Constitutive relations for flow down an inclined plane.

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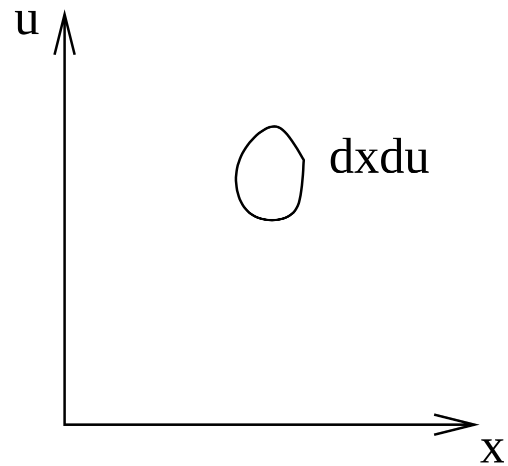
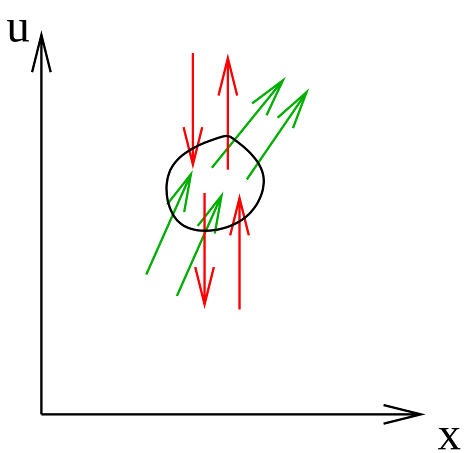
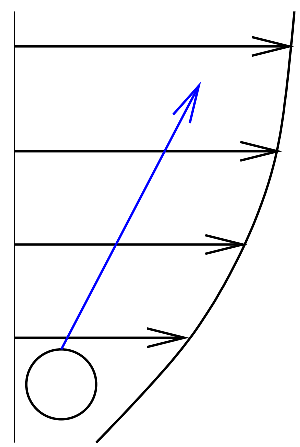
Kinetic theory — elastic hard spheres

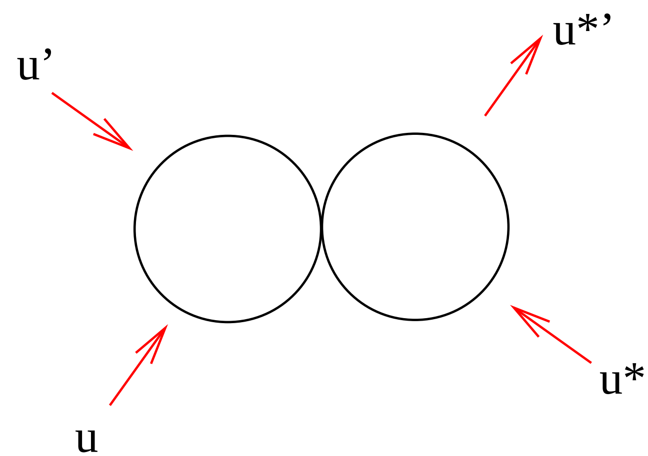
- Velocity distribution
  $f(\mathbf{x}, \mathbf{u}) d\mathbf{x} d\mathbf{u}$.
- Fluctuating velocity
  $\mathbf{c} = \mathbf{u} - \mathbf{U}$



Boltzmann eq $$\frac{\partial(\rho f)}{\partial t} + \frac{\partial(\rho c_i f)}{\partial x_i} + \frac{\partial(\rho a_i f)}{\partial c_i} - \frac{\partial U_i}{\partial x_j}\frac{\partial(\rho c_j f)}{\partial c_i} = \frac{\partial_c(\rho f)}{\partial t}$$







Boltzmann equation: $\frac{\partial(\rho f)}{\partial t} + \frac{\partial(\rho c_i f)}{\partial x_i} - \frac{\partial U_i}{\partial x_j}\frac{\partial(\rho c_j f)}{\partial x_i} = \frac{\partial_c(\rho f)}{\partial t}$

Equilibrium (no gradients)

$$\frac{\partial_c f}{\partial t} = 0$$

Solution — Maxwell-Boltzmann distribution

$$f = (2\pi T)^{-3/2} \exp\left(-mu^2/2T\right)$$

Non-equilibrium — Chapman-Enskog procedure:

$$\frac{\partial(\rho f)}{\partial t} + \frac{\partial(\rho c_i f)}{\partial x_i} - \frac{\partial U_i}{\partial x_j}\frac{\partial(\rho c_j f)}{\partial c_i} = \frac{\partial_c(\rho f)}{\partial t}$$

$$\frac{T^{1/2}\rho f}{L} \qquad G_{xy}\rho f \qquad \frac{T^{1/2}\rho(f - f_{eq})}{\lambda}$$

Asymptotic expansion in parameter $\epsilon = (\lambda/L)$; $f = f_0 + \epsilon f_1 + \ldots$

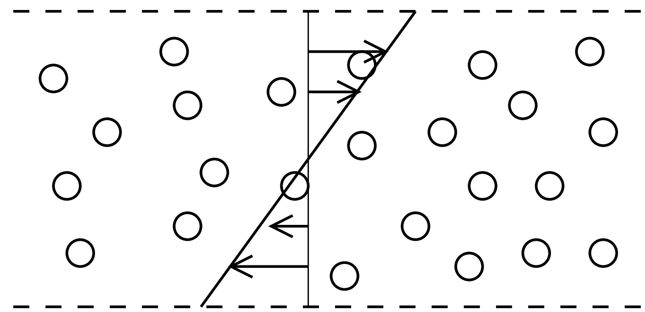
Leading order $\frac{\partial_c(\rho f)}{\partial t} = 0 \rightarrow f = f_{MB}$.

First correction

$$\frac{\partial(\rho f_0)}{\partial t} + \frac{\partial(\rho c_i f_0)}{\partial x_i} - \frac{\partial U_i}{\partial x_j}\frac{\partial(\rho c_j f_0)}{\partial c_i} = \frac{\partial_c(\rho f_1)}{\partial t}$$

Steady homogeneous shear flow of inelastic particles:

$$-G_{ij}\frac{\partial(\rho c_j f)}{\partial c_i} = \frac{\partial_c(\rho f)}{\partial t}$$



Nearly elastic collisions:

$e_n \ll 1 \rightarrow$ Dissipation $\ll$ Particle energy

Expand in $\varepsilon_n = (1 - e_n)^{1/2}$.

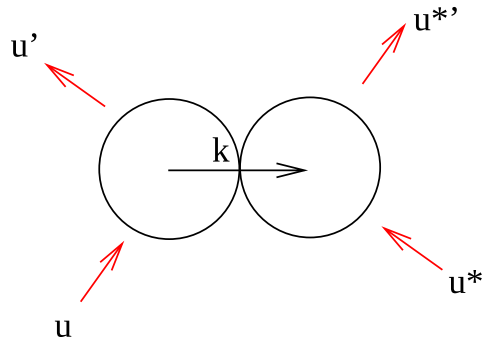
Leading order $\frac{\partial_c(\rho f_0)}{\partial t} = 0 \rightarrow f = f_{MB}$.

Rate of energy production $\sim \mu G_{xy}^2 \sim (T^{1/2}/d^2)G_{xy}^2$.

Rate of energy dissipation $\sim \rho^2 T^{3/2}(1 - e_n^2)^{1/2}$.

$\rightarrow G_{xy} \sim (1 - e_n^2)^{1/2} T^{1/2} \sim \varepsilon_n T^{1/2}$.

Collision rules — smooth particles



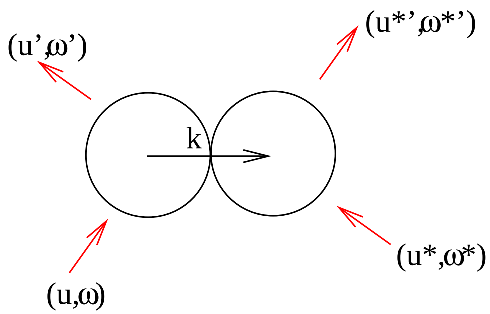
Relative velocity $\mathbf{w} = \mathbf{u} - \mathbf{u}^*$

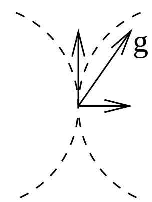
$$w'_k = -e_n w_k = -(1 - \varepsilon_n^2) w_k$$

$$w'_t = w_t$$

Energy conserved for $\varepsilon_n = 0$.

Collision rules — rough particles





$$\mathbf{g}'.\mathbf{k} = -e_n \mathbf{g}.\mathbf{k}$$

$$(\mathbf{I} - \mathbf{kk}).\mathbf{g}' = -e_t(\mathbf{I} - \mathbf{kk}).\mathbf{g}$$

Energy conserved for $e_n = 0$ and $e_t = \pm 1$.

