TREATMENT OF BOUNDARIES IN NUMERICAL RELATIVITY

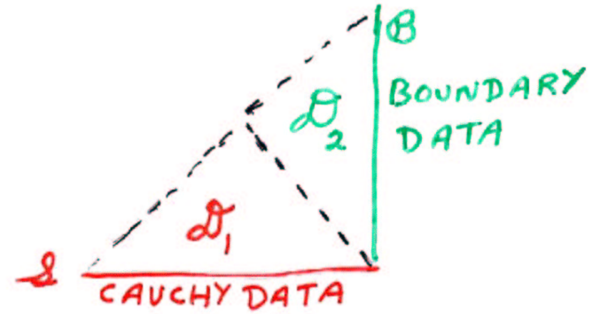
Jeff Winicour

based upon collaborative work with

Béla Szilágyi

THE INITIAL-BOUNDARY VALUE PROBLEM FOR GENERAL RELATIVITY

EVOLUTION-CONSTRAINT SYSTEM: $\left.\begin{array}{l}\mathcal{E}=0\\ \mathcal{C}=0\end{array}\right\} \Longrightarrow G_{\mu\nu}=0$



ASSUME CAUCHY IS OK.
WHAT CAN GO WRONG IN $\mathcal{D}_2$?

MATHEMATICALLY:

- NOT WELL-POSED
- CONSTRAINTS NOT SATISFIED

NUMERICALLY:

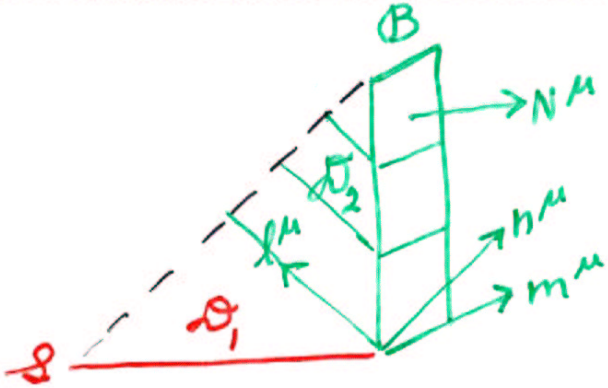
- UNSTABLE

PHYSICALLY:

- BOUNDARY DATA WRONG
- CAN'T EXTRACT WAVEFORM

THE DETAILS DEPEND UPON THE SYSTEM

SIMPLE EXAMPLE: NULL EVOLUTION IN $\mathcal{D}_2$



INTRODUCE NULL TETRAD ASSOCIATED WITH FOLIATION OF $\mathcal{B}$

$$g_{\mu\nu} = -\ell_{(\mu}n_{\nu)} + m_{(\mu}\bar{m}_{\nu)}$$

ASSUME $\mathcal{B}$ EITHER EXPANDS OR CONTRACTS IN ℓ^μ DIRECTION

EVOLUTION SYSTEM $\mathcal{E}$: $G_{\mu\nu}\ell^\nu = 0 \quad G_{\mu\nu}m^\mu m^\nu = 0$

INTRODUCE PROJECTOR ASSOCIATED WITH UNIT SPATIAL NORMAL N^μ TO $\mathcal{B}$

$$h^\mu_\nu = \delta^\mu_\nu - N^\mu N_\nu$$

CONSTRAINTS (3) $\mathcal{C}$: $h^\rho_\nu G_{\rho\sigma}N^\sigma = 0$

CONSTRAINT PROPAGATION

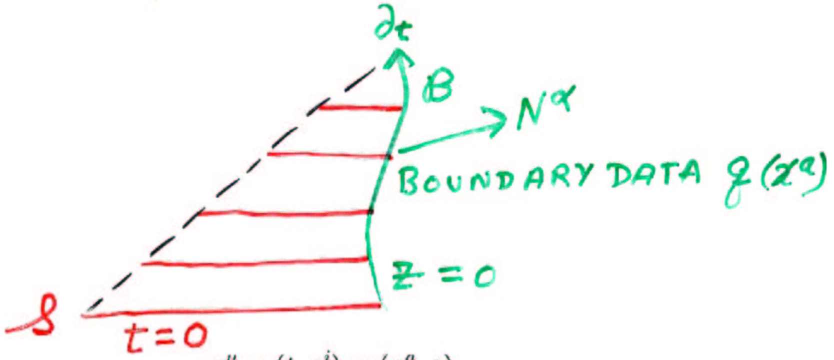
$$\mathcal{C} = \frac{1}{r^2}[r^2\mathcal{C}]_{\mathcal{B}}$$

WELL-POSED ???

