Interdisciplinary viewpoints of resonance and its role in cochlear mechanics

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Workshop on the Mathematics of Hearing

The ear is a remarkable detector, encoding sound pressure into neural signals that carry myriad pieces of information about the world around us to the brain. It is also highly selective, decomposing sound into constituent frequency components by virtue of acting as a hydrodynamic Fourier analyzer. Interestingly, not only does the ear respond to sound, but emits it as well, a facet that has revolutionized pediatric audiology.

Yet much still remains unknown about this system we all have a pair of built in to our heads. This one-day workshop aims to bring together a variety of researchers from different backgrounds to explore, from both mathematical and biological viewpoints, a wide range of topics related to the ear.

Invited Speakers
Bard Ermentrout (University of Pittsburgh) – nonlinear dynamics
Andre Longtin (University of Ottawa) – neural coding
Laura Miller (University of North Carolina) – fluid dynamics
Christopher Shera (University of Southern California) – inverse problems
Sarah Verhulst (Ghent University) – cochlear neurobiology and psychoacoustical modeling
George Zweig (MIT) – cochlear modeling

Organizing Committee: Christopher Bergevin (York University) and Sunil Puria (Harvard Medical School)

For registration and more information, please visit: www.fields.utoronto.ca/activities/16-17/mathhearing
13th Mechanics of Hearing conference

♦ June 19-24, 2017

♦ Brock University (St. Catherines, Ontario, Canada)

Niagara region

Waves

Hair cells

... & much more

www.mechanicsofhearing.org/moh2017/
To Do
• Resonance
• Highlight “interdisciplinary” approaches

Tangents
• Conceptual details (e.g., convolutions)
• Resonance in the inner ear?
• Nonlinear/active oscillators
• Other examples in biology
Job(s) of the inner ear:
• transduction
• spectral decomposition
An Acoustic Prism

High frequencies
Mid frequencies
Low frequencies

Stapes
Basilar membrane
Traveling wave

Zweig et al. (1976)
Tonotopy (re speech)
Theme/Question: What is the (basic) physical basis for “tonotopy”?

Hint: It ain’t a traveling wave per se…. (though such provides a useful framework)
Tonotopy & Traveling waves

(one possible) **Model:** Non-uniform transmission line
Big picture theme/question here:

What is the (basic) physical basis for “tonotopy”?

Foundation: Harmonic Oscillator

\[ \ddot{x} = -\omega_0^2 x - \gamma \dot{x} + A \cos(\omega t) \]

Key (steady-state) principle: Resonance
Foundation: Resonance

Amplitude

\[ \text{small } \gamma \]

\[ 1 \]

\[ 10 \]

\[ \omega/\omega_0 \]

\[ 0.5 \]

\[ 1 \]

\[ 2 \]

\[ \text{large } \gamma \]

Phase

\[ 0 \]

\[ 0 \]

\[ -0.5 \]

\[ \text{cycles} \]

Fig. 4-9 (a) Amplitude as function of driving frequency for different values of \( Q \), assuming driving force of constant magnitude but variable frequency. (b) Phase difference \( \delta \) as function of driving frequency for different values of \( Q \).
Aside: Spectral analysis

Time and frequency are “separated”.....

..... but you don’t have one without the other
Focal Point

Relatively simple Matlab code...
Exploring a handful of approaches to demonstrate these characteristics of resonance

Talk is really just an interdisciplinary crash course on linear systems theory...

Nota bene: The inner ear is not really linear per se (i.e., it is complex)
Aside

Resonance comes in a variety of “flavors”, e.g.,:

- Externally forced 2\textsuperscript{nd} order “systems” (i.e., energy is being input into them)

- Standing waves

- NMR/MRI

http://www.acs.psu.edu/drussell/demos/membranecircle/circle.html
Stevens (2000)
Numerous NMR principles relevant to cochlear mechanics
(e.g., micro/macroscopic states, phase coherence)

Fig. 3.147a–d. Classical representation of the NMR experiment. a In equilibrium the nuclear spins are distributed in the states \( \alpha \) and \( \beta \) according to the Boltzmann distribution. b At resonance and with a sufficiently strong RF field, the populations of \( \alpha \) and \( \beta \) are equalized and the spins precess in phase at the Larmor frequency \( \omega_L \). c Longitudinal relaxation restores the equilibrium distribution of the spins. d The phase coherence of the spins is lost by transverse relaxation. In reality the processes c and d proceed simultaneously.
**Resonance:** The power to evoke enduring images, memories, and emotions — *Oxford dictionary, online*
Several basic approaches:
(all arriving at the same answer)

1. Numerically solve the ODEs and extract the relevant magnitudes and phases (via an FFT)

2. Analytic solution I (via Fourier transforms)

3. Impulse response I (and associated transfer function)

4. Impulse response II (convolve in the time domain)

5. Analytic solution II (via eigensolutions)

Note: Via linear systems theory, these different “approaches” are not necessarily mutually exclusive (e.g., convolution theorem directly links #s 3 & 4)
Interdisciplinary connections (i.e., basic concepts this code demonstrates)

