From the Renormalization Group to Quantum Gravity:
Celebrating the science of Joe Polchinski

February 27 – 28, 2014

Spacetime Entanglement

R. Myers

One of Joe's other sports:

“Canada hands U.S. crushing OT loss, wins women's hockey gold”

“Canada destroys Sweden 3-0 for men's hockey gold”

“US men's hockey falls to Canada”

PERIMETER INSTITUTE
FOR THEORETICAL PHYSICS

Spacetime Entanglement

R. Myers

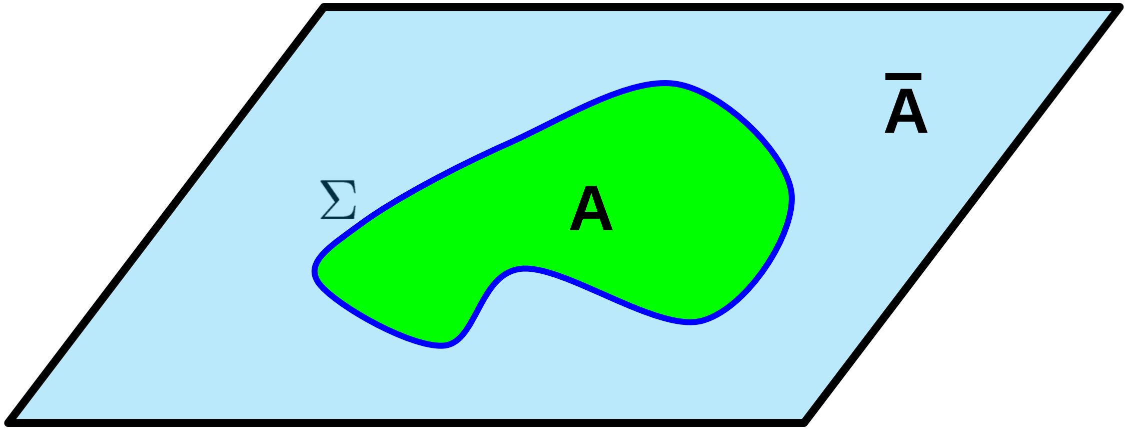
with M. Headrick, J. Rao, S. Sugishita & J. Wien

Proposal: Spacetime Entanglement

(Bianchi & RM)

- in a theory of quantum gravity, for any sufficiently large region A in a smooth background, consider entanglement entropy between dof describing A and $\bar{A}$; contribution describing short-range entanglement is finite and described in terms of geometry of entangling surface with leading term:

$$S_{\rm EE} = \frac{\mathcal{A}_\Sigma}{4G_N} + \cdots$$



- higher order terms similar to Wald entropy (RM, Pourhasan & Smolkin)

AdS/CFT Correspondence:

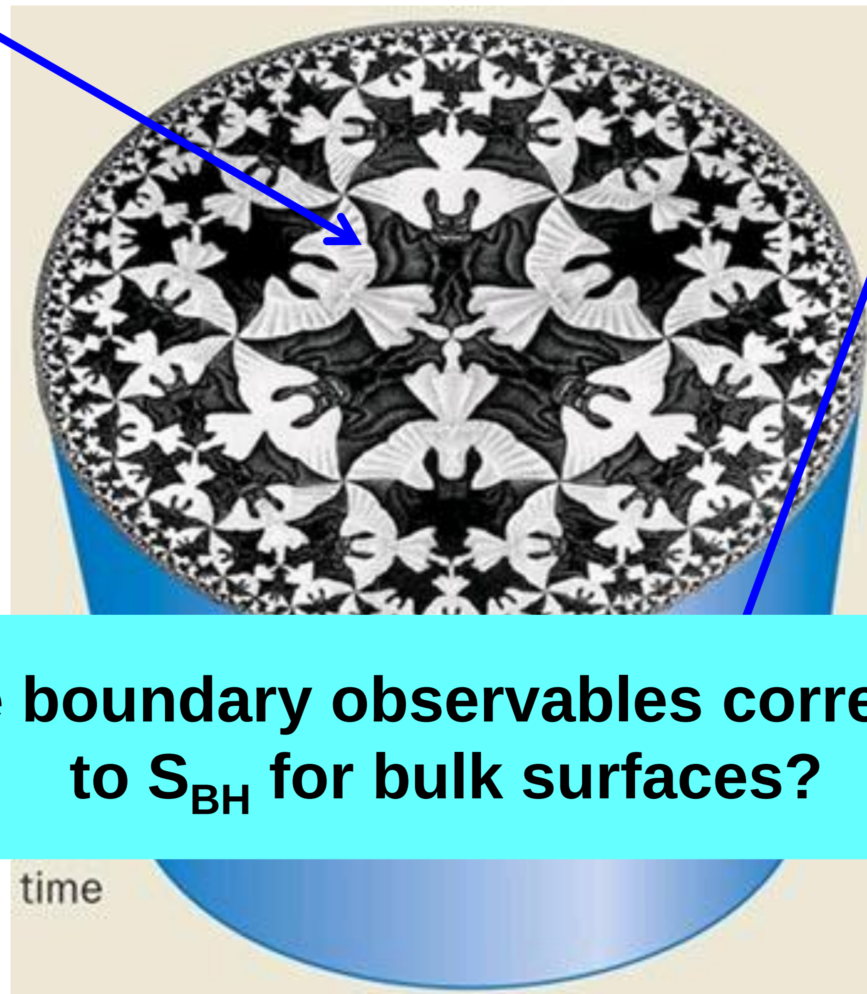
Bulk: gravity with negative Λ in **d+1** dimensions

