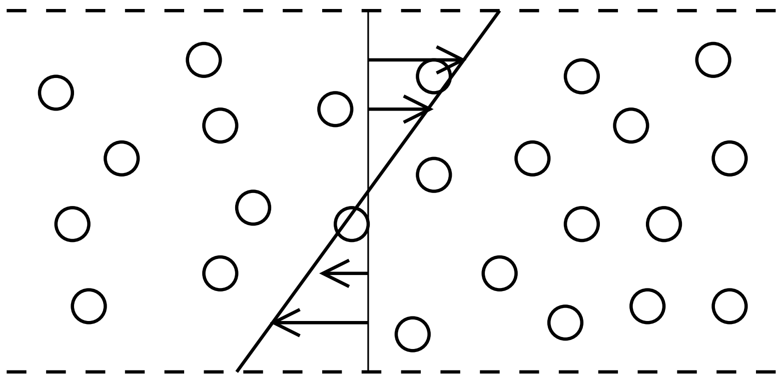
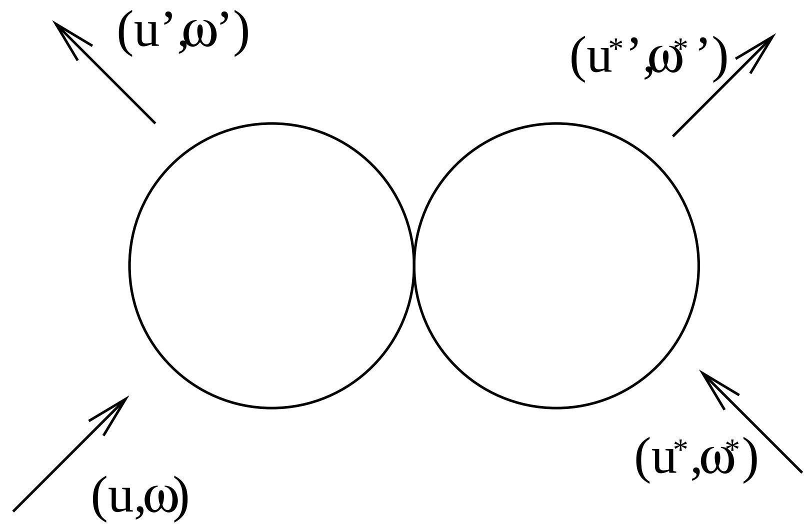
Granular flows at high Knudsen number

V. Kumaran
Department of Chemical Engineering
Indian Institute of Science
Bangalore 560 012

Rapid granular flows:





Dimensional variables

- Particle mass m,
- Particle diameter d,
- Strain rate $\dot{\gamma}$.

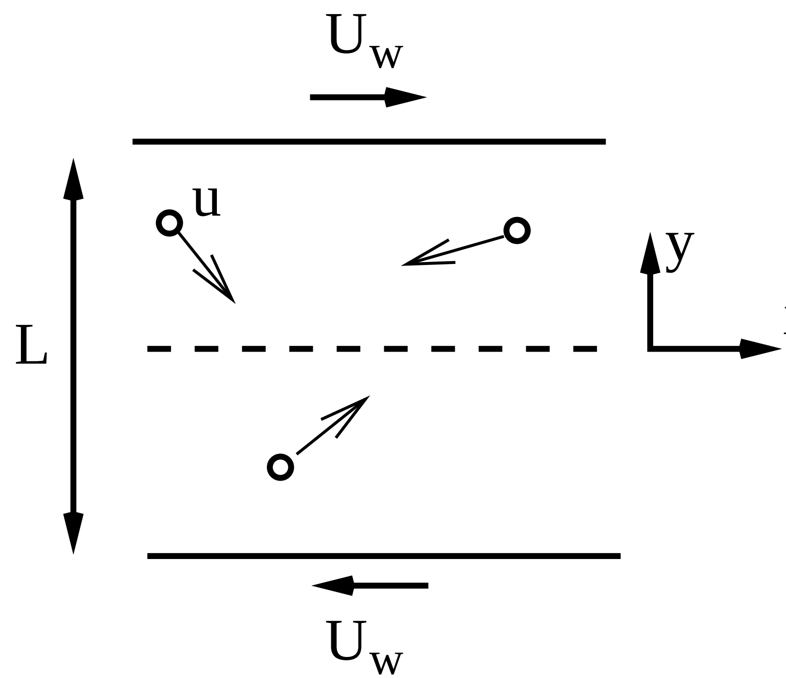
Dimensionless variables

- Volume fraction ϕ,
- Coefficients of restitution e_n, e_t.

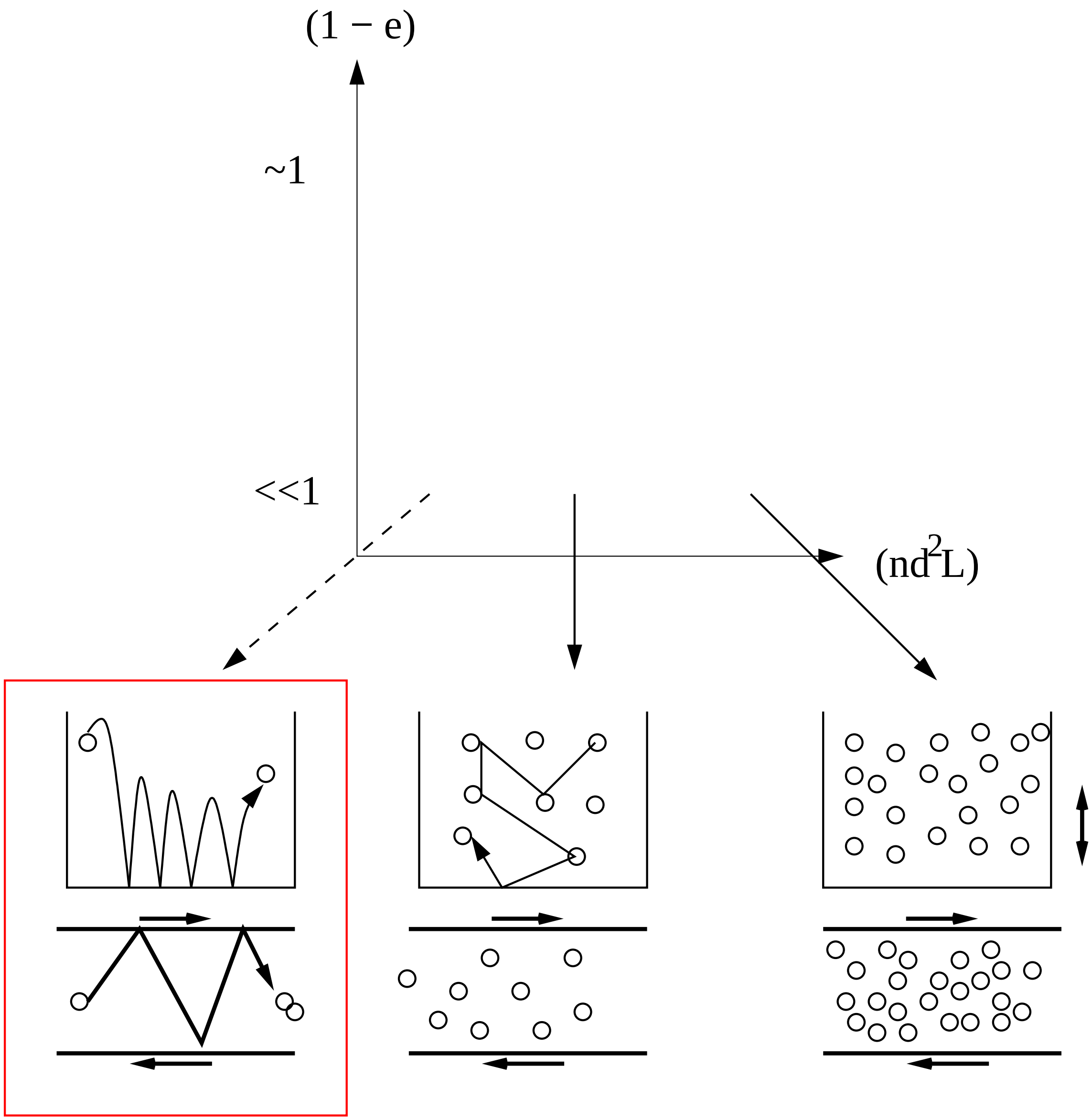
Constitutive relation $\sigma_{xy} = md^{-1}\dot{\gamma}^2 F(\phi, e)$.

Knudsen number:

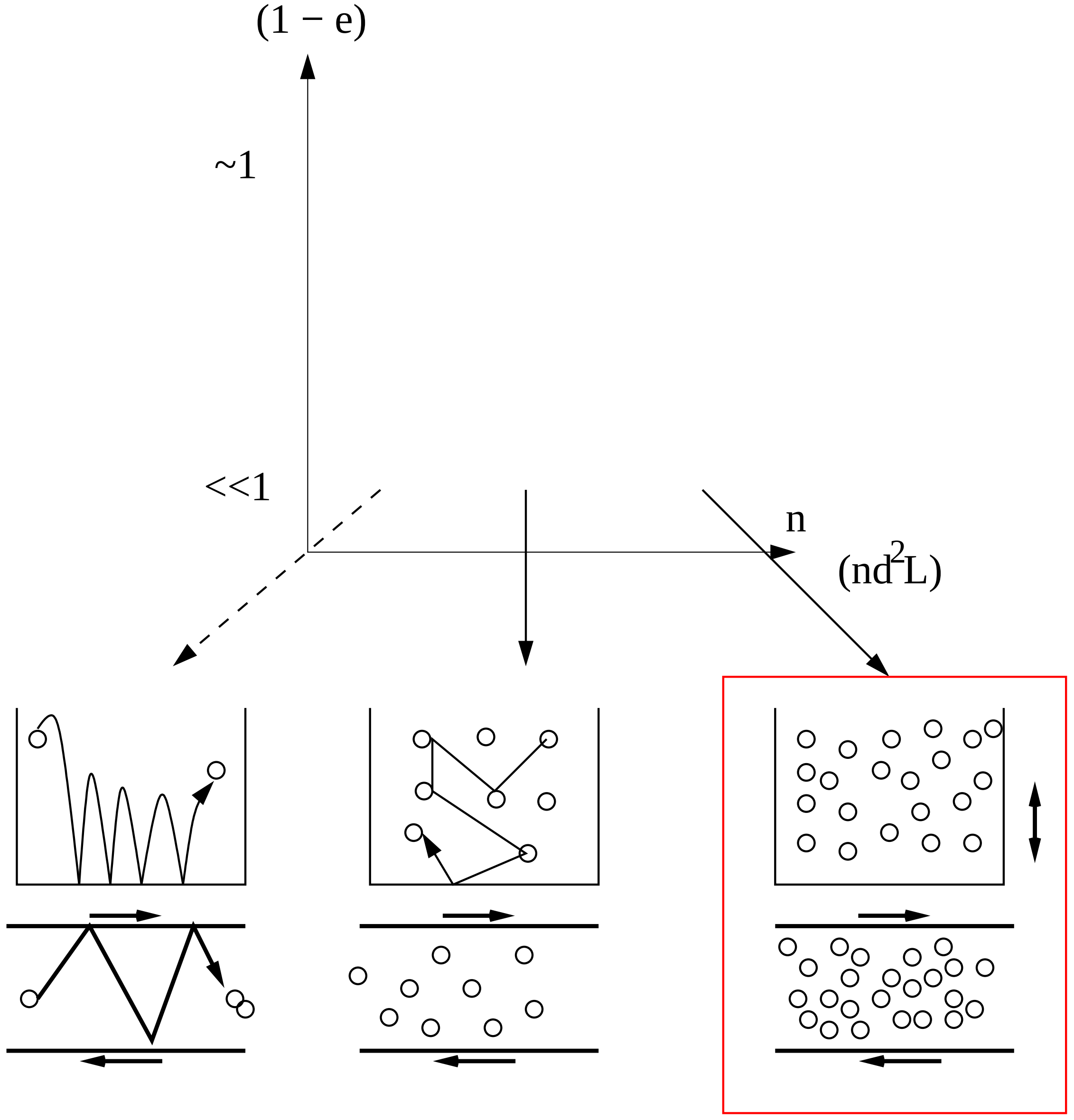
- $Kn = (\lambda/L)$.
- Number density n, fluctuating velocity u.
- $\nu_{P \to W} \sim nu$
- $\nu_{P \to P} \sim n^2 d^2 u L$.
- $(\nu_{P \to P}/\nu_{P \to W}) \sim n d^2 L$.
- High Knudsen number $\to \epsilon = n d^2 L \ll 1$.
- Open vibrated bed $\epsilon = N d^2$, where N is number of particles per unit area.



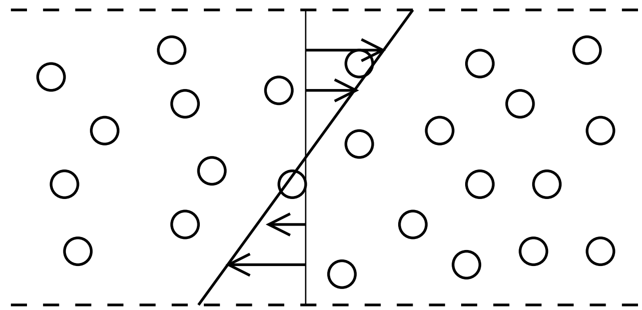
High Knudsen number limit:



Low Knudsen number limit:



Low Knudsen number limit:



- Local rheology independent of macroscopic scale L.
- Conservation equations for conserved fields (density, momentum, ?energy?).
- Differential equations subject to prescribed boundary conditions.