**Physics**
- ODEs (e.g., Newton’s 2\textsuperscript{nd}, Hooke’s Law)
- Resonance
- Notion of “steady-state”

**Mathematics**
- Fourier transforms
- Complex #s
- Eigenvalues
- Phase space

**Engineering**
- Linear systems theory
- Convolutions
- Impedance/Admittance
- Impulse response
- Transfer functions

**Numerical**
- Discrete Fourier transforms (FFT)
- Numerically solving ODEs (e.g., Euler, RK4, adaptive step-size and associated problems)
- Matlab syntax

**Basic physical intuition:**
- 2\textsuperscript{nd} order system: two reactive elements (i.e., energy-storing)
- energy transferred back & forth between
- there is an optimal rate for such (i.e., resonant frequency)
To Do
• Resonance
• Highlight “interdisciplinary” approaches

Tangents
• Conceptual details (e.g., convolutions)
• Resonance in the inner ear?
• Nonlinear/active oscillators
• Other examples in biology
% Code to solve the damped (sinusoidally-) driven harmonic oscillator (DDHO)
% for a variety of driving freqs. so to buildup the "resonance curve" via
% computation of the mag/phase of the Fourier transform of the steady-state
% response. Furthermore, the analytic solution for the DDHO as well as the
% transfer function are shown to be equivalent (Fig.1)

% Damped driven Harmonic Oscillator (DDHO)
% d^2x/dt^2 = -(P.wo^2)*x - P.gamma*dx/dt + (P.A)*sin(P.w*t)

% Regs:
% EXhoResonanceFunc.m (re ode45), rfft.m

clear;
% ==============================================================
% ---
% Oscillator params. and ICs
P.p0 = 0.0;  % Initial position {0}
P.v0 = 0.0;  % Initial velocity {0}
P.wo = 10;   % resonant (angular) freq {10}
P.gamma = 0.5;  % damping coefficient {0.5}
% ---
% Sinusoidal driving term params.
P.A = 10;    % Driving force amplitude {10}
P.wDrive = [5 15];  % start and end angular drive freqs. {[5 15]}
P.wDriveN = 25;  % # of drive freqs. to run {25}
% ---
P.tmax = 200; % Maximum time to solve [s; arb] {200}
P.SR = 150;   % sample rate for time step [Hz; arb] {150}
P.Npoints = 8192; % Number of points in time series for FFT, must be 2^n {8192}
% ---
P.plotN = 1;   % boolean re plotting the waveform and spectra for one driving freq. {1}
P.plotNnum = round(P.wDriveN/2);  % driving freq. index to plot {round(P.wDriveN/2)}
% ---
P.solveType = 1; % 0-ode45, 1-hard-coded RK4 {1}
P.stepF = 0;    % boolean re using a fixed step-size for ode45 {0}
% ==============================================================

Note: This is a slightly older version of the code (does not include methods 4 & 5)
dt = 1/P.SR; % spacing of time steps
init0 = [P.p0 P.v0]'; % Column vector of initial conditions.
tspan = [0:dt:P.tmax]; % time interval for entire computation
tW = [0:1/P.SR:(P.Npoints-1)/P.SR]; % (shorter/later) time interval for FFT window
L = length(tspan); TW = L-(P.Npoints-1); % create offset point extracting FFT window

freq = [0:P.Npoints/2]; % Note: these values are not angular (i.e., [freq]= 1/s, not rads/s)
df = P.SR/P.Npoints; % freq. spacing between bins
wDT = linspace(P.wDrive(1),P.wDrive(2),500); % create ang. freq. array for plotting

Q = P.wo/P.gamma; % "quality factor" (Note: tau=1/P.gamma=Q/P.wo, where tau is time const. of build-up)

lambdaP = 0.5*(-P.gamma + sqrt(P.gamma^2-4*P.wo^2)); % Eigenvalues, for x=0 (undriven)
lambdaM = 0.5*(-P.gamma- sqrt(P.gamma^2-4*P.wo^2));

% Note - Can also get eigenvalues via command: eig([0 1;-P.wo^2 -P.gamma])

Z = P.gamma + i*(wDT- P.wo^2./wDT); % impedance (see notes above; assumes mass is unity)
Y = 1./Z; % admittance (reciprocal of impedance)