anti-de Sitter space

Boundary: quantum field theory in **d** dimensions

conformal field theory

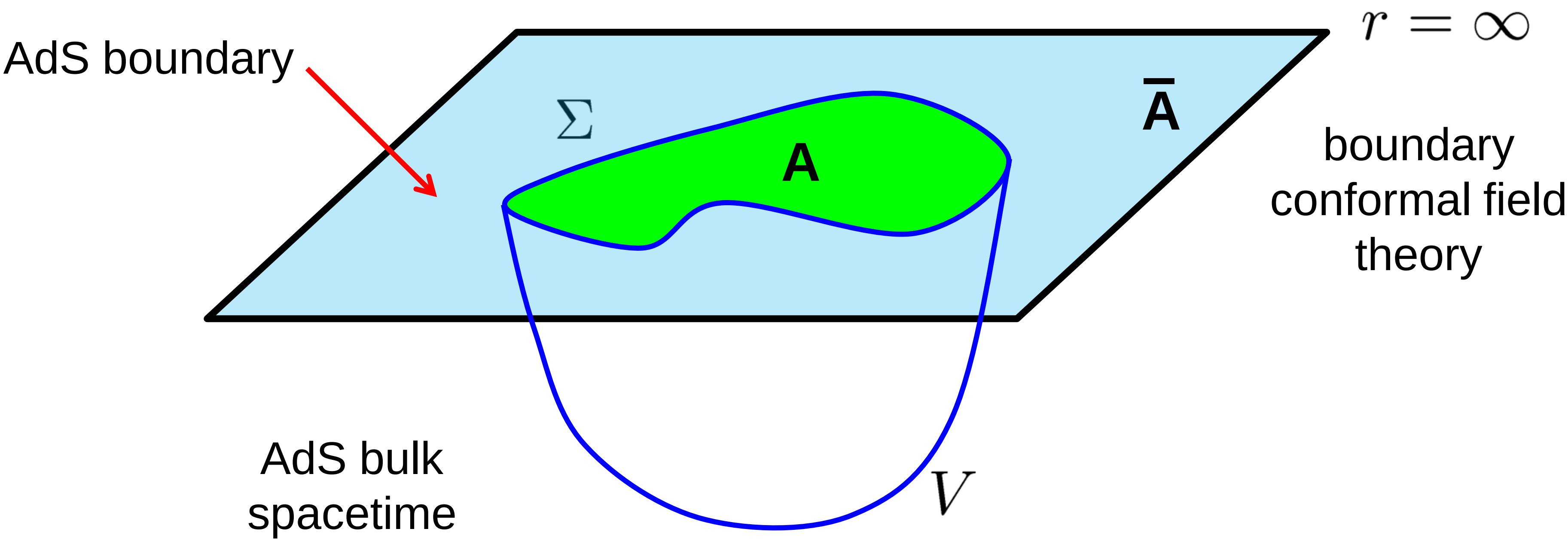
“holography”



Are there boundary observables corresponding to S_{BH} for bulk surfaces?

(Ryu & Takayanagi `06)

Holographic Entanglement Entropy:



$$S(A) = \min_{\partial V = \Sigma} \frac{A_V}{4G_N}$$

Lessons from Holography:

- R&T construction certainly assigns entropy $S_{BH} = \mathcal{A}/(4G_N)$ to bulk regions with "unconventional" boundaries:

  not black hole! not horizon! not boundary of causal domain!

- **indicates S_{BH} applies more broadly but more examples?**

- S_{BH} on other surfaces speculated to give new entropic measures of entanglement in boundary theory

  ⟶ causal holographic information
  (Hubeny & Rangamani; H, R & Tonni; Freivogel & Mosk; . . .)

  ⟶ entanglement between high and low scales
  (Balasubramanian, McDermott & van Raamsdonk)

  ⟶ hole-ographic spacetime
  (Balasubramanian, Chowdhury, Czech, de Boer & Heller)

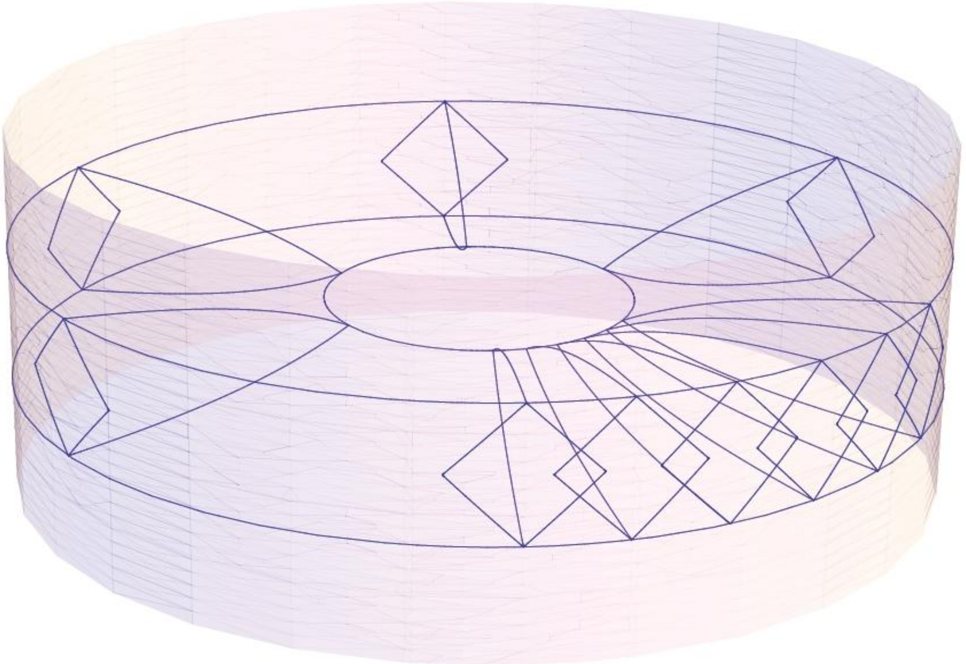
(Balasubramanian, Chowdhury, Czech, de Boer & Heller)
arXiv:1310.4204 [hep-th]

“hole-ographic spacetime”:

- calculate “gravitational entropy” of curves inside of d=3 AdS space, in terms of entropies of bounded regions in boundary

