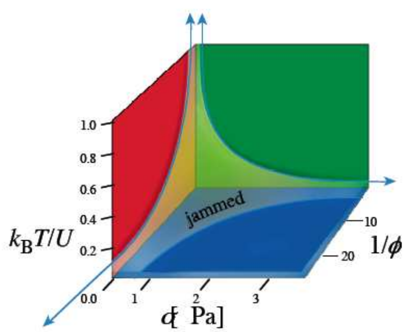
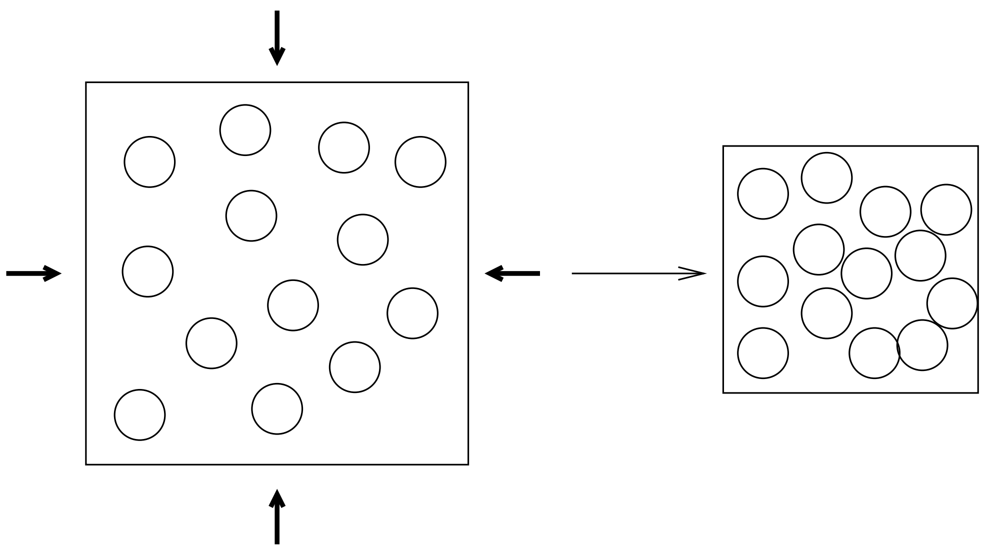
Load
Loose grains,
bubbles, droplets etc.
1/Density


1.0
0.8
0.6
0.4
0.2
0.0
kBT/U
jammed
10
20
1/ϕ
1
2
3
σ[Pa]

Jamming by compression:

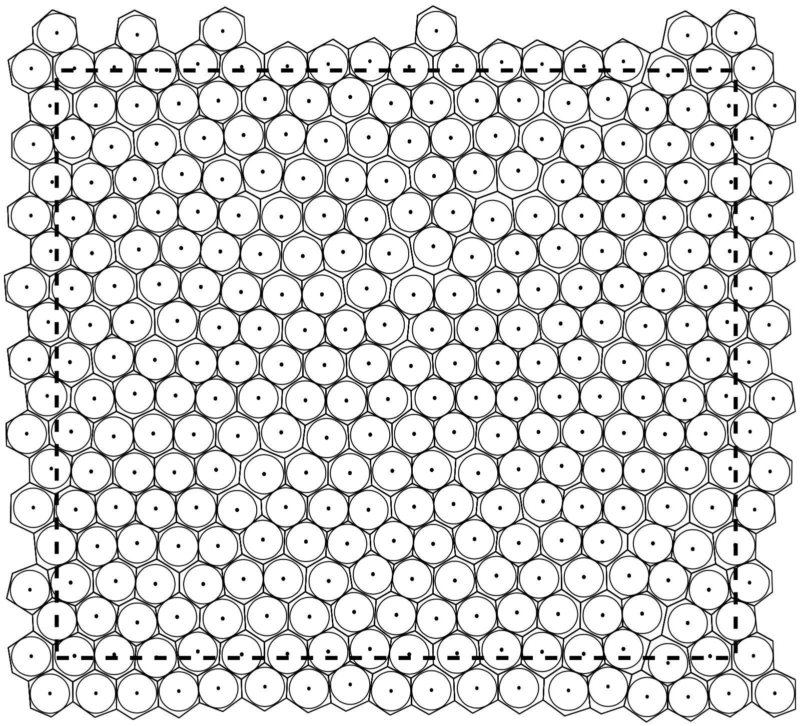
Soft particle fluids:
Quench (lower temperature) without permitting system to equilibrate to a crystal.

Hard particle fluids:
Compress (reduce volume) without permitting system to equilibrate to a crystal.
L. V. Woodcock, J. Chem. Soc., Farady Trans. 2 **72** 1667 (1976).

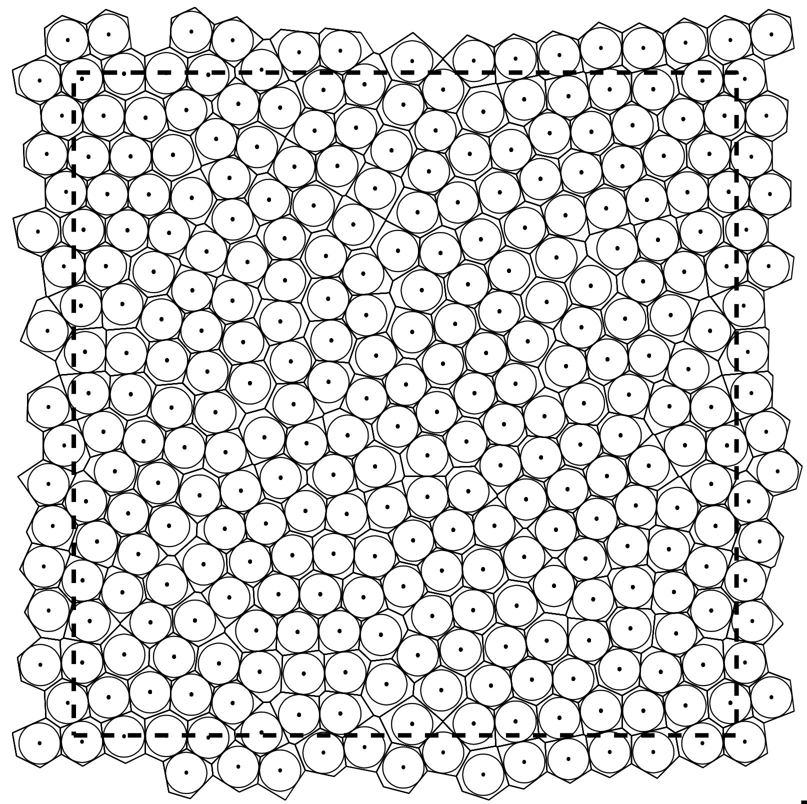


Hard disks 2D:

Ordered state $\nu = 0.78$:



Disordered state $\nu = 0.78$:

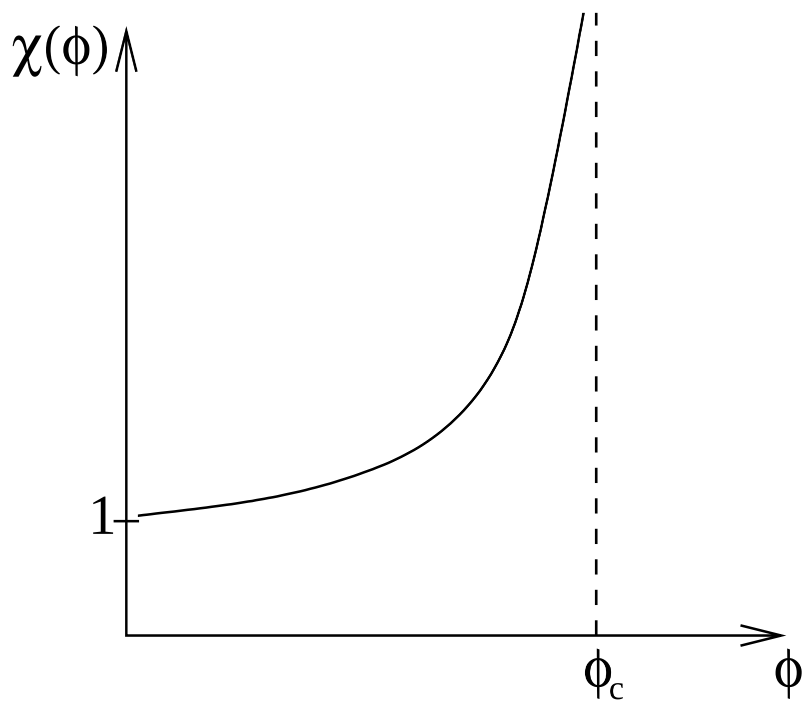


Random states in 3D:

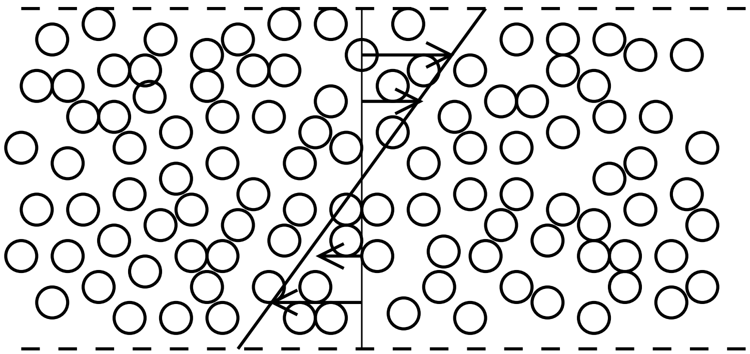
- Random close packing volume fraction $\phi_{rcp} = 0.64$.
- Empirical pair distribution function at constact (Torquato 1995).

$$\chi(\phi) = \frac{(2-\phi_f)}{2(1-\phi_f)^3}\frac{\phi_c-\phi_f}{\phi_c-\phi}$$

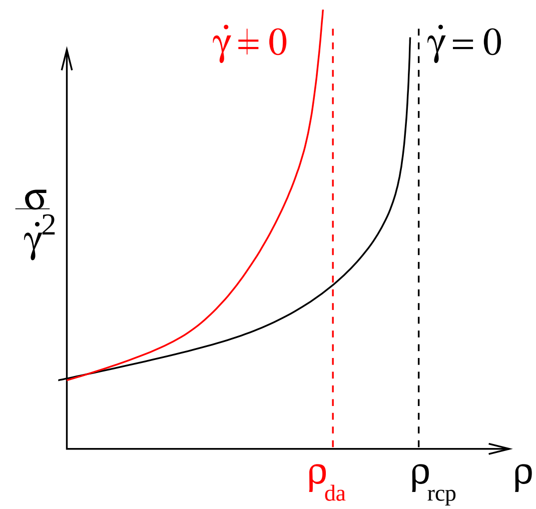
$$\chi(\phi) \sim \frac{1}{\phi_c-\phi}$$



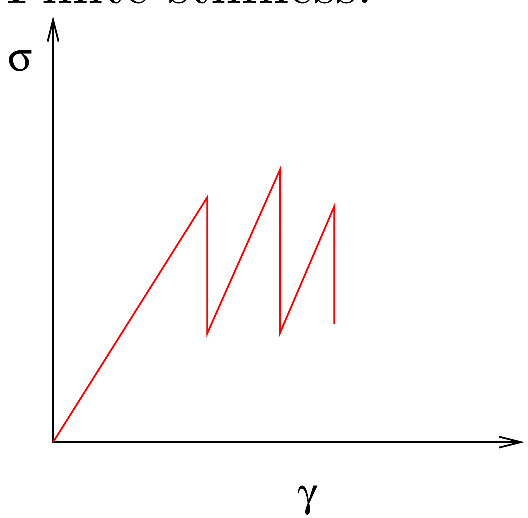
Jamming by shear: Dynamical arrest.

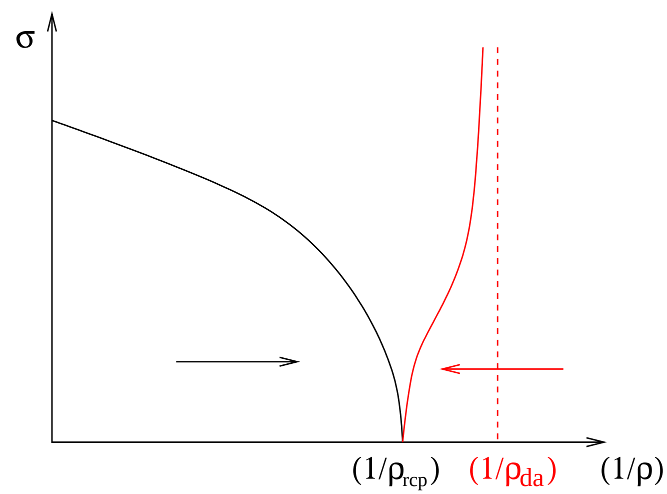


Hard particles:



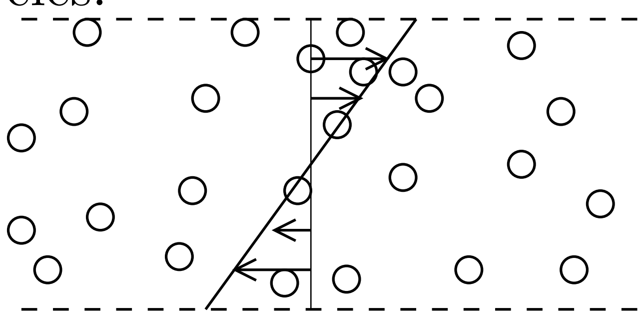
Finite stiffness:





Dynamical equivalent of Mode-coupling theories.