% grabbing driving freqs. from freq array
indx = find(freq>=P.fDrive(1) & freq<=P.fDrive(2)); % find relevant indicies
indx = find(freq>=P.wDrive(1)/(2*pi) & freq<=P.wDrive(2)/(2*pi)); % find relevant indicies
indxB = round(linspace(indx(1),indx(end),P.wDriveN)); % one means to get the desired subset
freqD = 2*pi*freq(indxB); % array of driving angular freqs
for mm=1:numel(freqD)
    P.w= freqD(mm); % extract driving freq.
    % *** Solve in one of two ways ***
    if P.solveType==0
        % use Matlab's ode45
        % ---
        % tell it to actually use the specified step-size
        if(P.stepF==1), options = odeset('MaxStep',1/P.SR); else options=[]; end
        [t,y] = ode45(@EXhoResonanceFunc,tspan,init0,options,P);
    else
        % use 4th order Runge-Kutta code
        xPoints(1) = P.p0; vPoints(1) = P.v0; % initialize ICs into dummy arrays
        x= P.p0; v= P.v0; % kludge
t= 1/P.SR; % time step
        for nn=1:(length(tspan)-1)
            % ---
            % Current time.
            % step1
            xk1= v;
vk1= -((P.wo)^2)*x - P.gamma*v + (P.A)*sin(P.w*t);
            % step 2
            xk2 = v + (dt/2)*vk1;
vk2= -((P.wo)^2)*(x + (dt/2)*xk1) - P.gamma*(v + (dt/2)*vk1)...
            + (P.A)*sin(P.w*(t+(dt/2)));
            % step 3
            xk3 = v + (dt/2)*vk2;
vk3= -((P.wo)^2)*(x + (dt/2)*xk2) - P.gamma*(v + (dt/2)*vk2)...
            + (P.A)*sin(P.w*(t+(dt/2)));
            % step 4
            xk4 = v + dt*vk3;
vk4= -((P.wo)^2)*(x + (dt)*xk3) - P.gamma*(v + dt*vk3)...
            + (P.A)*sin(P.w*(t+(dt/2)));
            % apply RK4 weighting
            x = x + (dt/6)*(xk1 + 2*xk2 + 2*xk3 + xk4);
v = v + (dt/6)*(vk1 + 2*vk2 + 2*vk3 + vk4);
% store away position and velocity
xPoints(nn+1) = x; vPoints(nn+1) = v;
end
    y(:,1)= xPoints'; y(:,2)= vPoints'; % repackage output
end
% steady-state portion of waveform for FFT
ySPEC = y(TW:TW+P.Npoints-1,1);  
% store away driving freqs.
wDrive(mm) = 2*pi*freq(indxB(mm));  
% store away SS mag.
mag(mm) = abs(sigSPEC(indxB(mm)));  
% need to correct the phase re the duration of the window allowed for settling into steady-state
phase(mm) = tPhase;  
if mm == P.plotNnum
    % integrated waveform and segment extracted for spectral analysis
    figure(2); clf;
    h1 = plot(tspan,y(:,1)); hold on; grid on;
    xlabel('Time'); ylabel('Position');
    title('Time Waveform of integrated solution to damped driven HO equation')
    L = length(tspan); TW = L-(P.Npoints-1);  
    ySPEC = y(TW:TW+P.Npoints-1,1);  
    h2 = plot(tspan(TW:TW+P.Npoints-1),ySPEC,'r.','MarkerSize',3);
    legend([h1 h2],'Entire waveform','Steady-state portion (used for FFT)'
    % phase space for waveform (entire and steady-state)
    figure(3); clf;
    hPS1 = plot(y(:,1),y(:,2)); hold on; grid on;
    hPS2 = plot(y(TW:TW+P.Npoints-1,1),y(TW:TW+P.Npoints-1,2),'r-');
    xlabel('Position'); ylabel('Velocity'); title('Phase plane');
    legend([hPS1 hPS2],'Entire waveform','Steady-state portion (used for FFT)'
    % plot spectra of steady-state waveform
    figure(4); clf;
    hS1 = plot(2*pi*freq,db(sigSPEC)); hold on; grid on;
    xlabel('Freq [rads/s]'); ylabel('Spectral amplitude [dB]');
    hS2 = plot(2*pi*freq(indxB(mm)),db(mag(mm)),'rs');  
    legend([hS1 hS2],'Steady-state spectra','Driving freq.');
% ++++++++++++++
% [Fig.1]** Mags/phases extracted from the numeric steady-state responses **
figure(1); clf;
subplot(211); hh1= plot(wDrive/P.wo,mag,'ko', 'MarkerSize',6,'LineWidth',2); hold on; grid on;
ylabel('Magnitude');
subplot(212); hh2= plot(wDrive/P.wo,unwrap(phase)/(2*pi), 'ko', 'MarkerSize',6,'LineWidth',2); hold on; grid on;
xlabel('Normalized (angular) angular freq (w/wo)'); ylabel('(unwrapped) Phase [cycs]');