Constitutive law:

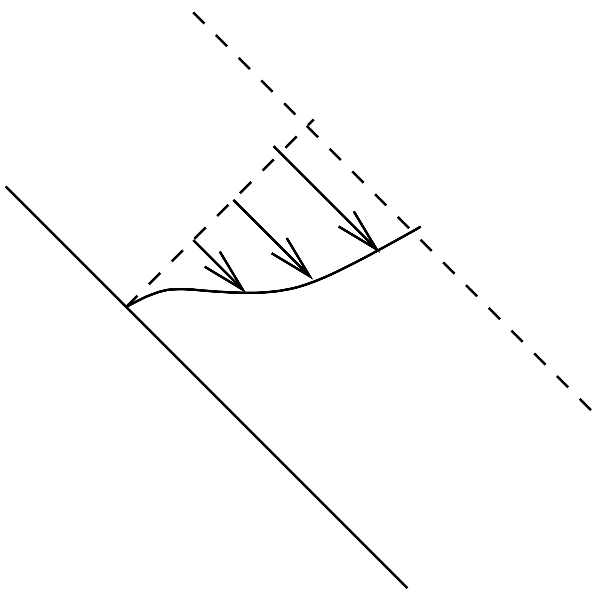
$$\sigma_{ij} = B^L_{ij}(n, d, e_t, e_n)\dot{\gamma}^2$$

Low density $B^L_{ij} \sim (nd^4)^{-1}$
High density $B^L_{ij} \sim d^{-1}$.

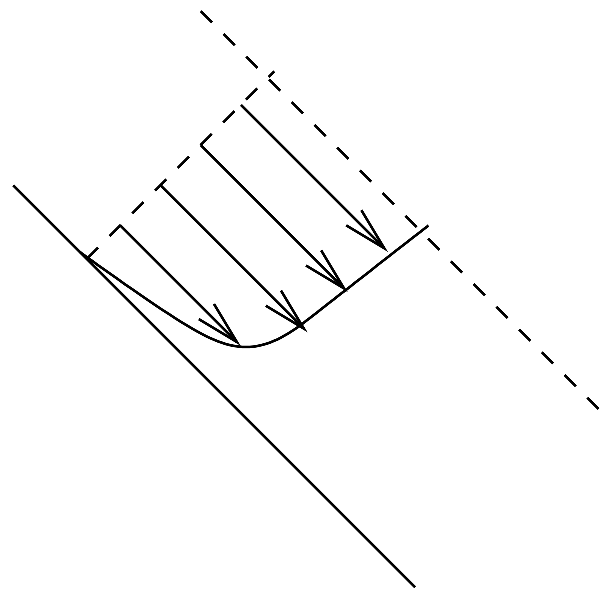
High Knudsen number limit:

$\sigma_{ij} = B^H_{ij}(n, d, L, e_t, e_n)\dot{\gamma}^2$

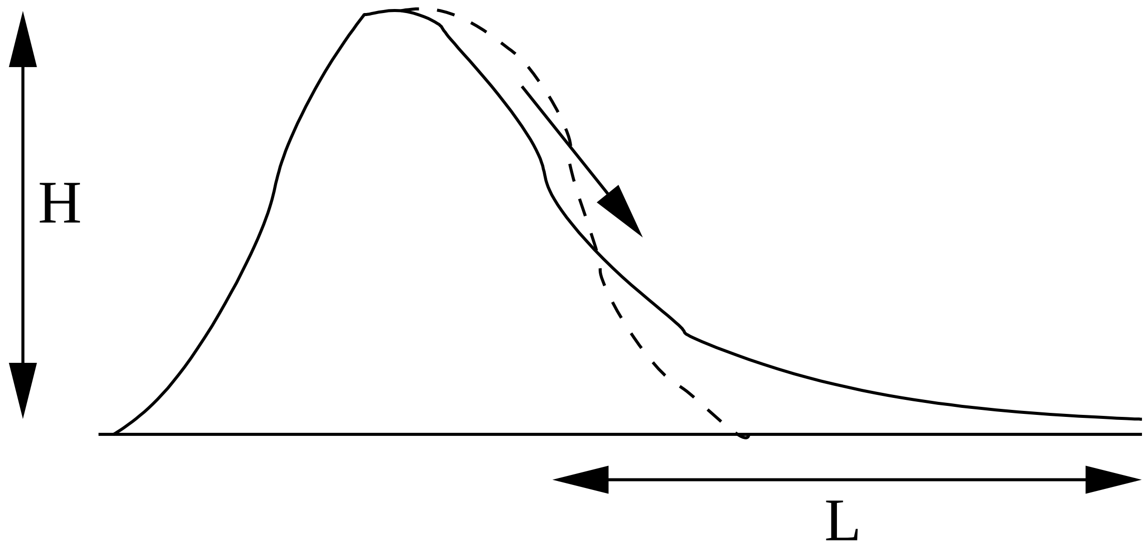
$B^H ? \geq ? B^H$

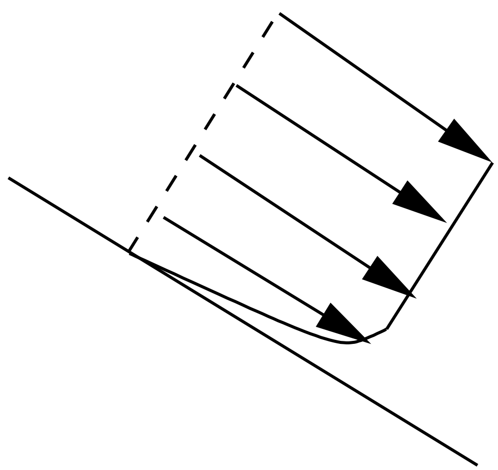


$B^H ? << ? B^L$

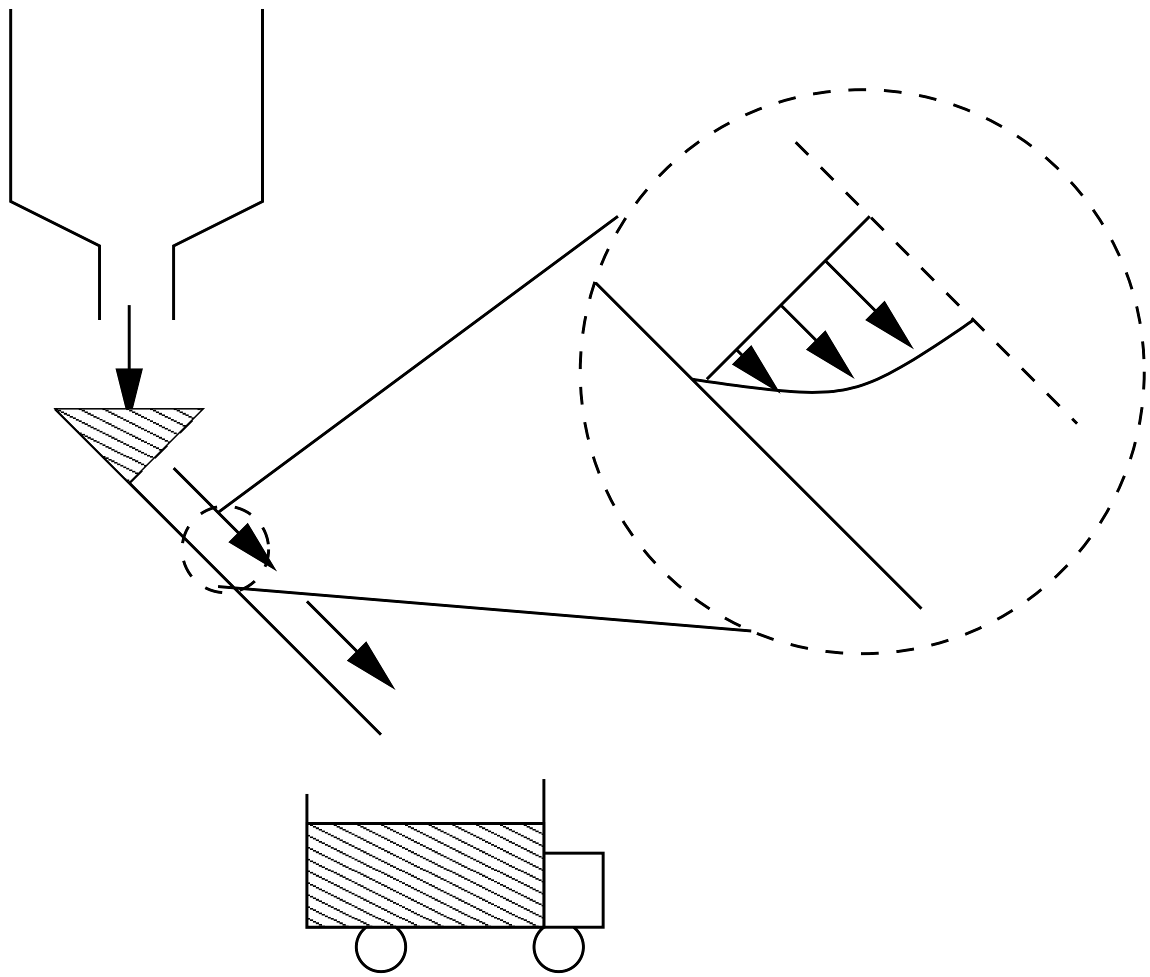


Long run-out avalanches:

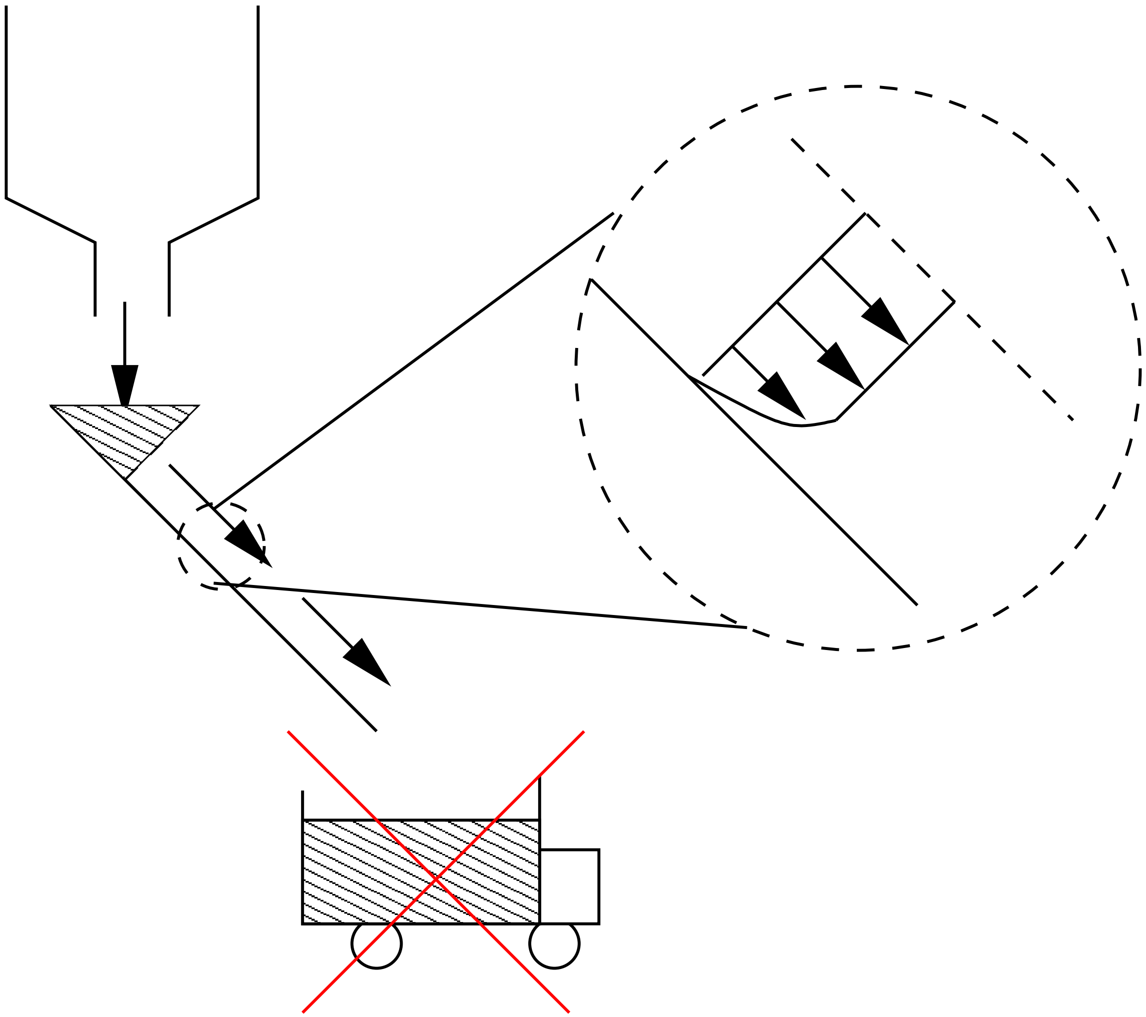




Filling from silos:



Filling from silos:



Analytical solutions

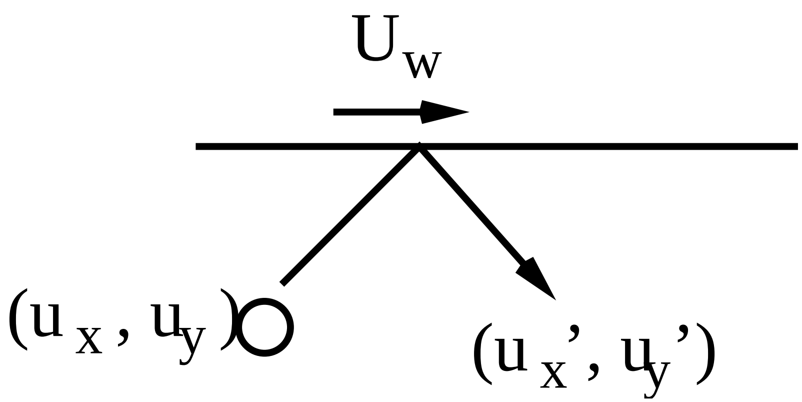
Low Knudsen number

- Energy dissipation in a collision $\ll$ particle energy $(1-e) \ll 1$.
- Flow scale $\gg$ mean free path.
- Locally distribution function close to MB distribution.
- Temperature determined by balance between source and sink.

