Geometric Character Theory

Note Title 5/26/2011

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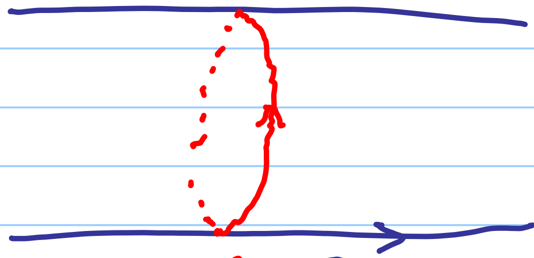
General theme:

Representation theory of G	⟷	Gauge theory of G
Algebraic structure	⟷	topological field theory structure
Geometric constructions of representations	⟷	Gauged sigma models
Representations, operators	⟷	Singularities, domain walls
traces / characters	⟷	compactification on circle
Langlands duality	⟷	S-duality

Today: dimensions, traces & characters

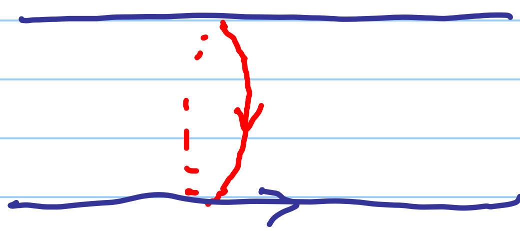
1. Dimensions in TFT

Consider an n-dimensional TQFT Z compactified on M^{n-1}



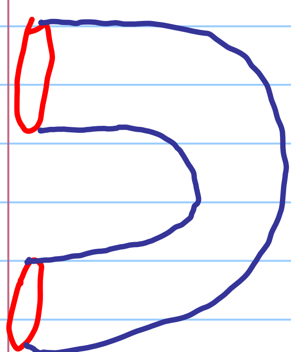
$M \times \mathbb{R}$

$\rightsquigarrow$ vector space $Z(M)$

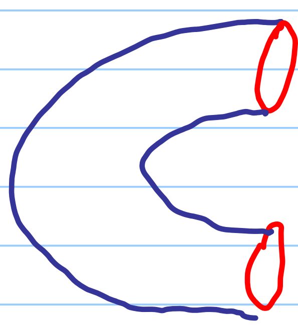


$M^{op} \times \mathbb{R}$

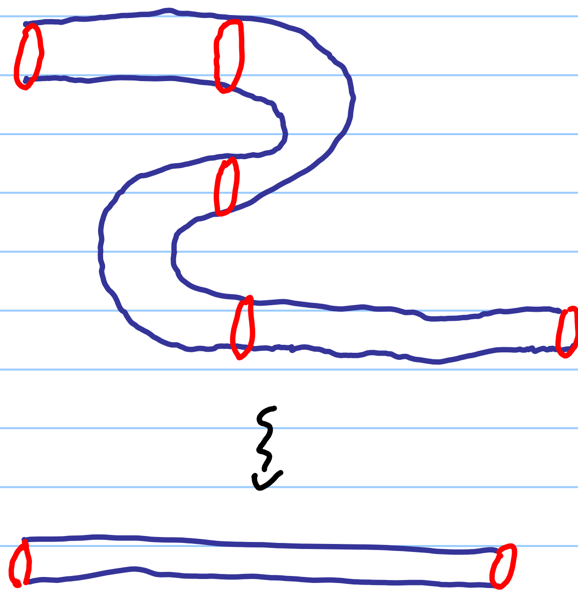
$\rightsquigarrow Z(M^{op})$



$\rightsquigarrow \mathrm{ev}: Z(M) \otimes Z(M^{op}) \longrightarrow \mathbb{C}$



$\rightsquigarrow \mathrm{coev}: \mathbb{C} \longrightarrow Z(M) \otimes Z(M^{op})$



$$\begin{array}{lcll} Z(M) & \longrightarrow & Z(M) & \\ & & \otimes & \Big\} \; ev \\ & & Z(M^{op}) & \\ coev \; \Big\{ & & \otimes & \\ & & Z(M) & \longrightarrow \; Z(M) \end{array}$$