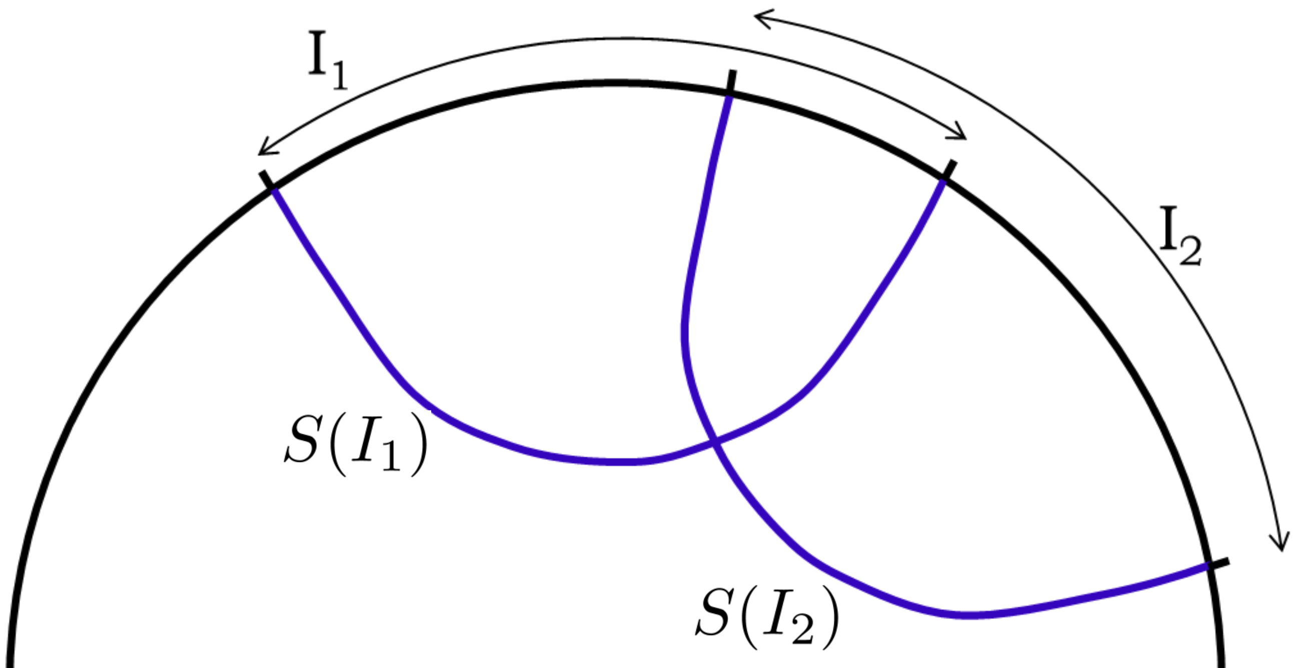
$$S_{\text{res}} \leq \sum \left(S(I_j) - S(I_j \cap I_{j+1})\right)$$

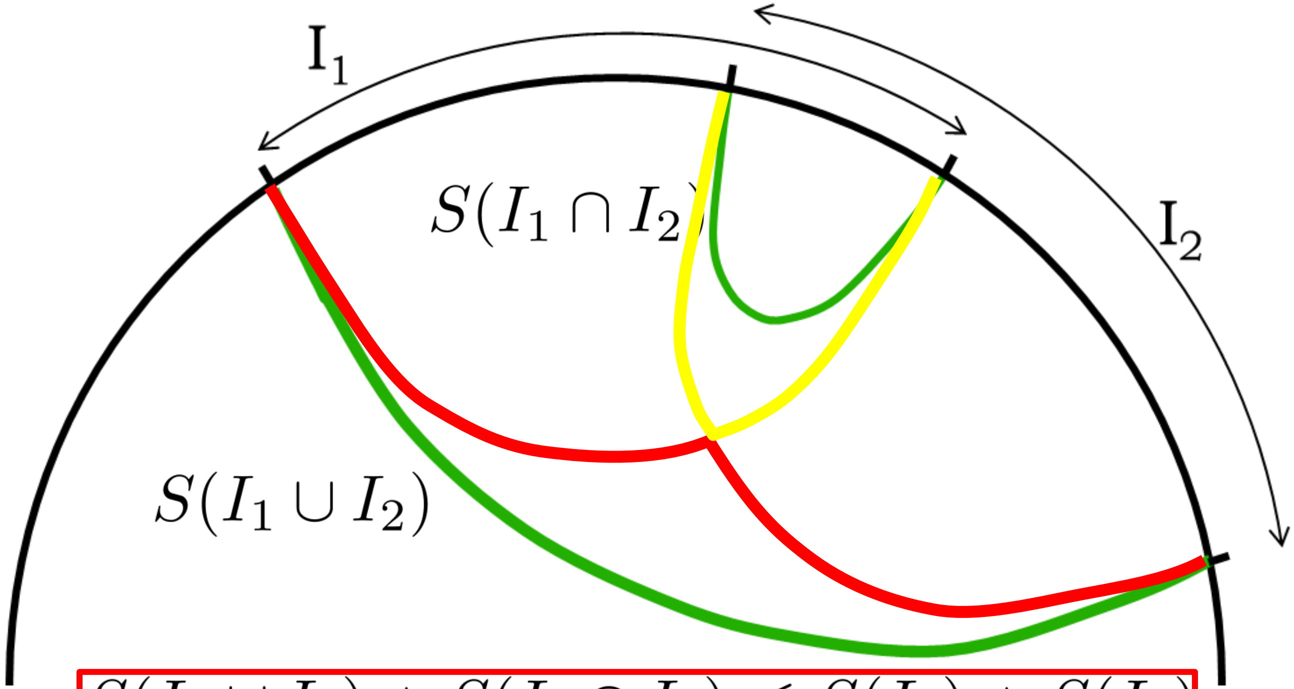
- “residual entropy”: maximum entropy of a density matrix consistent with all measurements of finite-time local observers