% ++++++++++++++
% [Fig.1]** Analytic solution ** (see French, 1971; as noted above, these expressions are
% equivalent to using Fourier transforms, which implicitly assume sinusoidal steady-state, to
% solving the main ODE)
magT= P.A./sqrt((P.wo^2-wDT.^2).^2 + ((P.gamma*wDT).^2));  % mag (theory)
phaseT= atan((P.gamma*wDT)./(-P.wo^2+wDT.^2));                % phase (theory; note sign change in denom. re
% convention)
phaseT= phaseT+ phase(1)+ abs(phaseT(1));                     % (kludge) correct for (arb?) phase offset in numeric solution
figure(1);
subplot(211); hh3= plot(wDT/P.wo,magT, 'r-', 'LineWidth',2);
subplot(212); hh4= plot(wDT/P.wo,unwrap(2*phaseT)/(4*pi), 'r-', 'LineWidth',2);  % kludge to get unwrapping working

% ++++++++++++++
% [Fig.1]** "Transfer function" ** re linear systems theory (i.e., the Fourier transform of the
% impulse response of the DHO)
init0 = [0 10]';  % set ICs such that there is an "impulse" at t=0
P.w= 0;          % make sure to "turn off" drive
options= []; [t,yI] = ode45(@EXhoResonanceFunc,tspan,init0,options,P);
specI= rfft(yI(1:P.Npoints));
magI= abs(specI);
magI= magI* (max(mag)/max(magI));  % scale impulse mag. re max. value of driven case
phaseI= angle(specI);
figure(1);
subplot(211); hh5= plot(2*pi*freq/P.wo,magI, 'b--', 'LineWidth',2); xlim([wDT(1) wDT(end)]/P.wo);
subplot(212); hh6= plot(2*pi*freq/P.wo,unwrap(2*phaseI)/(4*pi), 'b--', 'LineWidth',2); % kludge to get unwrapping working
xlim([wDT(1) wDT(end)]/P.wo);
% ++++++++++++++
% [Fig.1] Make a legend to put it all together (re Fig.1)
figure(1); subplot(211); legend([hh1 hh3 hh5],'
'...'Numeric solution re steady-state FFT',...
'Analytic solution','Transfer function');

% ++++++++++++++
% [Fig.5] Plot the impulse response and comparison to admittance
figure(5); clf;
subplot(221); plot(tW,yI(1:P.Npoints)); hold on; grid on; xlabel('Time [s]'); ylabel('x');
title('Impulse response (no drive; P.w=0, P.p0=0, P.v0=10'); xlim([0 tW(round(numel(tW)/3))]);
subplot(222); hI2= plot(freq,db(specI),'LineWidth',2); grid on; hold on; ylabel('Amplitude [dB]');
title('Transfer function (mag. of FFT of IR)'); xlim(P.wDrive/(2*pi));
subplot(224); hI3= plot(freq,angle(specI)/(2*pi),'LineWidth',2); grid on; hold on;
xlabel('Frequency [Hz]'); ylabel('Phase [cycles]');
title('Transfer function (phase of FFT of IR)'); xlim(P.wDrive/(2*pi));
subplot(223);
hZa= plot(wDT,abs(Y),'k-'); grid on; hold on; hZb= plot(wDT,abs(Z),'r.');
grid on; hold on; ylabel('Amplitude'); xlabel('Ang. frequency [rad/s]'); legend([hZa hZb],'admittance','impedance');

% ---
% for reference, also include (scaled) admittance to indicate (near?) equivalence
offset= max(db(Y))- max(db(specI)); % scaling (in dB) to match up
%magY= fliplr(db(Y)- offset); % kludge
magY= (db(Y)- offset);
subplot(222); hI2b= plot(wDT/(2*pi),magY,'r--');
legend([hI2 hI2b],'Transf. func.','(scaled) Admittance','Location','SouthWest');
angleY= angle(Y)/(2*pi) - angle(Y(1))/(2*pi); % there will be a slight vert. offset re angle(specI)/(2*pi)
subplot(224); hI3b= plot(wDT/(2*pi),angleY,'r--');