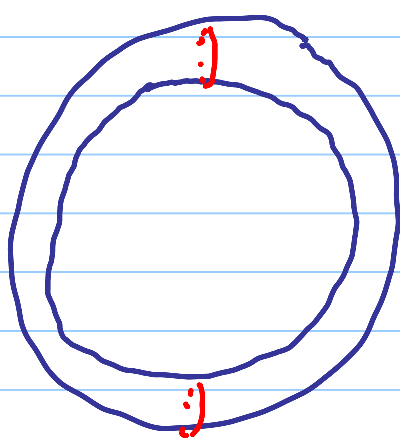
$$\rightsquigarrow$$

$$Z(M) \xrightarrow{\quad Id \quad} Z(M)$$

$\Rightarrow$ $Z(M)$ <u>dualizable</u> vector space,

[a.k.a. finite dimensional]

with dual $Z(M^{op})$



$\dim Z(M) =$

$Z(M \times S^1) =$

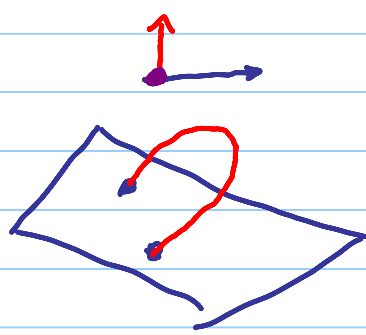
$ev(coev(1))$

$$Z(M) \otimes Z(M^{op}) \simeq End\, Z(M)$$

$coev(1) = Id_{Z(M)}$ $\qquad ev = Tr : End\, Z(M) \to \mathbb{C}$

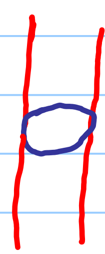
$$\dim Z(M) = Tr\,(Id_{Z(M)})$$

Extended TFT : compactify on lower-dimensional manifolds & obtain richer structure: OPE algebras, categories of branes, higher categories of defect operators, ...



$pt \times \mathbb{R}^2$ in 2d TFT

$\rightsquigarrow$ category of D-branes



$S^1 \times \mathbb{R}^2$ in 3d TFT

$\rightsquigarrow$ category of line operators

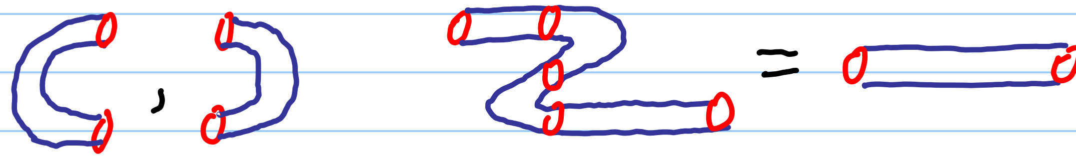
$S^2 \times \mathbb{R}^2$ in 4d TFT

$\rightsquigarrow$ category of line operators

$S^1 \times \mathbb{R}^3$ in 4d TFT

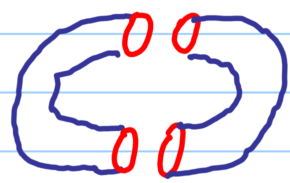
$\rightsquigarrow$ 2-category of surface operators

Each of these objects is <u>dualizable</u>



& has notion of <u>dimension</u> :

$$\dim Z(N) = Z(N \times S^1)$$

$=$ 

– compactify on an additional circle

$\rightsquigarrow$ decategorification :

dim : Vector spaces $\rightsquigarrow$ numbers

categories $\rightsquigarrow$ vector spaces

2-categories $\rightsquigarrow$ categories

. . . .