Strong Sub-Additivity: $S(I_1 \cup I_2) + S(I_1 \cap I_2) \leq S(I_1) + S(I_2)$

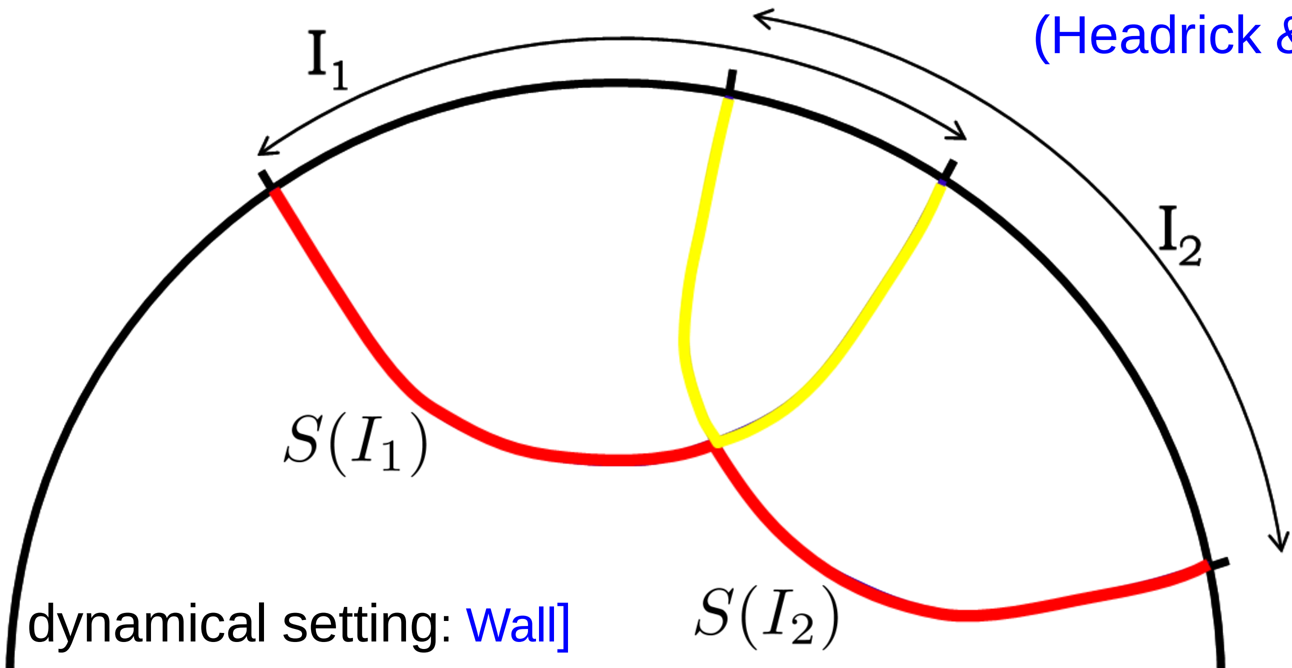
- recall "holographic proof" (Headrick & Takayanagi)



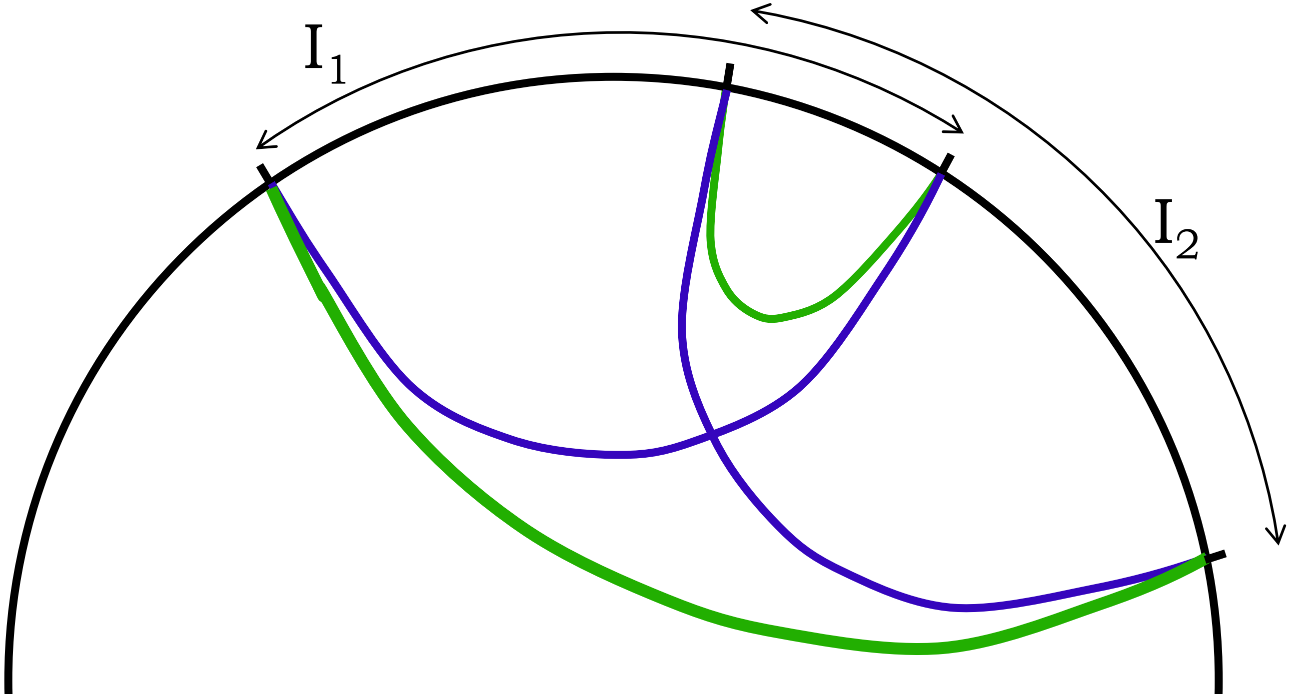


$$S(I_1 \cup I_2) + S(I_1 \cap I_2) \leq S(I_1) + S(I_2)$$

(Headrick & Takayanagi)



[extended to dynamical setting: Wall]

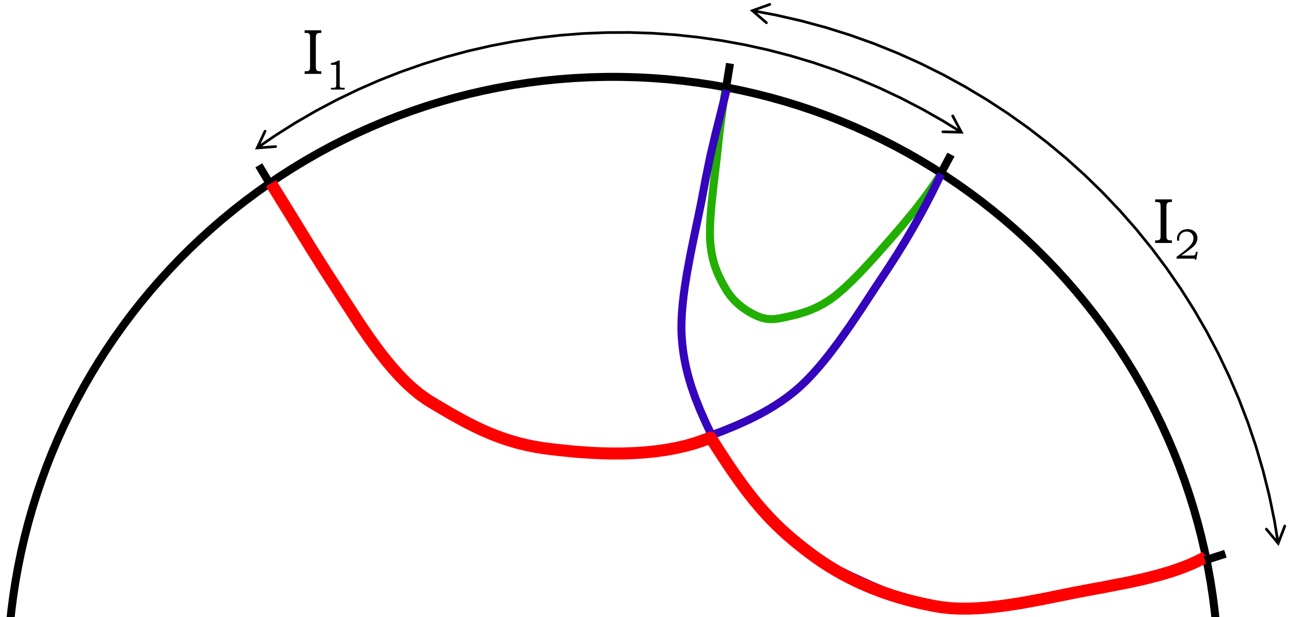


can write strong subaddivity as:

$$S(I_1 \cup I_2) \leq \underbrace{S(I_1) + S(I_2) - S(I_1 \cap I_2)}_{\text{Differential Entropy}}$$

Differential Entropy

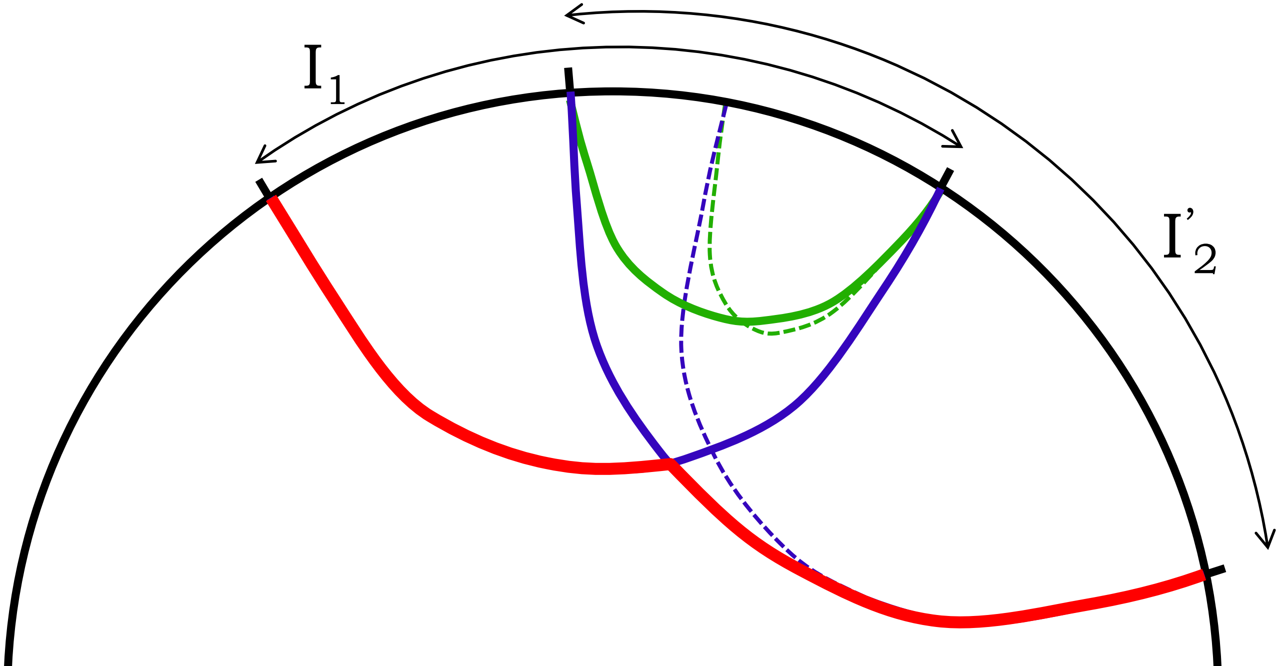
consider AdS_3 in global coordinates with two intervals, I_1 and I_2, on bdry



as a geometric inequality:

$$\hat{S}(I_1, I_2) \leq S(I_1) + S(I_2) - S(I_1 \cap I_2)$$

where $\hat{S}(I_1, I_2)$ corresponds to
(length of **outer envelope**)/$4G_N$



as a geometric inequality:

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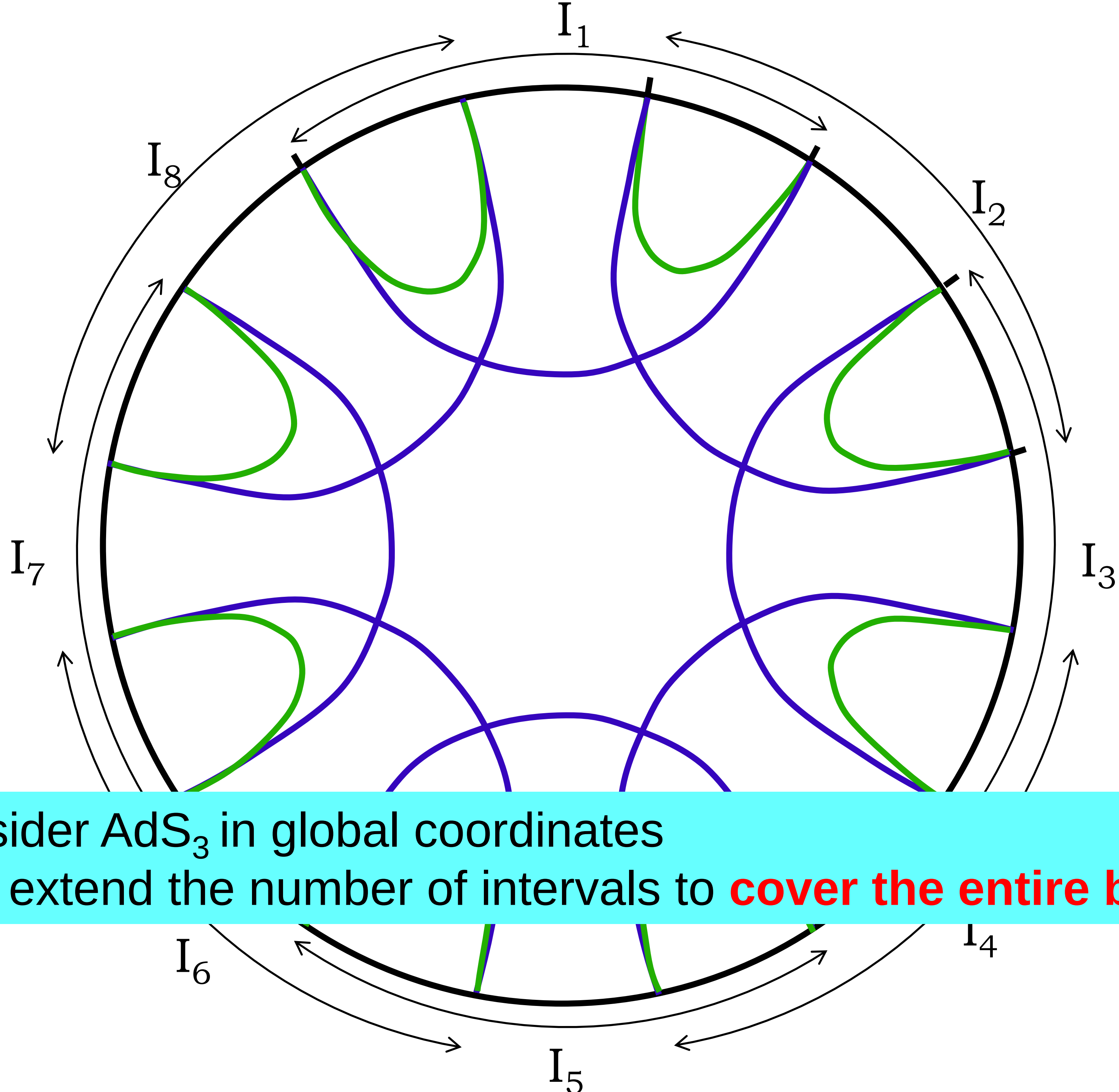
where $\hat{S}(I_1, I_2)$ corresponds to (length of **outer envelope**)/$4G_N$