% ++++++++++++++
% display some relevant #s to screen
disp(['Quality factor (P.wo/P.gamma)= ',num2str(Q)]);
disp(['Eigenvalues (for x=0, undriven case): ',num2str(lambdaP),' and ',num2str(lambdaM)]);
Notes
- To solve this numerically, need to turn 2nd order ODE into series of 1st order ODEs:
  \[ \frac{dx}{dt} = y \]
  \[ \frac{dy}{dt} = -P.\omega^2 x - P.\gamma y + (P.A) \sin(P.\omega t) \]
- For autonomous case (i.e., no drive), can rewrite in matrix form such that
  \[ A = \begin{bmatrix} 0 & 1 \\ -P.\omega^2 & -P.\gamma \end{bmatrix} \]
  straight-forward to find associated eigenvalues (see below)
- via P.solveType, user can solve either via ode45 or a hard-coded RK4
  (both should yield the same solution!); Note that (surprisingly) ode45 actually seems
  slower than the RK4 (the slowest is ode45 w/ the fixed step-size), possibly due to
  the passing of the large-ish structure P; also note that the default ode45 routine
  (i.e., adaptive step-size) introduces harmonic distortions in the spectra
  due to its nonlinear nature
- For the analytic solution (below manifest as magT and phaseT), the
  expression used below, as derived in French (1971) for the steady-state,
  is exactly the same as if one simply put in the Fourier transform and
  solved for the resulting magnitude and phase [confirmed on the back of an
  envelope; let \( x(t) = X(\omega) \exp(i\omega t) \) and plug back in, solving for \( X(\omega) \); note then
  that \( \text{magT} = \text{abs}(X) \) and \( \text{phaseT} = \text{angle}(X) \)]
- There are a few minor kludges below [e.g., vertical adjustment of the
  analytic solution so to match the (arbitrary?) ref. phase of the numeric
  solution]
- Impedance (\( Z \)) for DDHO is (by definition) the complex ratio of the driving
  force and the (steady-state) velocity (see 4080W2016L10REF.pdf). Real part of \( Z \) (resistance)
  describes energy loss while imaginary part (reactance) describes energy storage
- Comparison of the mags. for the transfer function and admittance
  (Fig.5, top right) are a bit kludgy (unsure why fliplr was needed) and
  off (worser overlap as you move away from \( \omega \))
function dy = EXhoResonanceFunc(t,y,P)
% Damped driven HO
% d^2x/dt^2 = -((P.wo)^2)*x - P.gamma*dx/dt + (P.A)*sin(P.w*t)
% Note: y(1) = x, y(2)= dx/dt
dy = zeros(2,1); % A column vector to be returned

   dy(1) = y(2);
   dy(2) = -((P.wo)^2)*y(1) - P.gamma*y(2) + (P.A)*sin(P.w*t);
% RFFT: scaled real FFT, X=rfft(x)
% Returns the positive-frequency half of the transform X=FFT(x).
% The transform X is normalized so that if \{x\} is a sine wave of
% unit amplitude and frequency n*df, then X[n]=1.
% Usage:   X=rfft(x)
% If x is N points long, NF=N/2+1 complex points are returned.
% See also IRFFT, FAST, FSST, FFT, IFFT,

function X=rfft(x)
    [m,n]=size(x);
    if (m==1 | n==1)
      % original...
      N=length(x)/2+1;
      xc=fft(x);
      X=xc(1:fix(N));
    else
      % do it column-wise...
      N=m/2+1;
      xc=fft(x);
      X=xc(1:fix(N),:);
    end

    X = X / (length(x)/2);
    return
Approach 1 – Numeric + FFT

\[ \ddot{x} = -\omega_0^2 x - \gamma \dot{x} + A \cos(\omega t) \]

Fig. 4–11 (a) Response of an undamped harmonic oscillator to a periodic driving force, as described by Eq. (4–19). This beat pattern would continue indefinitely. (b) Transient behavior of a damped oscillator with a periodic driving force off resonance. (c) Transient behavior at exact resonance, showing smooth growth toward steady amplitude. (Photos by Jon Rosenfeld, Education Research Center, M.I.T.)

French (1971)
Approach 1 – Numeric + FFT

\[ \ddot{x} = -\omega_0^2 x - \gamma \dot{x} + A \cos(\omega t) \]
Approach 1 – Numeric + FFT

\[
\chi(t) = A(\infty) \left[ 1 - e^{(-t/\tau)} \right]
\]

\[
\tau = \frac{1}{\gamma} = \frac{Q}{\omega_0}
\]

Note: Resonance takes time...

\[
\begin{align*}
\text{Note: Resonance takes time...} \\
\chi(t) &= A(\infty) \left[ 1 - e^{(-t/\tau)} \right] \\
\tau &= \frac{1}{\gamma} = \frac{Q}{\omega_0}
\end{align*}
\]

\[
\rightarrow \text{Hence the importance of “steady-state”}
\]
Approach 1 – Numeric + FFT

\[ \ddot{x} = -\omega_0^2 x - \gamma \dot{x} + A \cos(\omega t) \]

Compute the FFT to extract the magnitude (and phase; not shown)

Can also plot in phase space...
Approach 1 – Numeric + FFT

End up w/ the black circles....
Approach 2 – Analytic solution

\[ A(\omega) = \frac{F_0/m}{\left[ (\omega_0^2 - \omega^2)^2 + (\gamma \omega)^2 \right]^{1/2}} \]

\[ \delta(\omega) = \arctan \left( \frac{\gamma \omega}{\omega^2 - \omega_0^2} \right) \]

Can arrive here in a variety of ways (including Fourier transforms; see notes at end)

→ End up w/ red line
Approach 3 – Transfer Function

Impulse response (no drive; $P_w=0$, $P_p=0$, $P_v=10$)

Transfer function (mag. of FFT of IR)

Transfer function (phase of FFT of IR)

Amplitude vs. Time [s]

Amplitude vs. Frequency [Hz]

Admittance vs. Angular frequency [rad/s]
Impulse Response

Input: Incoming “signal” (arbitrary)

Output: “filtered” signal (combination of the system and the input)