Smooth inelastic particles:
$e_t = 1; (1 - e_n) = \varepsilon_n^2 \ll 1$

Rough inelastic particles:
$(1 + e_t) = \varepsilon_t^2 \ll 1;$
$(1 - e_n) = \varepsilon_n^2 \ll 1$

Boltzmann Collision integral — dilute limit:

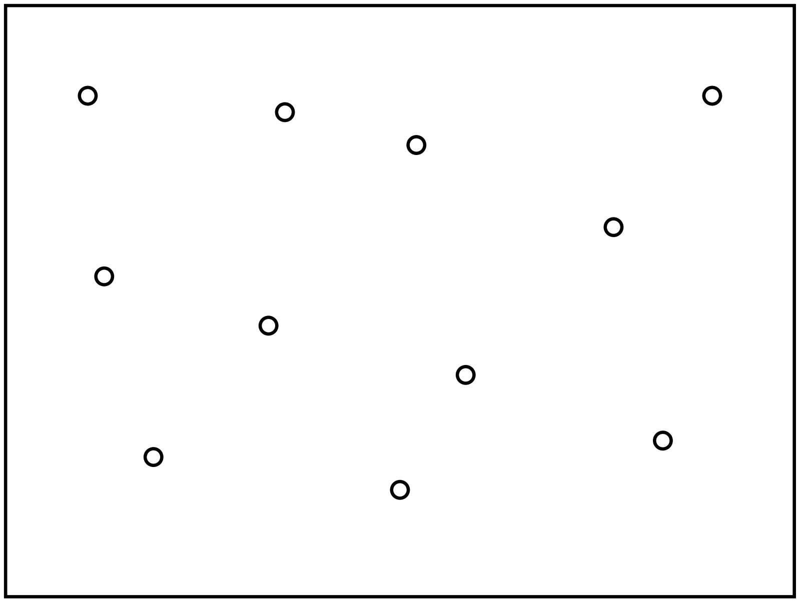
$$\frac{\partial_c \rho f}{\partial t} = \int_{\mathbf{k}} \int_{\mathbf{c}^*} \left(f(\mathbf{c}')f(\mathbf{c}^{*\prime}) - f(\mathbf{c})f(\mathbf{c}^*)\right) \left((\mathbf{u} - \mathbf{u}^*).\mathbf{k}\right)$$

Boltzmann collision integral — dense gases

Enskog approximation:

$$\frac{\partial_c \rho f}{\partial t} = \chi(\phi) \int_{\mathbf{k}} \int_{\mathbf{c}^*} \left(f(\mathbf{c}') f(\mathbf{c}^{*\prime}) - f(\mathbf{c}) f(\mathbf{c}^*) \right) \left((\mathbf{u} - \mathbf{u}^*).\mathbf{k} \right)$$

Pair distribution function $\chi(\phi)$
Accounts for the finite volume of partilcles.



Boltzmann collision integral — dense gases

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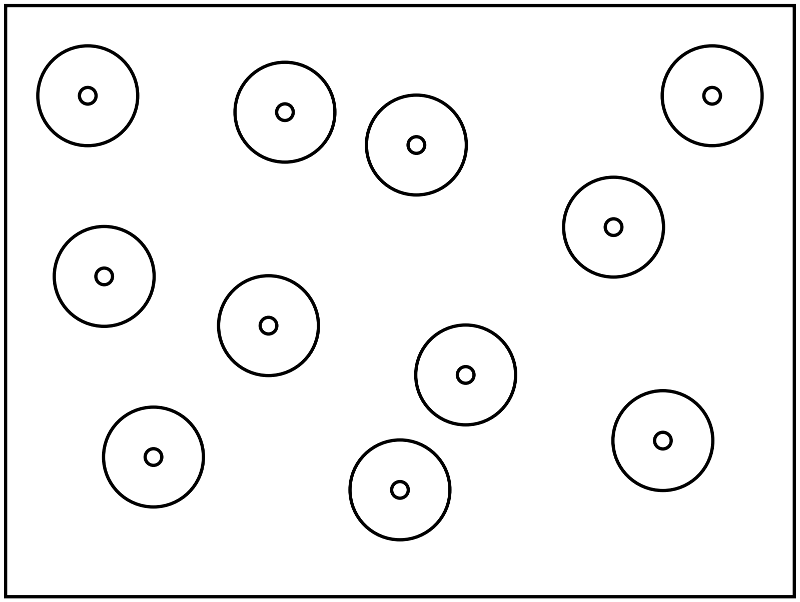
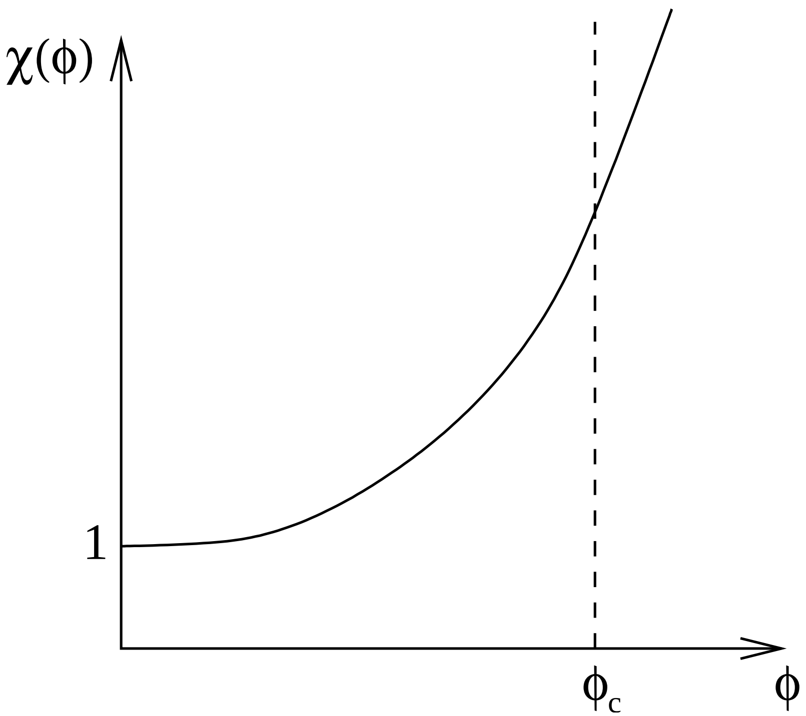


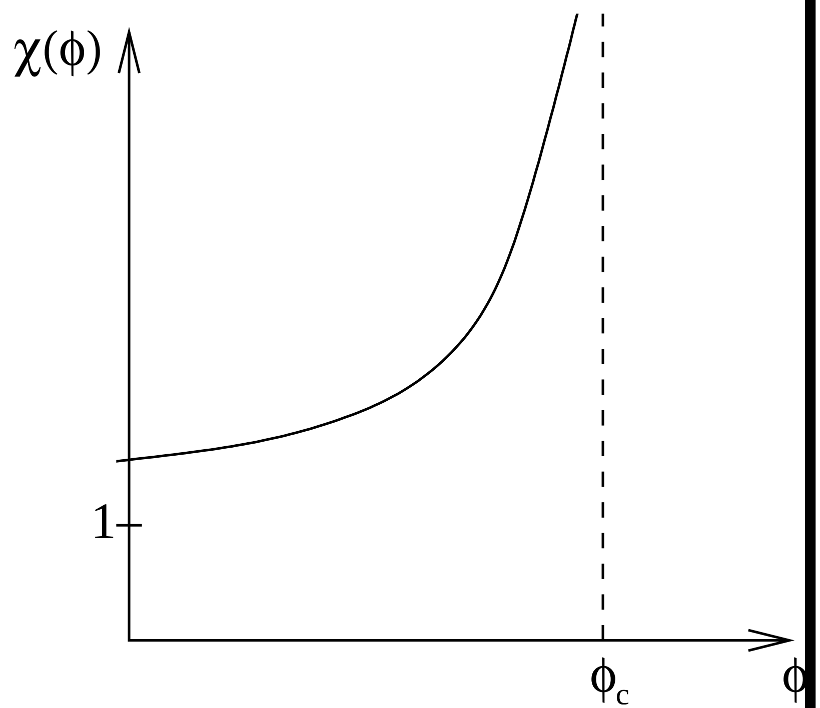
Pair distribution function

Carnahan-Starling pair distribution:

$$\chi(\phi) = \frac{2-\phi}{2(1-\phi)^3}$$



$$\chi(\phi) = \frac{1}{(\phi_c - \phi)^{1/3}}$$

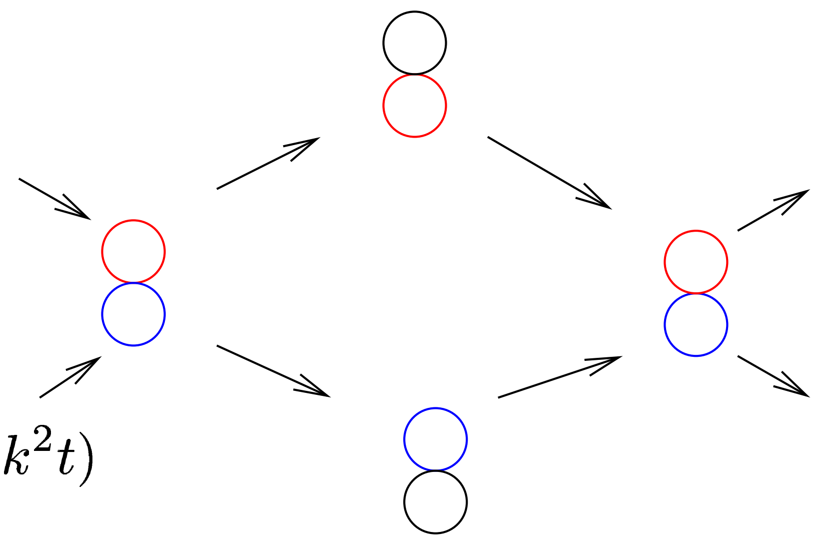


Correlated collisions:

Conservative systems:

$$\sigma_{xy} = \eta G_{xy} + \eta' G_{xy} |G_{xy}|^{1/2}$$

Related to long time tail in auto-correlation function:

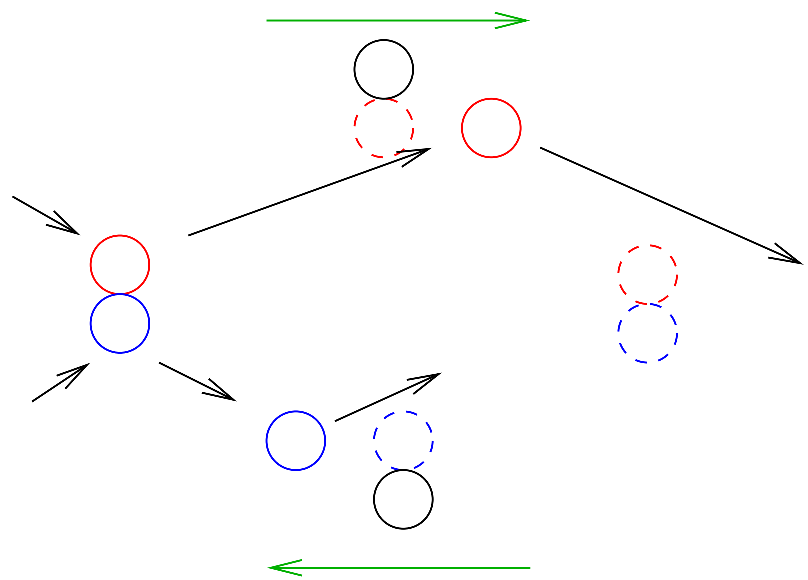


$$\langle v(\mathbf{x},t) v(\mathbf{x},0) \rangle \sim \int d\mathbf{k} \exp\left(-\eta k^2 t\right) \sim t^{-3/2}$$

Correlated collisions:

Sheared systems:
No long time tail for long wavelengths.

$$\langle v(\mathbf{x},t)v(\mathbf{x},0) \quad \sim \quad t^{-3}$$



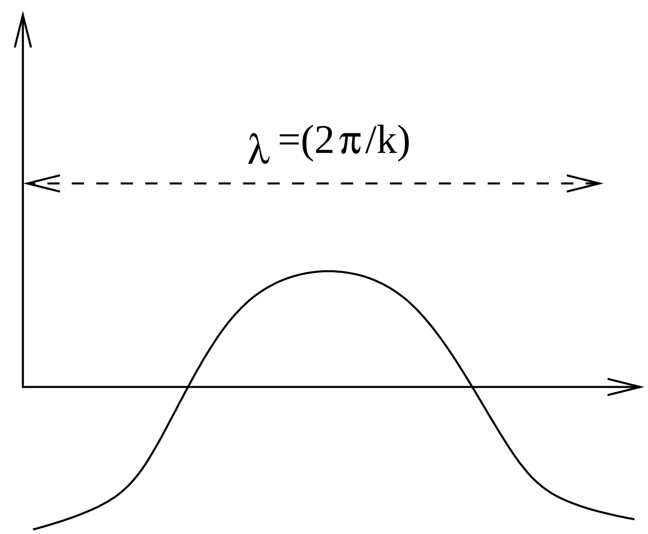
Expect regular coefficients for large length scales.

Moments of Boltzmann equation

- 'Slow' Mass, Momentum & Energy, conserved in collisions.
- Other 'fast' moments decay over time scales $\sim$ collision time.

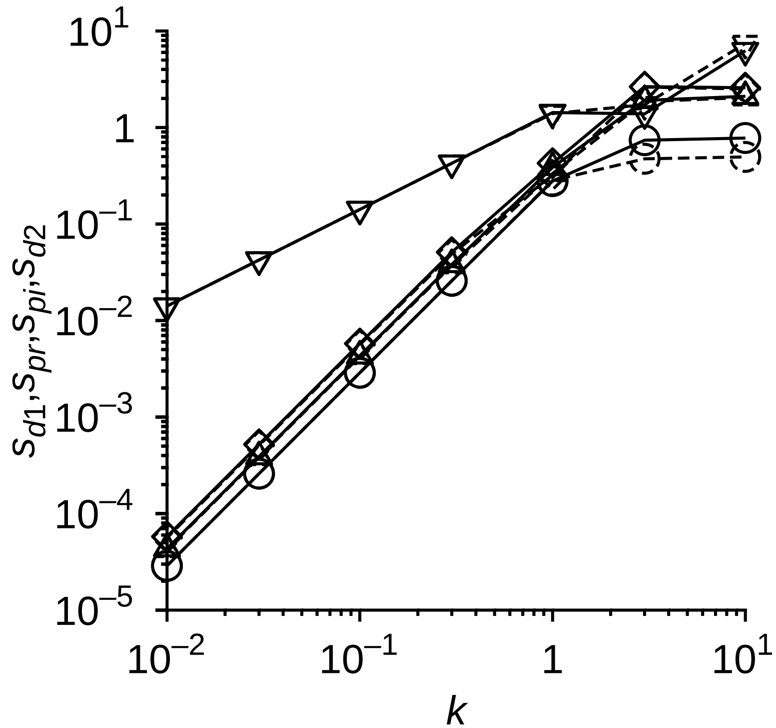
Linear response

- $f(\mathbf{c}) = f_0(\mathbf{c}) + \tilde{f}(\mathbf{c}) e^{(st + \imath kx)}$
- Linearised Boltzmann equation
  $$\left[s + \imath k c_x - G_{ij} \frac{\partial c_i}{\partial c_j} \right] \tilde{f} = L[\tilde{f}]$$
- $\tilde{f}(\mathbf{c}) = \sum_{i=1}^{N} A_i \psi_i(\mathbf{c})$
- $(sI_{ij} + \imath k X_{ij} - G_{ij} - L_{ij}) A_j = 0$



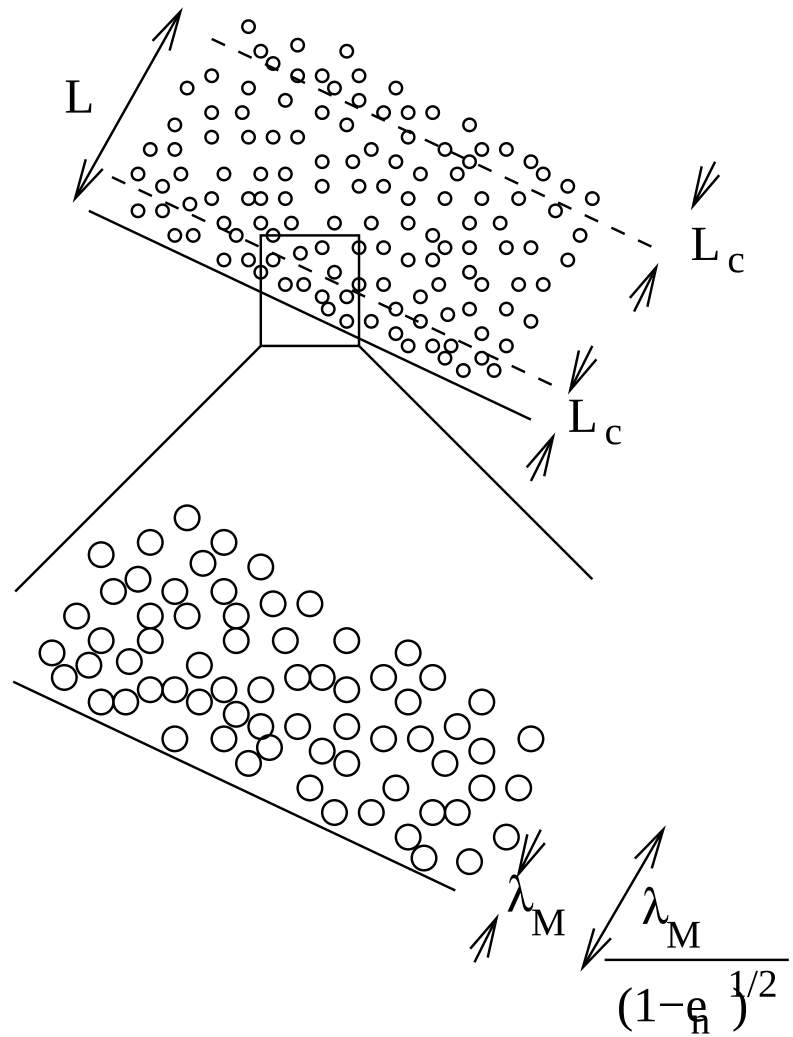
Hydrodynamic modes for elastic system

- Number of eigenvalues depends on number of basis functions chosen.
- For $k \to 0$,
  Transverse momenta $s_t = -(\mu/\rho)k^2$.
  Energy $s_e = -D_T k^2$.
  Mass & longitudinal mom.
  $s_l = \pm \imath k \sqrt{p_\rho} - \rho^{-1}(\mu_b + 4\mu/3)k^2$.
- All other modes with negative eigenvalues, indicating that other transients decay.