High Knudsen number

- Energy dissipation in collision $\sim$ particle energy $(1-e) \sim 1$.
- Flow scale $\sim$ mean free path.
- Distribution function very different from MB distribution.
- ???
- Non - equilibrium non - perturbative solutions.

High Knudsen number gas dynamics:



Maxwell boundary condition: Stochastic distribution of reflected velocity with the same temperature and mean velocity as the surface.

High Knudsen number limit

- Existence of steady solutions in the absence of binary collisions ($\epsilon = 0$).
- Sensitivity of these solutions to the collision conditions at the wall.
- Effect of collisions in the limit $\epsilon \to 0$.
- Scaling of velocity moments (stress components) with ϵ.

Particle-wall interaction models

- Smooth particle, wall imparts a fraction of its translational velocity to the particle.
- Rough particle-wall interactions.
- More sophisticated sliding and sticking collision models.

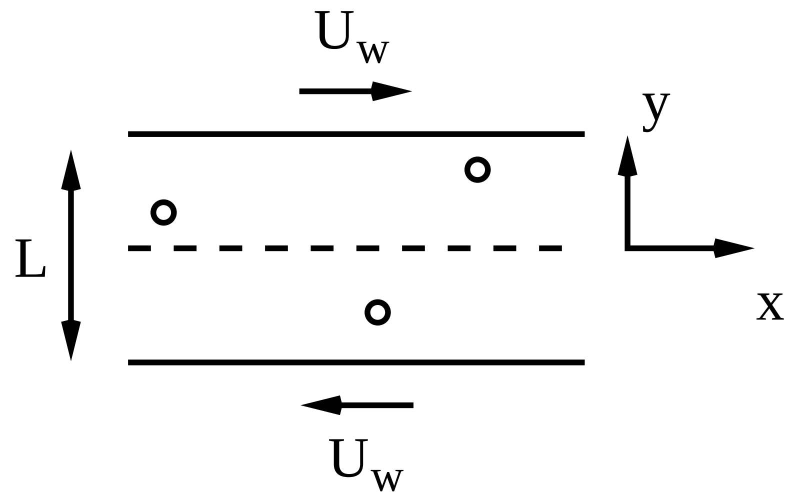
Shear flow smooth particles

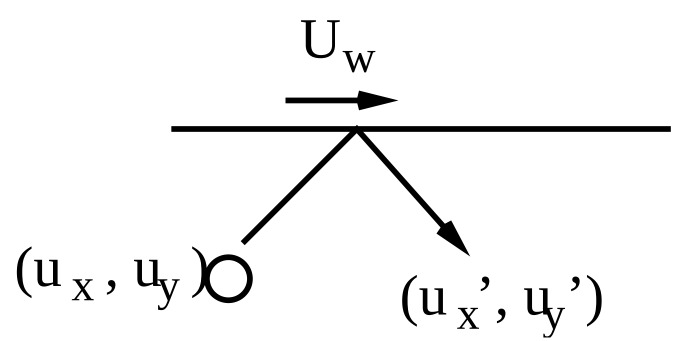
- Wall velocity $\pm U_w$
- Particle - wall collisions

$$u'_x = u_x + e_t(\pm U_w - u_x)$$

$$u'_y = -e_n u_y$$

- Particle - particle collisions inelastic.
- Parameter $\epsilon = (nd^2L) \ll 1$.





- Leading approximation — neglect interparticle collisions.
- Particle with initial velocity (u_x, u_y) tends to final state with velocity $(\pm U, 0)$

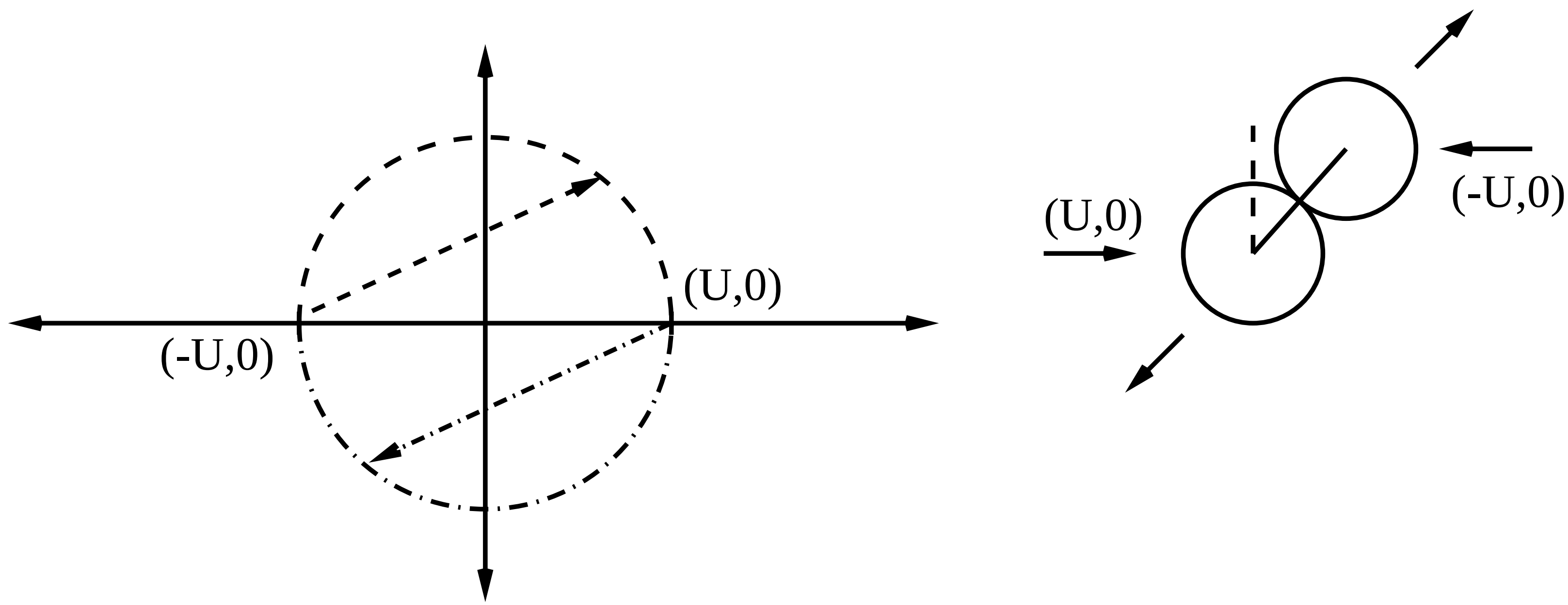
$$U = \frac{(1 - e_t)U_w}{1 + e_t}$$

- After successive collisions $(U, 0) \to (-U, 0) \to (U, 0) \ldots$
- Recurrence relation for velocity after i collisions

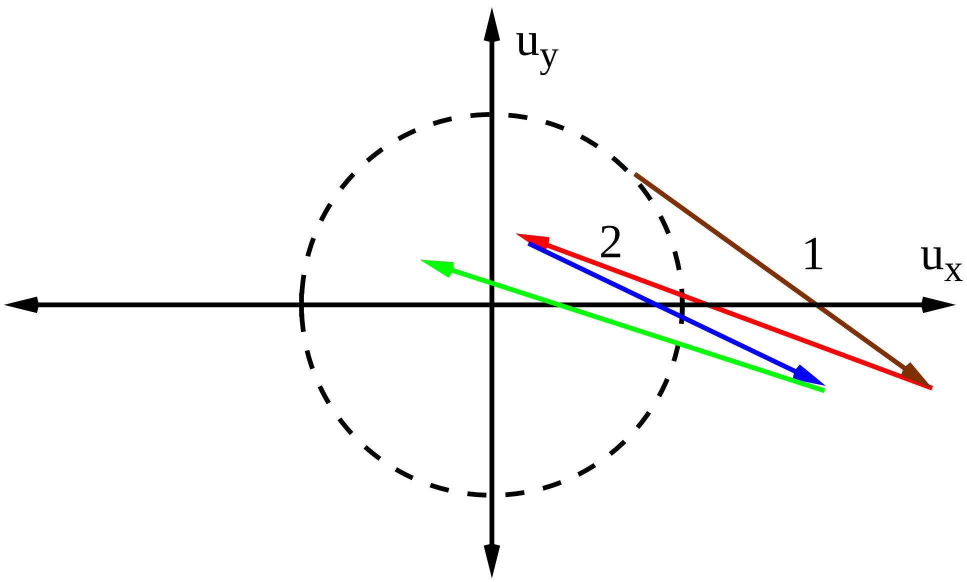
$$u_x + (-1)^i (1 + (-1)^{i-1} e_t^i) U = e_t^i u_x^{(0)} \quad u_y = (-1)^i e_n^i u_y^{(0)}$$

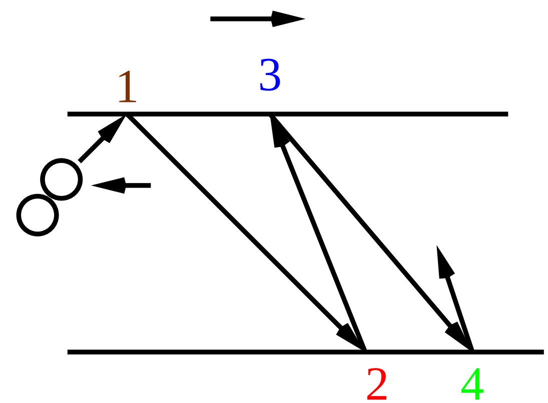
where $(u_x^{(0)}, u_y^{(0)})$ is velocity before first collision.

