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"Verlinde" & Bethe Ansatz via DAHA

Ivan Cherednik

FT — algebra Heis, Weyl_q — $E =$ geometrically unramified covers of E^n

$$\boxed{FT} \begin{cases} \frac{d}{dx_i} \to x_i \\ x_i \to -\frac{d}{dx_i} \end{cases}$$

$\uparrow$

$\mathcal{H}_{q,t}$ = DAHA — deformations deforms W_q ~ —"— $E^n \setminus \mathrm{diag}$

$q < 1$, $|q| = 1$

$q^N = 1$ — $\mathcal{H}_{q,t}$ = simplest irrep — nonsym generalized Verlinde $\boxed{S, T, \text{mult} \quad \langle , \rangle}$

$S \leftrightarrow$ Fourier Transform

reproducing kernel solves

Macdonald – Ruijsenaars eigenvalue problem

$\downarrow t = 0$

Gaitsgory – Lurie

q-Whittaker Givental – Lee

$\updownarrow$

Affine Grassmanian

"Verlinde" & Bethe ansatz (coordinate) (Ivan Cherednik)

(1) Elliptic configuration space

(2) DAHA in rank 1

(3) Verlinde algebras

(4) Bethe ansatz

(1) $H_2 = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + \sum_{i \neq j} k(k-1)\wp(x_i - x_j)$

$\wp \sim (2\pi i, \tau)$ $E = \mathbb{C}/(2\pi i\mathbb{Z} + \tau\mathbb{Z})$

$H_3, H_4, \ldots$ $\mathcal{E} = (E^n \setminus \mathcal{D}iag)/S_n$ $z_i \neq z_j$

$\sim Bun_{GL_n}(E)^{reg} \sim \mathcal{O}(z_1) \oplus \mathcal{O}(z_2) \oplus \ldots \oplus \mathcal{O}(z_n)$

90% of

How to extend big cell to the boundary

Algebraic Geometry Gelfand - MacPherson

$\mathbb{C}\pi_1(\mathcal{E}) / (T_i - a)(T_i - b) = 0$

$T_i = z_i \leftrightarrow z_{i+1}$

Braid relations a, b same for all i

Cherednik - Matsuo theorem

$\{\alpha\} \subset R$ root system $\quad (,)$

$x_\alpha = (\alpha, x) \qquad W = \text{Weyl group}$

$H_2 = \Delta + \sum k_{|\alpha|}(k_{|\alpha|-1}) / \wp(x_\alpha$

$$\mathcal{E} = \left(E^n \setminus R_{\alpha n}\, w\right) W$$

Don't remove full ram $\left(E^n \setminus D_0\right)$

$$\mathbb{C}\pi_1^{orb}(\mathcal{E}) / (T_i - a)(T_i - b)$$

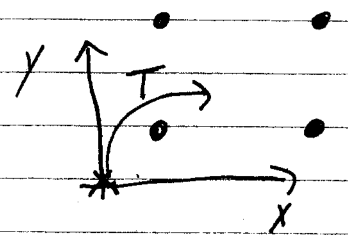
(2) DAHA in rk = 1

H_3, H_4

Ochiai - Oshima
Sekiguchi $\boxed{B, C}$

$$H_i \psi = \underbrace{p_i(x_1, \ldots, x_n)}_{\text{Bun}} \psi$$

$\mathbb{C}$ $\quad \pi_1^{orb}((E\setminus 0)/\{z \xrightarrow{S} -z\})$ $\qquad A_1 = SL(2)$