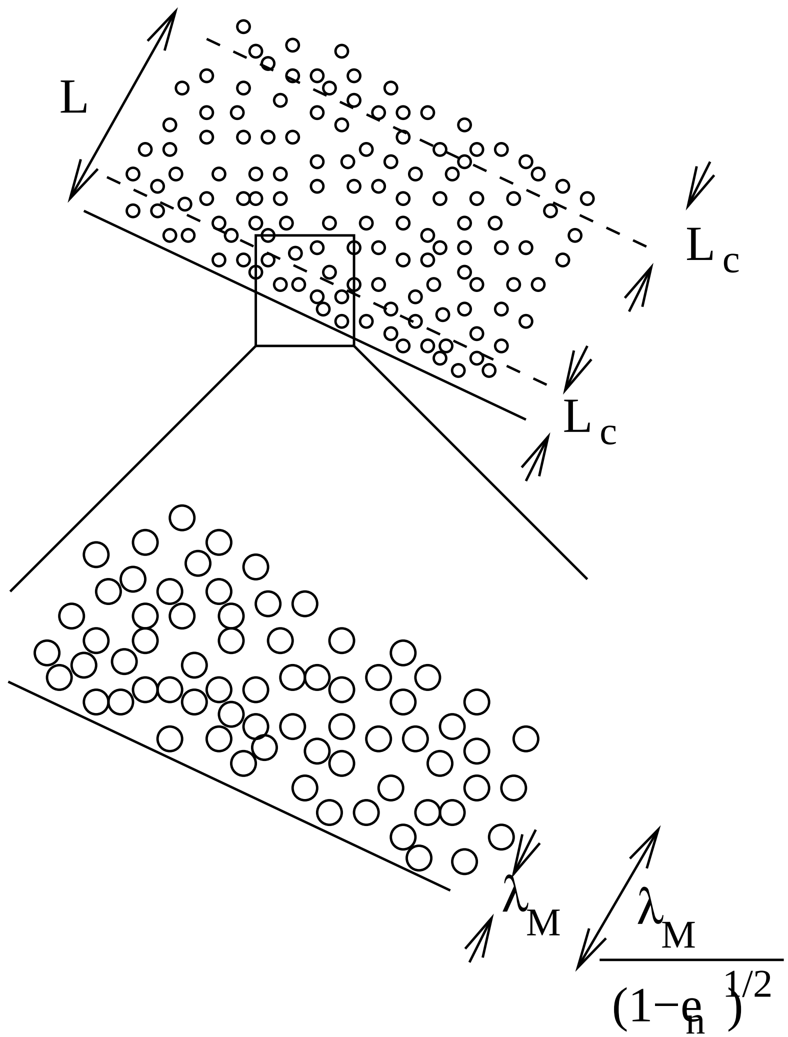
Hydrodynamic modes for smooth inelastic spheres

- Energy *not conserved.*
- Source of energy.
- Rate of conduction $(\lambda_M T^{1/2}/L^2)$.
- Rate of dissipation $((1-e)T^{1/2}/\lambda_M)$.
- Conduction length $L_c = (\lambda_M/(1-e)^{1/2}$.
- Energy conserved $L \ll L_c$.
- *Adiabatic approx.* $L \gg L_c$. Local balance between source and dissipation.



Hydrodynamic modes for smooth inelastic spheres

- Anguler momentum *conserved* in a reference frame located at the point of contact.
- Angular momentum *not conserved* in a reference frame located at the particle center.
- Length scale for decay of angular momentum perturbations $\sim \lambda_M$ from the boundary.



Smooth nearly elastic particles $O(1)$ $O(\varepsilon_n)$ $O(\varepsilon_n^2)$

$$\begin{aligned}\sigma_{ij} &= -p(\phi, S_{ij}, G_{ii})\delta_{ij} + 2\mu(\phi, S_{ij}G_{ii})S_{ij} + \mu_b(\phi, S_{ij}, G_{ii})\delta_{ij}G_{kk} \\ &+ (\mathcal{A}(\phi)(S_{ik}S_{kj} - (\delta_{ij}/3)S_{kl}S_{lk}) + \mathcal{B}(\phi)\delta_{ij}G_{kk}^2 + \mathcal{C}(\phi)S_{ij}G_{kk}) \\ &+ \mathcal{D}(\phi)(S_{ik}A_{kj} + S_{jk}A_{ki}) + \mathcal{F}(\phi)(A_{ik}A_{kj} - (\delta_{ij}/3)A_{kl}A_{lk}) \\ &- \frac{\mathcal{D}(\phi)}{2}\left(\frac{\partial}{\partial x_i}\left(\frac{1}{\rho}\frac{\partial p}{\partial x_j}\right) + \frac{\partial}{\partial x_j}\left(\frac{1}{\rho}\frac{\partial p}{\partial x_i}\right) - \frac{2\delta_{ij}}{3}\frac{\partial}{\partial x_k}\left(\frac{1}{\rho}\frac{\partial p}{\partial x_k}\right)\right)\end{aligned}$$

$$\begin{aligned}p &= \rho T(1 + (4 - 2\epsilon^2)\phi\chi(\phi)) \\ \mu(\phi) &= \frac{5T^{1/2}}{16\sqrt{\pi}\chi(\phi)}\left(1 + \frac{8\phi\chi(\phi)}{5}\right)^2 + \frac{48\phi^2\chi(\phi)T^{1/2}}{5\pi^{3/2}} \\ \mu_b(\phi) &= \frac{16\phi^2\chi T^{1/2}}{\pi^{3/2}}\end{aligned}$$

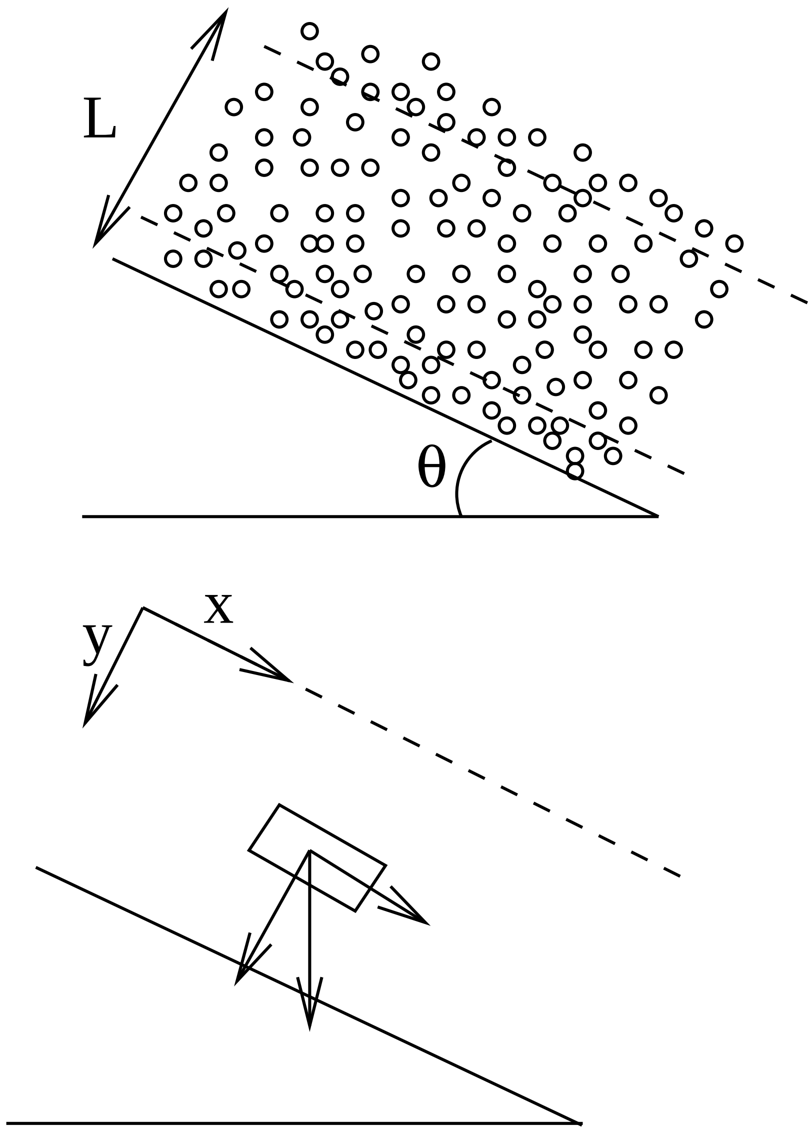
Coefficients $\mathcal{A}$ - $\mathcal{G}$ identical to Burnett expansion for $\varepsilon_n \to 0$.