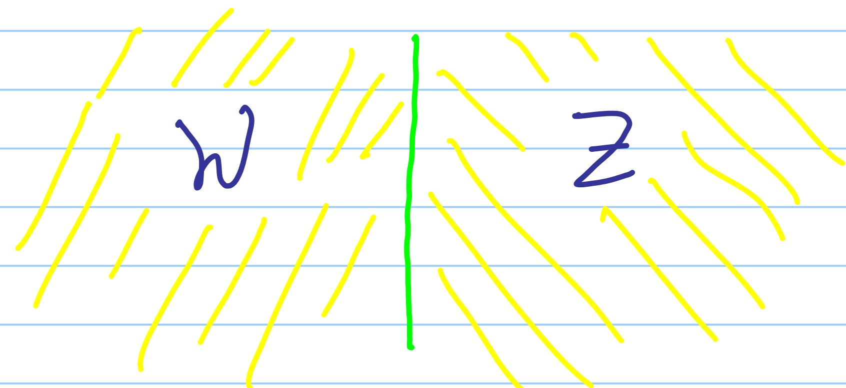
Mathematically : Hochschild homology

for dualizable objects in higher categories

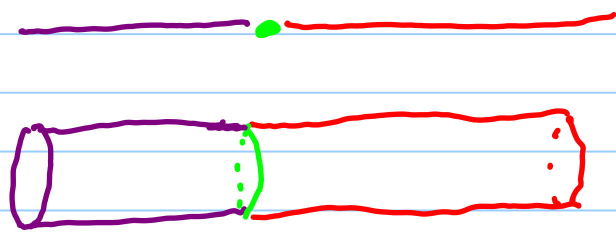
(cyclic homology : invariants under rotation of S^1
– we'll suppress distinction today)

What can one prove in this generality?

... preservation across domain walls



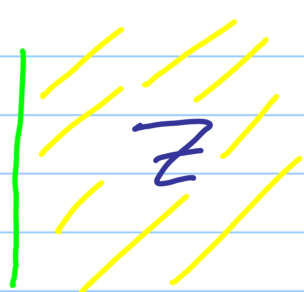
$W(M) \longrightarrow Z(M)$



$\dim W(M) \longrightarrow \dim Z(M)$

eg boundary conditions :

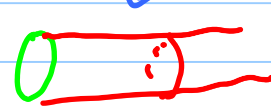
domain walls with trivial theory



boundary state

$v \in Z(M)$

$\mapsto$



$\dim v \in \dim Z(M)$

Mathematically:

$f_*: \mathcal{C} \to \mathcal{D}$ right-dualizable morphism of dualizable objects in a higher category $\Rightarrow$ $\dim f_*: \dim \mathcal{C} \to \dim \mathcal{D}$

.... dim is a symmetric monoidal functor on dualizable objects & right-dualizable morphisms

In particular $V: 1 \to \mathcal{C}$ dualizable object of $\mathcal{C}$ has character $[V] \in \dim \mathcal{C}$ which is functorial

$$\begin{tikzcd} \mathcal{C} \arrow[r, "f_*"] & \mathcal{D} \\ 1 \arrow[u] \arrow[ur] & \end{tikzcd}$$

$[f_* V] = (\dim f_*)\,[V]$

— special case of Jacob Lurie's Cobordism hypothesis with singularities

– treats all defects of extended TFTs & more!

[this case easy to prove directly, BZ-N]

2. Applications for 2d TFT

B-model:

X algebraic variety$/\mathbb{C}$ $\rightsquigarrow$

$Z_X(\cdot) := D(X)$ derived category of (quasi) coherent sheaves

- eg complexes of vector bundles

[X smooth, compact $\Longrightarrow$
Z_X full 2d TFT
(with framing anomaly)]

$$\dim D^b(X) = Z_X(S^1) = HH_*(X)$$
$$= R\Gamma(\Omega^\bullet(X))$$

Dolbeault cohomology

(cyclic homology $\rightsquigarrow$ de Rham cohomology)

V vector bundle $\in D^b(X)$

$\Longrightarrow$ $[V] \in \dim D^b(X)$

is the <u>Chern character</u> of V [D-brane charge]

Functoriality of Hochschild homology :

$f: X \longrightarrow Y$ any proper map of varieties,

(or general domain wall / integral transform

$f_*: D(X) \longrightarrow D(Y)$ continuous)

$\Longrightarrow$

$\dim f_* : HH_*(X) \longrightarrow HH_*(Y)$

V vector bundle on X

$$[f_* V] = (\dim f_*)([V])$$

- form of Grothendieck - Riemann - Roch

(or Hirzebruch-Riemann-Roch for $Y = pt$)

(Todd genus comes from comparison of Hochschild & de Rham ...)