Collision between particles with velocity $(\pm U, 0)$

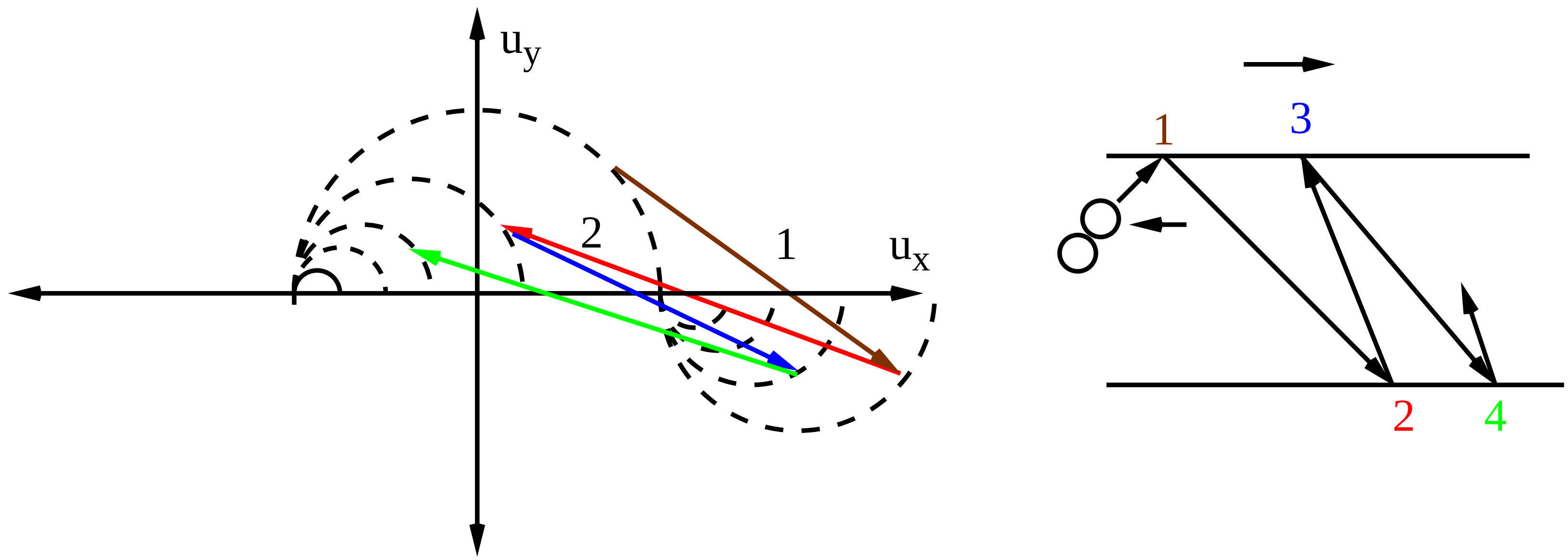


Subsequent particle collisions with the wall

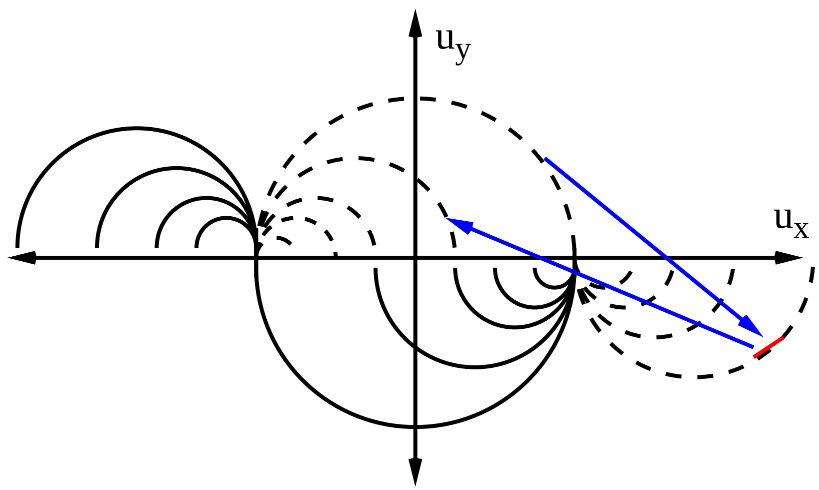




Subsequent particle collisions with the wall



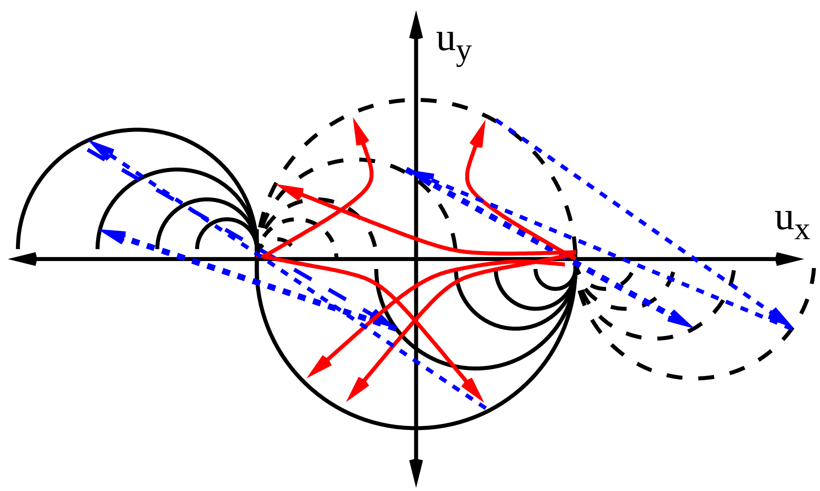
Contours in velocity space



Steady distribution function:

- Flux into a differential length on the contour is equal to flux out of a differential length.

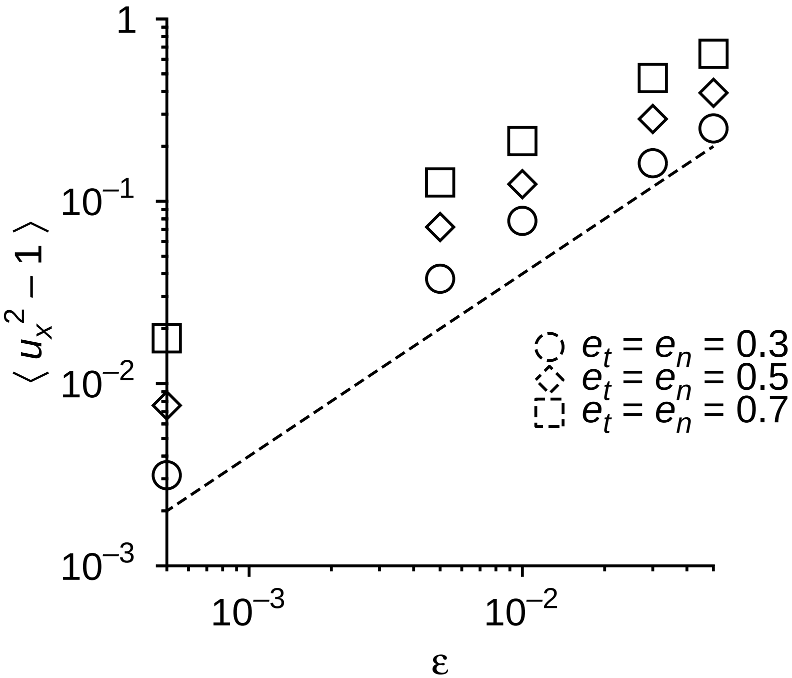
- **Binary collisions between particles** $(\pm U, 0)$**.**
- Error $\propto \epsilon^{1/2}$ for $u_y \sim \epsilon^{1/2} U$.
- Error $O(1)$ for $u_y \sim U$. However, frequency of binary collisions $O(\epsilon^{1/2})$ smaller.
- **Transfer between different contours only due to wall collisions.**
- Frequency of binary collisions $O(\epsilon^{1/2})$ smaller than wall collisions for $u_y \sim U$.

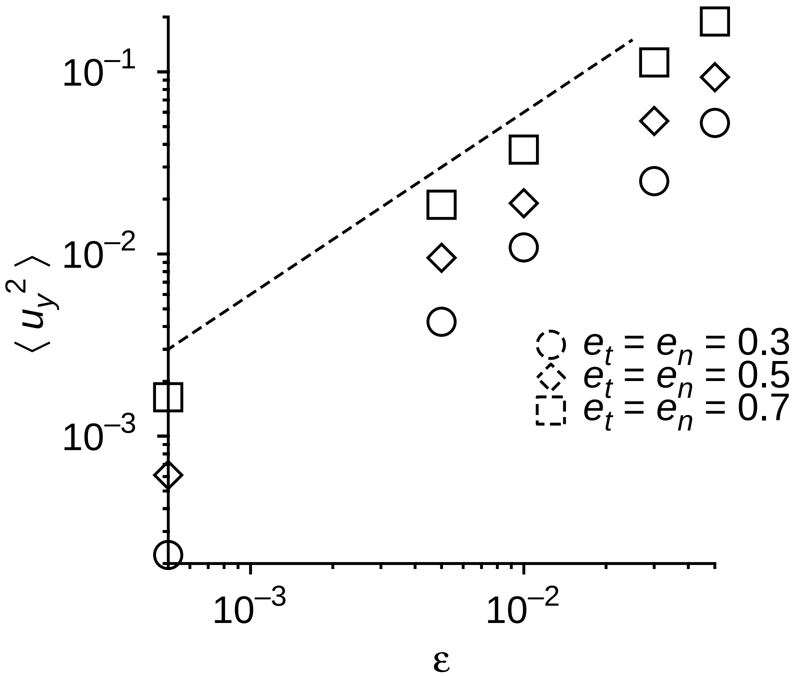


Results

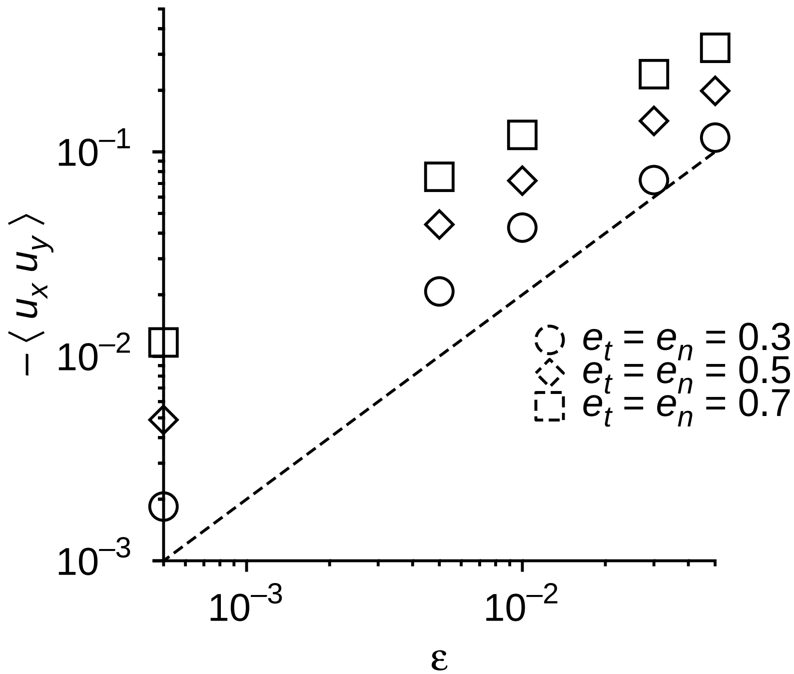
- $\langle u_x^2 - U^2 \rangle \sim \epsilon U^2$
- $\langle u_y^2 \rangle \sim \epsilon U^2$
- $\langle u_x u_y \rangle \sim \epsilon \log(\epsilon) U^2$
- Normal stress highly anisotropic $(\tau_{yy}/\tau_{xx}) \sim \epsilon$.
- (Shear stress / Normal stress) $\tau_{xy}/\tau_{yy} \sim \log(\epsilon)$.

Comparison with simulations:

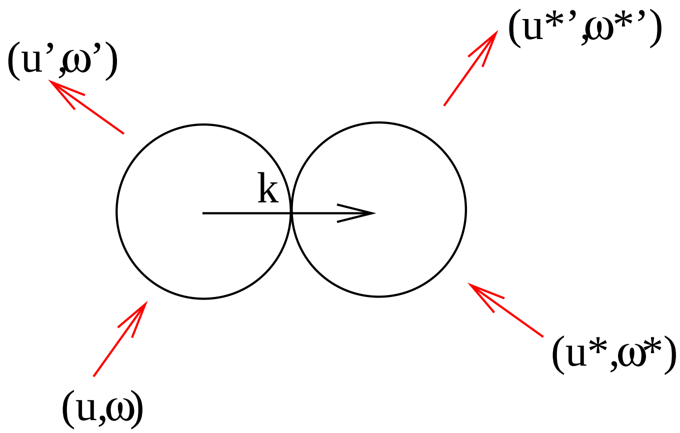


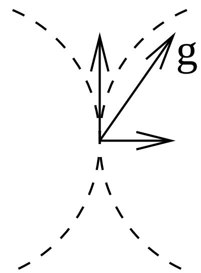


Comparison with simulations:



Rough particle-wall interactions





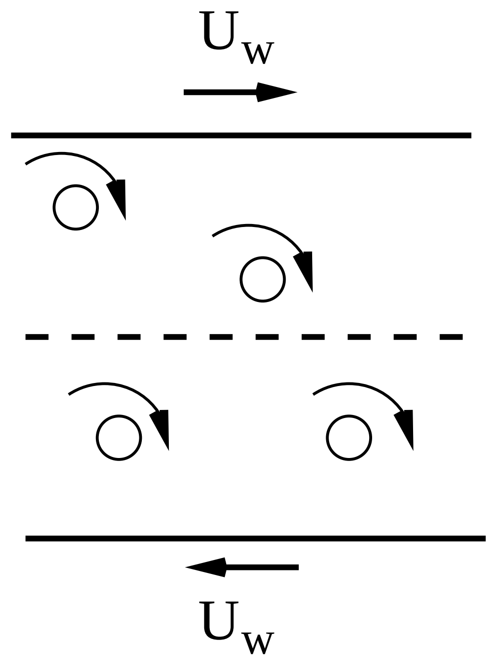
Collision rules:

$$g'_n = -e_n g_n$$

$$g'_t = -e_t g_t$$

Smooth elastic $e_n = 1, e_t = -1$; rough elastic $e_n = 1; e_t = 1$.

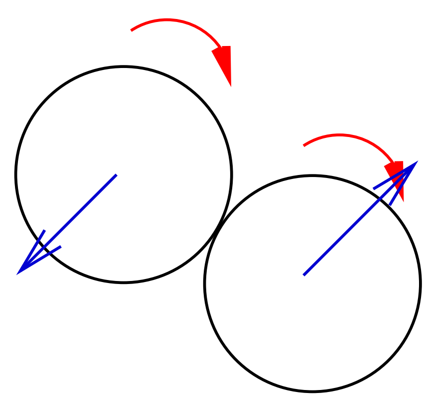
Rough particle-wall interactions



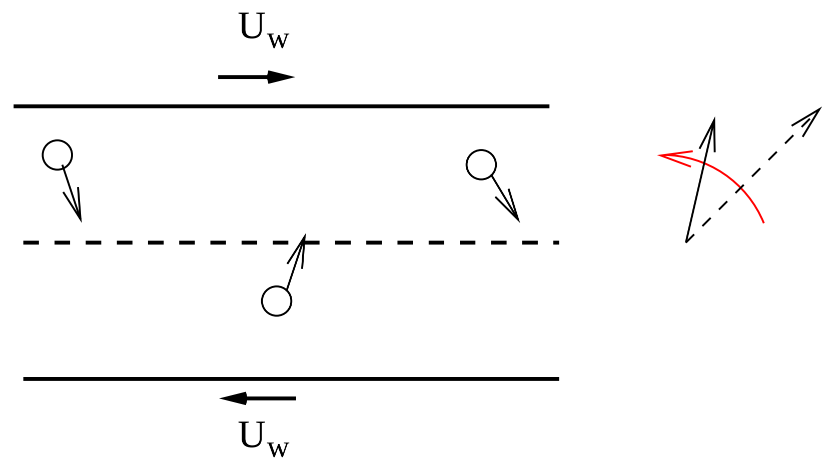
Steady state:

The Jacobian for the mapping of the velocity after two successive collisions is 1.

$u_x = 0, u_y = 0, \omega = (-2V_w/d)$.