Sheared flow of hard particles:



$\mu \dot{\gamma}^2 = D$

T = Mean sq. fluc. vel.

$\mu \sim (T^{1/2}/d^2)$

$D \propto (\rho^2 T^{1/2} d^2)(1 - e^2) T$

Energy balance

$\dot{\gamma} d \sim \epsilon T^{1/2}$

$\epsilon = (1 - e^2)^{1/2}$

Hard-particle fluid at equilibrium:

$$P(E) = \exp(-\beta E)/Q$$

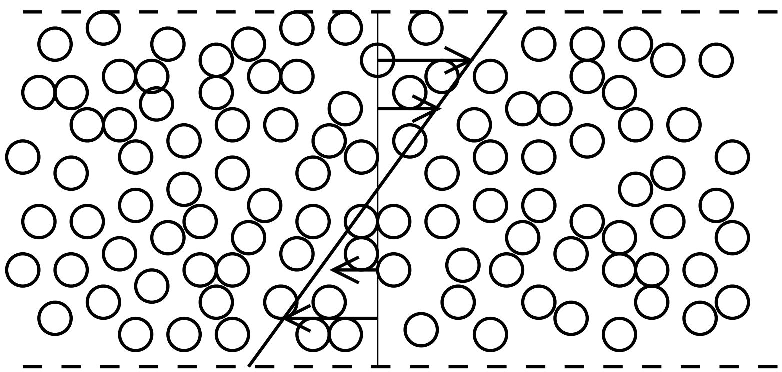
$$f(\mathbf{v}) = \left(\frac{m}{2\pi T}\right) \exp(-m v^2/2T)$$

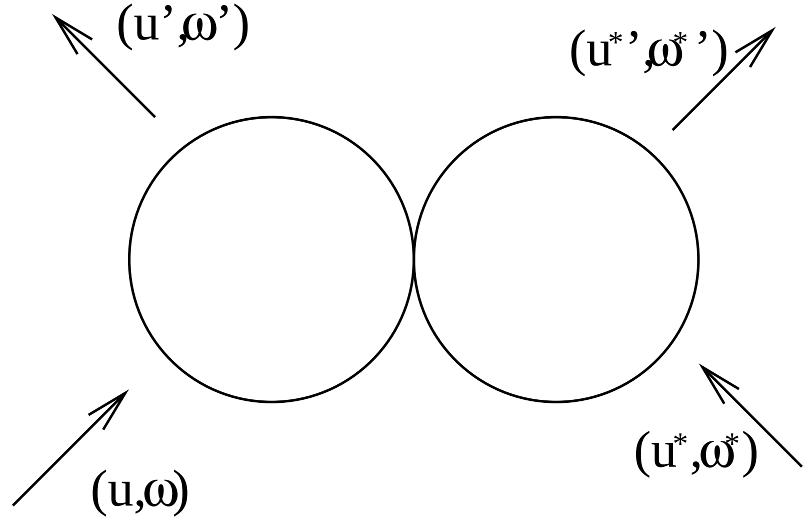
Near equilibrium:

$$\eta = \frac{\beta}{V} \lim_{k \to 0} \int_0^\infty dt \langle \sigma_{xy}(k,t) \sigma_{xy}(-k,0) \rangle$$

Structure — pair distribution function at contact $\chi(\phi)$.

Hard particles — instanteneous collisions:





Dimensional variables

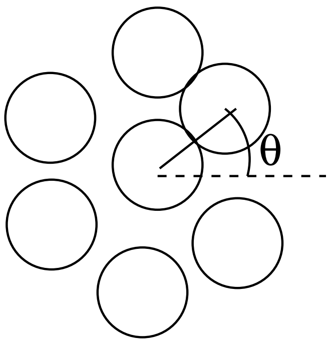
- Particle mass m,
- Particle diameter d,
- Strain rate G_{xy}.

Dimensionless variables

- Volume fraction ϕ,
- Coefficients of restitution e_n, e_t.

Constitutive relation $\sigma_{xy} = md^{-1}G_{xy}^2 F(\phi, e)$.

Structure:



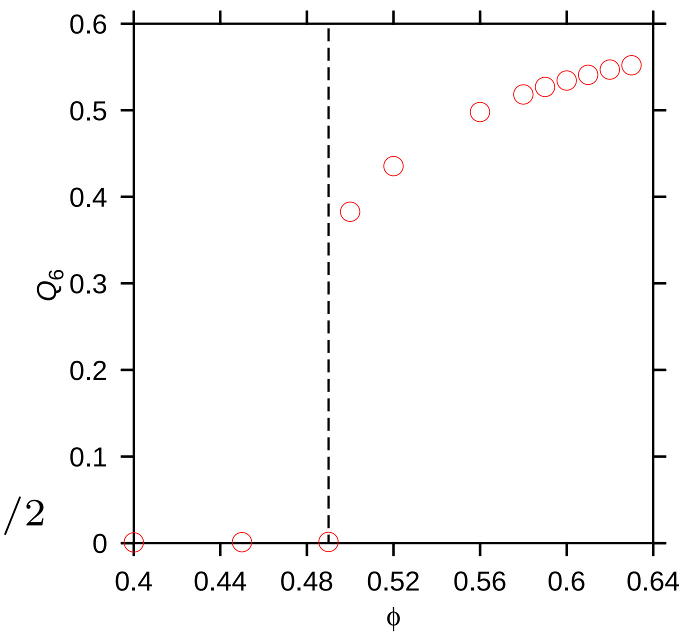
- 2D: $q_6 = \sum_{i=1}^{N} \exp(6\imath\theta)$

  $q_6 = 1$ for hexagonal packing.

- 3D Icosahedral order parameter:

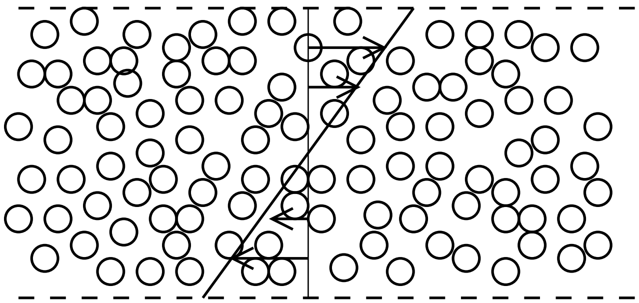
$$Q_l = \left(\frac{2l+1}{4\pi} \sum_{m=-l}^{l} |\langle Y_{lm}(\theta,\phi)\rangle|^2 \right)^{1/2}$$

$Q_6 = 0.6$ for FCC/HCP.

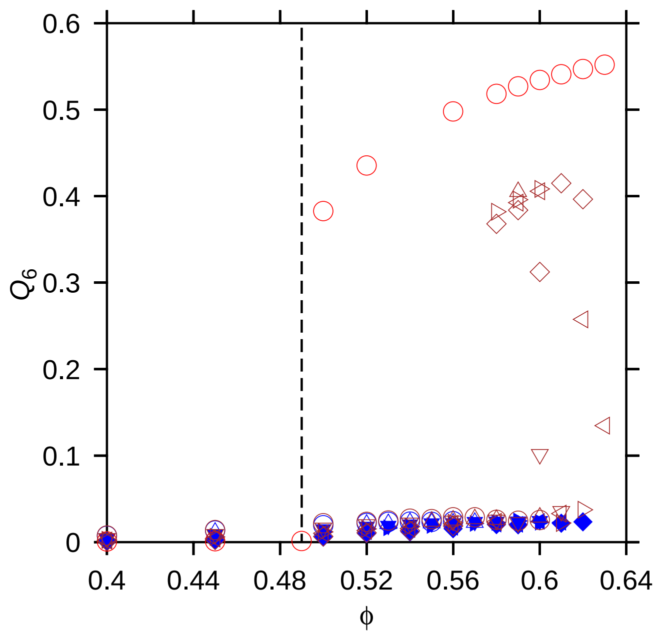


Structure:

Sheared inelastic fluid:



Balance between shear production and inelastic dissipation.



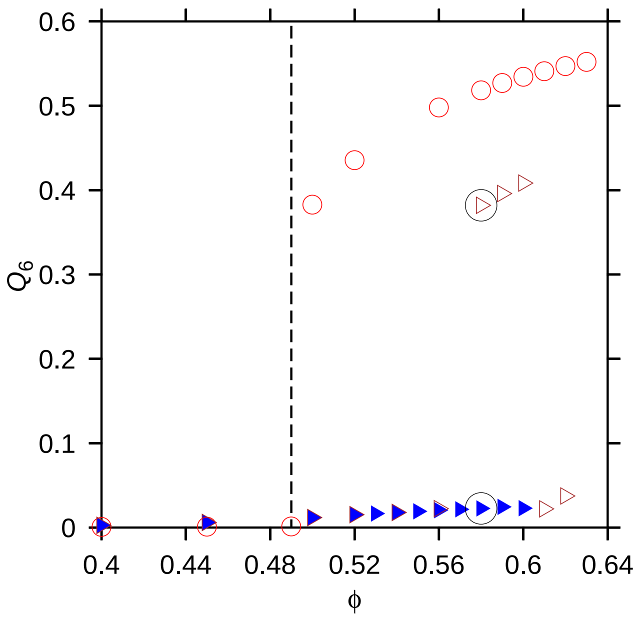
$$\mu \dot{\gamma}^2 = D$$

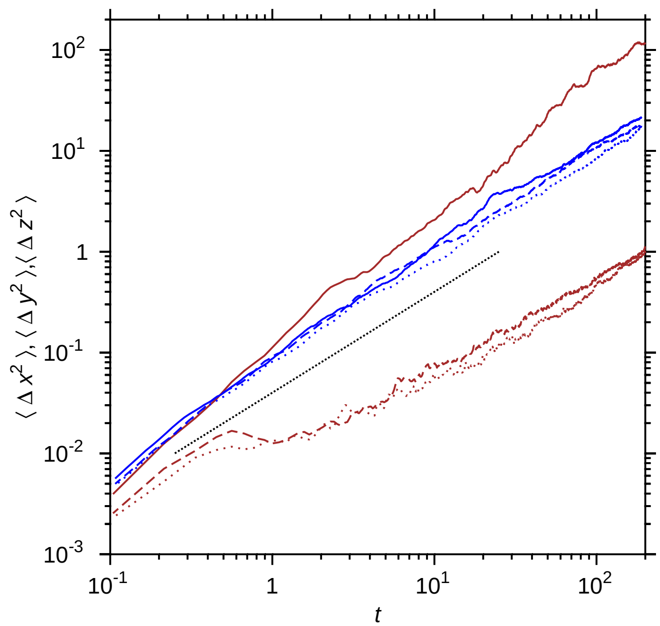
$$T = (d\dot{\gamma})^2 F(\phi, e_n, e_t)$$

∇ $e_n = 0.8$; $\triangleright$ $e_n = 0.9$; $\triangleleft$ $e_n = 0.95$; $\diamond$ $e_n = 0.98$; Box $e_n = 1.0$.

System size: $n = 256$, $n = 500$.

Diffusion:





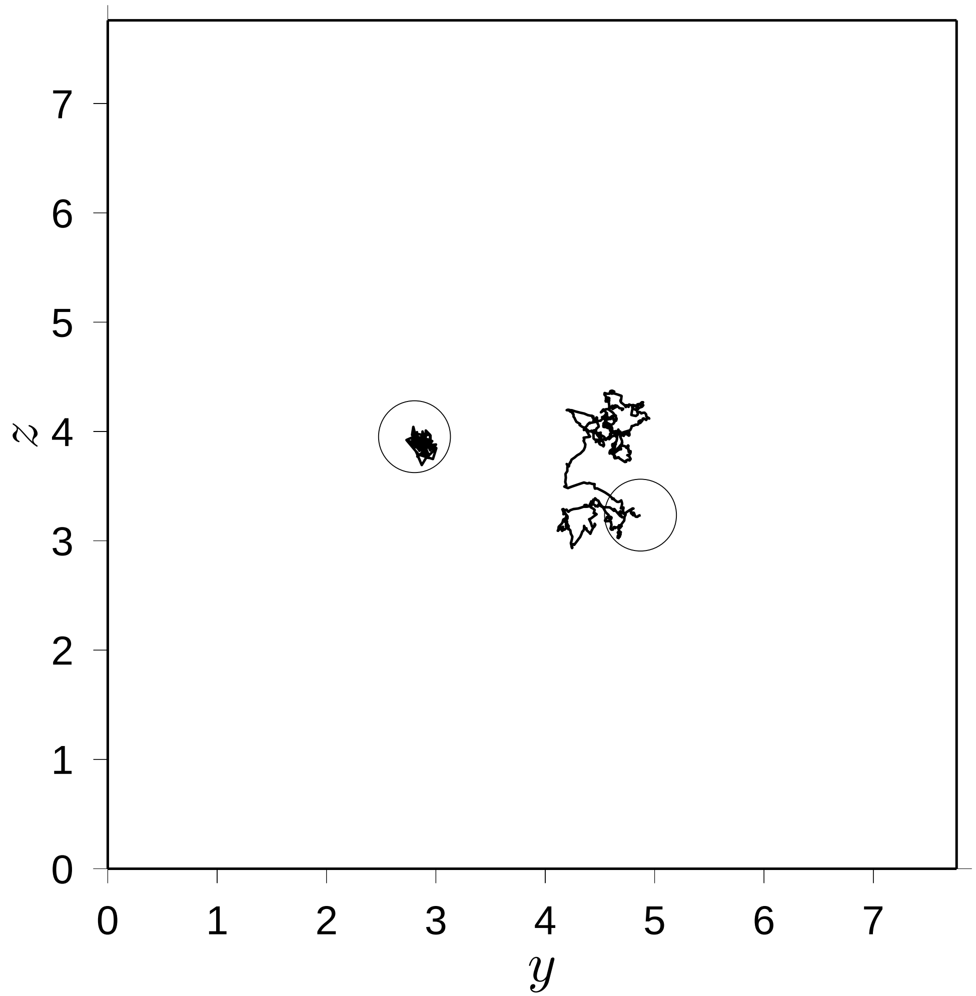
$e_n = 0.9$.