ROBUST $\surd$

CAUCHY BOUNDARY FOR A SCALAR FIELD ON A CURVED BACKGROUND

(SYMMETRIC) HYPERBOLIC SYSTEM: $g^{\mu\nu}\nabla_\mu\nabla_\nu\Phi = 0$



$$x^\mu = (t, x^i) = (x^a, z)$$

BOUNDARY FLUX: $\mathcal{F} = N^\mu T_{t\mu} = -(\partial_t\Phi)N^\mu\partial_\mu\Phi$

WELL-POSED BOUNDARY CONDITIONS: $\mathcal{F} \geq 0$

HOMOGENEOUS DIRICHLET BOUNDARY CONDITION:

$$\partial_t\Phi = 0$$

HOMOGENEOUS SOMMERFELD BOUNDARY CONDITION:

$$\partial_t\Phi + \sqrt{-g_{tt}}N^\mu\partial_\mu\Phi = 0$$

HOMOGENEOUS NEUMANN BOUNDARY CONDITION:

$$N^\mu\partial_\mu\Phi = 0$$

INHOMOGENEOUS BOUNDARY DATA $q(x^a)$

$$\partial_t\Phi = q(x^a), \qquad \partial_t\Phi + \sqrt{-g_{tt}}N^\mu\partial_\mu\Phi = q(x^a), \qquad N^\mu\partial_\mu\Phi = q(x^a)$$

WELL-POSED INITIAL-BOUNDARY VALUE PROBLEM FOR THE LINEARIZED EINSTEIN EQUATIONS IN THE HARMONIC GAUGE

$$g_{\mu\nu} = \eta_{\mu\nu} + \delta g_{\mu\nu}$$

$$\gamma^{\mu\nu} = \delta(\sqrt{-g} g^{\mu\nu})$$

EVOLUTION SYSTEM $\mathcal{E}$

$$\Box \gamma^{ij} = 0$$

$$\partial_\mu \gamma^{\mu\nu} = \partial_t \gamma^{t\nu} + \partial_i \gamma^{i\nu} = 0$$

CONSTRAINT SYSTEM $\mathcal{C}$

$$\Box \gamma^{t\nu} = 0$$

THE INITIAL-BOUNDARY VALUE PROBLEM FOR THIS SYSTEM IS WELL POSED FOR FREE DIRICHLET, SOMMERFELD OR NEUMANN BOUNDARY DATA FOR THE COMPONENTS γ^{ij}.

THE CORRESPONDING EVOLUTION CODE IS ROBUST

THE BAD NEWS:

- IT IS NOT NOT KNOWN HOW TO GENERALIZE THIS SYSTEM TO THE NONLINEAR CASE.
- NO KNOWN NONLINEAR METRIC-CONNECTION SYSTEM HAS THIS FLEXIBILITY OF BOUNDARY DATA.

BOUNDARIES IN LINEARZIED APPROXIMATION

- **J. Stewart,** *Class. Quantum Grav.*, **15, 2865 (1998)**
- **B. Szilágyi, B. Schmidt and J. Winicour,** *Phys. Rev.*, **D65, 064015 (2002).**
- **G. Calabrese, L. Lehner and M. Tiglio G. Calabrese, L. Lehner and M. Tiglio,** *Phys. Rev. D*, **65, 104031 (2002).**
- **G. Calabrese, L. Lehner, D. Neilsen, J. Pullin, O. Reula, O. Sarbach, M. Tiglio, gr-qc**
- **G. Calabrese, J. Pullin, O. Reula, O. Sarbach, M.l Tiglio , gr-qc**
- **S. Frittelli and R. Gómez, gr-qc**