[X, Y smooth & proper : Markarian, Caldararu, Ramadoss, Shklyarov ..]

Topological gauge theory

G $\mathbb{C}$-semisimple (or generally <u>affine</u>)

$\mathcal{Z}_G(\cdot) = \mathrm{Rep}\, G$ category of reps of G

[G finite $\Longrightarrow$ full 2d TFT,
Dijkgraaf - Witten]

$\dim \mathrm{Rep}\, G = \mathcal{Z}_G(S^1) = \mathbb{C}\left[\frac{G}{G}\right]$ class fns.

$V \in \mathrm{Rep}\, G \rightsquigarrow [V] \in \dim \mathrm{Rep}\, G$
is the <u>character</u> of V, a class fn.

~ formally $\mathcal{Z}_G$ can be considered
as a σ-model, with target the
orbifold / stack $\bullet/G$:

$\mathrm{Rep}\, G =$ Vector bundles on pt/G

$\mathbb{C}\frac{G}{G} =$ Hochschild homology of pt/G

Gauged B-models

Combine these two: $\mathfrak{X} = X/G$

algebraic stack (orbifold for G finite)

$$\mathcal{L}\mathfrak{X} = \{x \in X, g \in G : g \cdot x = x\}/G$$

inertia stack ("derived loop space")

= locally constant loops in $\mathfrak{X}$

(eg $\mathcal{L}\,{}^{\circ}/G = \frac{G}{G}$ adjoint quotient)

<u>Theorem</u> $\dim D(\mathfrak{X}) = R\Gamma(\Omega^{\bullet}(\mathcal{L}\mathfrak{X}))$

– gauged B-model reduced on circle

- collects all twisted sectors

("orbifold cohomology")

$$X \longrightarrow \bullet \qquad G\text{-equivariant} \Rightarrow$$

$$\mathcal{X} = X/G \xrightarrow{\pi} \bullet/G$$

domain wall between gauged σ-model & pure gauge theory: integrate out X fields

$$\mathcal{L}\mathcal{X} \xrightarrow{\mathcal{L}\pi} \mathcal{L}(\bullet/G) = \frac{G}{G}$$

$$(x,g) \longmapsto [g]$$

$$\mathcal{L}\pi^{-1}([g]) = X^g/G :$$

$\mathcal{L}\pi$ organizes all fixed points in X.

Theorem (BZ-N) $\mathcal{X}$ proper, smooth
G affine

$$\begin{tikzcd} \dim \mathcal{D}(\mathcal{X}) \arrow[r, "\dim \pi_*"] \arrow[d, phantom, "\wr"] & \dim \operatorname{Rep} G \arrow[d, phantom, "\wr"] \\ R\Gamma(\Omega^\bullet(\mathcal{L}\mathcal{X})) \arrow[r, "\mathcal{L}\pi_*"] & R\Gamma(\Omega^\bullet(\frac{G}{G})) \end{tikzcd}$$

ie pushforward on Hochschild homology given geometrically by integration along $\mathcal{L}\pi : \mathcal{L}\mathcal{X} \longrightarrow \frac{G}{G}$.