Note: $\hat{S}(I_1, I_2)$ is not a function of $I_1 \cup I_2$ alone; instead depends on details of partition

eg, $I_1 \cup I_2 = I_1 \cup I_2'$ but $\hat{S}(I_1, I_2) \neq \hat{S}(I_1, I_2')$

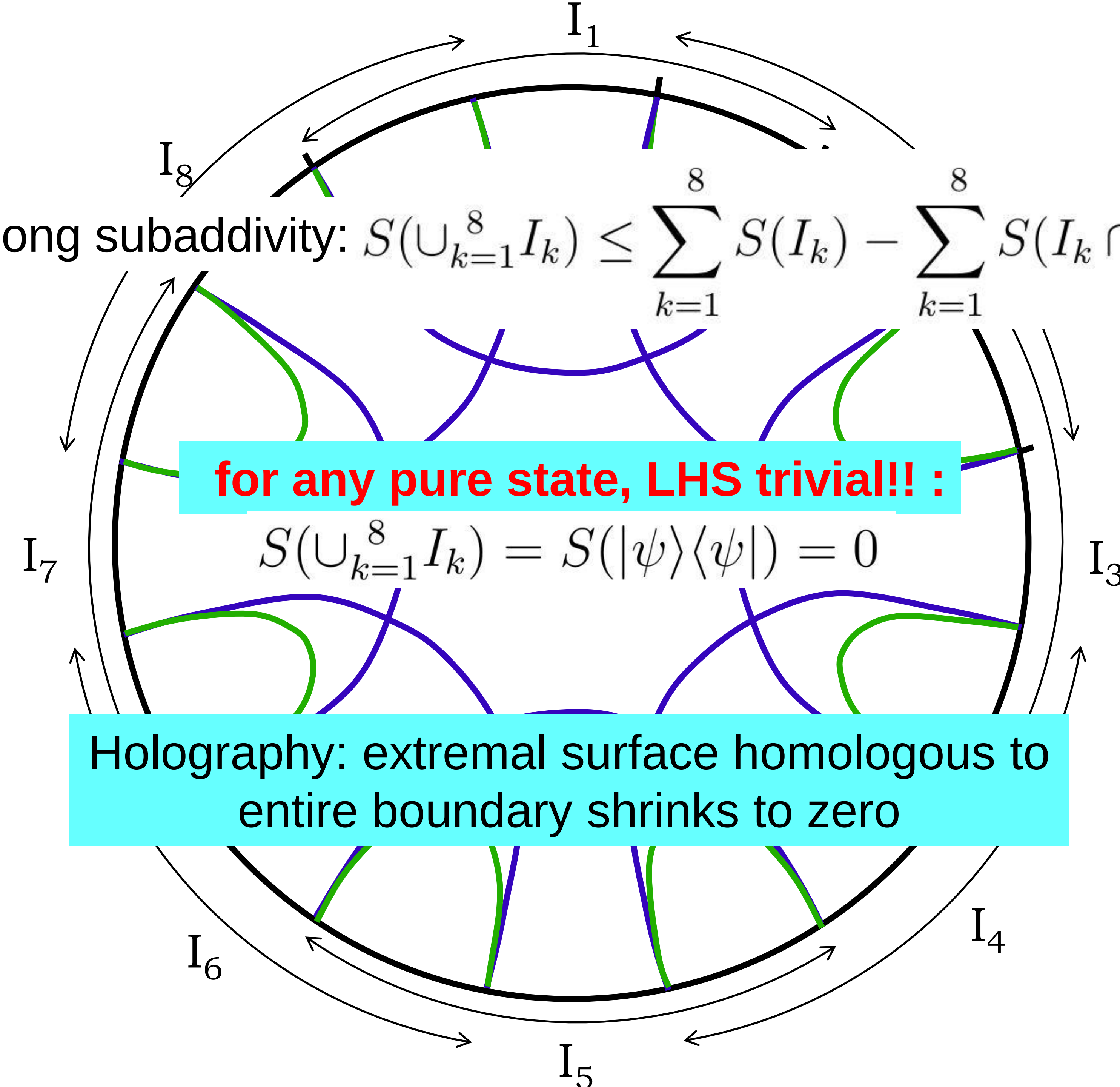


consider AdS_3 in global coordinates
now extend the number of intervals to **cover the entire bdry**