“Impulse response” (this fully characterizes the “system”)

Fig. 4.11. Transmission of a signal. The transmitted signal is given by the convolution of the signal $s(x)$ with the system’s impulse response $h(x)$
Aside: Impulse

**Resonance**

**CARL A. LUDEKE**

**JOURNAL OF APPLIED PHYSICS**

**VOLUME 13, JULY, 1942**

![Graph showing oscillatory behavior with time](image)

**Fig. 4.** The discontinuous force $F$ supplied by the motor in Fig. 3, as a function of time $t$.

$$md^2x/dt^2 + \beta dx/dt + kx = \sum A_n \sin n\omega t + \sum B_n \cos n\omega t.$$

$$x = \sum \frac{A_n \sin (n\omega t - \varphi_n)}{\left[ (k - mn^2\omega^2)^2 + \beta^2 n^2\omega^2 \right]^{\frac{1}{2}}}$$

$$+ \sum \frac{B_n \cos (n\omega t - \alpha_n)}{\left[ (k - mn^2\omega^2)^2 + \beta^2 n^2\omega^2 \right]^{\frac{1}{2}}}$$
Aside: Impulse

Impulse (i.e., a “click”) has a flat magnitude
(this is also a good place to mention the notion of a ‘group delay’)
Ex. Acoustic Impulse Response

Room response \( g(x) \) “filters” an input sound \( s(x) \)

\[
g(x) = \mathcal{L}\{s(x)\}
\]

Room response \( g(x) \) is just “convolution” between \( s \) and room’s impulse response \( h(x) \)

\[
g(x) = s(x) \ast h(x) = \int_{-\infty}^{+\infty} s(\xi)h(x - \xi) \, d\xi
\]

→ All the relevant bits of the room’s acoustics are contained in \( h \) (which we can easily measure!)
Transfer functions

→ The “transfer function” is simply the Fourier transform of the impulse response.

Buzug (2008)
 Aside: Impedance & Pole/zero descriptions

\[ \ddot{x} + \gamma \dot{x} + \omega_0^2 x = \frac{F_0}{m} e^{i\omega t} \]

\[ Z \equiv \frac{F_{ext}}{\ddot{x}} = \frac{x \cdot (-m\omega^2 + k + i\omega b)}{x \cdot i\omega} = b + i \left[ m\omega - \frac{k}{\omega} \right] \]

Real part of $Z$ (resistance) describes energy loss while imaginary part (reactance) describes energy storage.

**Poles** and **Zeros** of a transfer function are the frequencies for which the value of the denominator and numerator of transfer function becomes zero respectively. The values of the poles and the zeros of a system determine whether the system is stable, and how well the system performs. Control systems, in the most simple sense, can be designed simply by assigning specific values to the poles and zeros of the system.

https://en.wikibooks.org/wiki/Control_Systems/Poles_and_Zeros


**Note:** Electrical engineers commonly use complex frequency ($s$) representation, tied back to Laplace transforms.
Approach 3 – Transfer Function

→ End up w/ dashed blue curve
Approach 4 – Convolve the impulse response

Convolve (in time domain) input drive sinusoids w/ system's impulse response, then compute the FFT

Note: When convolving the impulse response and the drive, the initial transient is apparent, so we use the "long" time window and extract the "steady-state" portion of the convolved response.
Aside: Convolutions et al....
Aside: e.g., Microscope Imaging

“Within some quite general limitations, the object (specimen) and image are related by an operation known as convolution. In a convolution, each point of the object is replaced by a blurred image of the point having a relative brightness proportional to that of the object point. The final image is the sum of all these blurred point images. The way each individual point is blurred is described by the point spread function (PSF), which is simply the image of a single point.”

FIGURE 23.1. Diagram showing how a single point is imaged as the PSF by a microscope, and thus that the image of an extended object is the convolution of the object with the PSF.
Aside: Connections to tomography (e.g., CT) & Radon transforms

Fig. 5.15. Schematic summary of the relations among the spatial object domain (shown as an axial abdomen tomogram), the Radon space (given over an interval of $180^\circ$ from the object), and the Fourier space (only absolute values are shown). The Fourier domain results directly from the spatial domain by a two-dimensional Fourier transform of the object, but can also be obtained by the Fourier slice theorem using a set of one-dimensional Fourier transforms of the projection profiles in the Radon space.
Approach 4 – Convolve the impulse response

→ Now do this for a variety of different drive frequencies...

→ End up w/ the green +....
Approach 5 – Eigensolutions

Determine eigenvalues for $x = \dot{x} = 0$ (either numerically or analytically) as the resulting eigensolution, which is equivalent to the impulse response, then compute Fourier transform via FFT.
Approach 5 – Eigensolutions

End up w/ cyan dashed line....
Several basic approaches:
(all arriving at the same answer)

1. Numerically solve the ODEs and extract the relevant magnitudes and phases (via an FFT)

2. Analytic solution I (via Fourier transforms)

3. Impulse response I (and associated transfer function)

4. Impulse response II (convolve in the time domain)

5. Analytic solution II (via eigensolutions)

Note (re still to do #6)
stochastic differential equation (SDE) for a purely noise-driven case
Aside (still to do)

Noise-driven systems are very common physically....