Effect of binary collisions

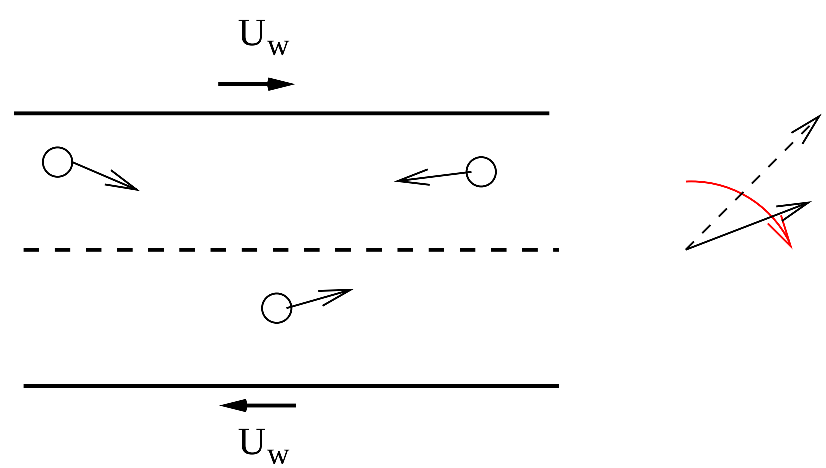


Wall-particle collisions

$\propto (nu_y)$

Particle-particle collisions

$\propto n^2 L\sigma(u_x^2 + u_y^2)^{1/2}$



Ratio

$\propto (n\sigma L(u_x^2 + u_y^2)^{1/2}/u_y)$

$\propto \epsilon(u_x^2 + u_y^2)^{1/2}/u_y)$

Ratio $\propto 1 \rightarrow$

$(u_x/u_y) \sim \epsilon^{-1}$

Transfer matrix for particle-wall collisions:

i = Pairs of particle-wall collisions.

Coupled u_x & $\Omega = \omega + (2V_w/d)$.

$$\begin{pmatrix} u_x^{(i+1)} \\ \Omega^{(i+1)} \end{pmatrix} = A \begin{pmatrix} u_x^{(i)} \\ \Omega^{(i)} \end{pmatrix}$$

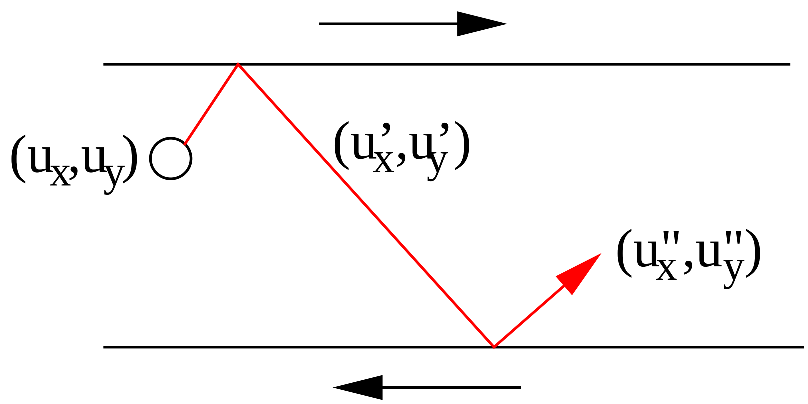
Eigenvalues of $A = \lambda_1 \geq \lambda_2$.

$\lambda_x = |\lambda_1|$.

$$u_y^{(i+1)} = e_n^2 u_y^{(i)}$$

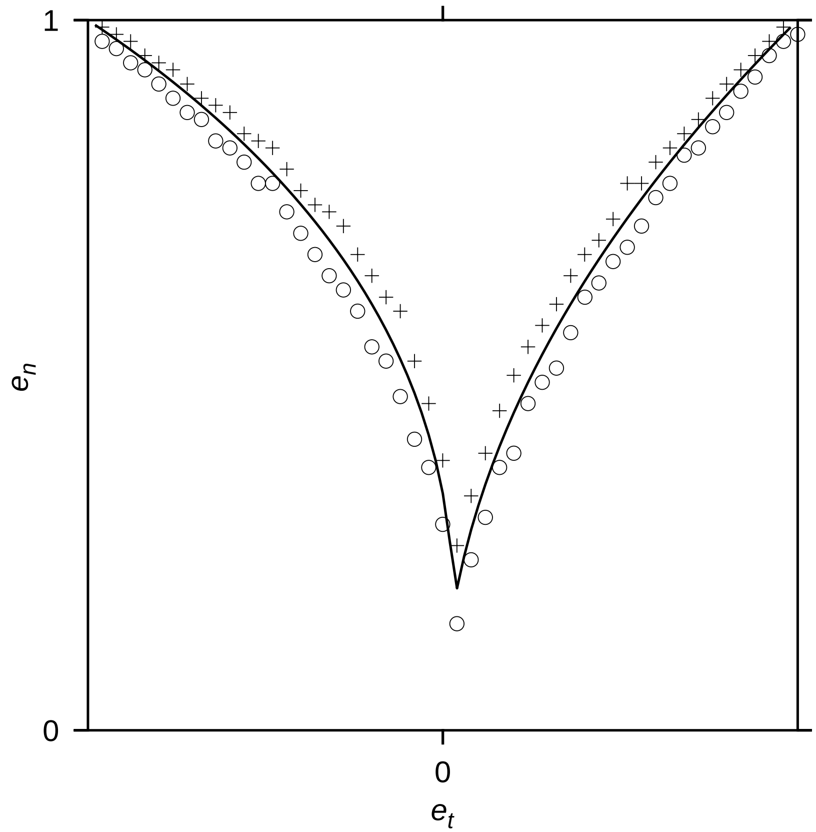
Dynamical steady state for $\lambda_x > \lambda_y$

Static steady state for $\lambda_x < \lambda_y$.



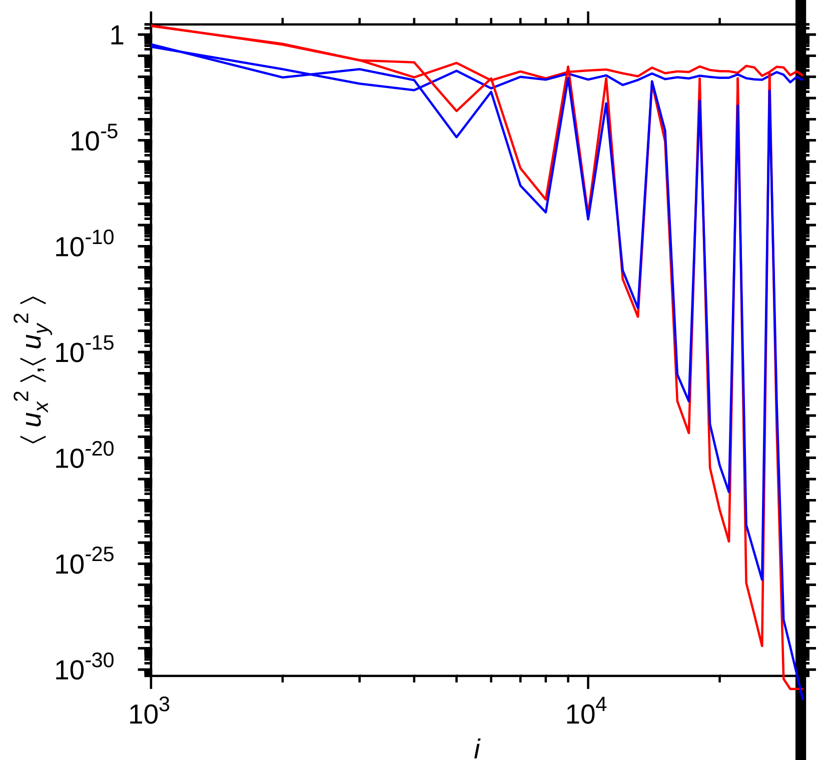
Comparison with simulations:

Static & dynamical steady states.

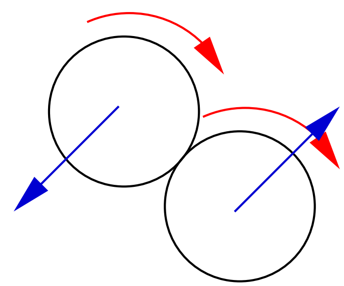


Lines theory, + static, ∘ dynamical.

Evolution of mean square velocity.

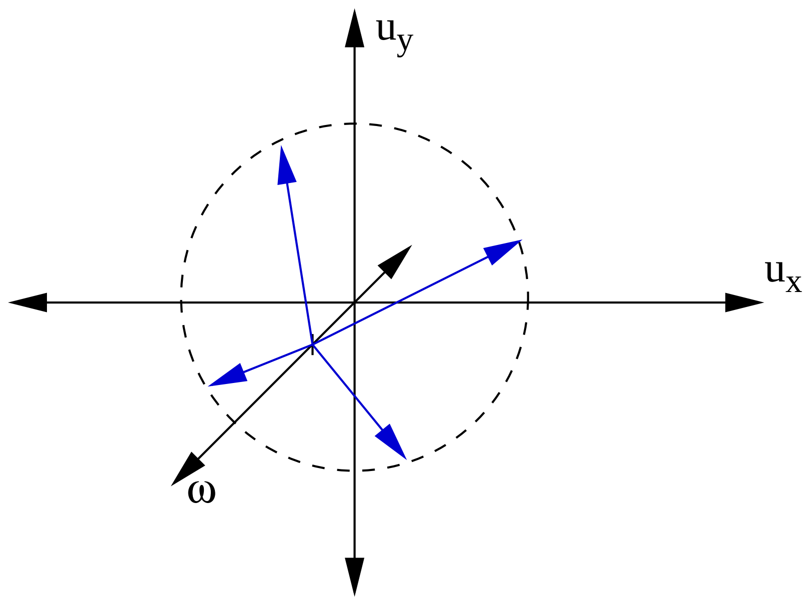


$e_t = 0.84$



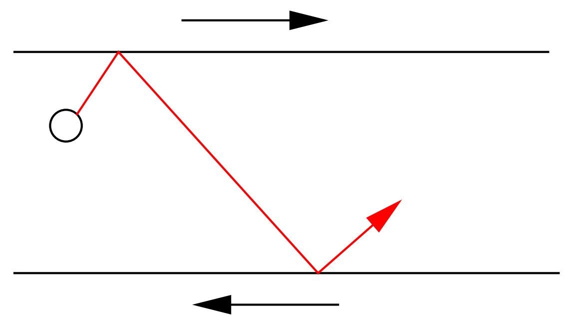
Dynamical steady state:

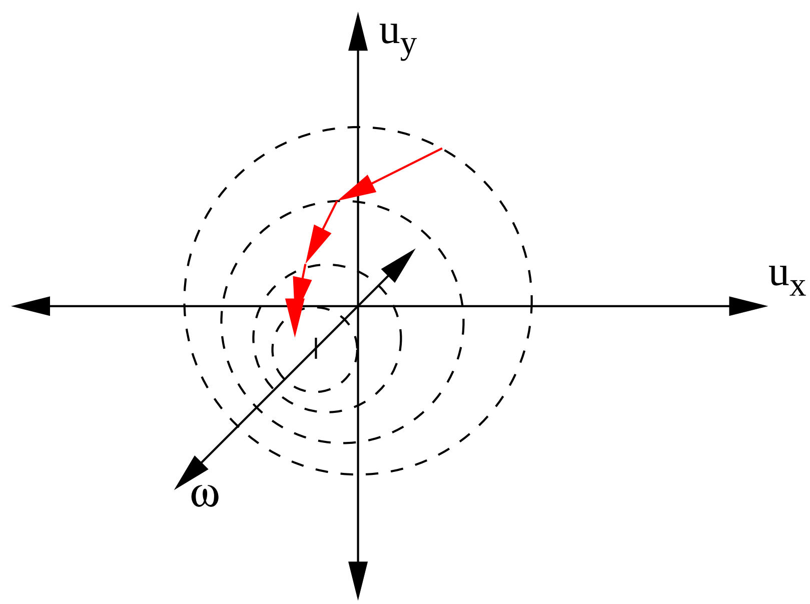
- Particle velocity scattered in binary collision.



Dynamical steady state:

- Subsequent wall collisions reduce translational velocity for $|u_x/u_y| \sim 1$.
- When $|u_x/u_y| \sim \epsilon^{-1}$, undergoes binary collision again.

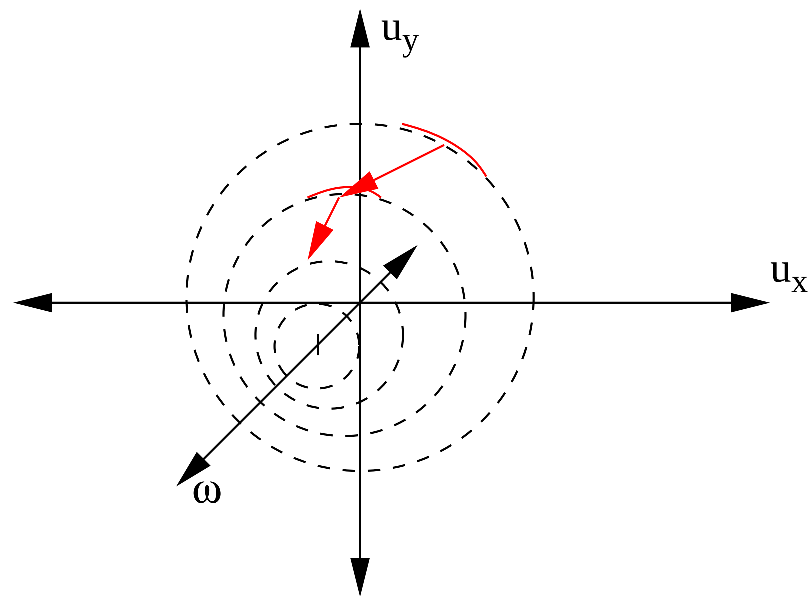




Distribution in dynamical steady state:

Evolution of velocity

$$u_x^{(i)} = \lambda_x u_x^{(i-1)} = \lambda_x^i u_x^{(0)}$$
$$u_y^{(i)} = \lambda_y u_y^{(i-1)} = \lambda_y^i u_y^{(0)}$$
$$\Omega^{(i)} = \lambda_x \Omega^{(i)} = \lambda_x^i \Omega^{(0)}$$



Flux balance:

$$u_y^{(i-1)} f_{i-1} = u_y^i f_i$$

Solution:

$$f_i = \lambda_y^{-1} f_{i-1} = \lambda_y^{-i} f_0$$

Determine f_0 by normalisation condition.