$\mathcal{H}\ell =$

$$q^{1/2}\, Y^{-1}X^{-1}YXT^2 = 1$$

$$XTXT = 1$$

$$Y^{-1}TY^{-1}T = 1$$

$$(T - t^{1/2})(T + t^{-1/2}) = 0$$

Monodromy of Lame

$\mathbb{C}P^1$ w/ 4 points removed Heun

Cor σ S: $X \to Y^{-1}$
$Y \to XT^2$

τ_+ T: $X \to X$
$Y \to q^{-1/4}XY$

τ_- $\quad Y \to Y, \quad X \to q^{1/4}YX$

Projective $PSL_2(\mathbb{Z}) = B_2$

$$\tau_+\tau_-^{-1}\tau_+ = \sigma = \tau_-^{-1}\tau_+\tau_-^{-1}$$

Birmann & Scott

<u>Th^m</u> $\quad T \to t^{1/2} s + \dfrac{t^{1/2} - t^{-1/2}}{X^2 - 1}(s - 1)$

$\mathbb{C}[X, X^{-1}]$
Fock $\quad X \to X$

$$Y \to s \Gamma^{-1} T$$

$$s(X) = X^{-1}$$
$$\Gamma(X) = q^{1/2} X$$ difference operator

generic q, t irreducible etc

$$Y(E_n) = q^{-n_*/2} E_n$$

$$E_n = X^n + \sum c_m X^m$$

$$|m| \leq |n|$$

$$\boxed{n > 0 \quad \text{no } X^{-n}}$$

$$n_* = \frac{n+k}{2}, \quad n > 0$$

$$\frac{n-k}{2} \quad n \leq 0$$

$$Y + Y^{-1}\big|_{sym} = \frac{t^{1/2} Y - t^{-1/2} X^{-1}}{Y - X^{-1}} \Gamma$$

$$+ \quad t^{1/2} X^{-1} - t^{-1/2} X \quad \Gamma^{-1}$$

3 Verlinde Algebras

$X \to X$

$T \to$

$Y \to$

$\Downarrow$

$\xi^N = 1$ $\qquad 0 < k < N/2$ avoids ~~[illegible]~~ [illegible]

$\mathcal{P}$

Thm $\quad \mathcal{P}/(X^{2N} + X^{-2N} - 2)$ $\qquad$ always a faithful rep

$V_{2N+\cancel{4k}\,4k} \qquad\qquad V_{2N-\cancel{4k}\,4k}$

V_{2N-4k} is perfect

S, T $\qquad$ Weyl algebra at root of unity has unique rep of dim n

$\mathcal{P}/\text{rad} < f, g, u >$

~~Simplest case~~

$t = 1$

$T^2 = 1 \quad t = q$

Next $\quad \boxed{t = q}$

V^{sym}_{2N-4k} $\qquad$ dim $= N - 2k + 1$

$T(f) = t^{1/2} f$

<u>Verlinde</u> Integrable modules of $\widehat{sl_2}$

of level $= c = N-2$

$$V_{N-1} \begin{cases} S, T & (\text{Kac, Peterson}) \\ \text{Fusion} & (\text{Cherednik, Dolan}) \\ \langle\ ,\ \rangle & \end{cases}$$

<u>Verlinde</u> alg

Axioms for Fourier-Transform

$$V^{sym} \qquad \left.\begin{matrix} k = 1 \\ Sym \end{matrix}\right\} \Rightarrow \text{Verlinde alg}$$

k measures ramification

$$\left(V^{(k=1}_{2N-4k} \right)^{sym} = \text{Verlinde}$$

$$\Lambda \subset T^*(\hat{G}/\hat{B} \times \hat{G}/\hat{B})$$

$$K_T(\Lambda)$$

(4) Bethe Ansatz

$$1 = \prod_{\alpha>0} \left(\frac{t^{N/2} - X_\alpha^N t^{-N/2}}{t^{-N/2} - \Lambda_\alpha^N t^{N/2}} \right)^{(a,\alpha)}$$

determines C's

such that rep is unitary

$$P / (X^{2N} + X^{-2N} - C)$$

Gerasimov & Shatashvili
added Higgs field to gauged WZW

Bethe ansatz will collapse at t a root of unity

Controls when modules are unitary