Helmholtz resonator

... but harder to deal w/ analytically and computationally

in-situ earphone calibration
**Model:** Non-uniform transmission line

Several key ingredients:
- resonance
- longitudinal (e.g., fluid) coupling
- WKB approximation

→ Now in much better shape to understand this model!
Theme/Question:
What is the (basic) physical basis for “tonotopy”?

Hint: It ain’t a traveling wave per se.... (though such provides a useful framework)

→ This picture may be a bit more complicated....
Call it what you will (or will not), but all the basic ingredients for “resonance” are there (e.g., elements that trade energy back and forth on a cycle-by-cycle basis, “stiffness gradient”, etc...)
Tangent I: Resonance in the inner ear....

\[ \ddot{x} = -\omega_0^2 x - \gamma \dot{x} + A \cos(\omega t) \]

Resonance is chiefly a combination of two reactive forces:
- spring/stiffness
- inertial (i.e., moving mass)

Can we model a hair cell bundle using this basic formulation?
Tangent I: Resonance in the inner ear....

The role of fluid inertia in mechanical stimulation of hair cells

Dennis M. Freeman and Thomas F. Weiss


→ Hair cell bundles seem to operate in the region where viscous forces become relatively large....
Tangent I: Resonance in the inner ear....

Comparative Aspects of Hearing in Vertebrates and Insects with Antennal Ears

Joerg T. Albert\textsuperscript{1} and Andrei S. Kozlov\textsuperscript{2}

Box 2. How liquid in the inner ear has shaped the hair bundle.

A hair bundle operates at small Reynolds numbers on the order of $10^{-4}$. The Reynolds number (Re) is defined by: $Re = \frac{uL\rho}{\mu}$, where $u$ is the velocity, $L$ is a linear dimension (e.g., a hair bundle’s size), $\rho$ is the density and $\mu$ is the dynamic viscosity of the fluid. The Reynolds number indicates the relative importance of inertia over viscous forces for a particular type of flow. For the hair bundle, a Reynolds number of much lower than 1 indicates the relative importance of viscous forces.

Life at low Reynolds number

E. M. Purcell

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(Received 12 June 1976)

American Journal of Physics, Vol. 45, No. 1, January 1977
Tangent I: Resonance in the inner ear....

But things are a bit more complicated than just “what is the Reynold’s # of a hair cell bundle?”....
**Tangent I: Resonance in the inner ear....**

**Chicken basilar papilla** (i.e., auditory hair cells)

Note: Be careful! This picture can be misleading. In-vivo, there is a massive tectorial membrane (TM) overlying these hair cells....
Tangent I: Resonance in the inner ear....
Tangent I: Resonance in the inner ear....

Tangent I: Resonance in the inner ear....

Chicken basilar papilla (i.e., auditory hair cells)

Note: Be careful! This picture can be misleading. In-vivo, there is a massive tectorial membrane (TM) overlying these hair cells....

→ Coupling between hair cells affects mechanical properties (e.g., tuning) how?
Tangent I: Resonance in the inner ear....
Tangent I: Resonance in the inner ear....

[in collaboration w/ Bob Harrison]
Tangent I: Resonance in the inner ear....

Note: These hair cells do not have an overlying TM (for the most part)
Tangent I: Resonance in the inner ear....

- “phalanx” of hair cells
- Implications for inter-cell coupling?
- Longitudinal propagations? (e.g., traveling/standing waves)
Tangent I: Resonance in the inner ear....

These are classic measurements from *dead ears*....

Fig. 11-58. Phase displacement and resonance curves for four low tones.

Bekesy (1960)

.... so clearly “tuned” responses can arise in the absence of an active process (see tangent #2)

Frishkopf & DeRosier (1983)
This figure leads us into tangent #2 (i.e., the ear ain’t passive...)
Tangent II: The ear isn’t just passive.....

- Healthy ears actually emit sound

Spontaneous otoacoustic emissions (SOAEs)
SOAEs directly tied to forward auditory transduction (i.e., neural responses)
Tangent II: SOAEs & ANF responses

Evidence for an Active Process and a Cochlear Amplifier in Nonmammals

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➢ SOAE “suppression” related to auditory nerve fiber tuning....

➢ ... probably in a complicated fashion
Simple model to explain an SOAE peak: \[
\ddot{x} = -\omega_0^2 x - \gamma \dot{x} + A \cos(\omega t)
\]
Passive, linear case doesn’t do the trick.