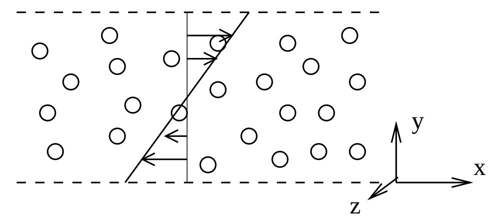
Solid — flow direction; Dashed — gradient direction; Dotted — vorticity direction. $e_n = 0.9, \phi = 0.58$

$n = 256$ $n = 500$.

Dense flowing granular material is a *gas*, not a solid/liquid.

No cage effect in shear flow:





$\phi = 0.55,\ e_n = 0.9$

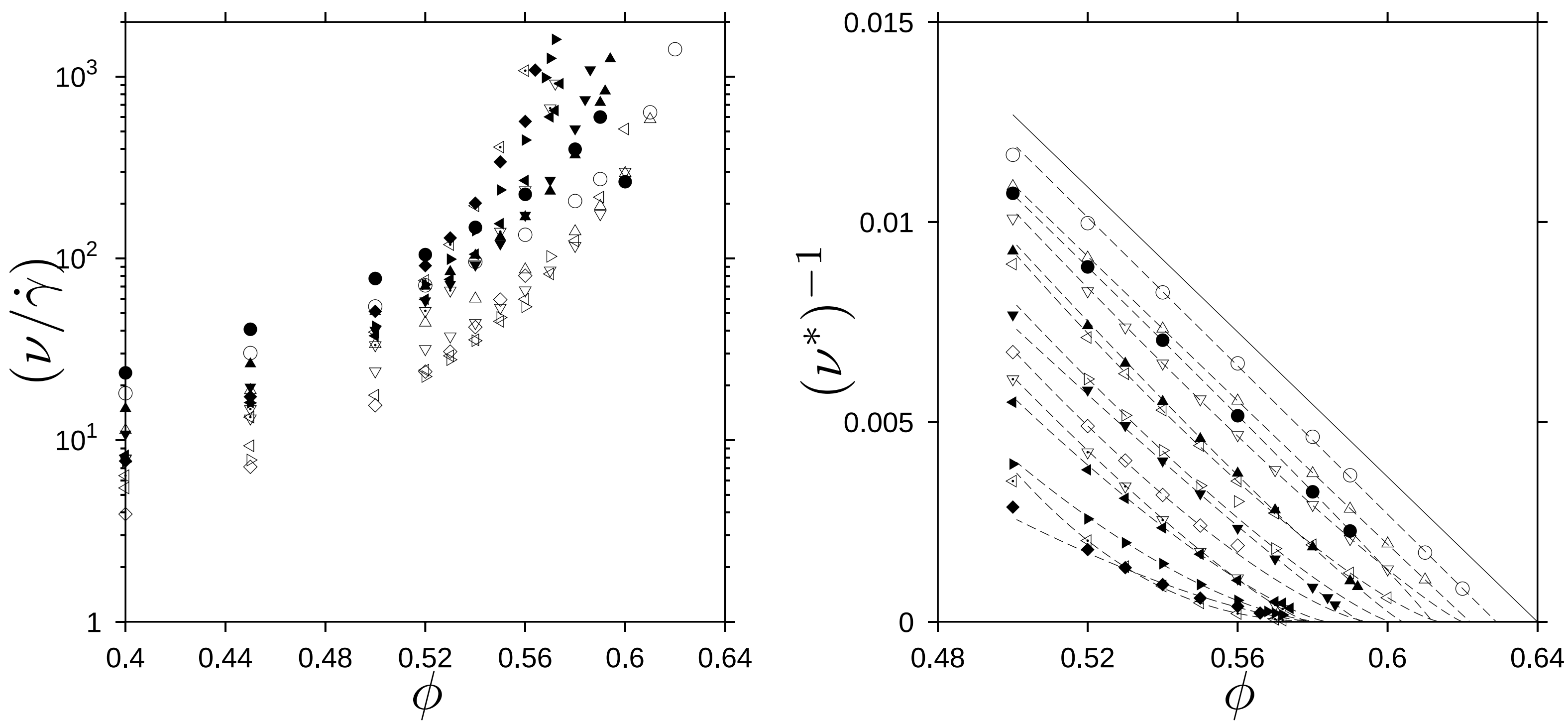
$$t = 10.6(d/T^{1/2}) = 13.2\dot{\gamma}^{-1}$$

Elastic

Sheared inelastic

Sheared steady state:

Collision freq. & stress diverge at lower volume fraction than RCP (0.64).



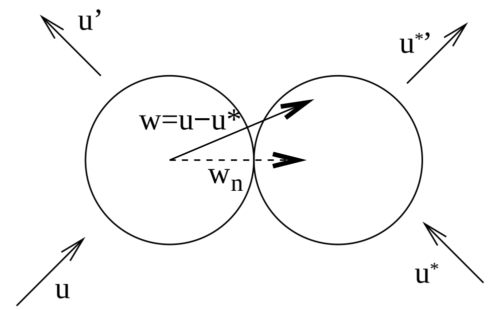
$e_n = 0.98$ ($\circ$), $e_n = 0.95$ ($\triangle$), $e_n = 0.9$ (∇), $e_n = 0.8$ ($\triangleleft$), $e_n = 0.7$ ($\triangleright$), $e_n = 0.6$ ($\diamond$),

Smooth particles (open symbols), rough particles (filled symbols).

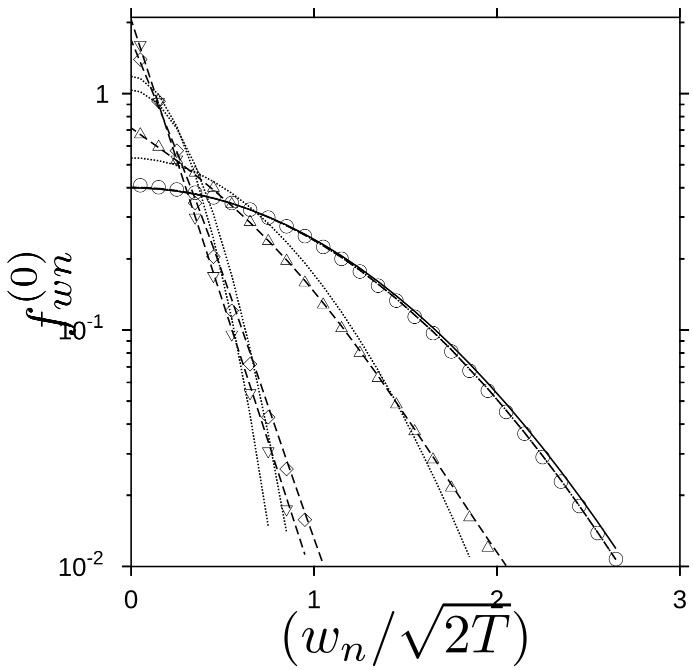
Important correlation effect

Distribution of relative velocities $w = u - u^*$

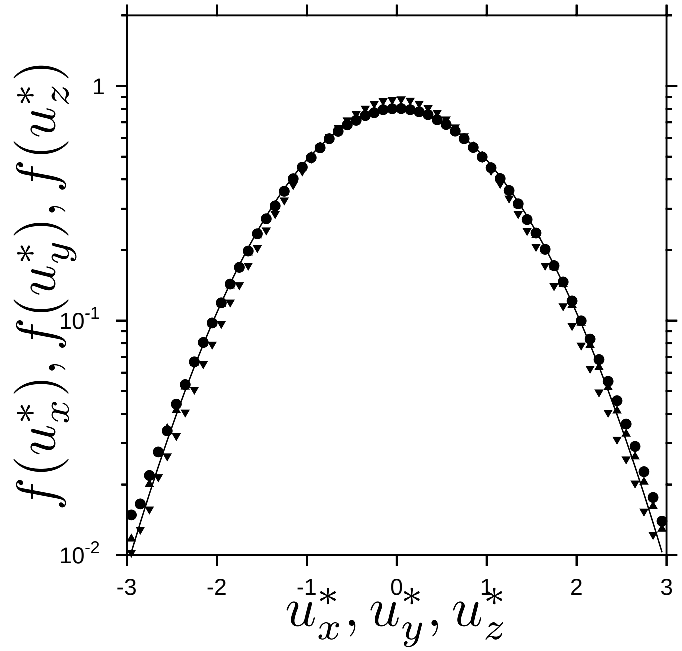
Not a Gaussian!



Relative velocity distribution:



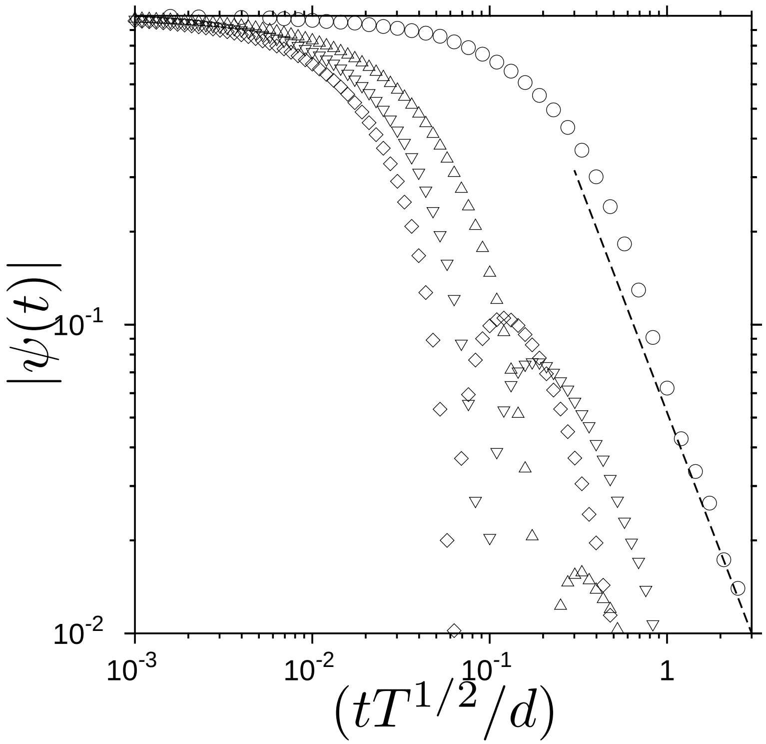
Single-particle distribution



Rough particles, $\phi = 0.56$, $e_n = 0.98$ ($\circ$), $e_n = 0.8$ ($\triangle$), $e_n = 0.6$ (Box).

Fast decay of velocity correlations:

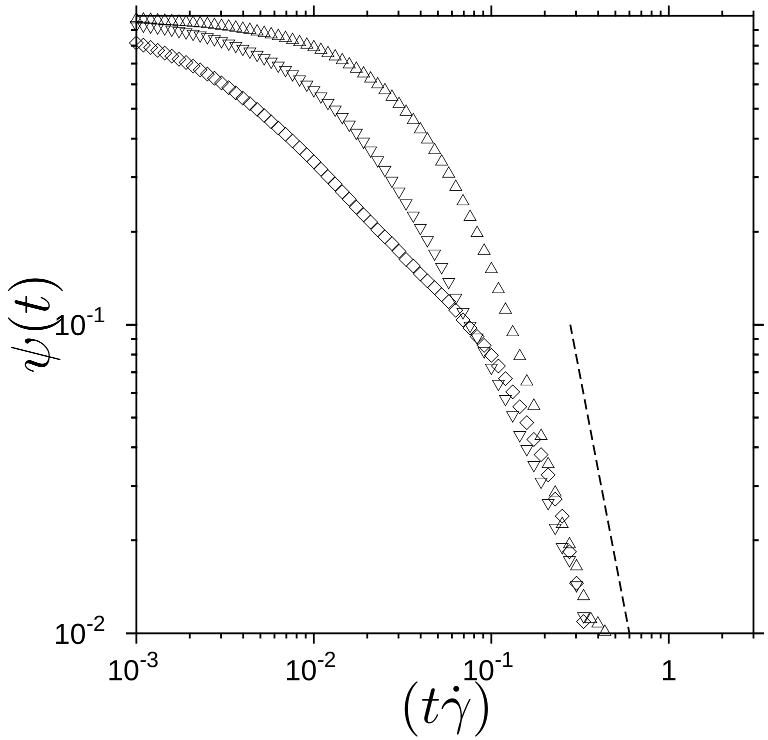
Elastic



$\triangle \phi = 0.45; \nabla \phi = 0.5, \diamond \phi = 0.55$

$\psi(t) \sim t^{-3/2}$.