BOUNDARIES FOR NONLINEAR EINSTEIN EQUATIONS

- **H. Friedrich and G. Nagy,** *Commun. Math. Phys.*, **201, 619 (1999).**

EVOLUTION VARIABLLES: e_μ, $\Gamma^\rho_{\mu\nu}$, $C^\rho_{\mu\nu\sigma}$

THIS IS THE ONLY NONLINEAR SYSTEM WHICH IS KNOWN TO ADMIT PHYSICALLY GENERAL BOUNDARY CONDITIONS: ONLY THE WEYL CURVATURE REQUIRES BOUNDARY CONDITION AND ANALOGUES OF DIRICHLET, NEUMANN AND SOMMERFELD ARE ALLOWED.

POOR MAN'S VERSION

- **B. Szilágyi and J. Winicour gr-qc/0205044**

BASED ON

CAUCHY PROBELM IN HARMONIC COORDINATES:
Y. Foures-Bruhat, *Acta. Math.*, **88, 141 (1955).**
A. E. Fisher and J. E. Marsden, *Comm. Math. Phys.*, **28, 1 (1972).**

WELLPOSED INITIAL-BOUNDARY VALUE PROBLEM FOR NONLINEAR SYSTEMS WITH CHARACTERISTIC BOUNDRIES:
P. Secchi, *Arch. Rational Mech. Anal.*, **134, 155 (1996).**

HARMONIC INITIAL-BOUNDARY VALUE PROBLEM

REDUCED EVOLUTION SYSTEM: $\gamma^{\mu\nu} = \sqrt{-g} g^{\mu\nu}$

$$\gamma^{\alpha\beta}\partial_\alpha\partial_\beta\gamma^{\mu\nu} + S^{\mu\nu}(\gamma, \partial\gamma) = 0$$

WELL-POSED FOR ANY DISSIPATIVE BOUNDARY CONDITIONS, e.g. DIRICHLET, SOMMERFELD, NEUMANN

CONSTRAINTS: $H^\mu = \partial_\nu \gamma^{\mu\nu} = \hat{H}^\mu(x^\rho, \gamma)$
FOR BREVITY SET $\hat{H}^\mu(x^\rho, \gamma) = 0$
REDUCED EQUATIONS IMPLY

$$\gamma^{\alpha\beta}\partial_\alpha\partial_\beta H^\mu + C^{\mu\alpha}_\beta \partial_\alpha H^\beta + D^\mu_\beta H^\beta = 0$$

UNIQUENESS IMPLIES $H^\mu = 0$ **IF IT SATISFIES A DISSIPATIVE HOMOGENEOUS BOUNDARY CONDITION.**

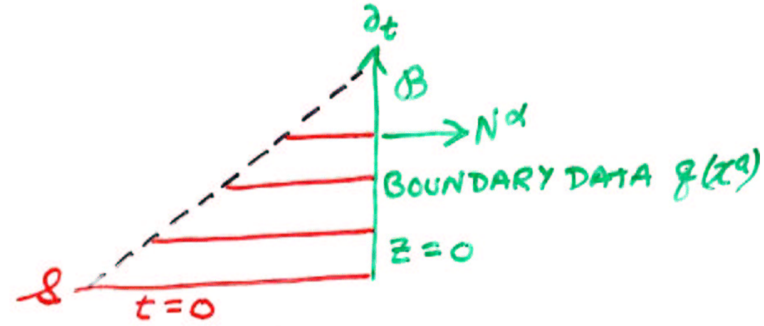
THIS IS NOT EASY TO ARRANGE.
EXAMPLE: DIRICHLET CONDITION $H^\mu|_\mathcal{B} = 0$.
LET BOUNDARY BE AT $z = 0$ **WITH** $x^\mu = (x^a, z)$. **THEN**

$$H^z = \underline{\partial_a\gamma^{za}} + \partial_z\gamma^{zz} = 0$$

$$H^a = \partial_b\gamma^{ab} + \underline{\partial_z\gamma^{az}} = 0$$

NAIVE BOUNDARY DATA FOR $\gamma^{\mu\nu}$ **IMPROPERLY POSES BOTH DIRICHLET AND NEUMANN CONDITIONS ON** γ^{az}

WELL-POSED HOMOGENEOUS BOUNDARY DATA



$x^\mu = (x^a, z)$
$x^a = (t, x, y)$

ONE CHOICE THAT WORKS:

$$\left.\begin{array}{l} \gamma^{za}|_\mathcal{B} = 0 \\ \partial_z\gamma^{zz}|_\mathcal{B} = 0 \\ \partial_z\gamma^{ab}|_\mathcal{B} = 0 \end{array}\right\} \implies \left\{\begin{array}{l} H^z|_\mathcal{B} = 0 \\ \partial_z H^a|_\mathcal{B} = 0 \end{array}\right.$$

$\implies$ **CONSTRAINTS SATISFIED**

INHOMOGENEOUS BOUNDARY DATA $q(x^a)$

BOUNDARY HARMONIC GAUGE FREEDOM (SHIFT)

$$\gamma^{za}|_\mathcal{B} = q^a(x^b)\gamma^{zz}|_\mathcal{B}$$

NOTE: BOUNDARY DATA FOR $\gamma^{za}|_\mathcal{B}$ **DEPENDS ON DATA FOR** $\gamma^{zz}|_\mathcal{B}$ **WHICH CAN ONLY BE DETERMINED BY CARRYING OUT THE EVOLUTION**

BOUNDARY NORMAL: $\partial^n = \frac{1}{N^z}N^\mu\partial_\mu = \partial_z + q^a\partial_a$
NEUMANN DATA: $q^{zz} = \partial^n\gamma^{zz}|_\mathcal{B}$

$$H^z|_\mathcal{B} = 0 \implies q^{zz} = -\partial_a q^a \gamma^{zz}|_\mathcal{B}$$

REMAINING NEUMANN BOUNDARY DATA

$$q^{ab} = \partial^n \gamma^{ab}|_{\mathcal{B}}$$

RELATED TO EXTRINSIC CURVATURE K^{ab} OF BOUNDARY.

$$\partial^n H_a|_{\mathcal{B}} = 0 \Longrightarrow$$

$$\sqrt{-h} D_b (K_a^b - \delta_a^b K) + \sqrt{g^{zz}} K_{ab} H^b - \frac{g^{zz}}{2} H_b \partial_a q^b = 0$$

HERE h_{ab} AND D_a ARE THE METRIC AND CONNECTION INTRINSIC TO $\mathcal{B}$

This forms a symmetric hyperbolic system which determines the 6 pieces of Neumann data q^{ab} in terms of 3 free functions, as well as the free (boundary gauge) data q^a and boundary values of the variables γ^{zz}, γ^{ab} and $\partial_z \gamma^{za}$ which must be determined by the evolution.

ANY SOLUTION OF THE REDUCED EQUATIONS WITH THIS BOUNDARY DATA SATISFIES THE CONSTRAINTS.

IS THE INITIAL-BOUNDARY PROBLEM WELL-POSED???

NUMERICAL IMPLEMENTATION

SOME DIFFICULT CHOICES

- **FIRST DIFFERENTIAL ORDER OR SECOND**
  SECOND ORDER IN TIME FIRST ORDER IN SPACE
- **CUBIC BOUNDARY OR SPHERICAL**
  CUBIC
- **GENERAL BOUNDARY GAUGE OR $\gamma^{za}|_{\mathcal{B}} = 0$**
  $\gamma^{za}|_{\mathcal{B}} = q(x^b)\gamma^{zz}|_{\mathcal{B}}$
- **HARMONIC FORCING TERMS OR $\partial_\mu \gamma^{\mu\nu} = 0$**
  $\partial_\mu \gamma^{\mu\nu} = \hat{H}^\mu(x^\rho, \gamma)$
- **BOUNDARY ACCURACY**
  1ST ORDER IN NORMAL DIRECTION, 2ND ORDER TANGENTIALLY
- **NUMERICAL STENCILS, DISSIPATION, ...**
  BAG OF "TRICKS"