Simple model to explain an SOAE peak: \[
\ddot{x} = -x - \varepsilon(x^2 - 1) \dot{x}
\]
van der Pol oscillator

Note: This equation comes in different flavors/forms (e.g., “normal form”, complex)

\[
\dot{z} = -\mu z + i\omega_0 z - |z|^2 z
\]
where \(z \in \mathbb{C}, \mu, \omega_0 \in \mathbb{R}\)

2 A (linear, undriven) harmonic oscillator can be described by a single, first–order ODE in terms of a complex variable \(z\) (e.g., [10]):
\[
\dot{z} = -\mu z + i\omega_0 z.
\]
Via a change of variables, this can be re-expressed as a 2nd order (real–valued) ODE:
\[
\ddot{x} + 2\mu \dot{x} + (\omega_0^2 + \mu^2)x = 0.
\]

Thus, the two notations are essentially equivalent. Note that in this case, no matter what the sign of \(\omega_0\) is, the quantity \((\omega_0^2 + \mu^2)\) will always be positive. Thus the system will always have a positive stiffness, though the damping can be positive or negative (depending upon the sign of \(\mu\)).

Binder et al (2011 MoH)
Tangent II: The ear isn’t just passive.....

\[ \ddot{x} = -x - \varepsilon (x^2 - 1) \dot{x} \]

**Limit cycles:**
- Negative damping for small displacements injects energy into system
- Nonlinearity stabilizes
- Self-sustained oscillation!
Tangent II: The ear isn’t just passive.....

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Narrow band filters and active resonators.

Psychophysical, Physiological and Behavioural Studies in Hearing

Proceedings of the 5th International Symposium on Hearing

Noordwijkerhout, The Netherlands
April, 8-12, 1980

Are spontaneous otoacoustic emissions generated by self-sustained cochlear oscillators?

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Frequency Clustering in Spontaneous Otoacoustic Emissions from a Lizard’s Ear

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FIGURE 1  Mechanical equivalent of the model. Sallets are represented as inertial oscillators (mass $M_j$, spring $K_j$), coupled to their neighbors by elastic (constant $k$) and damping (constant $\gamma$) elements. In addition, there exists an active driving mechanism within each oscillator (not shown).
This figure leads us into tangent #2 (i.e., the ear ain’t passive...)

Bergevin, Manley & Koppl (PNAS 2015)
Bergevin, Verhulst & van Dijk (SHAR 2017)
Tangent III: Other examples of “oscillators” in biology...

Vibrissa Resonance as a Transduction Mechanism for Tactile Encoding

Maria A. Neimark,1* Mark L. Andermann,1,2* John J. Hopfield,3 and Christopher I. Moore4

The Journal of Neuroscience, July 23, 2003 • 23(16):6499 – 6509

vi·bris·sa

/nəˈbrisə/ (ˈ)
noun ZOOLOGY

any of the long stiff hairs growing around the mouth or elsewhere on the face of many mammals, used as organs of touch; whiskers.

ORNITHOLOGY

each of the coarse bristlelike feathers growing around the gape of certain insectivorous birds that catch insects in flight.
Oscillations in cell biology
Karsten Kruse and Frank Jülicher

Current Opinion in Cell Biology 2005, 17:20–26

Schematic representation of Min oscillations in *E. coli*. MinD (green) is localized on the inner bacterial membrane (yellow) on one side of the cell, where it aggregates. MinE (red) induces disassembly of the MinD aggregates and detachment of MinD molecules into the cytoplasm. MinD then assembles on the membrane of the opposite side of the cell and the process is repeated.
Tangent III: Other examples of “oscillators” in biology...

Pressure source (lungs)

Vibration source (vocal folds)

Filter (vocal tract)

Output (mouth)

Stevens (2000)
Tangent III

- **Source**: Vibrating vocal folds make ‘broadband’ sound

  → Harmonics commonly referred to as "overtones"

- **Filter**: Vocal tract shapes source sound

  → Resulting ‘shape’ emphasizes features (e.g., formants)

- **Formants**: Filtered harmonics; basis for vowels

---

**Figure 3.1**: Sketches indicating components of the output spectrum $|p_r(f)|$ for a vowel and a fricative consonant. The output spectrum is the product of a source spectrum $S(f)$, a transfer function $T(f)$, and a radiation characteristic $R(f)$. The source spectra are similar to those derived in figures 2.10 and 2.33 in chapter 2. For the periodic source, $S(f)$ represents the amplitudes of spectral components; for the noise source, $S(f)$ is amplitude in a specified bandwidth. See text.

Stevens (2000)
Fini

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Ex. RLC circuit = Damped, Driven Harmonic Oscillator

**Mechanical**

- $F$ (force) $\leftrightarrow V$ (potential)
- $v$ (velocity) $\leftrightarrow I$ (current)
- $x$ (position) $\leftrightarrow q$ (charge)
- $m$ (mass) $\leftrightarrow L$ (inductance)
- $b$ (damping) $\leftrightarrow R$ (resistance)
- $k$ (spring) $\leftrightarrow 1/C$ (capacitance)

**Electrical**

**state variables**

**system properties**