Mirror symmetry at higher genus

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2. Introduce a new approach to the higher-genus closed string B-model (joint with Si Li, Harvard). This is based on the Bershadsky-Cecotti-Ooguri-Vafa quantum field theory, and my work on renormalization. (Preprint available on my homepage, also Li's thesis). So far, we can only construct the quantum theory in dimension 1.

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3. State a theorem of Li, that mirror symmetry holds for the elliptic curve: the generating function of Gromov-Witten invariants of the elliptic curve coincides with the partition function of the BCOV quantum field theory of the mirror elliptic curve.

Mirror symmetry pre-Kontsevich

Mirror symmetry was first formulated around 1990 (Candelas, de la Ossa, Green, and Greene-Plesser).

Original form of the conjecture: $X, X^\vee$ a mirror pair of Calabi-Yau three-folds. Then, the conjecture states

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Answer : (Givental, Barannikov). Both sides are encoded in a pair (V, L) where

- V is a symplectic vector space.
- $L \subset V$ is a conic Lagrangian submanifold.

B-model small Lagrangian cone

There are two versions of the story: small (without descendents) and large (includes descendents).

Let X be a Calabi-Yau 3-fold (equipped with holomorphic volume form). Let

$$V_B^{small}(X) = H^3(X, \mathbb{C})$$
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There's a map

$$\begin{aligned} \mathcal{M}_X &\to V_B^{small}(X) \\ Y &\mapsto [\Omega_Y] \in H^3(X). \end{aligned}$$

$L_B^{small}(X)$ is the image of this map.

A-model small Lagrangian cone (Givental)

Let

$$V_A^{small} = \oplus_{p=0}^{3} H^{p,p}(X) \otimes \mathbb{C}((q))$$

$$\langle \alpha^{p,p}, \beta^{3-p,3-p} \rangle = (-1)^p \int_X \alpha \wedge \beta$$

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Let $\mathbf{F}_0$ be the generating function of genus 0 Gromov-Witten invariants:

$$\mathbf{F}_0 : H^{0,0} \oplus H^{1,1} \to \mathbb{C}((q))$$
$$\left(\frac{\partial}{\partial \alpha_1} \dots \frac{\partial}{\partial \alpha_k} \mathbf{F}_0 \right)(0) = \sum q^d \int_{[\overline{\mathcal{M}}_{0,k,d}(X)]^{virt}} \mathrm{ev}_1^* \alpha_1 \dots \mathrm{ev}_k^* \alpha_k .$$

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Note:

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Let

$$L_A^{small}(X) = 1 + \text{graph of } \mathrm{d}\mathbf{F}_0 \subset V_A^{small}$$

Formal germ of Lagrangian cone at $1 \in V_A^{small}$.

Polarizations

A-model: Lagrangian cone in V_A is defined over $\operatorname{Spec} \mathbb{C}((q))$.

B model to match this, we need to take a family of varieties over $\operatorname{Spec} \mathbb{C}((q))$.

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Correct polarization: $X \to \operatorname{Spec} \mathbb{C}((q))$, $M : H^3(X) \to H^3(X)$ monordomy. Look at $\operatorname{Ker}(M-1)^2 \subset H^3(X)$.

Descendents

If $\alpha_1, \ldots, \alpha_n \in H^*(X)$, define

$$\langle \tau_{k_1}(\alpha_1), \ldots, \tau_{k_n}(\alpha_n) \rangle_{g,n,d} = \int_{[\overline{\mathcal{M}}_{g,n,d}(X)]^{virt}} \psi_1^{k_1} \, \mathrm{ev}_1^*(\alpha_1) \ldots \psi_n^{k_n} \, \mathrm{ev}_n^*(\alpha_n).$$

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The generating function

$$\mathbf{F}_g \in \mathscr{O}(H^*(X)[[t]]) \otimes \mathbb{C}[[q]]$$

is defined by

$$\left(\frac{\partial}{\partial(t^{k_1}\alpha_1)} \cdots \frac{\partial}{\partial(t^{k_n}\alpha_n)} \mathbf{F}_g \right)(0) = \sum q^d \langle \tau_{k_1}(\alpha_1), \ldots, \tau_{k_n}(\alpha_n) \rangle_{g,n,d}$$

A-model symplectic formalism with descendents (Givental)

Let

$$V_A^{big}(X) = H^*(X)((t))$$

with symplectic pairing

$$\langle \alpha f(t), \beta g(t) \rangle = \left(\int_X \alpha\beta \right) \operatorname{Res} f(t) g(-t) \mathrm{d}t.$$

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Polyvector fields

X a CY of dimension d. Let

$$\mathrm{PV}^{i,j}(X) = \Omega^{0,j}(X, \wedge^i TX).$$

Contracting with $\Omega \in H^0(X, K_X)$ gives an isomorphsim

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$\mathrm{PV}^{*,*}(X)$ has graded-commutative product, and trace

$$\mathrm{Tr} : \mathrm{PV}^{3,3}(X) \to \mathbb{C} \quad \mathrm{Tr}(\alpha) = \int_X \Omega(\alpha \vee \Omega).$$

B-model Lagrangian cone with descendents (Barannikov)

Let

$$V_B^{big} = \mathsf{PV}(X)((t)).$$

Differential $Q = \overline{\partial} + t\partial$, symplectic pairing

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L_B^{big} is moduli of "extended" CY deformations of X. (Equivalent to deformations of $\mathrm{Perf}(X)$ as a dg Calabi-Yau category).