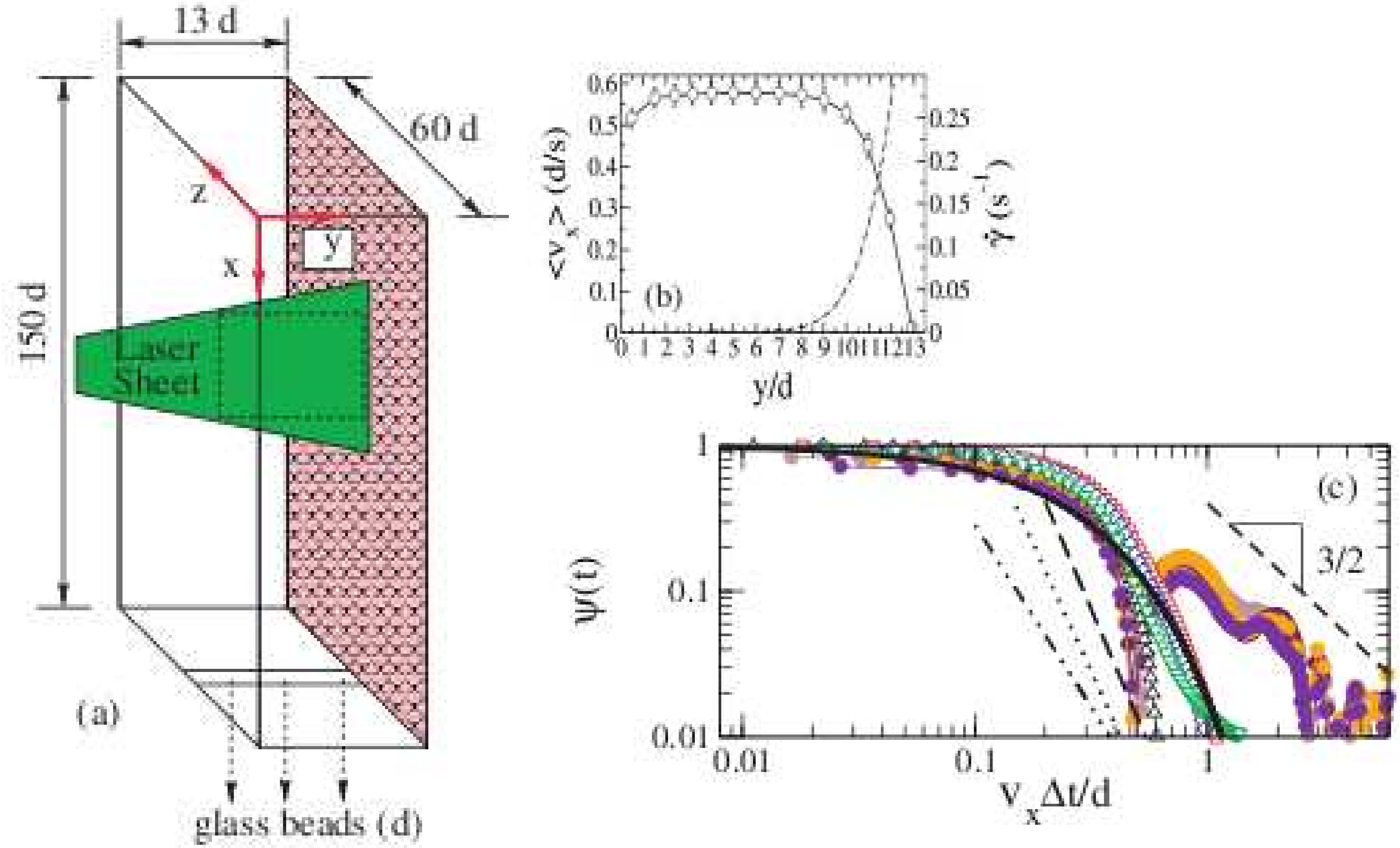
Sheared inelastic $e = 0.9$



$\triangle \phi = 0.45; \nabla \phi = 0.5, \diamond \phi = 0.55$

$e = 0.9$; $\psi(t) \sim t^{-15/4}$?.

Fast decay of velocity correlations (Orpe et al 2007):



Dynamical arrest in sheared hard-particle fluids:

- Volume fraction $\phi_{da} < \phi_{rcp}$, function of e.
- Motion diffusive, fast decay of autocorrelation function.
- Strong correlation effect on relative velocity distribution of colliding particles.

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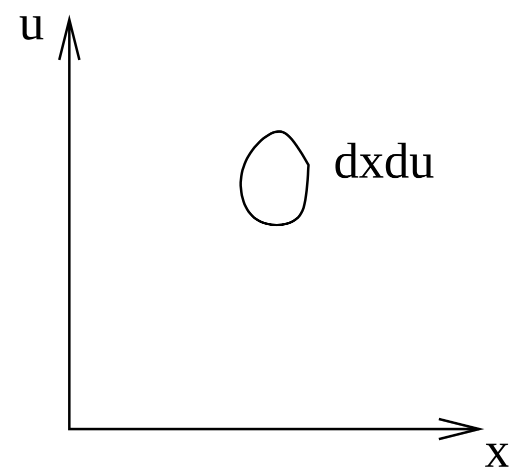
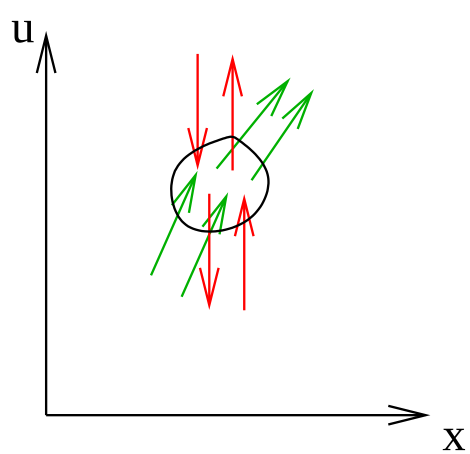
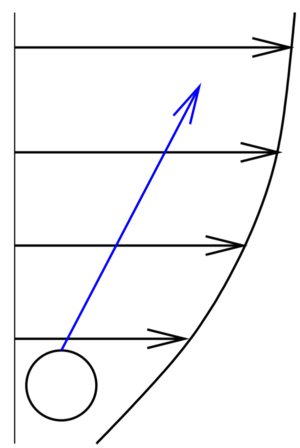
Kinetic theory — elastic hard spheres

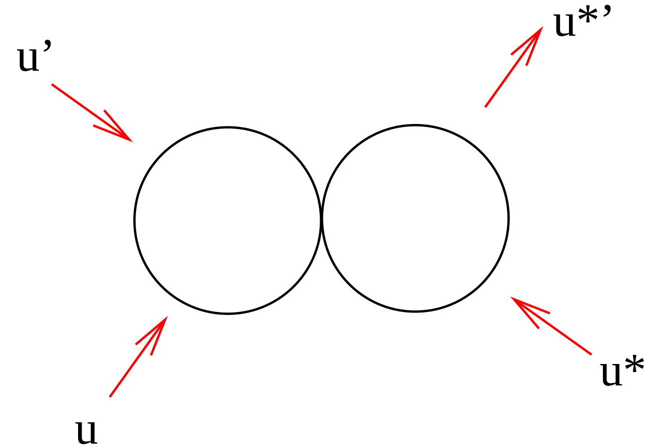
- Velocity distribution
  $f(\mathbf{x}, \mathbf{u}) d\mathbf{x} d\mathbf{u}$.
- Fluctuating velocity
  $\mathbf{c} = \mathbf{u} - \mathbf{U}$



Boltzmann eq $\frac{\partial(\rho f)}{\partial t} + \frac{\partial(\rho c_i f)}{\partial x_i} + \frac{\partial(\rho a_i f)}{\partial c_i} - \frac{\partial U_i}{\partial x_j} \frac{\partial(\rho c_j f)}{\partial c_i} = \frac{\partial_c(\rho f)}{\partial t}$







Boltzmann equation: $\frac{\partial(\rho f)}{\partial t} + \frac{\partial(\rho c_i f)}{\partial x_i} - \frac{\partial U_i}{\partial x_j}\frac{\partial(\rho c_j f)}{\partial x_i} = \frac{\partial_c(\rho f)}{\partial t}$

Equilibrium (no gradients)

$$\frac{\partial_c f}{\partial t} = 0$$

Solution — Maxwell-Boltzmann distribution

$$f = (2\pi T)^{-3/2} \exp\left(-mu^2/2T\right)$$

Non-equilibrium — Chapman-Enskog procedure:

$$\frac{\partial(\rho f)}{\partial t} + \frac{\partial(\rho c_i f)}{\partial x_i} - \frac{\partial U_i}{\partial x_j}\frac{\partial(\rho c_j f)}{\partial c_i} = \frac{\partial_c(\rho f)}{\partial t}$$

$$\frac{T^{1/2}\rho f}{L} \qquad G_{xy}\rho f \qquad \frac{T^{1/2}\rho(f - f_{eq})}{\lambda}$$

Asymptotic expansion in parameter $\epsilon = (\lambda/L)$; $f = f_0 + \epsilon f_1 + \ldots$

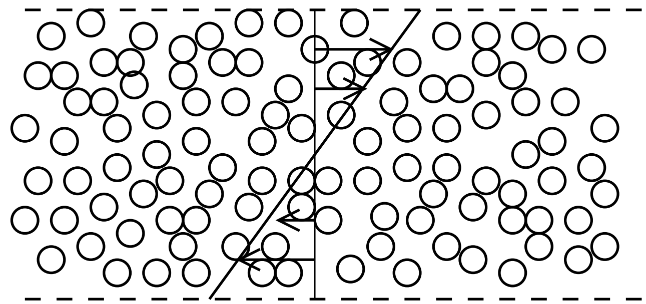
Leading order $\frac{\partial_c(\rho f)}{\partial t} = 0 \rightarrow f = f_{MB}$.

First correction

$$\frac{\partial(\rho f_0)}{\partial t} + \frac{\partial(\rho c_i f_0)}{\partial x_i} - \frac{\partial U_i}{\partial x_j}\frac{\partial(\rho c_j f_0)}{\partial c_i} = \frac{\partial_c(\rho f_1)}{\partial t}$$

Steady homogeneous shear flow of inelastic particles:

$$-G_{ij}\frac{\partial(\rho c_j f)}{\partial c_i} = \frac{\partial_c(\rho f)}{\partial t}$$



Nearly elastic collisions:

$e_n \ll 1 \rightarrow$ Dissipation $\ll$ Particle energy

Expand in $\epsilon = (1-e_n)^{1/2}$.

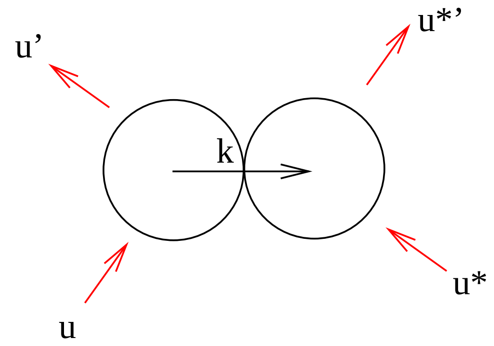
Leading order $\frac{\partial_c(\rho f_0)}{\partial t} = 0 \rightarrow f = f_{MB}$.

Rate of energy production $\sim \mu G_{xy}^2 \sim (T^{1/2}/d^2) G_{xy}^2$.

Rate of energy dissipation $\sim \rho^2 T^{3/2}(1-e_n^2)$.

$\rightarrow G_{xy} \sim (1-e_n^2)^{1/2} T^{1/2} \sim \epsilon T^{1/2}$.

Collision rules — smooth particles



Relative velocity $\mathbf{w} = \mathbf{u} - \mathbf{u}^*$

$$w'_k = -e_n w_k = -(1 - \epsilon^2) w_k$$

$$w'_t = w_t$$

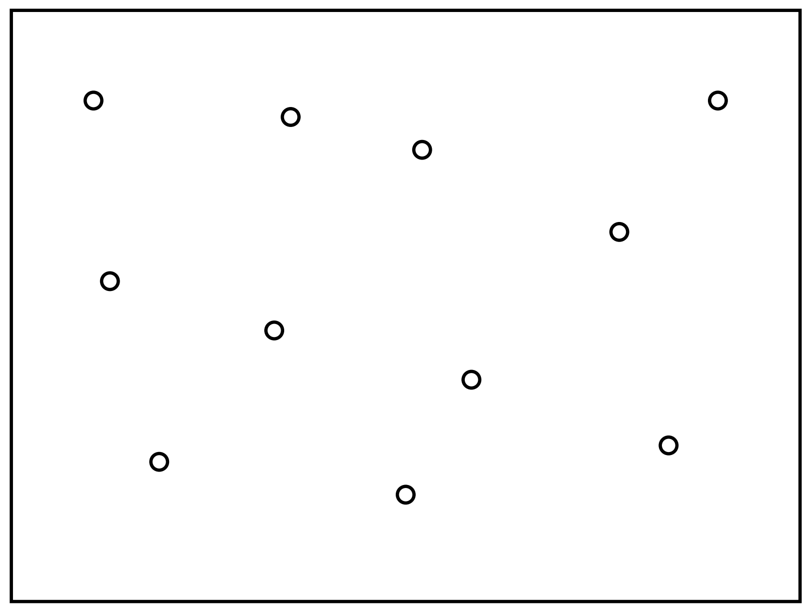
Energy conserved for $\epsilon = 0$.

Boltzmann collision integral — dense gases

Enskog approximation:

$$\frac{\partial_c \rho f}{\partial t} = \chi(\phi) \int_{\mathbf{k}} \int_{\mathbf{c}^*} \left(f(\mathbf{c}')f(\mathbf{c}^{*\prime}) - f(\mathbf{c})f(\mathbf{c}^*)\right)\left((\mathbf{u} - \mathbf{u}^*).\mathbf{k}\right)$$

Pair distribution function $\chi(\phi)$
Accounts for the finite volume of partilcles.