TESTS OF NAIVE ALGORITHM

- ROBUST STABILITY
- LINEARIZED WAVE CONVERGENCE TESTS
- NONLINEAR GAUGE WAVE CONVERGENCE TESTS

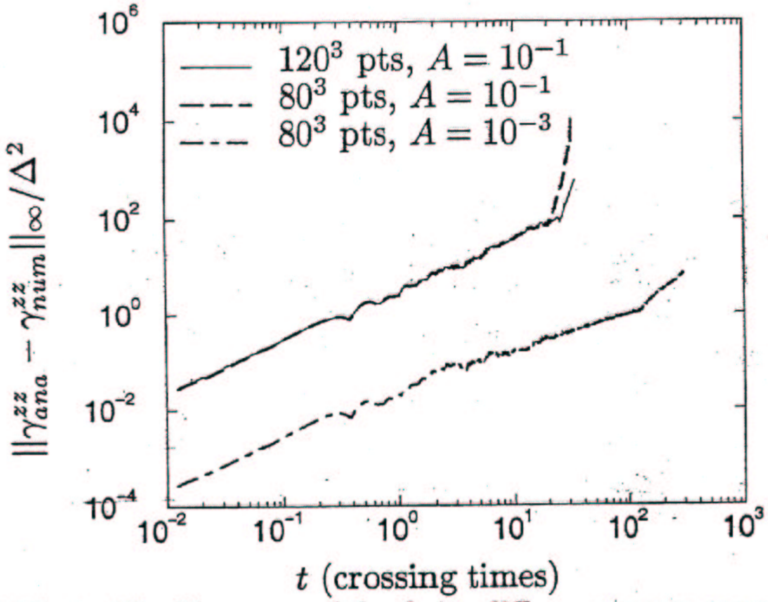
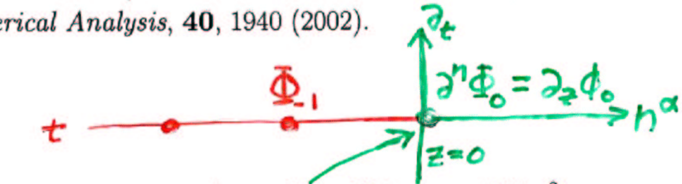


FIG. 2. The L_∞ norm of the finite-difference error, rescaled by a factor of $1/\Delta^2$, for a gauge-wave. The upper two (mostly overlapping) curves demonstrate convergence to the analytic solution for a wave with amplitude $A = 10^{-1}$ evolved for 30 crossing times with gridsizes 80^3 and 120^3. The lower curve represents evolution of the same gauge-wave with $A = 10^{-3}$ for 300 crossing times with gridsize 80^3.

NEUMANN BOUNDARY CONDITIONS FOR THE SCALAR WAVE EQUATION IN SECOND DIFFERENTIAL ORDER FORM

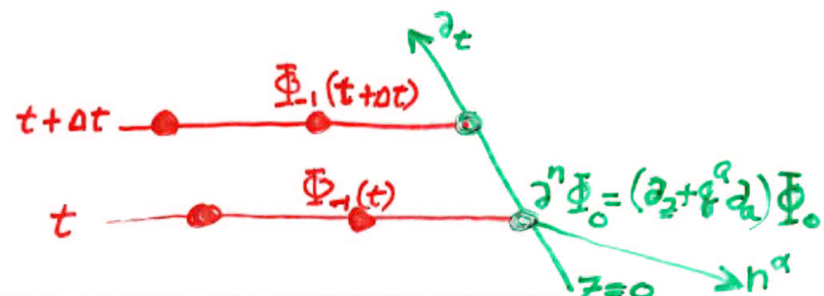
CURVED BOUNDARY AT REST IN A MINKOWSKI METRIC BACKGROUND

H.-O. Kreiss, N. A. Petersson and J. Yström, *SIAM Journal on Numerical Analysis*, **40**, 1940 (2002).