Rough nearly elastic particles $O(1)$ $O(\varepsilon_n)$ $O(\varepsilon_n^2)$

$$\sigma_{ij} = -p(\phi, S_{ij}, G_{ii})\delta_{ij} + 2\mu(\phi, S_{ij}G_{ii})S_{ij} + \mu_b(\phi, S_{ij}, G_{ii})\delta_{ij}G_{kk}$$
$$+ \mathcal{A}(\phi)(S_{ik} - (\delta_{ij}/3))S_{kl}S_{lk}) + \mathcal{B}(\phi)\delta_{ij}G_{kk}^2 + \mathcal{C}(\phi)S_{ij}G_{kk}$$
$$+ \mathcal{D}(\phi)(A_{ik}S_{kj} + A_{jk}S_{ki} - A_{ik}A_{kj} + (\delta_{ij}/3)A_{kl}A_{lk})$$
$$-\frac{\mathcal{D}(\phi)}{2}\left(\frac{\partial}{\partial x_i}\left(\frac{1}{\rho}\frac{\partial p}{\partial x_j}\right) + \frac{\partial}{\partial x_j}\left(\frac{1}{\rho}\frac{\partial p}{\partial x_i}\right) - \frac{2\delta_{ij}}{3}\frac{\partial}{\partial x_k}\left(\frac{1}{\rho}\frac{\partial p}{\partial x_k}\right)\right)$$
$$+\mathcal{E}(\phi)\left(A_{ik}S_{kj} - S_{ik}A_{kj} + \frac{1}{2}\left(\frac{\partial}{\partial x_j}\left(\frac{1}{\rho}\frac{\partial p}{\partial x_i}\right) - \frac{\partial}{\partial x_i}\left(\frac{1}{\rho}\frac{\partial p}{\partial x_j}\right)\right)\right)$$

$$\mu = \frac{408\phi^2\chi}{35\pi^{3/2}} + \frac{7(7+16\phi\chi)^2}{992\sqrt{\pi}\chi}$$
$$\mu_b = \frac{16\phi^2\chi}{\pi^{3/2}} + \frac{49(1+4\phi\chi)(3+16\phi\chi)}{320\sqrt{\pi}\chi}$$
$$\mathcal{E} = (\rho/4K)$$

Flow down inclined plane: Leading solution



- Momentum equations:

$$\begin{aligned}(d\sigma_{xy}/dy) &= -\rho g \sin(\theta) \\ (d\sigma_{yy}/dy) &= \rho g \cos(\theta)\end{aligned}$$

- Ratio $(\sigma_{xy}/\sigma_{yy}) = -\tan(\theta)$
- Dimensional analysis, $\sigma_{xy} = B_{xy}(\phi)\dot{\gamma}^2$, $\sigma_{yy} \sim B_{yy}(\phi)\dot{\gamma}^2$.

$$\tan(\theta) = -B_{xy}(\phi)/B_{yy}(\phi)$$

- ϕ is independent of height in adiabatic approximation.

Flow down inclined plane: Leading solution

Navier-Stokes approx:

$$\tan(\theta) = \frac{\mu\dot{\gamma}}{p} = \frac{\mu_\phi T^{1/2}\dot{\gamma}}{p_\phi T}$$

$$\tan(\theta) \sim \varepsilon_n$$

Burnett approximation:

$$\frac{\mu\dot{\gamma}}{p - \mathcal{B}\dot{\gamma}^2} = \tan(\theta)$$

$$\frac{\mu_\phi T^{1/2}\dot{\gamma}}{p_\phi T - \mathcal{B}\dot{\gamma}^2} = \tan(\theta)$$

Strain rate:

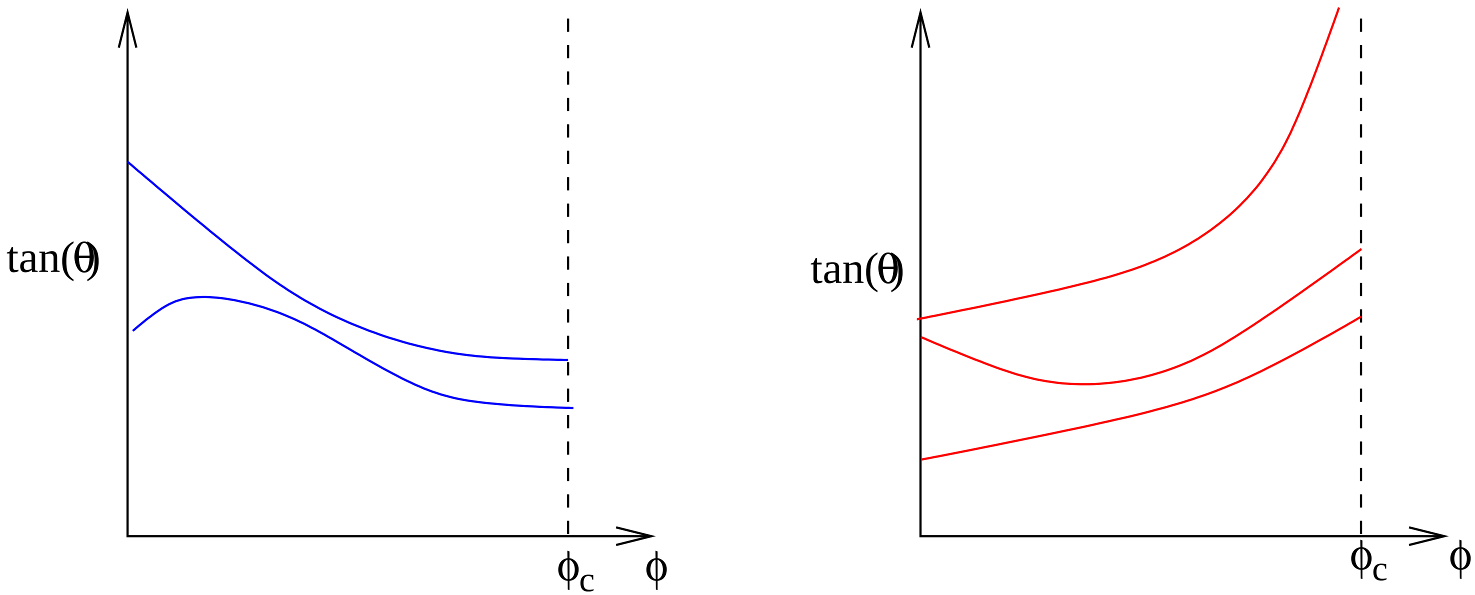
$$\frac{d\sigma_{xy}}{dy} = \mu\dot{\gamma} = \mu_\phi T^{1/2}\dot{\gamma}$$

$$\dot{\gamma} \sim y^{1/2}$$

Density dependence: $\mu_\phi \propto \chi(\phi)$.

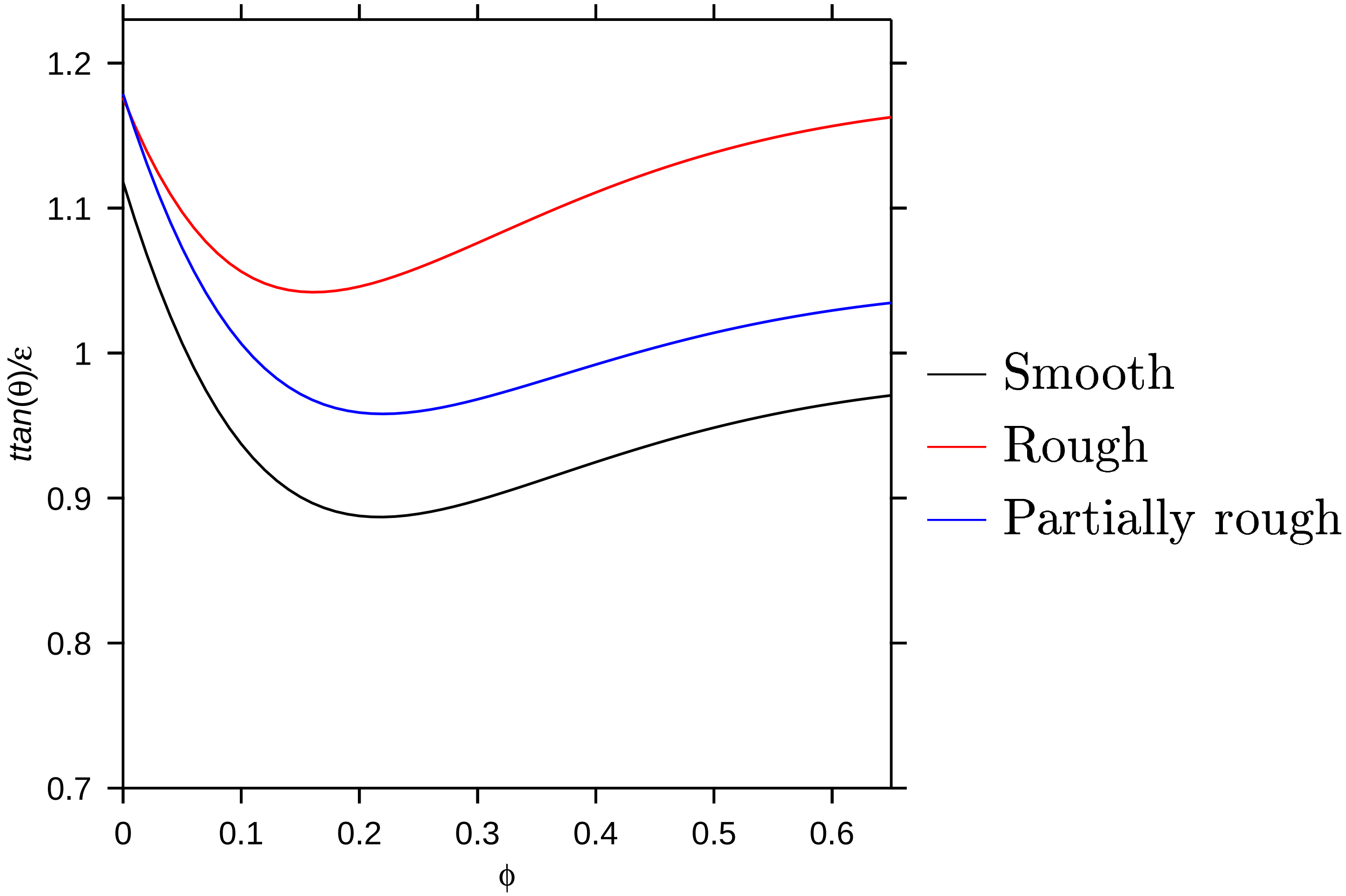
- Strain rate increases continuously from zero if $\chi(\phi) \to \infty$ for $\phi \to \phi_c$.
- Strain rate increases discontinuously if $\chi(\phi)$ finite for $\phi \to \phi_c$.