- easy consequence of explicit form of functoriality on Hochschild homology: same holds for any proper morphism $\mathcal{X} \to \mathcal{Y}$ geometric stacks: integrate over $\mathcal{L}\mathcal{X} \longrightarrow \mathcal{L}\mathcal{Y}$

Corollary: generalized Atiyah-Bott fixed point theorem

V equivariant vector bundle on X

$\iff$ $\mathcal{V}$ vector bundle on $\mathfrak{X} = X/G$

$\pi_*(\mathcal{V})$ = virtual representation $H^*(X, V)$

dim $\pi_*(\mathcal{V})$ = its character

identify with $\mathcal{L}\pi_*[\mathcal{V}]$, integral of form $[V]$ on $\mathcal{L}\, X/G$:

Value at g given by integral over X^g

– A-B with no transversality, in families, for orbifolds, for affine algebraic groups

Variant:

Dolbeault operators $\rightsquigarrow$ de Rham operators

B-model on X $\rightsquigarrow$ A-model on T^*X

$\mathcal{D}(X)$ = category of $\mathcal{D}$-modules

= sheaves with flat connection

= B-branes on quantized T^*X

" = " A-branes on T^*X

Theorem (BZ-N) $\mathfrak{X}$ nice stack $\Rightarrow$

$$\dim \mathcal{D}(\mathfrak{X}) = H^*_{dR}(\mathcal{L}\mathfrak{X})$$

($\Rightarrow$ eg $\dim \mathcal{D}(X) = H^*_{dR}(X)$ for varieties)

So equivariant flat bundles have Chern characters which are cohomology classes of inertia / loop space

Functoriality theorem works the same for D-modules $\implies$ Atiyah-Bott for de Rham operators in great generality.

eg $\gamma \in \operatorname{Aut} X$ autoequivalence

let $\mathcal{X} = X/\langle\gamma\rangle \longrightarrow \bullet/\mathbb{Z}$

$\pi_*[\underline{\mathbb{C}}] = H^*(X,\mathbb{C}) \circlearrowleft H^*(\gamma)$

$\dim \pi_*[\underline{\mathbb{C}}]|_\gamma = s\mathrm{Tr}(\gamma, H^*(X,\mathbb{C}))$

$$\mathcal{L}\mathcal{X} = \coprod_{\mathbb{Z}} X^{\gamma^n} \times B\mathbb{Z}$$
$$\downarrow$$
$$\coprod_{\mathbb{Z}} \{\gamma^n\} \times B\mathbb{Z}$$

$\implies$ Lefschetz fixed point formula giving trace of γ on $H^*(X)$ via count of fixed points.

3. Characters in 3d gauge theory

Frobenius/Atiyah-Bott character formula:

Take $X = G/K$ homogeneous space

$V = \underline{\mathbb{C}}_X$ is G-equivariant $\Longrightarrow$

$\pi_* V =$ functions on G/K $\in \mathrm{Rep}\, G$

induced representation.

$\mathcal{L}X/G =$

$$\{x \in G/K,\ g \in \mathrm{Stab}_G x\}/G \simeq \frac{K}{K}$$

$$\frac{K}{K} \xrightarrow{\mathcal{L}\pi} \frac{G}{G}$$

$[\mathbb{C}_X] = \underline{1}$

$[\mathrm{Fun}(G/K)] = \mathcal{L}\pi_* \underline{1}$

Character of induced representation at g counts $k \in K$ which are conjugate to g.

Weyl character formula:

G complex reductive $\supset B$ Borel

$X = G/B$ flag manifold

$$\mathcal{L}X/G = \{ g \in G,\ B' \in G/B \text{ flag} : g \in B' \}/G$$

$$\begin{array}{c} \downarrow \mathcal{L}\pi \\ \dfrac{G}{G} \end{array}$$

Grothendieck-Springer simultaneous resolution

- $\mathcal{L}\pi^{-1}$ (unipotents in G) $\simeq T^*G/B$

- $\mathcal{L}\pi^{-1}$ (g regular semisimple) $= |W|$ fixed points (permutations of eigenspaces)

$V = L_\lambda$ holomorphic line bundle on G/B
($\longleftrightarrow \lambda$ weight for G)

$\Rightarrow$ character of irreducible f.d. representation of h.w. λ as sum over W ..