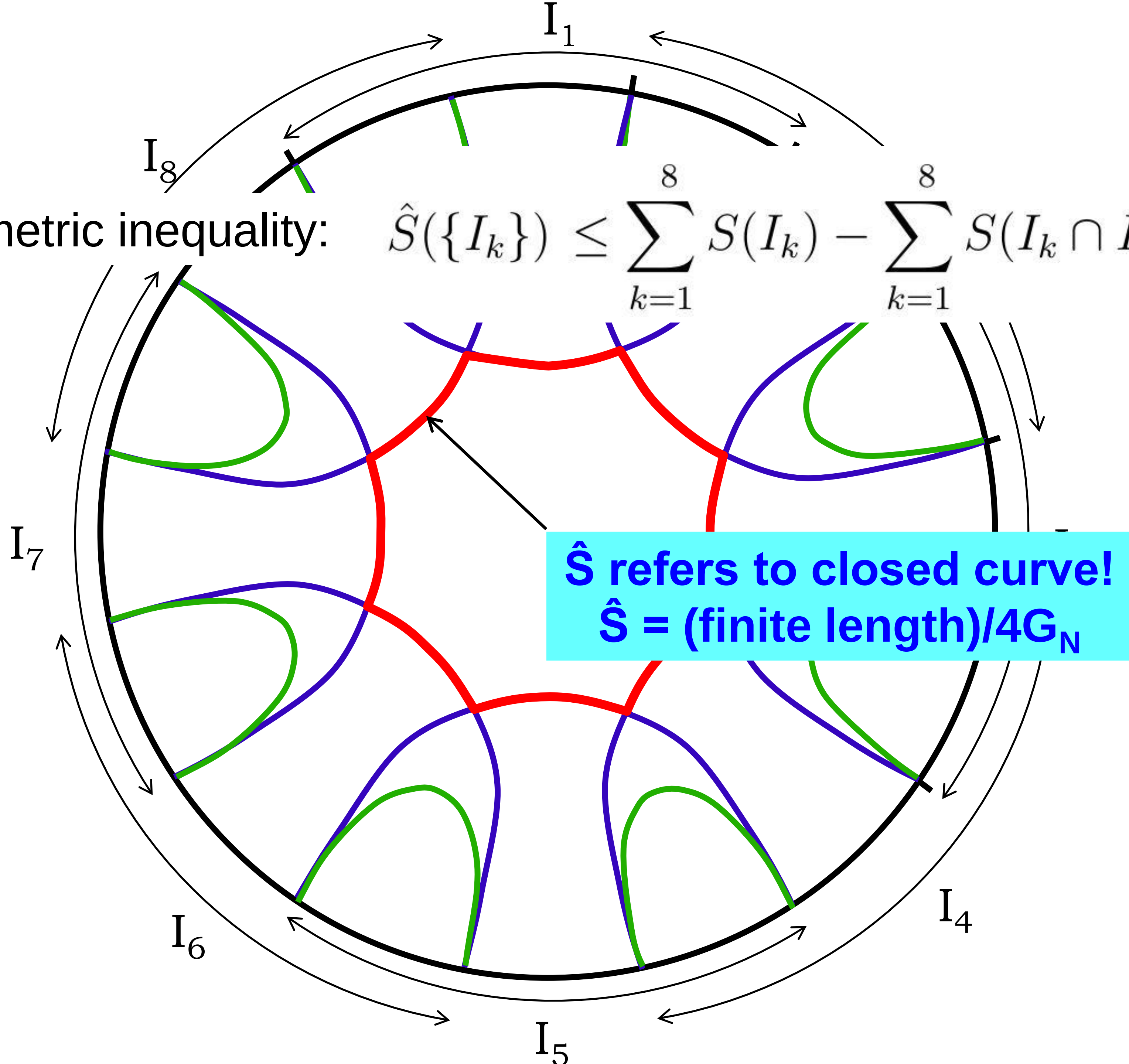
strong subaddivity: $S(\cup_{k=1}^{8} I_k) \leq \sum_{k=1}^{8} S(I_k) - \sum_{k=1}^{8} S(I_k \cap I_{k+1})$

for any pure state, LHS trivial!! :

$$S(\cup_{k=1}^{8} I_k) = S(|\psi\rangle\langle\psi|) = 0$$



Holography: extremal surface homologous to entire boundary shrinks to zero geometric inequality: $\hat{S}(\{I_k\}) \leq \sum_{k=1}^{8} S(I_k) - \sum_{k=1}^{8} S(I_k \cap I_{k+1})$