Flow down an inclined plane: Leading solution



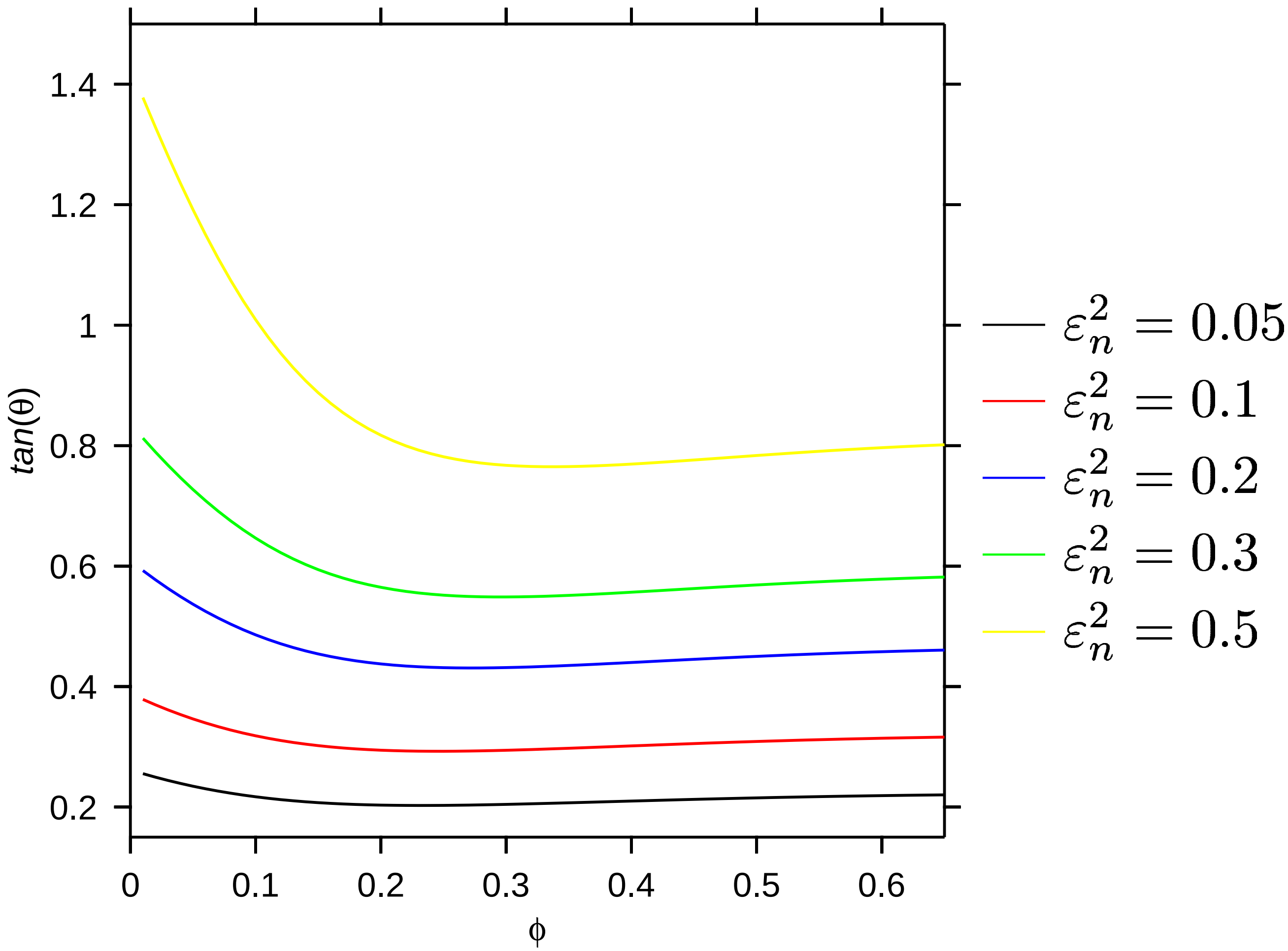
- Variation of θ with volume fraction in the flow.
- Minimum angle θ_c minimum angle at which flow ceases as inclination is decreased.
- Maximum angle θ_m is the maximum angle at which steady flow is sustained.

Flow down inclined plane — Navier-Stokes approx.



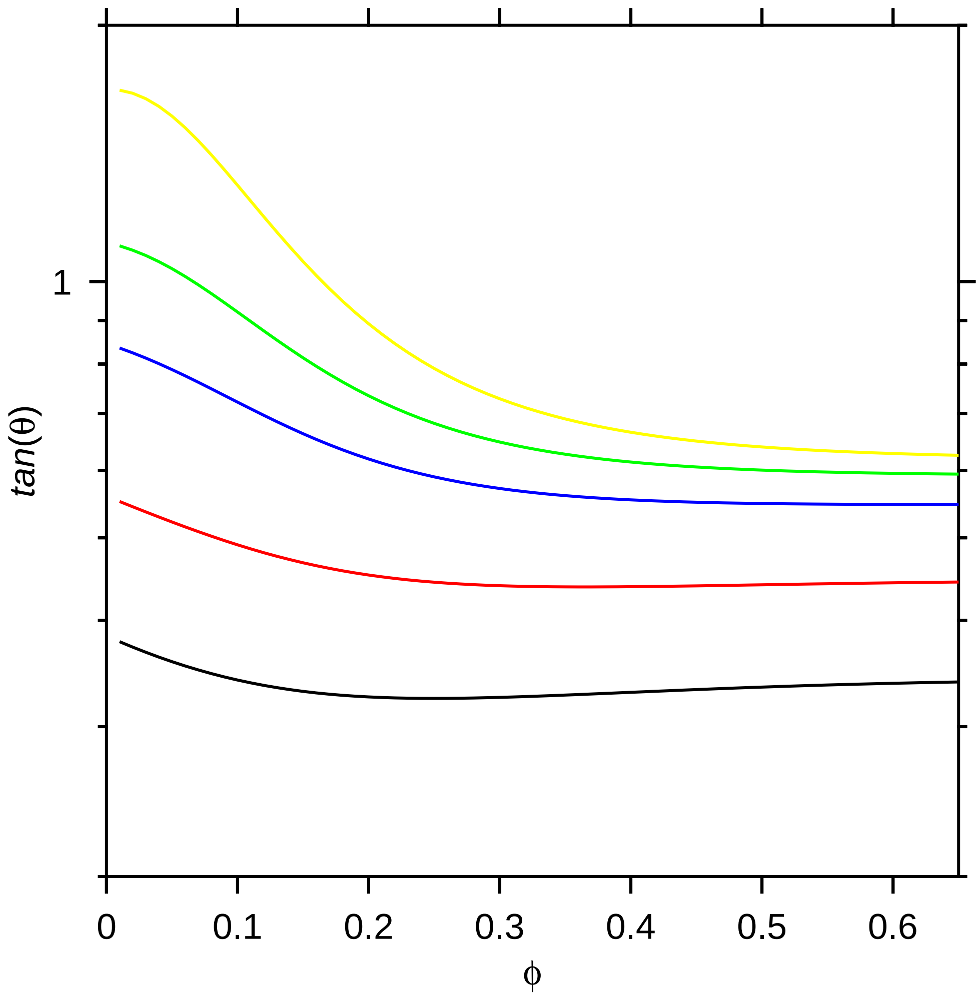
Flow down inclined plane — Burnett approx

Smooth



Flow down inclined plane — Burnett approx.