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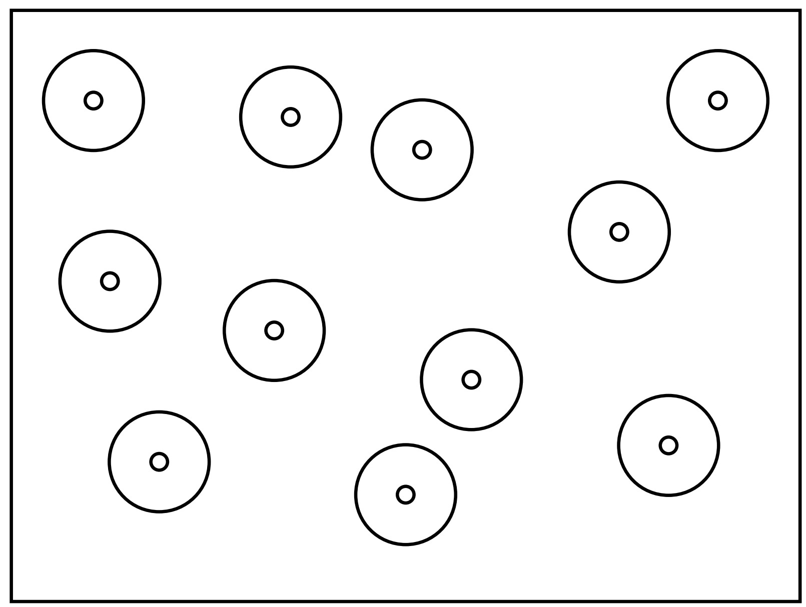


Conservation equations:

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \mathbf{u}) = 0$$

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}.\nabla \mathbf{u} \right) = -\nabla.\sigma$$

$$\rho C_v \left(\frac{\partial T}{\partial t} + \mathbf{u}.\nabla T \right) = -\nabla.\mathbf{q} + \sigma : (\nabla \mathbf{u}) - D$$

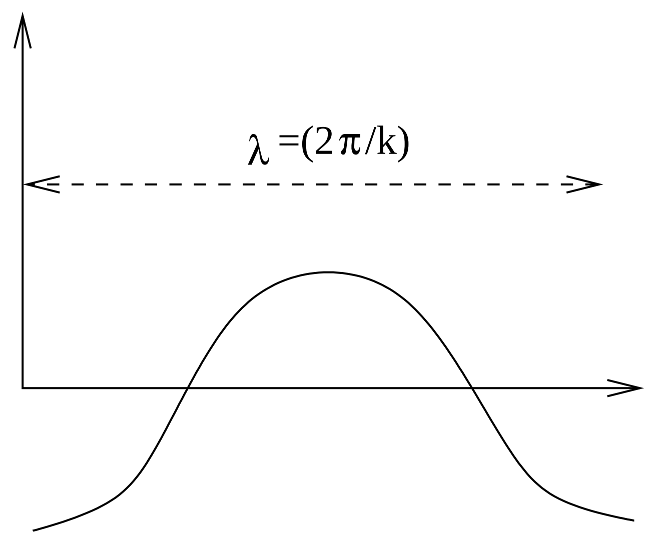
$$\sigma = -p_\phi T \mathbf{I} + \mu_\phi T^{1/2} \mathbf{S} + B_1 \mathbf{S}.\mathbf{S} + B_2(\mathbf{S}.\mathbf{A} - \mathbf{A}.\mathbf{S}) + B_3 \mathbf{A}.\mathbf{A}$$

Moments of Boltzmann-Enskog equation

- 'Slow' Mass, Momentum & Energy, conserved in collisions.
- Other 'fast' moments decay over time scales $\sim$ collision time.

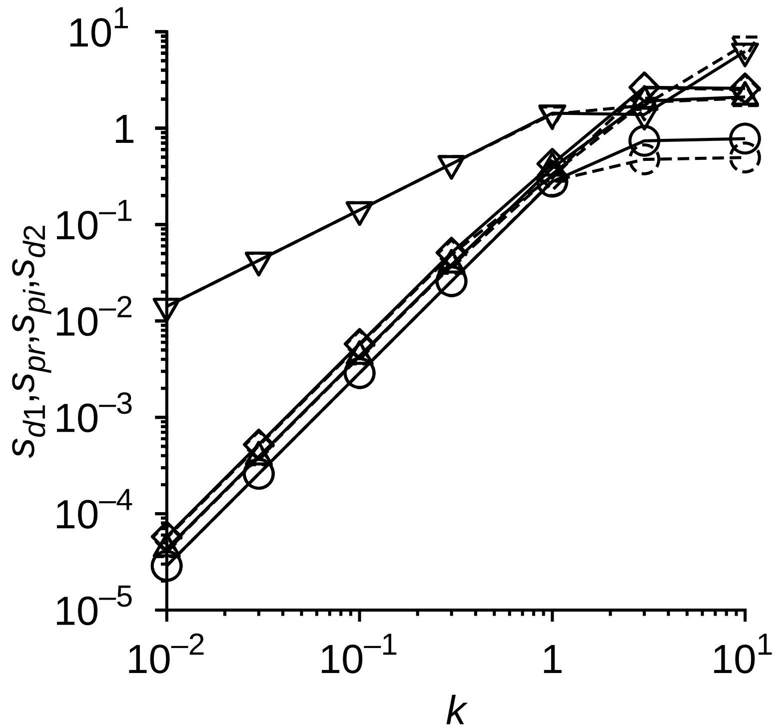
Linear response

- $f(\mathbf{c}) = f_0(\mathbf{c}) + \tilde{f}(\mathbf{c})e^{(st+\imath kx)}$
- Linearised Boltzmann equation
$$\left[s + \imath k c_x - G_{ij}\frac{\partial c_i}{\partial c_j}\right]\tilde{f} = L[\tilde{f}]$$
- $\tilde{f}(\mathbf{c}) = \sum_{i=1}^{N} A_i \psi_i(\mathbf{c})$
- $(sI_{ij} + \imath k X_{ij} - G_{ij} - L_{ij})A_j = 0$



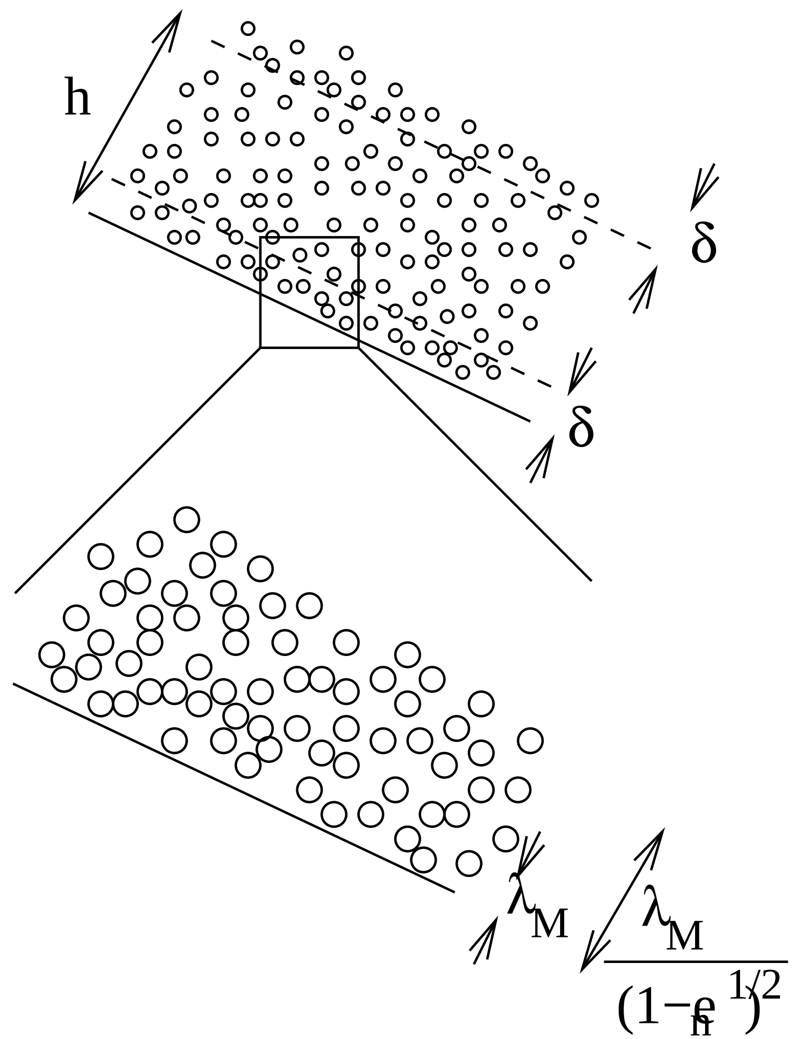
Hydrodynamic modes for elastic system

- Number of eigenvalues depends on number of basis functions chosen.
- For $k \to 0$,
  Transverse momenta $s_t = -(\mu/\rho)k^2$.
  Energy $s_e = -D_T k^2$.
  Mass & longitudinal mom. $s_l = \pm \imath k\sqrt{p_\rho} - \rho^{-1}(\mu_b + 4\mu/3)k^2$.
- All other modes with negative eigenvalues, indicating that other transients decay.

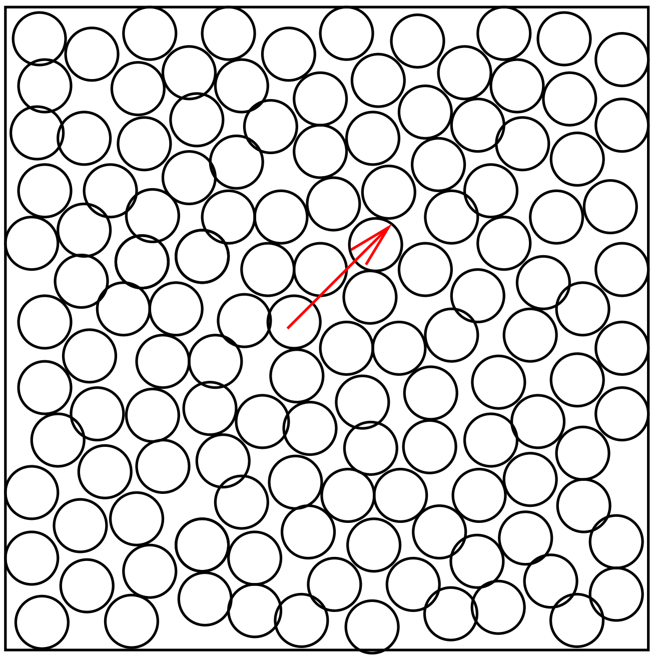


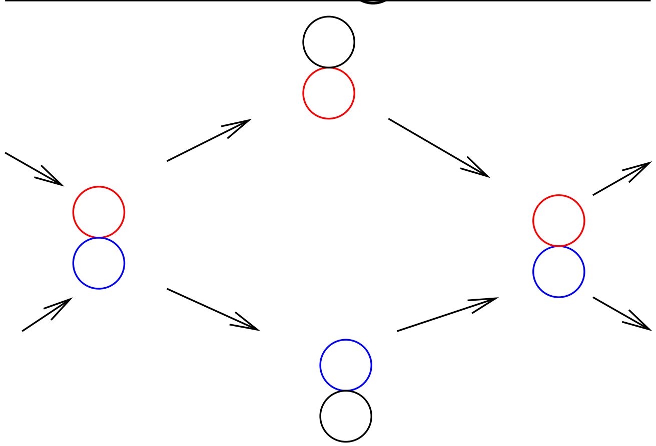
Hydrodynamic modes for smooth inelastic spheres

- Energy *not conserved*.
- Rate of conduction $(\lambda_M T^{1/2}/L^2)$.
- Rate of dissipation $((1-e)T^{1/2}/\lambda_M)$.
- Conduction length $L_c = (\lambda_M/(1-e)^{1/2})$.
- Energy conserved $L \ll L_c (k \gg \epsilon)$.
- *Adiabatic approx.* $L \gg L_c (k \ll \epsilon)$. Local balance between source and dissipation.



Correlations in dense elastic fluids:





Partly incorporated in the divergence of the pair correlation function as close packing is approached.