Three dimensional analog

$\mathcal{X}_G$ character theory of a complex group G,

BZ.N arXiv: 0904.1247

(a topological twist of $\mathcal{N}=8$ $d=3$ SYM,

Witten arXiv: 0905.4795)

Replace G actions on vector spaces

by G actions on categories

[flat actions: matrix elements are

$\mathcal{D}$-modules on G]

Prime example: $G \curvearrowright X$

$\Rightarrow G \curvearrowright \mathcal{D}(X)$

flat action on category of $\mathcal{D}$-modules

- give boundary conditions for

the character theory, ie $\mathcal{X}_G(pt)$

$$\dim \left\{ \begin{matrix} \text{flat } G\text{-} \\ \text{categories} \end{matrix} \right\} = \mathcal{D}\left(\frac{G}{G}\right)$$

"class fld connections" :

G-equivariant systems of linear PDE on G

[$\mathcal{X}_G(S^1)$]

Characters of flat categories are Lusztig's character sheaves.

[$\Rightarrow$ S-duality for character sheaves !]

"Frobenius character formula" :

$X = G/B$

$$\mathcal{L}X/G \xrightarrow{\mathcal{L}\pi} \frac{G}{G}$$ Grothendieck-Springer

Theorem (BZ.N) $\dim \mathcal{D}(G/B) = \mathcal{L}\pi_* \underline{\mathbb{C}}$

the Springer sheaf [cohomologies of fibers]

$\longleftrightarrow$

the Harish Chandra system

Harish Chandra: distributional characters of admissible Lie group representations solve this system

$\rightsquigarrow$ great regularity properties

Beilinson-Bernstein:

$$\mathcal{D}(G/B) \simeq \mathfrak{g}\text{-modules}_0$$

BZ-N: given nice $\mathfrak{g}$-module $M \in \mathcal{D}(G/B) \longmapsto$ character $[M]$, a section of the character sheaf $[\mathcal{D}(G/B)]$

$\longleftrightarrow$ solution of HC system!

Expect to recover Schmid-Vilonen's Harish Chandra - Weyl character formula

4. The trace formula & 4d gauge theory

Geometric Langlands program
(twisted $\mathcal{N}=4$ $d=4$ SYM) :

Study category $\mathcal{D}(\mathrm{Bun}_G \Sigma)$
$\mathcal{D}$-modules on moduli space of
G-bundles on Riemann surface Σ

(A-branes on Hitchin space of Σ
$\sim T^*\mathrm{Bun}_G \Sigma$)

<u>Theorem</u> (BZ-N)

$\dim \mathcal{D}(\mathrm{Bun}_G \Sigma)$ [as plain category, ie HH_*]

$\simeq H^*_{dR}(\mathcal{L}\,\mathrm{Bun}_G \Sigma)$

[= value of TFT on $\Sigma \times S^1$]

$$\mathcal{L}\,\mathrm{Bun}_G \Sigma = \mathrm{HIGGS}_G \Sigma =$$

$$\left\{ (P, \eta) : \begin{array}{l} P \text{ G-bundle}, \\ \eta \in \mathrm{Aut}\, P \end{array} \right\}$$

– "grouplike" version of Hitchin space.

[can add arbitrary ramification / twisting of Higgs field]

Cohomology of $\mathrm{HIGGS}_G \Sigma$ (or Lie alg. version)

central object in Ngô's work on

Fundamental Lemma (for Lie algebras)

– an identity needed to apply the

Arthur-Selberg Trace Formula,

principal tool of the Langlands program

Ngô deduces Fundamental Lemma
from study of cohomology of
Hitchin fibration
– ie of sheaf $\pi_* \underline{\mathbb{C}}$
of cohomologies of fibers

$$\begin{array}{c} \mathrm{HIGGS}_G\, \Sigma \\ \big\downarrow \pi \\ \{\text{spectral curves over } \Sigma\} \end{array}$$

BZ-N : Ngô's sheaf $\pi_* \underline{\mathbb{C}}$
expresses the character
$$[\, \mathcal{D}(\mathrm{Bun}_G\, \Sigma)\,]$$
as module for the Hecke algebra
($\longleftrightarrow$ 't Hooft line operator)

$\rightsquigarrow$ geometric version of trace formula
describes this character via S-duality!