Rough



$\varepsilon_n^2 = \varepsilon_t^2 = 0.05$

$\varepsilon_n^2 = \varepsilon_t^2 = 0.1$

$\varepsilon_n^2 = \varepsilon_t^2 = 0.2$

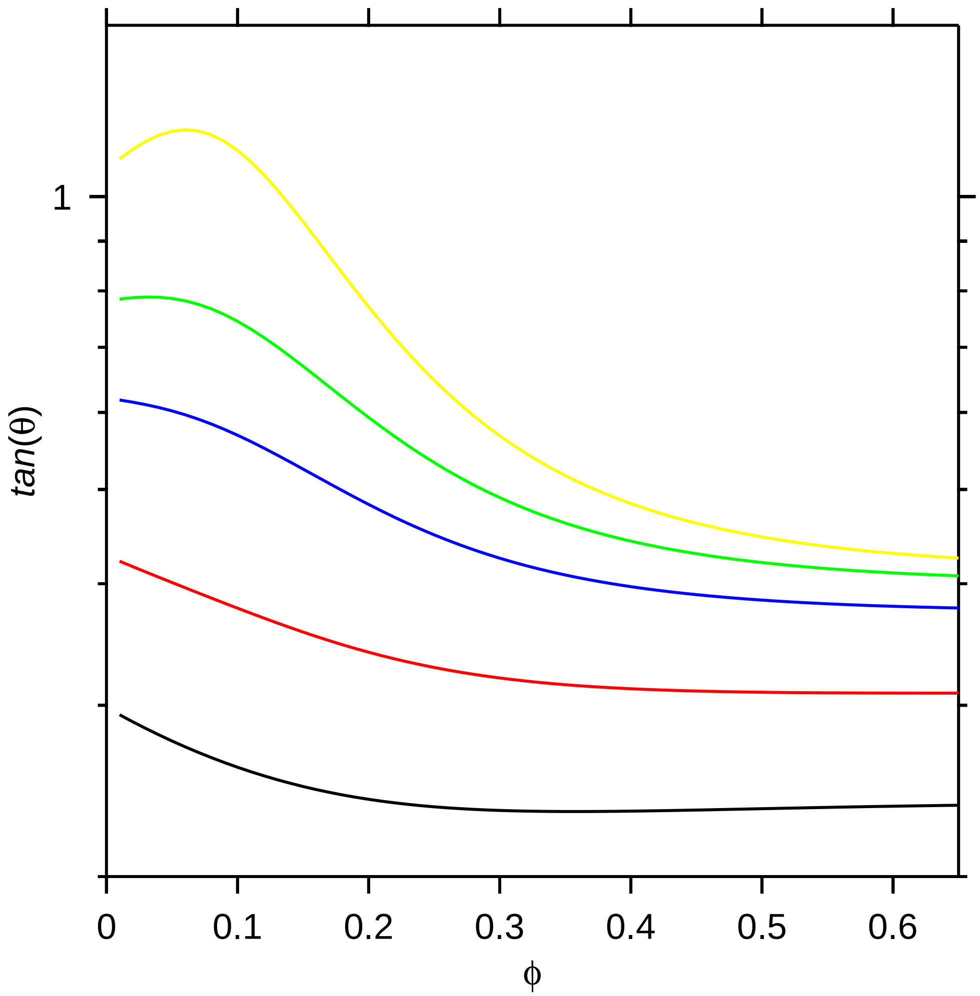
$\varepsilon_n^2 = \varepsilon_t^2 = 0.3$

$\varepsilon^2 = \varepsilon_t^2 = 0.5$

θ increases as ϕ decreases for $\varepsilon_n^2 > 0.192$.

Flow down inclined plane — Burnett approx.

Partially rough



— $\varepsilon^2 = 0.05$

— $\varepsilon_n^2 = \varepsilon_t^2 = 0.1$

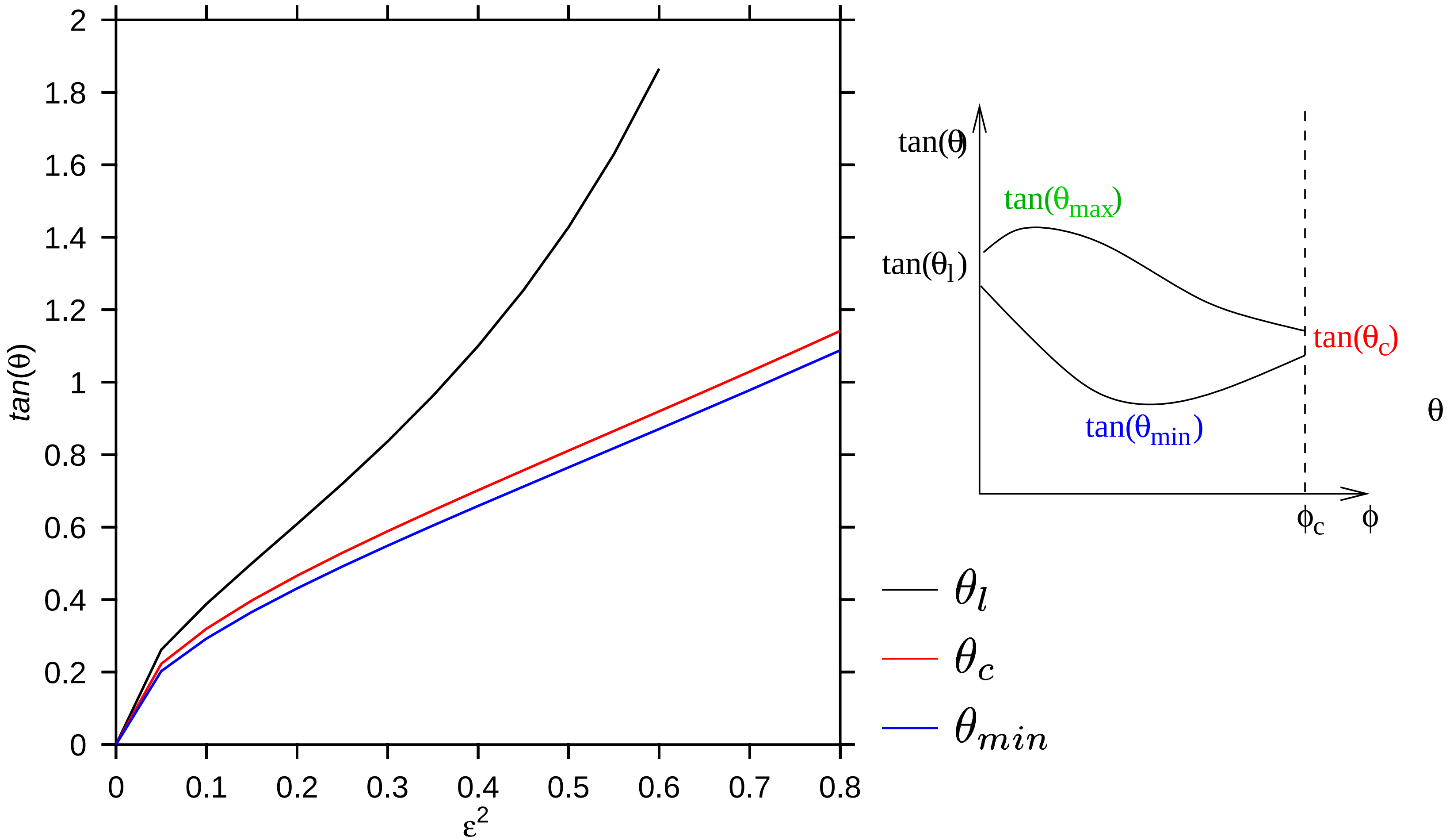
— $\varepsilon_n^2 = \varepsilon_t^2 = 0.2$

— $\varepsilon_n^2 = \varepsilon_t^2 = 0.3$

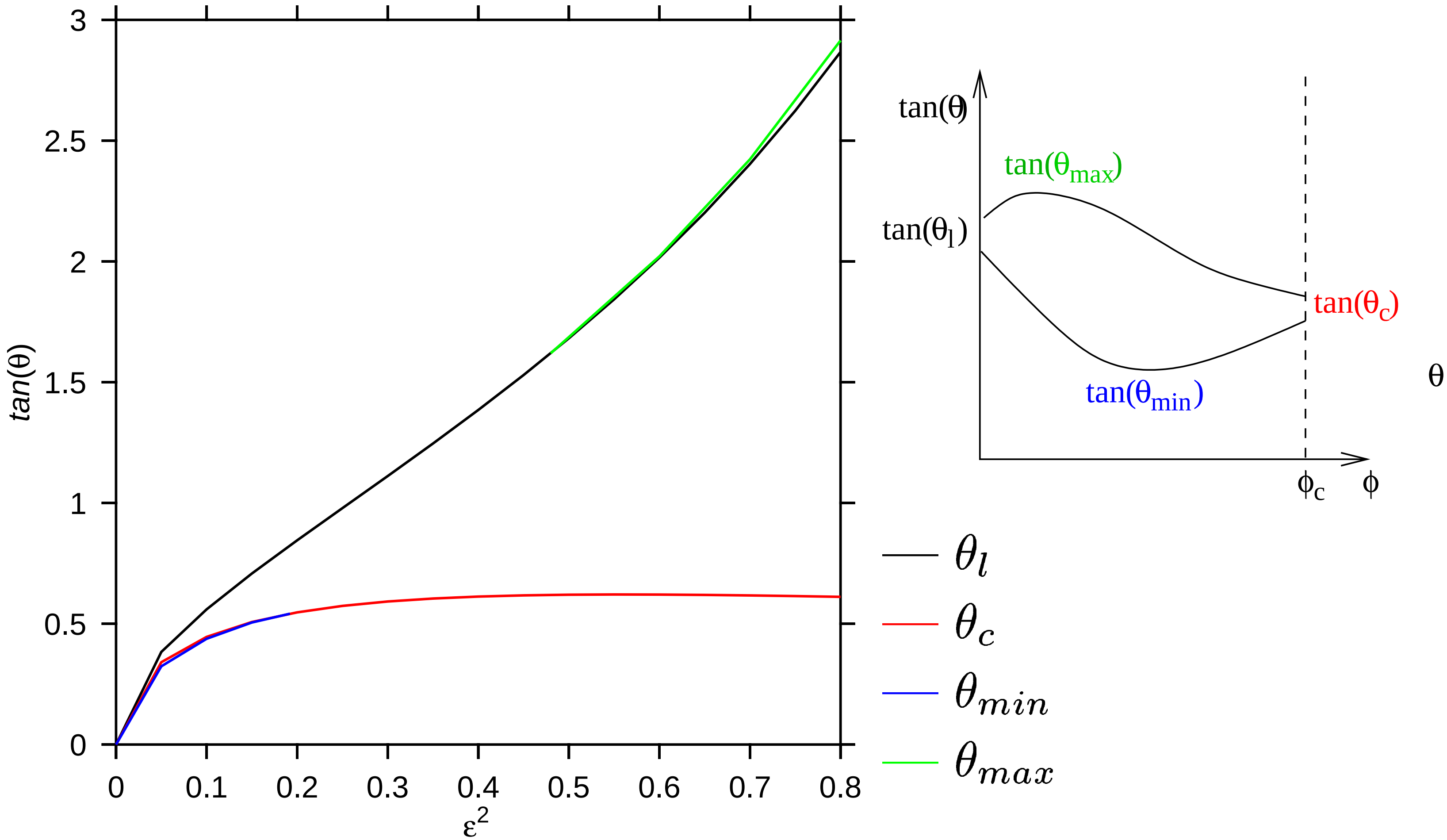
— $\varepsilon_n^2 = \varepsilon_t^2 = 0.5$