Time correlation — long time tail:

$$\partial_t u_x(\mathbf{k}) = -\eta k^2 u_x(\mathbf{k})$$

$$u_x(\mathbf{k}, t) = \exp(-\eta k^2 t) u_x(\mathbf{k}, 0)$$

Velocity correlation:

$$\int d\mathbf{k} \langle u_x(\mathbf{k}, t) u_x(-\mathbf{k}, 0) \rangle$$

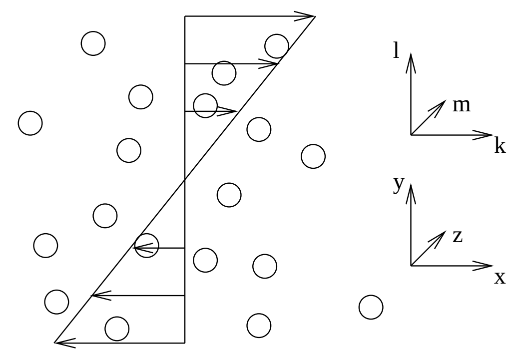
$$\sim \int d\mathbf{k} \exp(-\eta k^2 t)$$

$$\sim t^{-d/2}$$

Linear response

Infinite sheared granular material

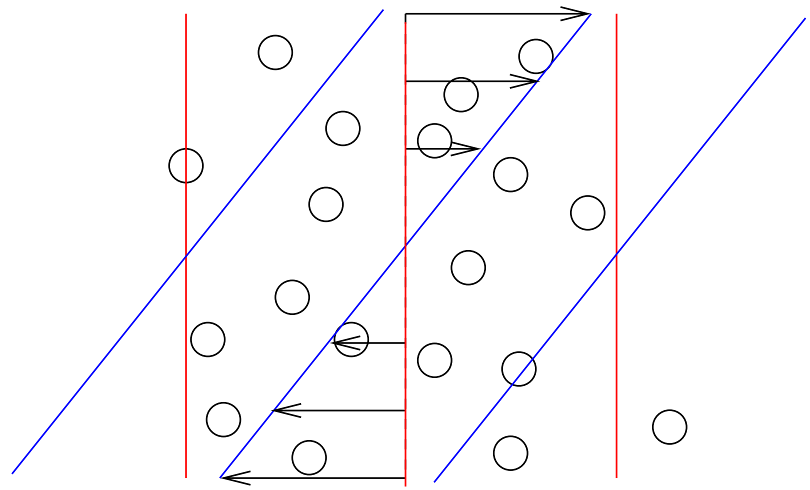
- Mean flow $\bar{u}_x = \bar{G}y$, $\bar{u}_y = 0$, $\bar{u}_z = 0$.
- Small dissipation $\epsilon = (1 - e_n)^{1/2} \ll 1$.
- Length $L \gg L_c (k \ll \epsilon)$.
- Mass conservation
  $\partial_t \rho + \nabla.(\rho \mathbf{u}) = 0.$
- Momentum conservation
  $\rho(\partial_t \mathbf{u} + \mathbf{u}.\nabla \mathbf{u}) = \nabla.\sigma.$
- Perturbations



$$\begin{pmatrix} \rho(\mathbf{x}, t) \\ \mathbf{u}(\mathbf{x}, t) \end{pmatrix} = \begin{pmatrix} \tilde{\rho}(t) \\ \tilde{\mathbf{u}}(t) \end{pmatrix} \exp\left(\imath kx + \imath ly + \imath mz\right)$$

Linear response

- Infinite shear flow — not homogeneous.
- Time dependent wave vector $k = k_0, l = l_0 - k_0\bar{G}t, m = m_0$.
- 'Linear' response equations



$$\partial_t \begin{pmatrix} \tilde{\rho}(t) \\ \tilde{\mathbf{u}}(t) \end{pmatrix} + (\mathcal{L}_0 + t\mathcal{L}_1 + t^2\mathcal{L}_2) \begin{pmatrix} \tilde{\rho}(t) \\ \tilde{\mathbf{u}}(t) \end{pmatrix} = 0$$

$$\begin{pmatrix} \tilde{\rho}(t) \\ \tilde{\mathbf{u}}(t) \end{pmatrix} = \exp\left(-t\mathcal{L}_0 - (t^2/2)\mathcal{L}_1 - (t^3/3)\mathcal{L}_2\right) \begin{pmatrix} \tilde{\rho}(0) \\ \tilde{\mathbf{u}}(0) \end{pmatrix}$$

For $k_0 = 0$, $\mathcal{L}_1 = 0$, $\mathcal{L}_2 = 0$.

Linear response — flow plane

Short time $t \ll \bar{G}^{-1}$:

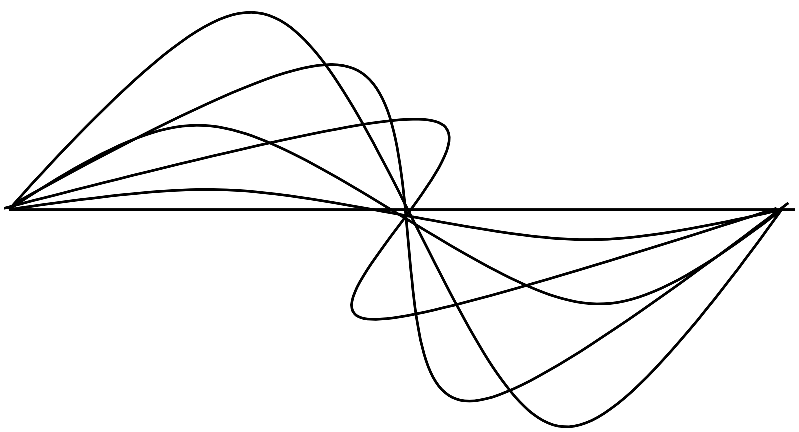
$$\begin{pmatrix} \tilde{\rho}(t) \\ \tilde{u}_x(t) \\ \tilde{u}_y(t) \end{pmatrix} = \exp\left(s_{\rho xy}\right) \begin{pmatrix} \tilde{\rho}(0) \\ \tilde{u}_x(0) \\ \tilde{u}_y(0) \end{pmatrix}$$

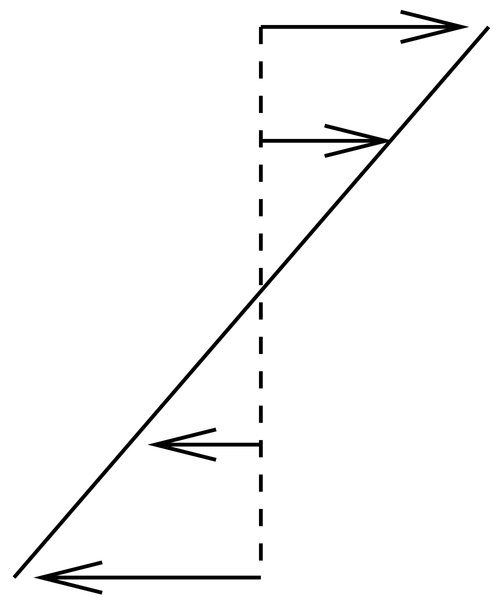
where

$$s_{\rho xy}^3 = -\bar{G}^2 k_0^2 \bar{\mu}_\rho + k_0 l_0 \bar{G} \left(\bar{p}_\rho - \frac{\bar{G}^2}{4} (\bar{\mathcal{A}}_\rho + 2\bar{\mathcal{C}}_\rho) \right)$$

- Three solutions — two propagating, one diffusive.
- For $l_0 = 0$, $s_{\rho xy} \propto -(-1, (-1)^{1/3}, (-1)^{2/3}) \bar{G}^{2/3} k_0^{2/3} \bar{\mu}_\rho^{1/3}$.
- For $l_0 \neq 0$, $s_{\rho xy} \propto (-1, (-1)^{1/3}, (-1)^{2/3}) k_0^{1/3} l_0^{1/3} \bar{p}_\rho^{1/2}$

Collective motion: growth and decay of fluctuations:





Linear response — flow plane

$$s_{\rho xy}\tilde{\rho} + \bar{\rho}\imath k_0 \tilde{u}_x + \bar{\rho}\imath l_0 \tilde{u}_y = 0$$

$$\bar{\rho}(s_{\rho xy}\tilde{u}_x + \bar{G}\tilde{u}_y) = 0$$

$$\bar{\rho} s_{\rho xy}\tilde{u}_y - (\imath \bar{G} k_0 \bar{\mu}_\rho \tilde{\rho} + \imath l_0 \bar{p}_\rho)\tilde{\rho} = 0$$

Summary — Flow plane: Vorticity direction

	$k \ll \epsilon$	$k \gg \epsilon$
Propagating s_{pr}	$-k^{2/3}$	$-k^2$
s_{pi}	$\pm k^{2/3}$	$\pm k$
Diffusive s_d	$+k^{2/3}$	$-k^2$
Transverse s_z	$-k^2$	$-k^2$
Energy s_T	$-k^0$	$-k^2$

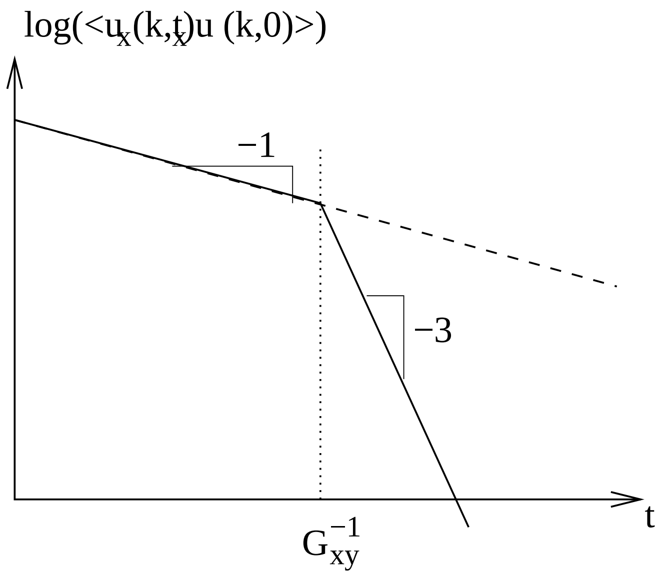
	$m \ll \epsilon$	$m \gg \epsilon$
Diffusive $s_{\rho z}$	$+m$	$-m^2$
$s_{\rho z}$	$-m$	$\pm \imath m$
Transverse s_{xy}	$+m$	$-m^2$
s_{xy}	$-m$	$-m^2$
Energy s_T	$-m^0$	$-m^2$

Effect of correlations:

Time correlation in elastic sheared fluid:

Sheared system:

$$(\partial_t + G_{xy} k_x \frac{\partial}{\partial k_y}) u_x = -\eta k^2 u_x$$

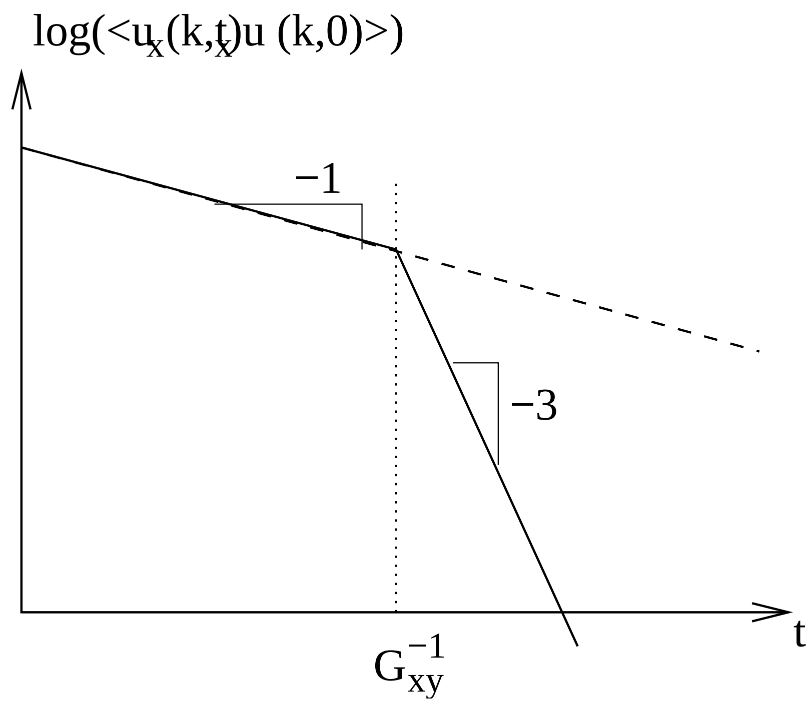


$$u_x(t) = u_x(0) \exp\left[-Dt\left(k^2 - G_{xy} t k_x k_y + \frac{1}{3} G_{xy}^2 t^2 k_x^2\right)\right]$$

$$u_x(t) \sim \exp\left(-1/3 G_{xy}^2 k_x^2 t^3\right) \quad .$$

Green-Kubo relation:

$$\eta = \frac{\beta}{V} \int d\mathbf{k}' \int_0^{G_{xy}^{-1}} dt\ t^{-d/2}$$



Two dimensions:

$$\eta = \eta_0 + \eta_1 \log (G_{xy})$$

Three dimensions:

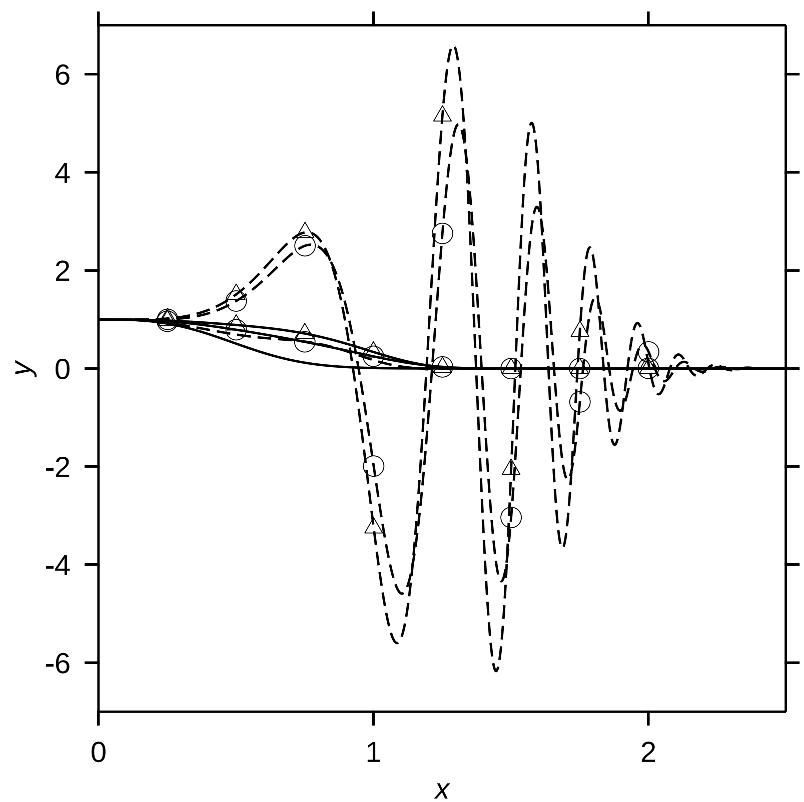
$$\eta = \eta_0 + \eta_1 |G_{xy}|^{1/2}$$

(First calculated Ernst et al 1978).