Velocity moments:

- $$\langle u_y^2 \rangle \sim \epsilon^{\alpha}$$

  $$-\langle u_x u_y \rangle \sim \epsilon^{\alpha}$$

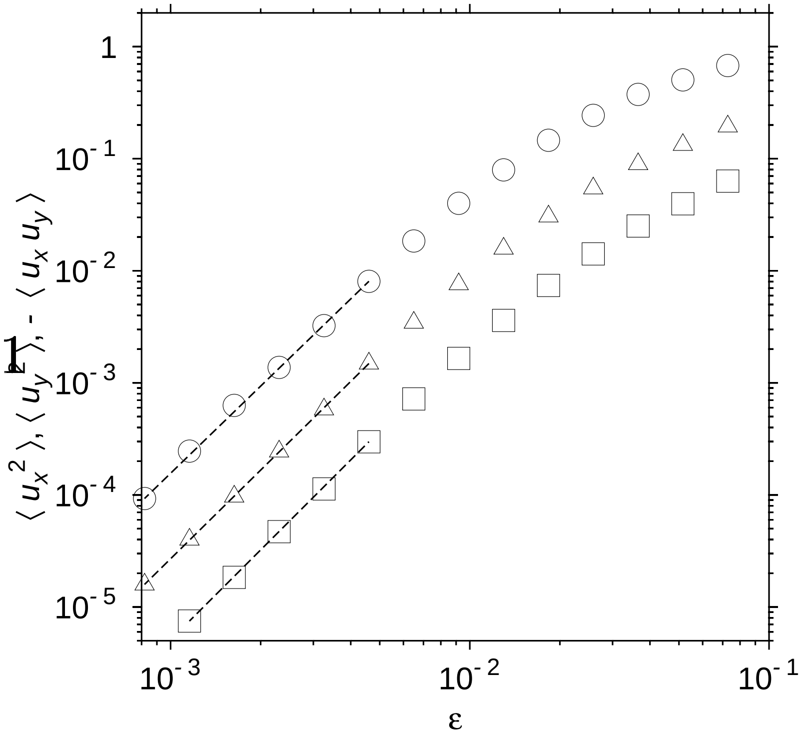
  $\alpha = (\log(\lambda_y)/\log(\lambda_y/\lambda_x)) > 1$

- $$\langle u_x^2 \rangle \sim \epsilon^{\alpha}$$

  for $\lambda_x^2 > \lambda_y$.

- $$\langle u_x^2 \rangle \sim \epsilon^{2\log(\lambda_x)/\log(\lambda_y/\lambda_x)}$$

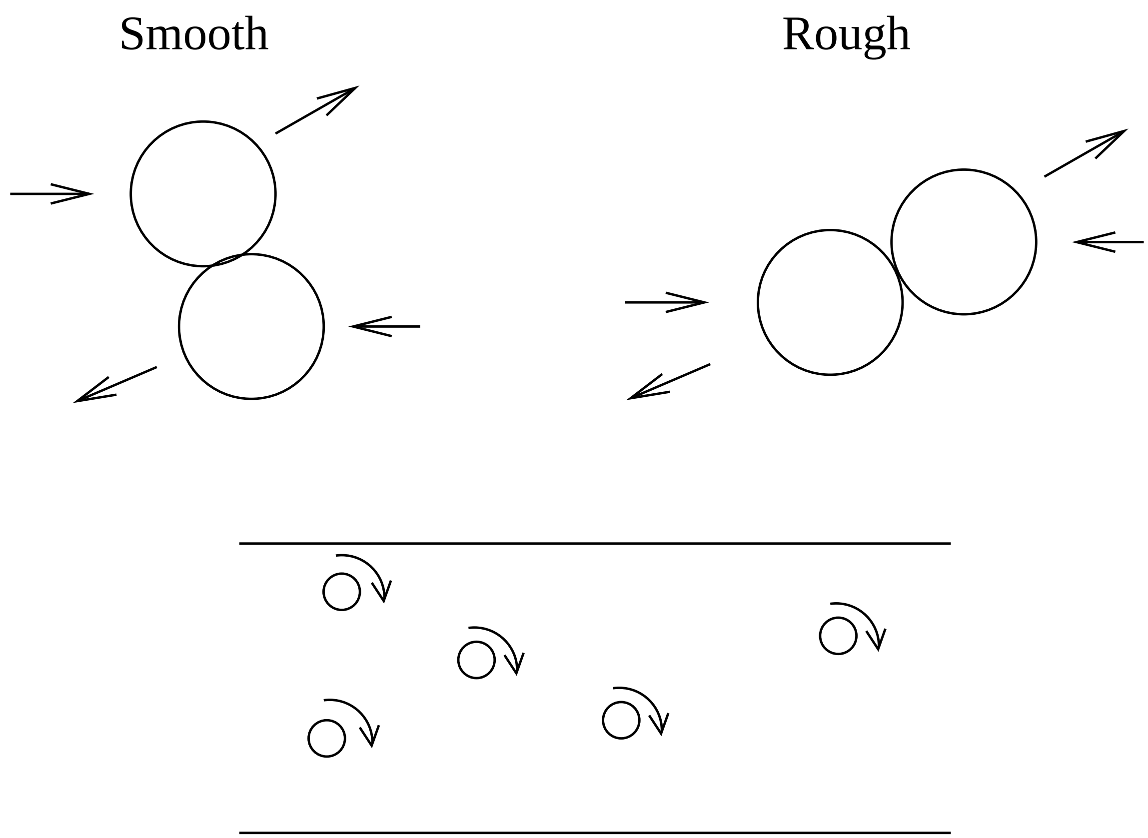
  for $\lambda_x^2 < \lambda_y$.



($\circ$) $\langle u_x^2 \rangle, e_t = 0.70, e_n = 0.75$; ($\triangle$) $\langle u_y^2 \rangle, e_t = 0.70, e_n = 0.75$; ($\square$) $-\langle u_x u_y \rangle, e_t = 0.70, e_n = 0.75$.

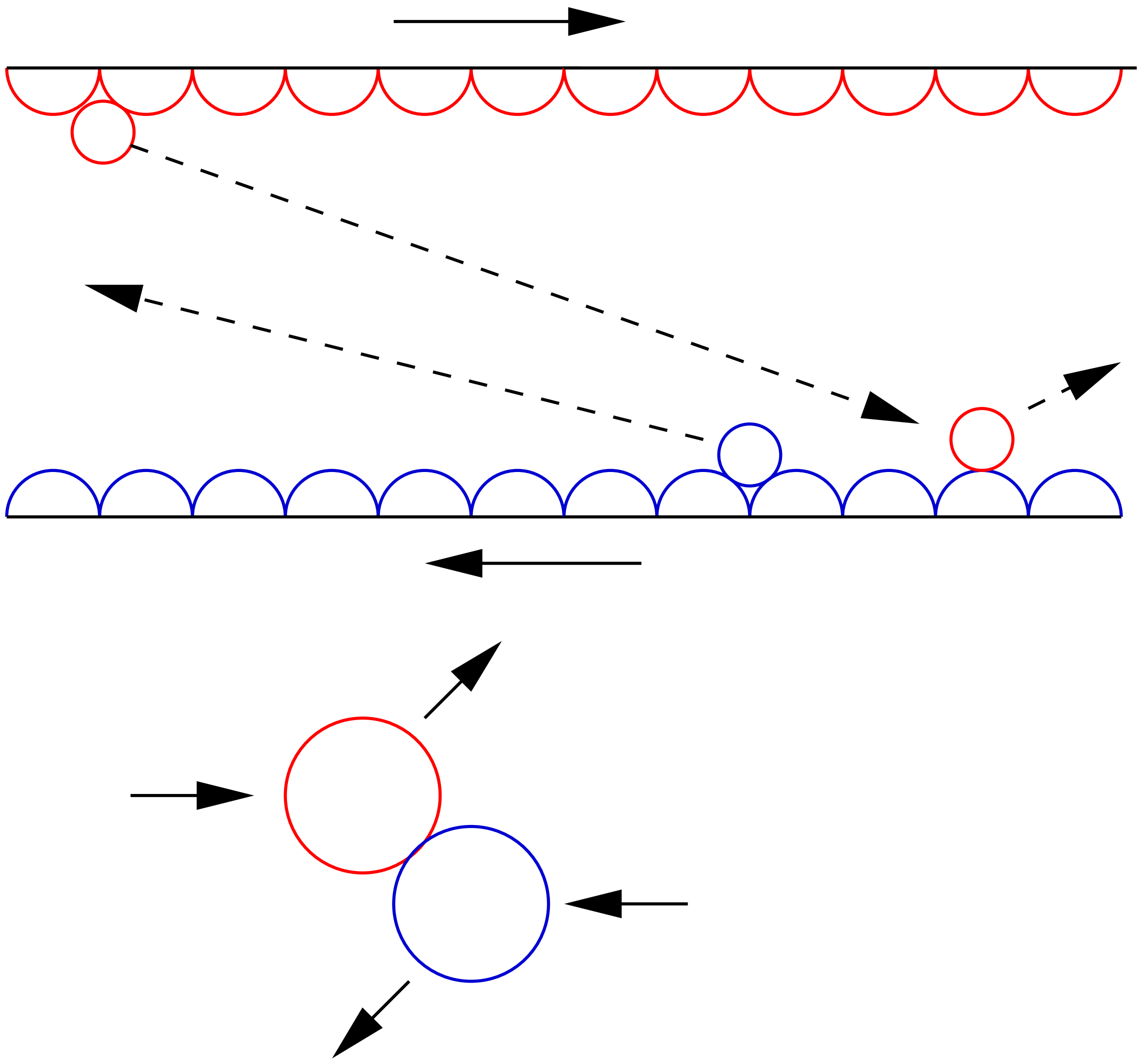
More realistic boundary conditions:

Flat walls; sticking & sliding friction:

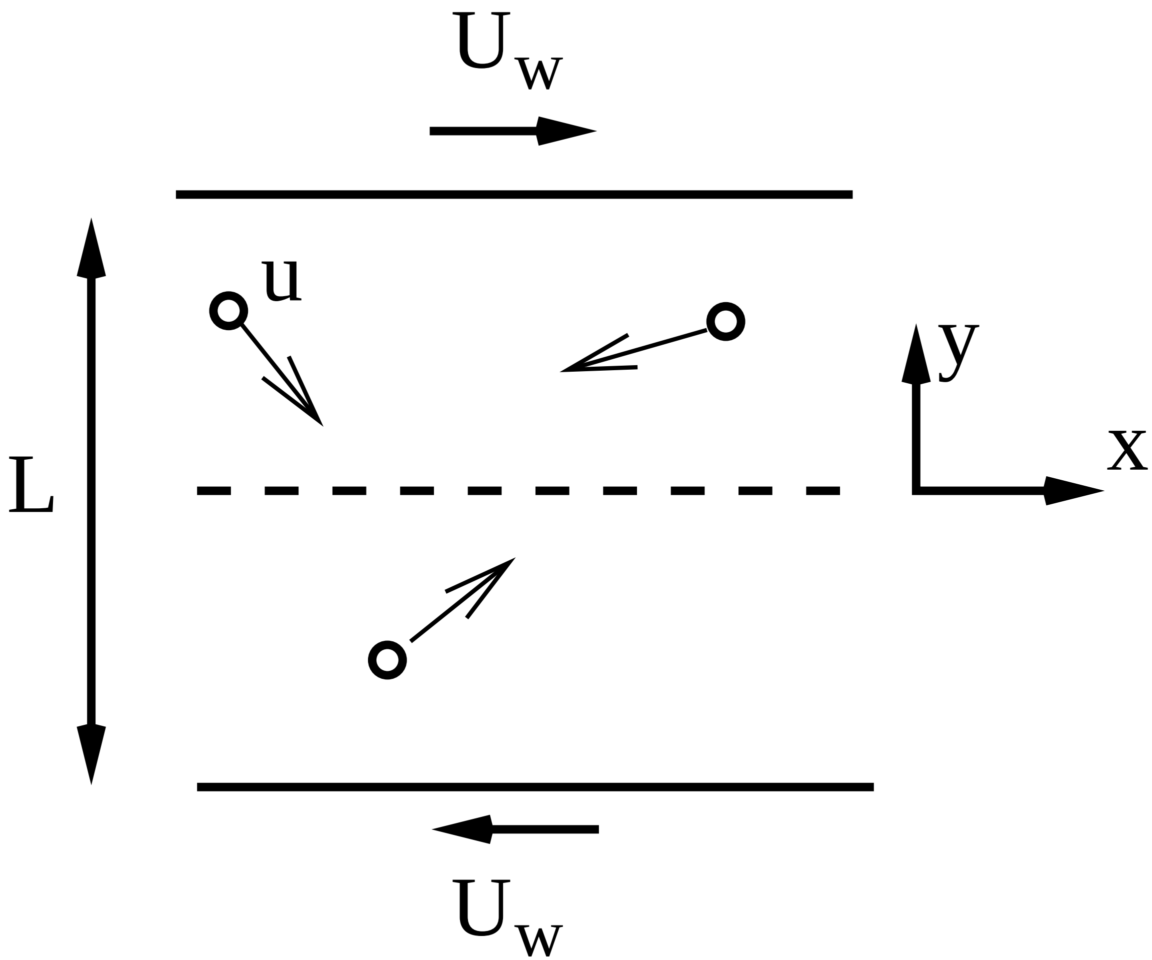


More realistic boundary conditions:

Bumpy walls; sticking & sliding friction:



Bagnold coefficients:



Smooth walls:

$$B^H_{xx} \sim (nL^2)$$

$$B^H_{yy} \sim (nL^2)(nd^2L)$$

$$B^H_{xy} \sim (nL^2)(nd^2L)\log{(nd^2L)}$$

Rough walls:

$$B^H_{xx} \sim (nL^2)(nd^2L)^\alpha$$

$$B^H_{yy} \sim (nL^2)(nd^2L)^\alpha$$

$$B^H_{xy} \sim (nL^2)(nd^2L)^\alpha$$

Smooth walls:

$$(B_{xx}^{H}/B_{xx}^{L}) \sim \varepsilon(d/L)$$

$$(B_{yy}^{H}/B_{yy}^{L}) \sim \varepsilon^{2}(d/L)$$

$$(B_{xy}^{H}/B_{xy}^{L}) \sim \varepsilon^{2}\log(\varepsilon)(d/L)$$

Rough walls:

$$(B_{xx}^{H}/B_{xx}^{L}) \sim \varepsilon^{\alpha}(d/L)$$

$$(B_{yy}^{H}/B_{yy}^{L}) \sim \varepsilon^{\alpha}(d/L)$$

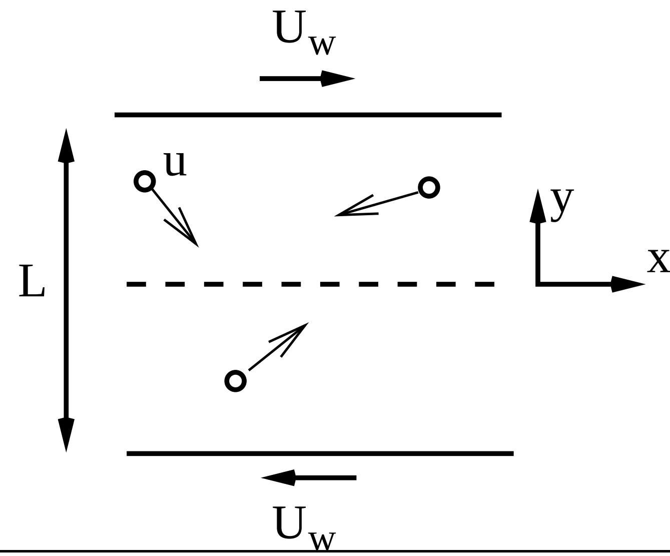
$$(B_{xy}^{H}/B_{xy}^{H}) \sim \varepsilon^{\alpha}(d/L)$$

Smooth: $(\sigma_{xx}/U_w^2) \sim n$

$(\sigma_{yy}/U_w^2) \sim n\varepsilon$

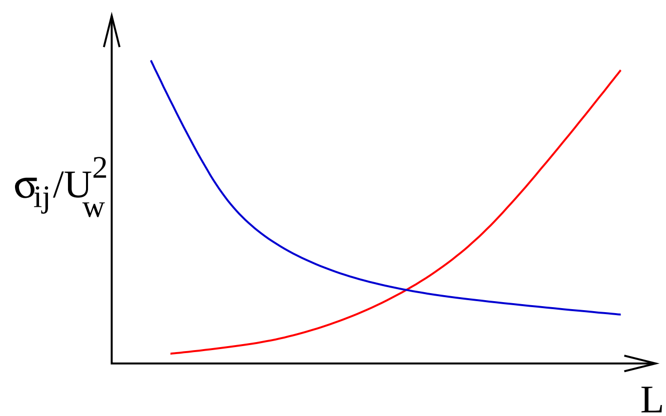
$(\sigma_{xy}/U_w^2) \sim n\varepsilon \log(\varepsilon)$

Rough: $(\sigma_{ij}/U_w^2) \sim n\varepsilon^{\alpha}$



Constant n

Constant nL



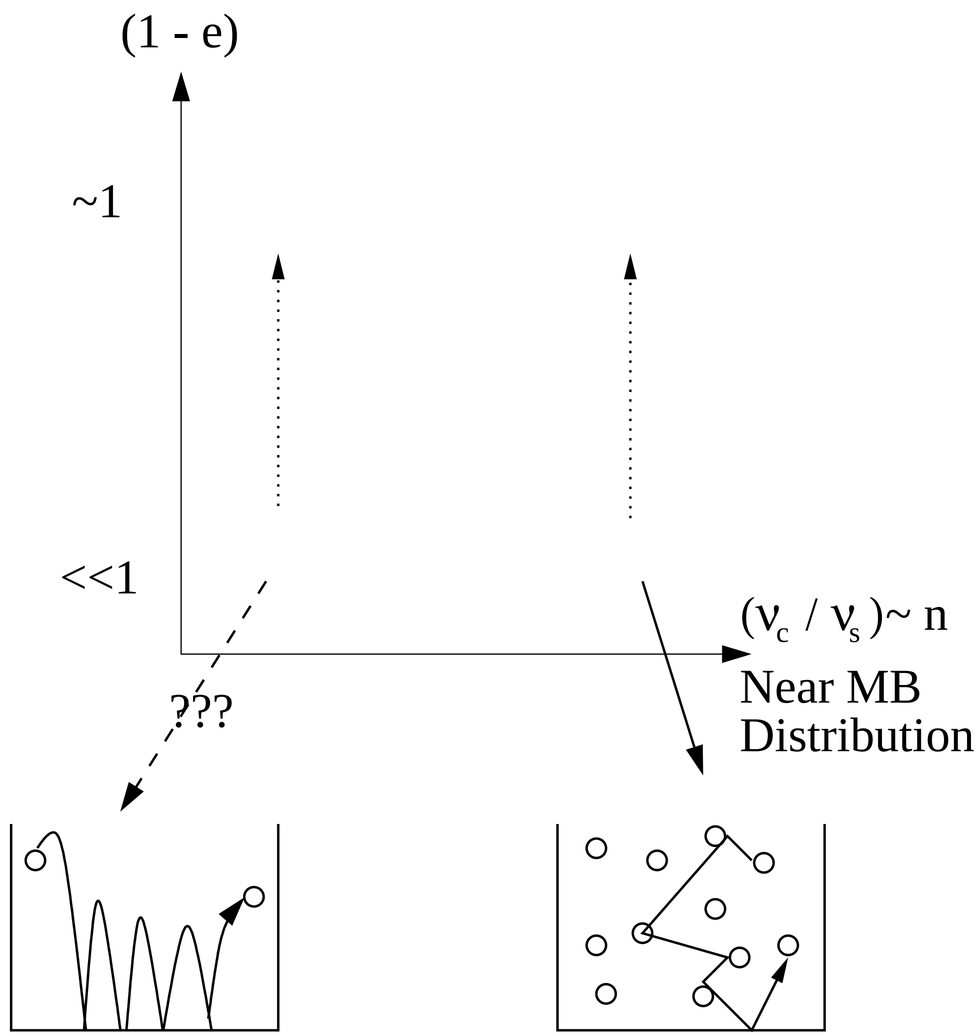
Conclusions:

- Asymptotic solutions of Boltzmann equation in high Knudsen number limit for smooth & rough particle-wall interactions.
- Stresses scale as a power of $\varepsilon \sim Kn^{-1}$.
- Bagnold coefficient at high Knudsen number could be much lower than that at low Knudsen number.

Vibrated granular materials:

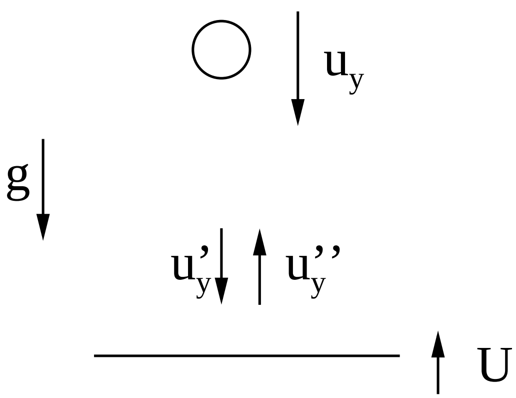
Two frequencies:

- Frequency of interparticle collisions ν_c.
- Frequency of collisions with surface ν_s.

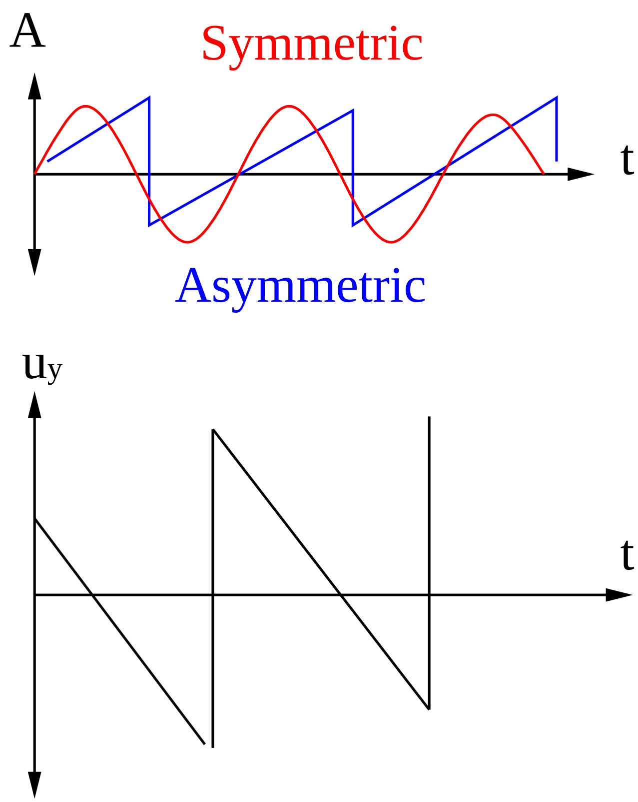


Dilute limit ($\nu_c \ll \nu_s$):

- Leading order — single particle on vibrating surface.



$$(u''_y - U) = -e\,(u'_y - U)$$

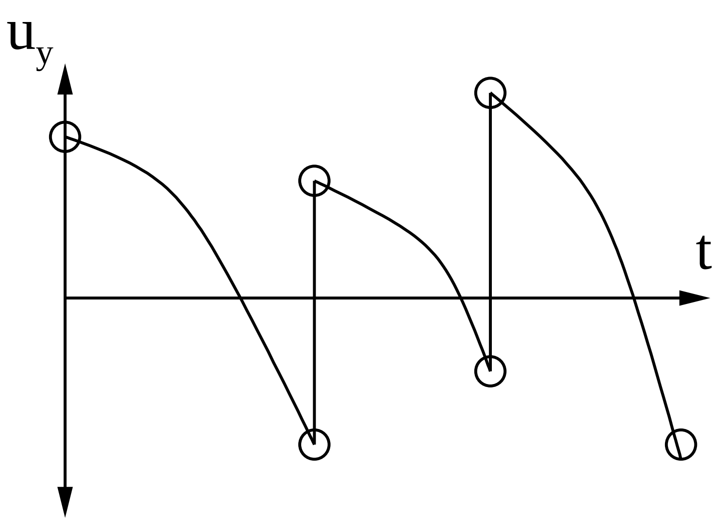


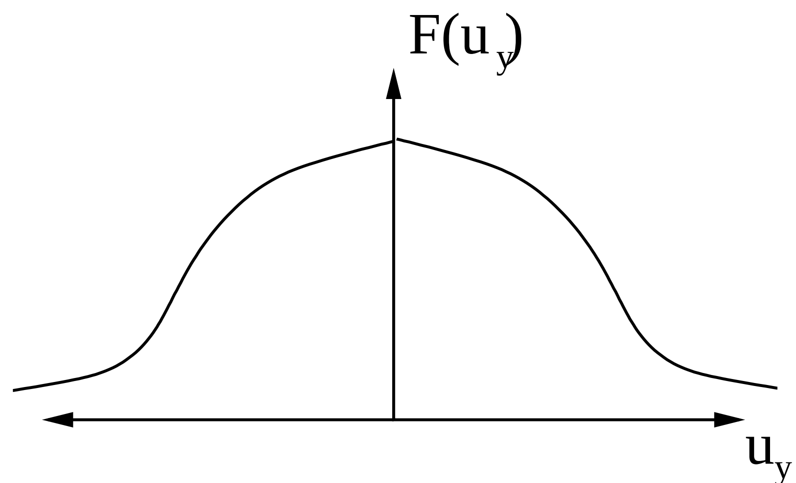
- For $(1 - e) \ll 1$, $T \gg \langle U_s^2 \rangle$.