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Easy to verify: L_B^{big} is preserved by the differential (and satisfies Givental's other axioms).

Genus 0 mirror symmetry conjecture with descendents

Conjecture

X a Calabi-Yau, $X^\vee \to \operatorname{Spec} \mathbb{C}((q))$ the mirror family.
Then there is a quasi-isomorphism of symplectic vector spaces

$$V_A^{big}(X) = H^*(X)((t)) \simeq \mathrm{PV}(X^\vee)((t)) = V_B^{big}(X)$$

taking L_A^{big} to L_B^{big}.

Proved in many cases by Givental, Lian-Liu-Yau, Barannikov.

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Genus 0 A and B-model: a Lagrangian submanifold L in a symplectic vector space V.

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Higher genus: we should quantize this picture. Symplectic vector space V quantizes to the Weyl algebra

$$\mathcal{W}(V) = \text{ free algebra over } \mathbb{C}[[\hbar]] \text{ generated by } \alpha \in V^\vee \\ \text{with relations } [\alpha, \beta] = \hbar \langle \alpha, \beta \rangle \,.$$

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Lagrangian submanifold $L \subset V$ quantizes to a vector in Fock(V), the Fock module for $\mathcal{W}(V)$.

A-model at higher genus (Givental)

$$V_A = T^* H^*(X)[[t]].$$

So,

$$\mathrm{Fock}(V_A) = \mathscr{O}(H^*(X)[[t]])$$

algebra of functions on $H^*(X)[[t]]$.

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$$Z_A = \exp\left(\sum \hbar^{g-1} \mathbf{F}_g\right) \in \mathrm{Fock}(V_A)[[q]]$$

A-model partition function.

Vector in Fock space which in $\hbar \to 0$ limit becomes $L_A \subset V_A$.

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Recall

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L_B is the extended moduli of deformations of X.

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We will discuss how to do this using QFT.

Quantum field theory and the B-model

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Fields include Beltrami differentials $\Omega^{0,1}(X, TX)$, the gauge group is the diffeomorphism group.

The space of solutions to the equation of motion is $\mathcal{M}_X$, formal moduli space of Calabi-Yaus near X.

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Witten : the BCOV partition function is a state in $\mathrm{Fock}(H^3(X))$.

Extended BCOV theory

We want to quantize extended moduli space

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Extended BCOV action is the functional

$$\mathbf{F}_0 \in \mathscr{O}(\mathrm{PV}(X)[[t]])$$

such that

$$\mathrm{Graph}(\mathrm{d}\mathbf{F}_0) = L_B.$$

Extended BCOV theory

Concretely:

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satisfies

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Here: dg *Poisson* manifold, with a potential $\mathbf{F}_0$ satisfying $\{\mathbf{F}_0,-\} = \mathrm{d}$. Can still be treated using usual techniques.

The classical master equation

Recall

$$\mathrm{PV}(X)((t)) = T^* \left(\mathrm{PV}(X)[[t]]\right)$$

as graded vector space but not as a cochain complex.

If $\Phi \in \mathscr{O}(\mathrm{PV}(X)[[t]])$ then

$$\mathrm{Graph}(\mathrm{d}\Phi) \subset \mathrm{PV}(X)((t))$$

is preserved by the differential on $\mathrm{PV}(X)((t)) \iff$

$$Q\Phi + \tfrac{1}{2}\{\Phi, \Phi\} = 0$$

$\{-,-\}$ a Poisson bracket on $\mathscr{O}(\mathrm{PV}(X)[[t]])$ of degree 1.

The classical master equation

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This equation is called *classical master equation*.

Since L_B^{big} is preserved by the differential, $\mathbf{F}_0$ satisfies classical master equation.

Interpreting the classical master equation

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Aim : quantize this classical field theory. My book *Renormalization and effective field theory* gives the definition of quantization we use, and allows one to construct quantizations by obstruction theory (term by term in $\hbar$).

Quantization (naive approach)

Naive idea: look for a series

$$\mathbf{F} = \sum \hbar^g \mathbf{F}_g \in \mathscr{O}(\mathrm{PV}(X)[[t]])[[\hbar]]$$

satisfying quantum master equation

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Problem : Δ is not defined (because of ultraviolet divergences of quantum field theory).

Definition of quantization

Solution (*Renormalization and effective field theory*, C. 2011): gives general definition of a perturbative QFT.