θ increases as ϕ decreases for $\varepsilon_n^2 > 0.099$.

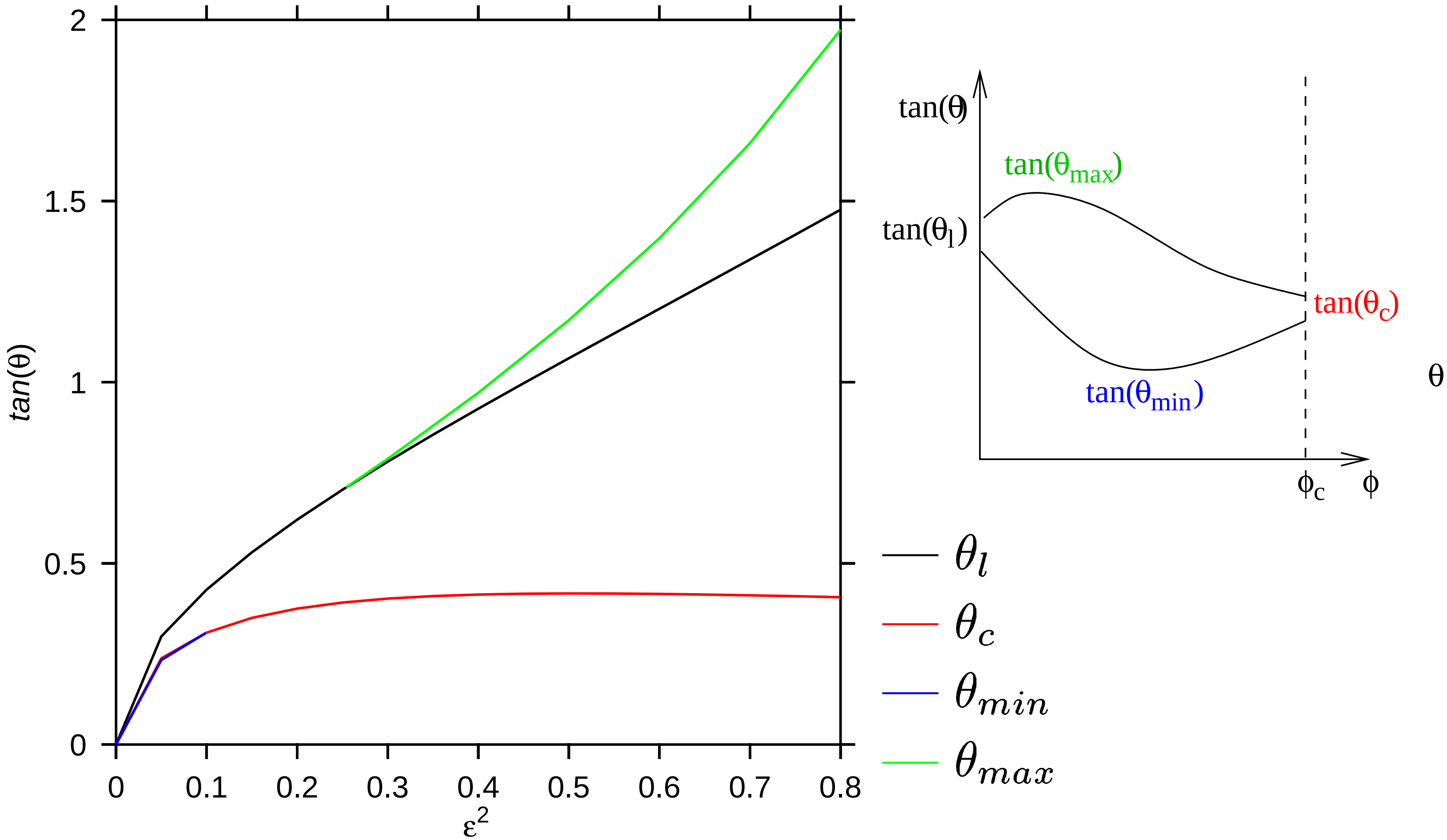
Flow down inclined plane — smooth particles



Flow down inclined plane — rough



Flow down inclined plane — partially rough



Flow down inclined plane:

Energy equation:

$$\frac{d}{dy} K \frac{dT}{dy} + \mu \dot{\gamma}^2 - D = 0$$

Dense gas: $K \sim \chi(T^{1/2}/d^2) = K_\phi(T^{1/2}/d^2)$.

$\mu \sim \chi(T^{1/2}/d^2) = \mu_\phi(T^{1/2}/d^2)$.

$D \sim \rho^2 \chi T^{3/2} d^2 (1 - e^2) \sim (\chi T^{3/2}/d^4) = D_\phi(T^{3/2}/d^4)$.

$\dot{\gamma} = G(\phi)(T^{1/2}/d)$

Scale $y^* = (y/H)$.

Scaled energy equation:

$$\delta^2 \frac{d}{dy^*} K_\phi T^{1/2} \frac{dT}{dy^*} = \mu_\phi \dot{\gamma}^2 - D_\phi (1 - e^2) T^{3/2}$$

where the small parameter $\delta = (d/H)$.

Flow down inclined plane:

Expansion $\phi = \phi^{(0)} + \delta\phi^{(1)} + \delta^2\phi^{(2)}$

Leading order $(T^{(0)})^{3/2}(\mu_\phi G^2 - D_\phi) = 0$

Leading solution $\phi = \phi^{(0)}$; $T^{(0)} = (\rho^{(0)} gH(1-y^*)\cos(\theta)/p_\phi^{(0)}(\phi^{(0)}))$

First correction $\phi^{(1)} = 0$

Second correction:

$$\frac{d}{dy^*} K_\phi (T^{(0)})^{1/2} d\frac{T^{(0)}}{dy^*} = \frac{d}{d\phi}\left(\mu\dot{\gamma}^2 - D\right)\Big|_{\phi=\phi^{(0)}} \phi^{(2)}$$

Analytical solution:

$$\phi^{(2)} = \frac{K_\phi}{2(1-y^{*2})}\left(\frac{d}{d\phi}\left(\mu\dot{\gamma}^2 - D\right)\Big|_{\phi=\phi^{(0)}}\right)^{-1}$$

Final solution:

$$\phi = \phi^{(0)} + \frac{d^2}{H^2}\frac{K_\phi}{2(1-y^{*2})}\left(\frac{d}{d\phi}\left(\mu\dot{\gamma}^2 - D\right)\Big|_{\phi=\phi^{(0)}}\right)^{-1}$$

Analytical estimates:

$K_\phi \sim K_c\chi$

$$\left(\frac{d}{d\phi}\left(\mu\dot{\gamma}^2 - D\right)\Big|_{\phi=\phi^{(0)}}\right)^{-1} \sim L_c(1-e^2)\frac{d\chi}{d\phi}$$

$$\phi = \phi^{(0)} + \frac{d^2}{H^2}\frac{K_c}{L_c(1-e^2)}\left(\frac{1}{\chi}\frac{d\chi}{d\phi}\right)^{-1}$$

Rough & partially rough particles: $K_c \sim 4; L_c \sim 40.$

Conclusions

- Constitutive relations for smooth particles — same form as those for dense gases.
- Constitutive relations for rough particles — antisymmetric part of rate of deformation tensor at Burnett order. Significant difference in coefficients.
- Hydrodynamic modes — non-analytic scaling, significantly different from those for a gas at equilibrium. Burnett coefficients have significant influence on structure of hydrodynamic modes.
- Steady state flow down inclined plane — sensitive to numerical coefficients in constitutive relation, realistic results obtained only when Burnett order terms are incorporated for restricted sets of parameter values.