Dilute limit $\nu_c \ll \nu_s$):

ł

- Distribution function $F(u_y)du_y$ = Probability of finding a particle in du_y about u_y at the surface.





Dilute limit $\nu_c \ll \nu_s$:

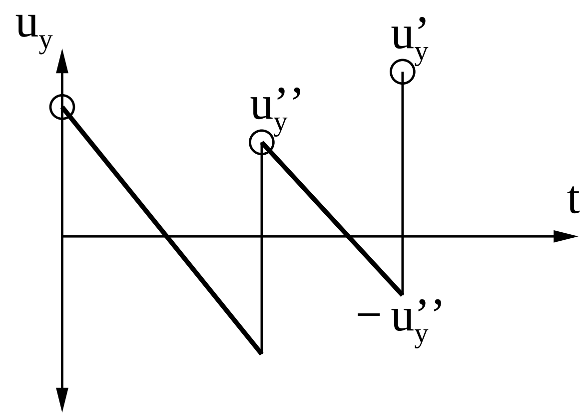
ł

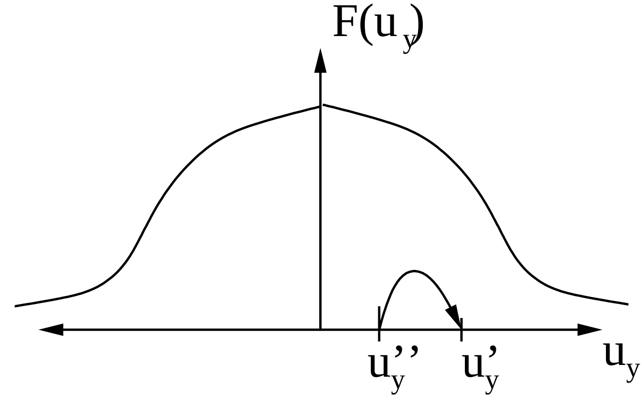
- Flux balance at steady state

$$\langle (u''_y + U) F(u''_y) du''_y \rangle_S = \langle (u'_y + U) F(u'_y) du'_y \rangle_S$$

- Expand in $\epsilon = (1 - e)$, average over U (*U uncorrelated at successive collisions*)

$$2\langle U^2 \rangle_U \left(u_y \frac{d^2 F}{du_y^2} + \frac{dF}{du_y} \right) - 2\langle U \rangle_U \left(u_y \frac{dF}{du_y} + 2F \right) + \epsilon \left(2F u_y + u_y^2 \frac{dF}{du_y} \right) = 0$$





- Symmetric base amplitude $\langle U \rangle_U = 0$

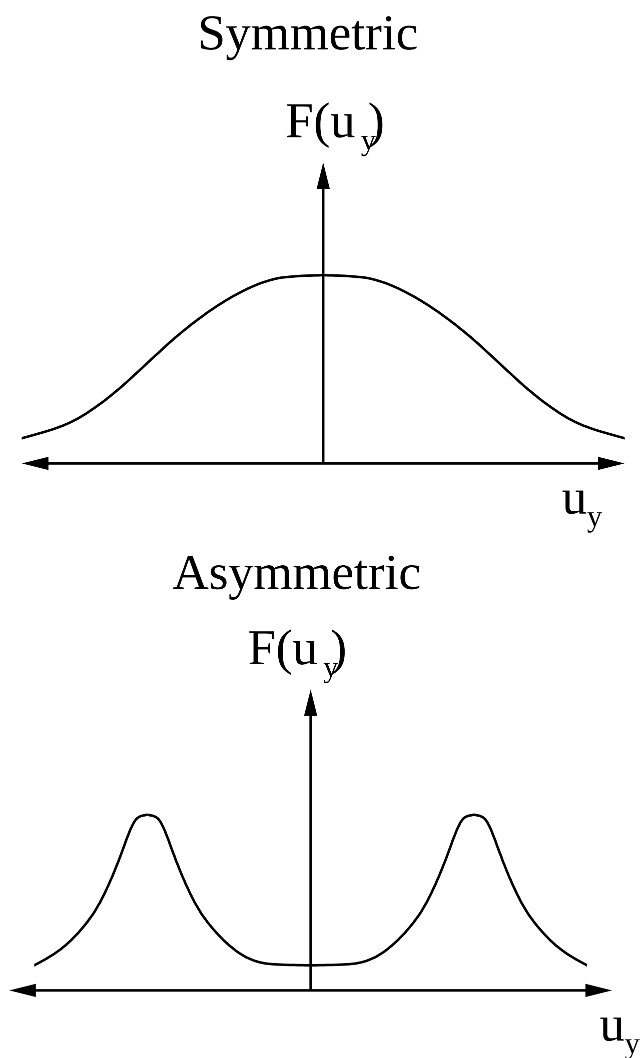
$$F(u_y) = \frac{1}{\sqrt{2\pi T_s}} \exp\left(\frac{-u_y^2}{2T_s}\right)$$

$$T_s = \frac{\langle U^2 \rangle_U}{2\epsilon}$$

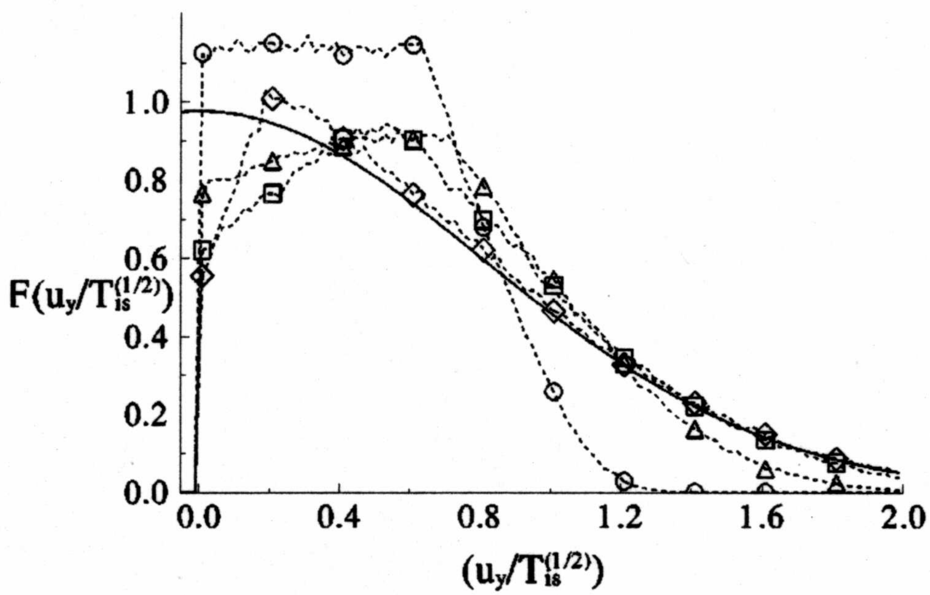
- Asymmetric base amplitude $\langle U \rangle_U \neq 0$

$$F(u_y) = \frac{1}{\sqrt{2\pi T_a}} \exp\left(\frac{-(u_y \pm u_s)^2}{2T_a}\right)$$

$$u_s = \frac{2\langle U \rangle_U}{\epsilon} \quad T_a = \frac{2}{\epsilon^2}\left(\frac{\langle U^2 \rangle_U}{\langle U \rangle_U^2} - 1\right)$$



Symmetric base amplitude



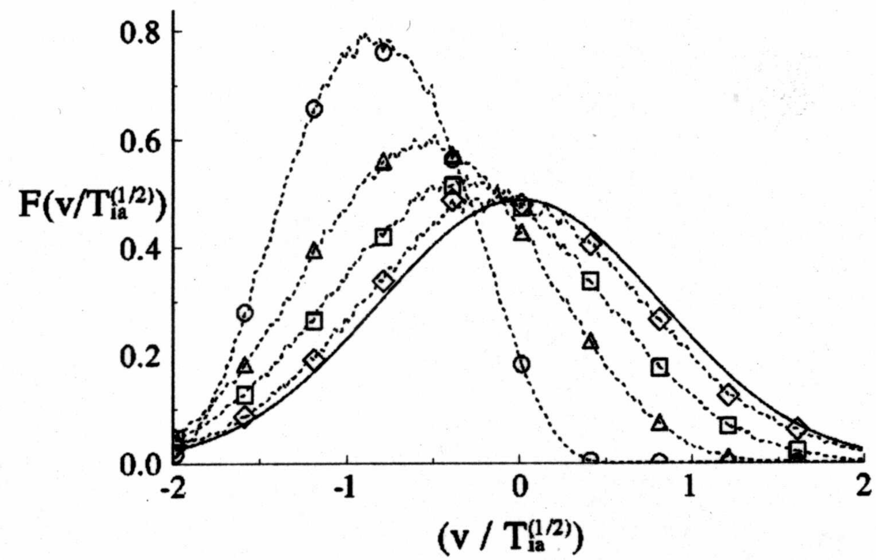
$\circ\ \varepsilon = 0.3$

$\triangle\ \varepsilon = 0.1$

$\square\ \varepsilon = 0.03$

$\diamond\ \varepsilon = 0.01$

Asymmetric base amplitude



○ $\varepsilon = 0.3$

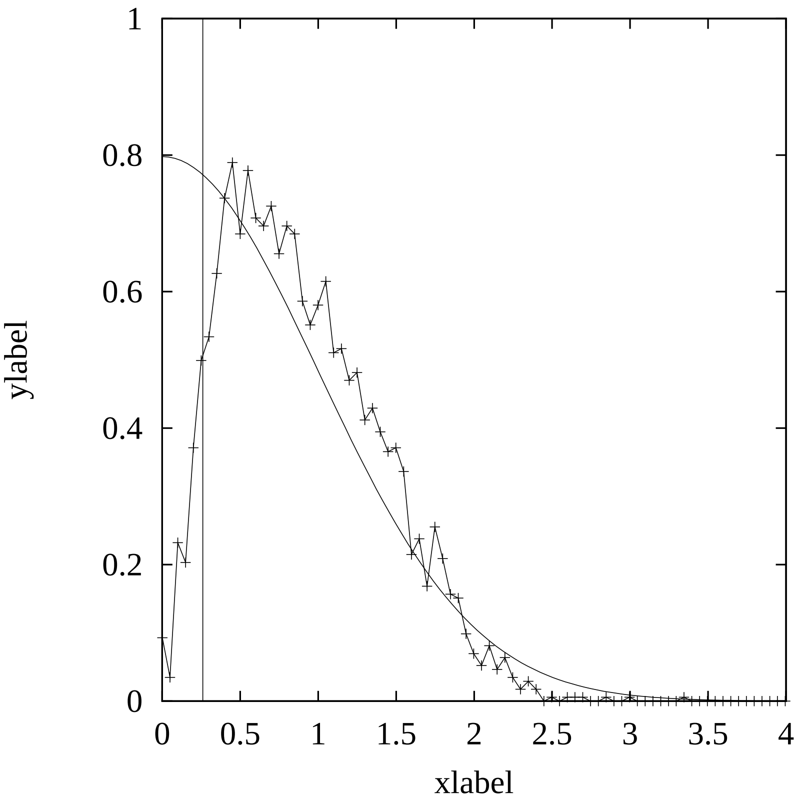
△ $\varepsilon = 0.1$

□ $\varepsilon = 0.03$

◇ $\varepsilon = 0.01$

Comparison with experiments

$f(u_z^*)$

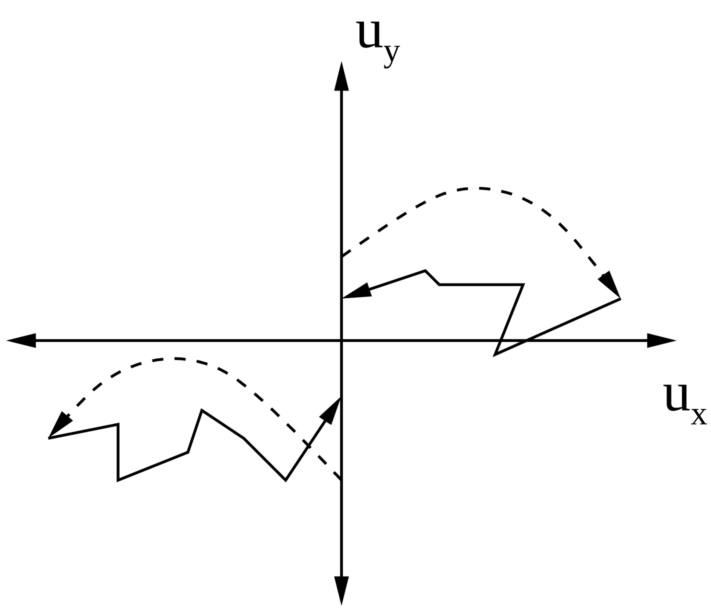


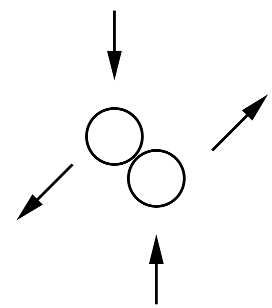
u_z^*

Points — experiments; line — theory. (No fitting parameters).

Dilute limit $\nu_c \ll \nu_s$

- Include effect of pair interactions due to collisions between particles near single particle distribution limit $\delta = (nd/\epsilon) \ll 1$





- Distribution function

$$f(u_x, u_y) = |u_x|^{c\delta-1} f(u_y)$$

Dissipative limit

Conclusions:

- Analytical solution for distribution function.
- Highly non - isotropic rheological properties.
- Good agreement with simulations.