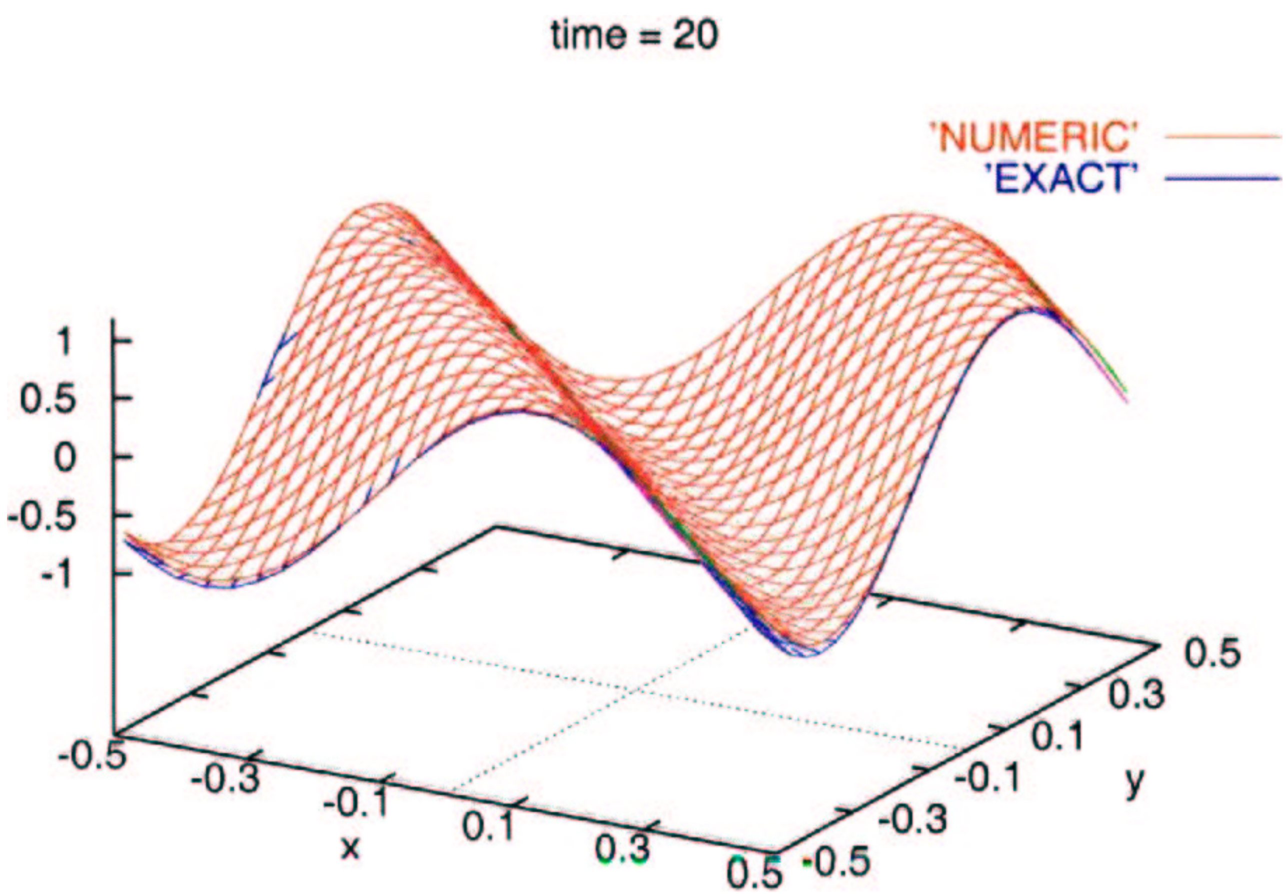


$$\Phi_{-1} = \Phi_0 - \partial^n \Phi_0 \, \Delta z + O(\Delta z^2)$$

MOVING BOUNDARY ON A DYNAMIC BACKGROUND SPACETIME



WHEN THE BOUNDARY MOVES TOWARD THE CAUCHY INTERIOR THE INTERPOLATION STENCIL FOR Φ_0 INVOLVES FUTURE TIME LEVELS.



THE BOTTOM LINE
IS TO COMPUTE WAVEFORMS
FROM BINARY BLACK HOLES ...

BUT YOUR CODE
HAS TO PASS BASIC TESTS
IF THE WAVEFORMS CAN BE TRUSTED.