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Definition

A quantization of the BCOV theory is a family of action functionals

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($\mathbf{F}[L]$ is "scale L effective action"). These must satisfy:

- Renormalization group equation: $\mathbf{F}[L]$ expressed in terms of $\mathbf{F}[\varepsilon]$ by (roughly) "integrating out modes of wave-length between ε and L".

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- Locality axiom : as $L \to 0$, $\mathbf{F}[L]$ approximated by the integral of a Lagrangian.

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The BCOV theory admits a unique quantization on an elliptic curve.

Proof: obstruction theory/ cohomological calculations.

The situation in higher dimensions is not so satisfactory.

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Quantum master equation and RGE imply we can construct a cohomology class

$$[\exp(\mathbf{F}[L]/\hbar)] \in H^*(\mathrm{Fock}(\mathrm{PV}(X)((t))))$$

independent of L.

This will be the partition function of the BCOV theory.

Mirror symmetry for the elliptic curve

E elliptic curve. Mirror family: $E_\tau^\vee$, $\tau \in \mathbb{H}$, $q = e^{2\pi i\tau}$.

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$$V_B^{big}(E_\tau) = H^*\left(\Omega^{0,*}(E, \wedge^* TE)((t))\right), \text{ differential } \overline{\partial} + t\partial.$$

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Under this isomorphism, the A-model partition function $Z_A(E) \in \mathrm{Fock}(V_A^{big}(E))$ *corresponds to* $Z_B(E^\vee) \in \mathrm{Fock}(V_B^{big}(E^\vee))$.

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This means all GW invariants of an elliptic curve E can be computed from quantum BCOV theory on the mirror elliptic curve $E^\vee$.

Correlators and the Hodge filtration

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But naive splitting $\overline{F}^1$ (complex conjugate to Hodge filtration) does not vary holomorphically with E. "Holomorphic anomaly".

Large complex structure splitting

If $\tau \in \mathbb{H}$, let E_τ be the elliptic curve. If $\sigma \in \mathbb{H}$ let $F^1_\sigma H^1(E_\tau)$ be Hodge filtration for structure σ: then $\overline{F}^1_\sigma H^1(E_\tau)$ splits Hodge filtration on $H^1(E_\tau)$.

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To match the A-model use splitting $\lim_{\sigma \to i\infty} \overline{F}^1_\sigma H^1(E_\tau)$.

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$$1 \in H^0(E, \mathscr{O}_E) \leftrightarrow 1 \in H^0(E^\vee)$$
$$\mathrm{d}\overline{z} \in H^1(E, \mathscr{O}_E) \leftrightarrow \mathrm{d}\overline{z} \in H^{0,1}(E^\vee)$$
$$\partial_z \in H^0(E, TE) \leftrightarrow \mathrm{d}z \in H^{1,0}(E^\vee)$$
$$\partial_z \mathrm{d}\overline{z} \in H^1(E, TE) \leftrightarrow \mathrm{d}z\mathrm{d}\overline{z} \in H^2(E^\vee).$$

Sketch of proof

1. Prove that Virasoro constraints hold on the B-model. Obstruction theory argument: they hold classically, there is a unique quantization, so they hold at the quantum level.

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$$\left\langle \omega t^{k_1}, \ldots, \omega t^{k_n} \right\rangle_{g,n}^{\tau,\infty}$$
where $\omega \in H^1(E, TE)$ has $\mathrm{Tr}(\omega) = 1$, so $\omega = \partial_z \mathrm{d}\bar{z}/2\,\mathrm{Im}\,\tau$.

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4. Implies there are operators $\{O_k \mid k \geq 0\}$ in the chiral free boson such that
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5. Okounkov-Pandharipande: A-model correlators are expectation values of a family of commuting operators in a system of 2 chiral free fermions. The operators are the same: essentially characterized by commutativity.

GW invariants of an elliptic curve are complicated (determined by Okounkov-Pandharipande, 2002).

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If

$$\begin{aligned} & F^E(z_1, \ldots, z_n; q) \\ & = z_1 \ldots z_n \prod_{m=1}^{\infty} (1 - q^m) \exp\left(\sum q^d \left\langle \tau_{k_1}(\omega), \ldots, \tau_{k_n}(\omega) \right\rangle_{g,n,d} z_1^{k_1} \ldots z^{k_n} \right) \end{aligned}$$

and

$$\theta(z) = \theta(z, q) = \sum_{n \in \mathbb{Z}} (-1)^n q^{(n+\frac{1}{2})^2/2} e^{(n+\frac{1}{2})z}$$

then

$$F^E(z_1, \ldots, z_n; q) = \sum_{\substack{\text{permutations of} \\ z_1, \ldots, z_n}} \frac{\det \left[\frac{\theta^{(j-i+1)}(z_1 + \cdots + z_{n-j})}{(j-i+1)!} \right]_{i,j=1}^n}{\theta(z_1)\theta(z_1 + z_2) \ldots \theta(z_1 + \cdots + z_n)}$$