Viscosity from Green-Kubo formula:

$$\eta = \frac{\beta}{V} \lim_{k \to 0} \int_0^\infty dt \langle \sigma_{xy}(k,t) \sigma_{xy}(-k,0) \rangle$$

However: Large transient growth.



Inelastic fluid:

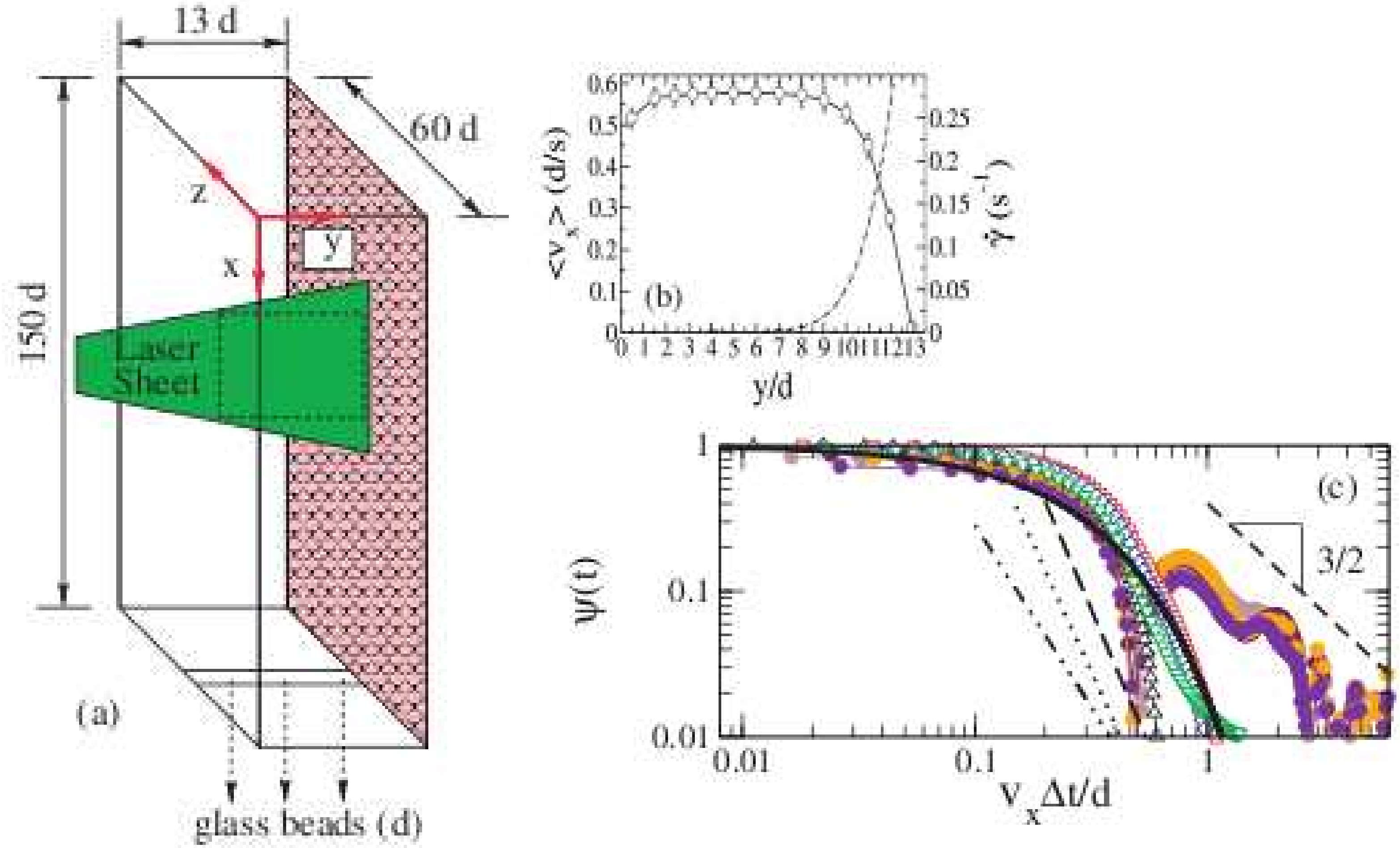
Decay rates $\lambda \propto (-k^{2/3}), (-k_z)$.

$$\eta = \sim \int dt \int d\mathbf{k} \exp(-\lambda t)$$
$$\sim (\int dt (t^{-5d/4}))$$

2D — $\eta \propto \dot{\gamma}^{3/2}$

3D — $\eta \propto \dot{\gamma}^{11/4}$.

Experimental evidence of fast decay (Orpe et al 2007):



Effect of correlations:

- Assume binary collisions.
- Include correlations between colliding particles.

$$f_\alpha = \bar{\rho} F_\alpha (1 + g_\alpha)$$

$$f_{\alpha\beta} = \bar{\rho}^2 F_\alpha F_\beta (1 + g_\alpha + g_\beta + g_{\alpha\beta})$$

$$f_{\alpha\beta\gamma} = \bar{\rho}^3 F_\alpha F_\beta F_\gamma (1 + g_\alpha + g_\beta + g_\gamma + g_{\alpha\beta} + g_{\beta\gamma} + g_{\alpha\gamma} + g_{\alpha\beta\gamma})$$

where F_α is the Maxwell-Boltzmann distribution.

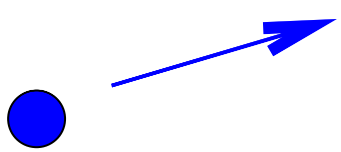
Effect of correlations:

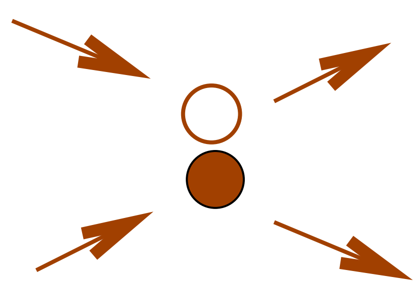
One-particle equation:

$$F_\alpha(\partial_t + S_\alpha)(1 + g_\alpha) = \int_\gamma C^E_{\alpha\gamma}[F_\alpha F_\gamma(1 + g_\alpha + g_\gamma + g_{\alpha\gamma})]$$

where $C^E_{\alpha\gamma}$ is the collision operator, and the streaming operator S_α is,

$$S_\alpha(\psi) = \mathbf{c}_\alpha . \frac{\partial \psi}{\partial \mathbf{x}_\alpha} + \epsilon\dot{\gamma}\left(y_\alpha \frac{\partial \psi}{\partial x_\alpha}\right) - \epsilon\dot{\gamma}\left(c_{\alpha y}\frac{\partial \psi}{\partial c_{\alpha x}} - \epsilon c_{\alpha x} c_{\alpha y}\right)$$





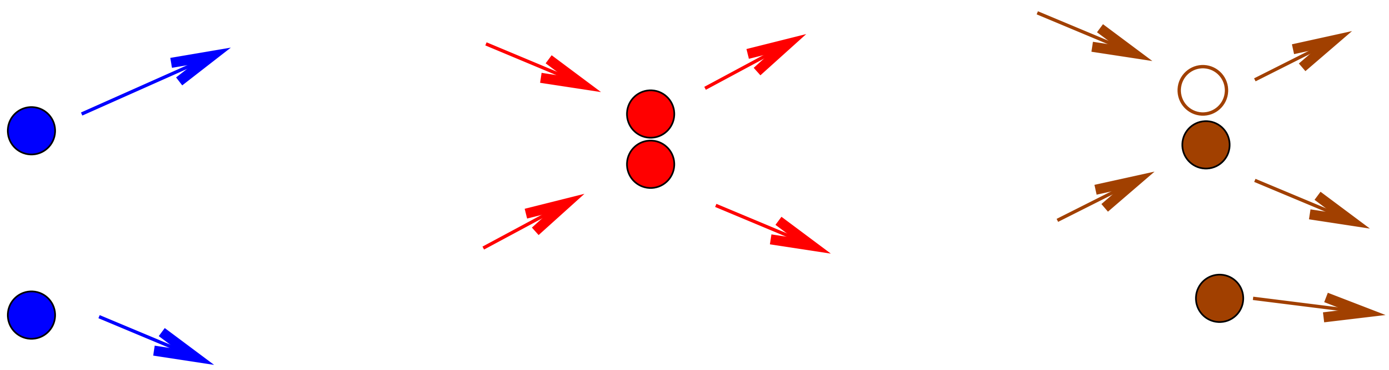
Effect of correlations:

Two-particle equation:

$$F_\alpha F_\beta \left(\frac{\partial}{\partial t} + \epsilon \dot{\gamma} y_{\alpha\beta} \frac{\partial}{\partial x_{\alpha\beta}} + \mathbf{c}_\alpha . \frac{\partial}{\partial \mathbf{x}_\alpha} + \mathbf{c}_\beta . \frac{\partial}{\partial \mathbf{x}_\beta} + \epsilon \dot{\gamma} (S_\alpha + S_\beta) \right) g_{\alpha\beta}$$

$$= \bar{\rho}^{-1} \delta(\mathbf{x}_{\alpha\beta}) C^E_{\alpha\beta}[F_\alpha F_\beta (1 + g_\alpha + g_\beta + g_{\alpha\beta})]$$

$$+ \int_\gamma C^E_{\alpha\gamma}[F_\alpha F_\beta F_\gamma (g_{\alpha\beta} + g_{\beta\gamma})] + \int_\gamma C^E_{\beta\gamma}[F_\alpha F_\beta F_\gamma (g_{\alpha\gamma} + g_{\alpha\beta})]$$



Effect of correlations:

Rewrite:

$$F_\alpha F_\beta \left(\frac{\partial}{\partial t} + \epsilon \dot{\gamma} y_{\alpha\beta} \frac{\partial}{\partial x_{\alpha\beta}} + L_\alpha + L_\beta \right) g_{\alpha\beta}$$
$$= \quad \bar{\rho}^{-1} \delta(\mathbf{x}_{\alpha\beta}) C^E_{\alpha\beta} [F_\alpha F_\beta (1 + g_\alpha + g_\beta + g_{\alpha\beta})]$$

where L_α and L_β are the linear operators

$$L_\alpha(\psi) = S_\alpha(\psi) - \int_\gamma C^E_{\alpha\gamma}((1 + P_{\alpha\gamma})(F_\gamma \psi))$$

Linear operator L_α is the operator for hydrodynamic mode equations,

$$F_\alpha(\partial_t + L_\alpha)\psi = 0$$

For conserved modes, $L_\alpha \psi \to 0$ for $k \to 0$.

Effect of correlations:

$$F_\alpha F_\beta \left(\frac{\partial}{\partial t} + \epsilon \dot{\gamma} y_{\alpha\beta} \frac{\partial}{\partial x_{\alpha\beta}} + L_\alpha + L_\beta \right) g_{\alpha\beta}$$
$$= \quad \bar{\rho}^{-1} \delta(\mathbf{x}_{\alpha\beta}) C^E_{\alpha\beta} [F_\alpha F_\beta (1 + g_\alpha + g_\beta + g_{\alpha\beta})]$$

Divergent contribution to $g_{\alpha\beta}$ from products of conserved modes,

$$g_{\alpha\beta} = \sum A_{IJ} \phi_{\alpha I} \phi_{\beta J}$$

Using this, can show Green-Kubo relation in leading approximation in small ϵ,

$$\eta = \frac{\beta}{V} \lim_{k \to 0} \int_0^\infty dt \langle \sigma_{xy}(k,t) \sigma_{xy}(-k,0) \rangle$$

Effect of correlations:

Define

$$\hat{L}^{\alpha}_{\mathbf{k}}(\xi_{\alpha M}) = \lambda_M(\mathbf{k})\xi_{\alpha M} = \lambda_M E_{MN}\phi_{\alpha N}$$

Dual of eigenfunctions

$$\int_{\alpha} F_{\alpha}\xi_{\alpha M}\eta_{\alpha N} = \delta_{MN}$$

Solution of the Chapman-Enskog equations:

$$f_{\alpha} = F_{\alpha}(1 + \epsilon h^{(1)}_{\alpha} + \epsilon^2 h^{(2)}_{\alpha} + \ldots)$$

$$\sigma^{(1)}_{xy} = \bar{\rho}^{-1} X_{IJ} K^{LM}_{IJ} X_{LM}$$

Correlation for conserved modes:

$$
\begin{aligned}
X_{IJ} &= \rho \int_\alpha F_\alpha c_{\alpha x} c_{\alpha y} \phi_{\alpha I} \phi_{\alpha J} \\
&+ \frac{\rho^2 \chi}{2} \int_\alpha \int_\beta \int_{\mathbf{a}} (\phi_{\alpha I} \phi_{\beta J} - (1/2) \phi_{\alpha I} \phi_{\alpha J} - \\
&(1/2) \phi_{\beta I} \phi_{\beta J})(c'_{\alpha x} - c_{\alpha x}) a_y (\mathbf{c}_\alpha - \mathbf{c}_\beta).\mathbf{a}
\end{aligned}
$$

Effect of correlations:

$$\sigma_{xy}^{(2)} = -\bar{\rho}^{-1} X_{IJ} K_{IJ}^{LM} X_{LM} - \bar{\rho}^{-1} X_{IJ} \Theta_{IJ}^{KL} K_{KL}^{MN} X_{MN}$$

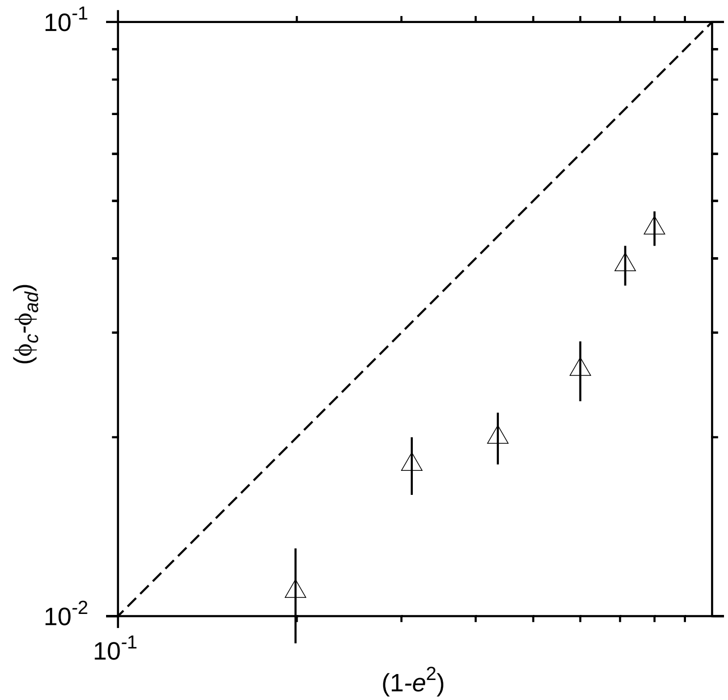
$$K_{IJ}^{LM} = \Bigg(\frac{\bar{\rho}^{-1}\epsilon\dot{\gamma}}{2} \int_{-\infty}^{t} dt' \int_{\mathbf{k}} E_{PI}^{-1}(\mathbf{k}(t),t) E_{QJ}^{-1}(-\mathbf{k}(t),t) \\ \times e^{-\int_{t'}^{t} dt''(\lambda_P(\mathbf{k}(t''))+\lambda_Q(-\mathbf{k}(t'')))} \\ E_{PL}^{+}(\mathbf{k}'(t'),t') E_{QM}^{+}(-\mathbf{k}'(t'),t') \Bigg)$$

$$\Theta_{IJ}^{KL} = \Bigg(\int_{\alpha} F_{\alpha}\phi_I\phi_J S_{\alpha}(\phi_K\phi_L) \frac{\bar{\rho}^2\chi}{2} \int_{\alpha}\int_{\beta} \phi_I\phi_J C_{\alpha\beta}^{(0)} [F_{\alpha}F_{\beta} \\ + (\phi_{\alpha K}\phi_{\beta L} - (1/2)\phi_{\alpha K}\phi_{\alpha L} - (1/2)\phi_{\beta K}\phi_{\beta L})] \Bigg)$$

Effect of correlations:

Dyson equation:

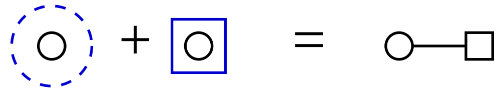
$$\sigma_{xy} = -\bar{\rho} X_{IJ} \left(\mathcal{I}_{IJ}^{MN} - \Theta_{IJ}^{KL} K_{KL}^{MN} \right)^{-1} X_{KL}$$



Divergent stress at
$|\mathcal{I}_{IJ}^{MN} - \Theta_{IJ}^{KL} K_{KL}^{MN}| = 0$.
$\Theta_{IJ}^{KL} \sim \dot{\gamma}, K_{KL}^{MN} \sim \chi \sim (\phi_c - \phi)^{-1}$
$(\phi_c - \phi) \sim \dot{\gamma} \sim (1 - e^2)^{1/2}$.

Single-particle:

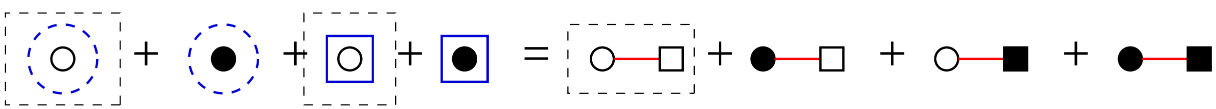
$$(\partial_t + S_\alpha) f_{CE\alpha} = \int_\beta C_{\alpha\beta}[f_{CE\alpha} f_{CE\beta}]$$

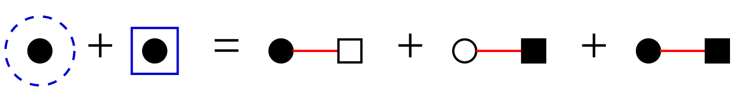


Single-particle: $f_\alpha = f_{CE\alpha}(1 + g_\alpha)$.

Equation:

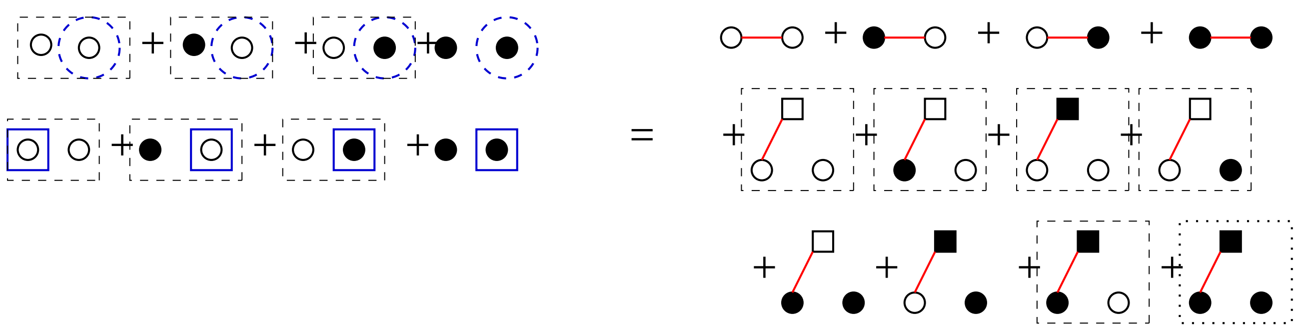
$$(\partial_t + S_\alpha)(f_{CE\alpha}(1 + g_\alpha)) = \int_\beta C_{\alpha\beta}[f_{CE\alpha} f_{CE\beta}(1 + g_\alpha + g_\beta + g_{\alpha\beta})]$$

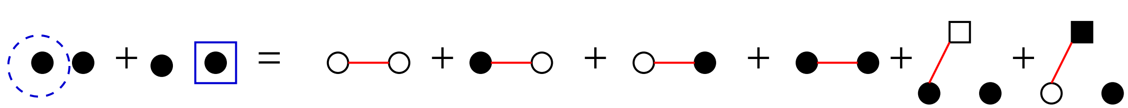




Two-particle: $f_{\alpha\beta} = f_{CE\alpha} f_{CE\beta}(1 + g_\alpha + g_\beta + g_{\alpha\beta})$

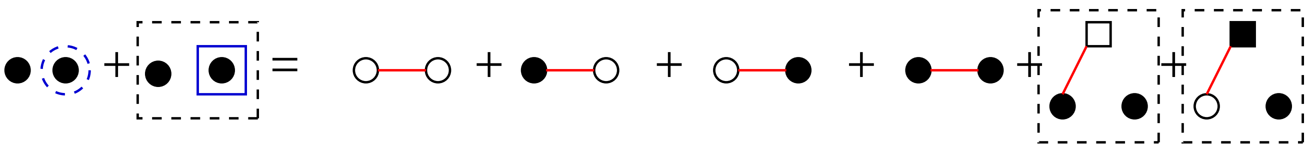
$$
\begin{aligned}
&(\partial_t + S_\alpha + S_\beta)(f_{CE\alpha} f_{CE\beta}(1 + g_\alpha + g_\beta + g_{\alpha\beta})) \\
&\quad = \quad \bar{\rho}^{-1} \delta(\mathbf{x}_{\alpha\beta}) C_{\alpha\beta}[f_{CE\alpha} f_{CE\beta}(1 + g_\alpha + g_\beta + g_{\alpha\beta})] \\
&\quad + \quad \int_\gamma (C_{\alpha\gamma} + C_{\beta\gamma})[f_{CE\alpha} f_{CE\beta} f_{CE\gamma}(1 + g_\alpha + g_\beta + g_\gamma \\
&\qquad\quad + g_{\alpha\beta} + g_{\alpha\gamma} + g_{\beta\gamma} + g_{\alpha\beta\gamma})]
\end{aligned}
$$

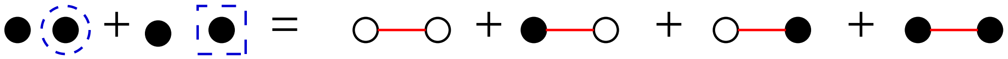




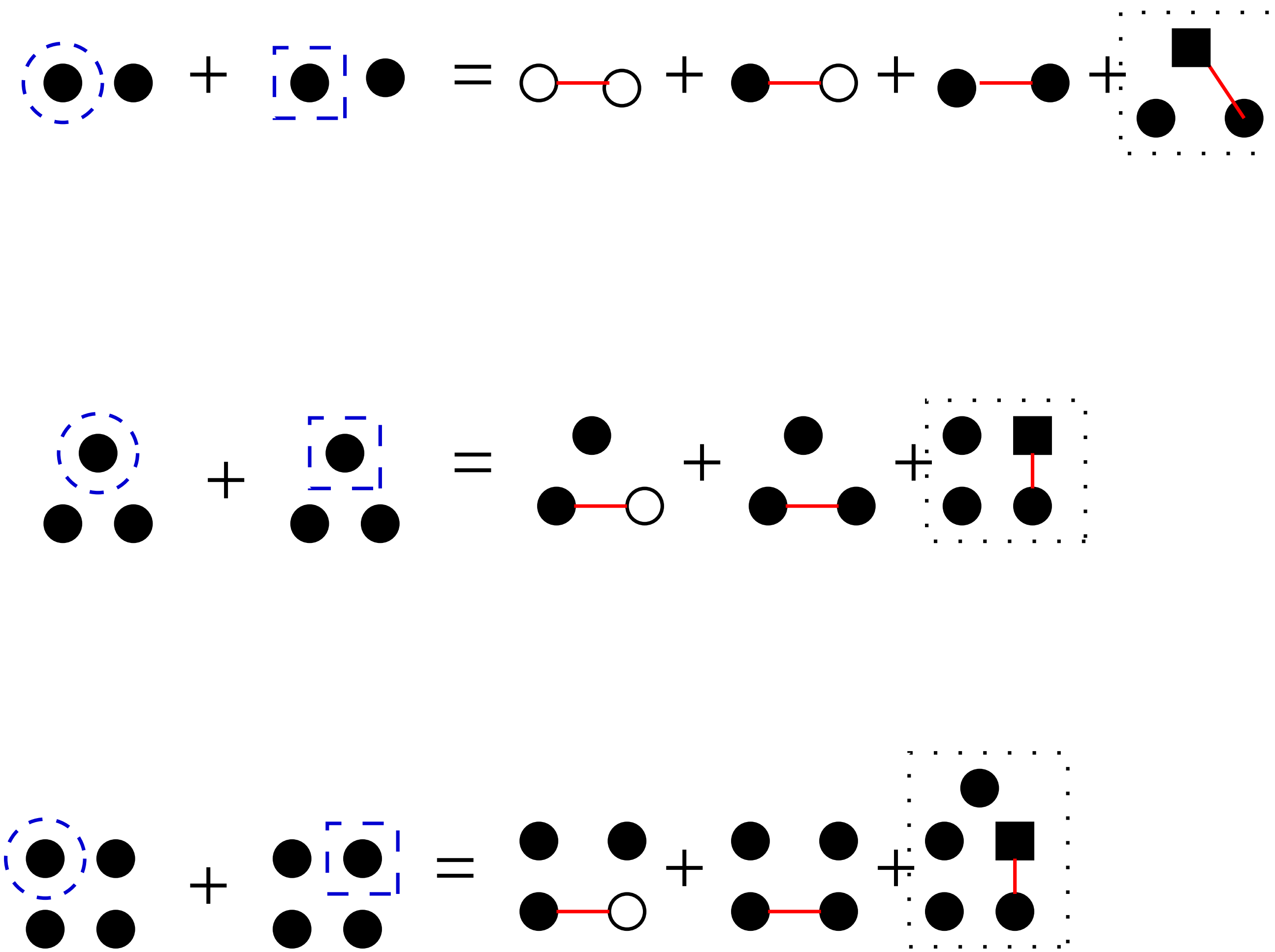
Two-particle: Express in terms of operator

$L_\alpha = S_\alpha - \int_\gamma (1 + P_{\alpha\gamma}) C_{\alpha\gamma}$:





Three-particle and higher:



Conclusions

Dynamics matters ...

... but we can still calculate some things.