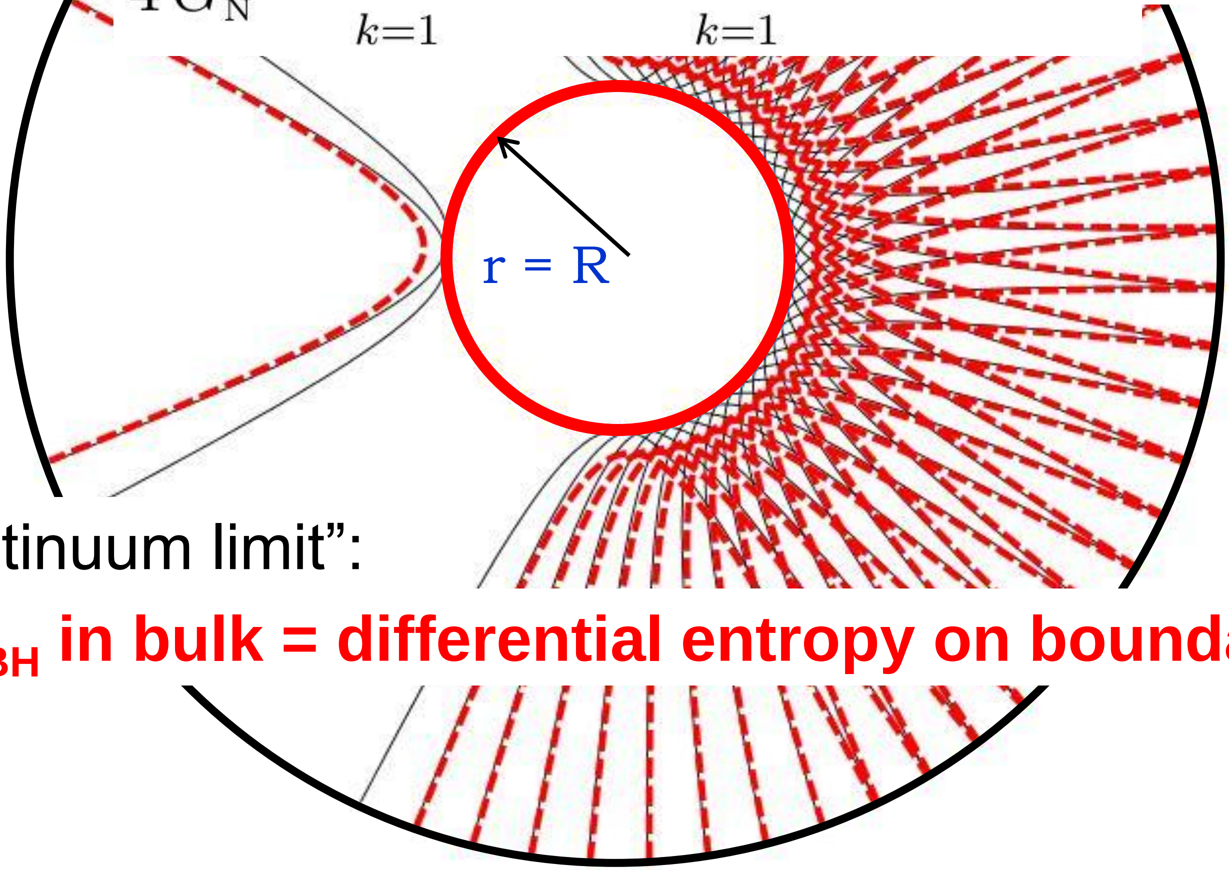


Ŝ refers to closed curve!
Ŝ = (finite length)/4G_N

(Balasubramanian, Chowdhury, Czech, de Boer & Heller)

- keep length of intervals is fixed but take number of intervals to infinity
- $\hat{S}$ becomes the length of a (smooth) circle of constant radius
- **surprise is that the geometric inequality is precisely saturated!**

$$\longrightarrow \quad \frac{2\pi R}{4\,G_{\mathrm{N}}} = \sum_{k=1}^{\infty} S(I_k) - \sum_{k=1}^{\infty} S(I_k \cap I_{k+1}) \quad \textbf{!!!}$$



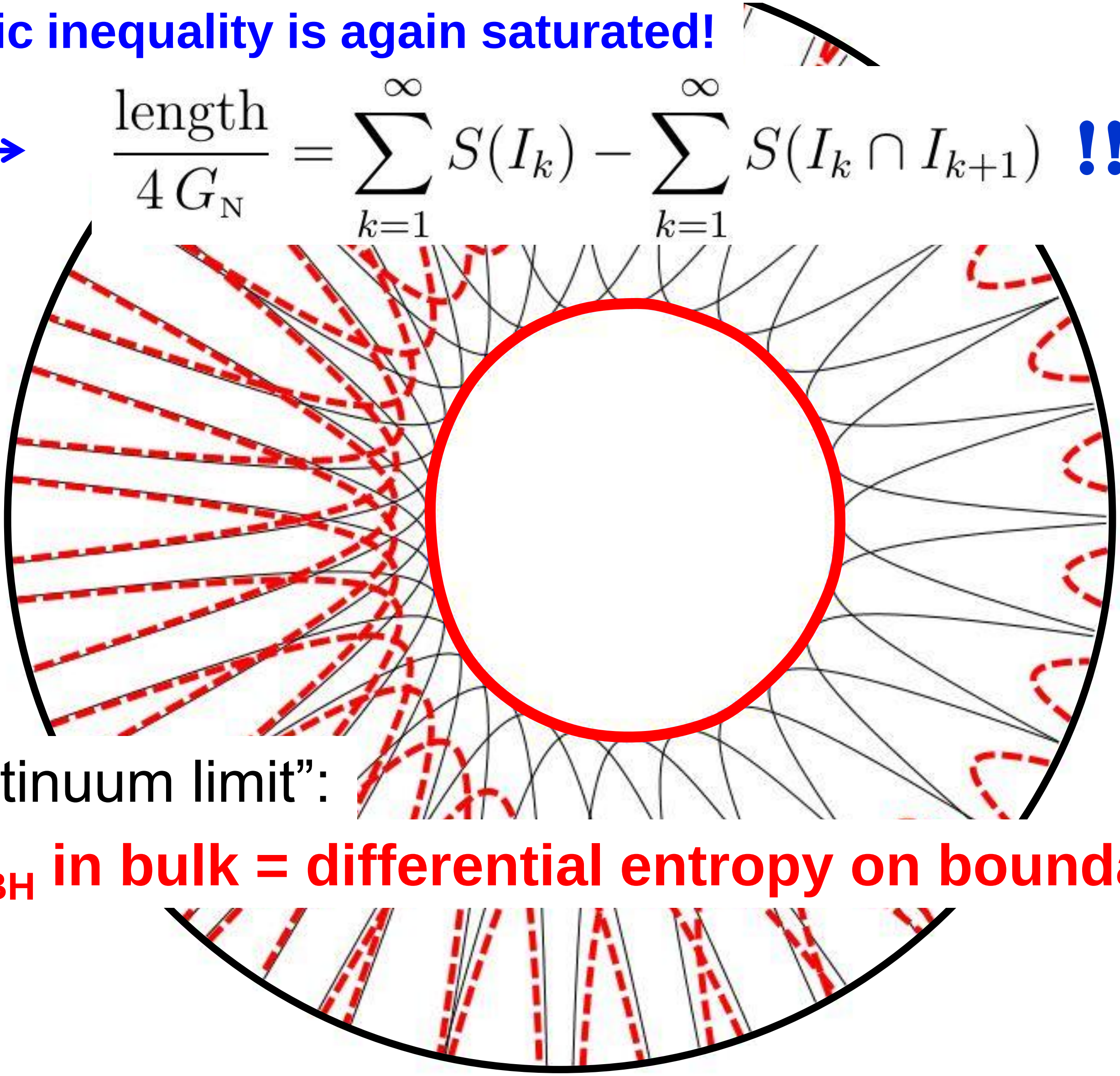
in "continuum limit":

S_{BH} in bulk = differential entropy on boundary

geometric inequality: $\hat{S}(\{I_k\}) \leq \Sigma_{k=1}^{\infty} S(I_k) - \Sigma_{k=1}^{\infty} S(I_k \cap I_{k+1})$

(Balasubramanian, Chowdhury, Czech, de Boer & Heller)

- prescription extends to arbitrary curves in the bulk with Ŝ = length of curve
- **geometric inequality is again saturated!**



$$\longrightarrow \quad \frac{\text{length}}{4\,G_N} = \sum_{k=1}^{\infty} S(I_k) - \sum_{k=1}^{\infty} S(I_k \cap I_{k+1}) \quad \textbf{!!!}$$

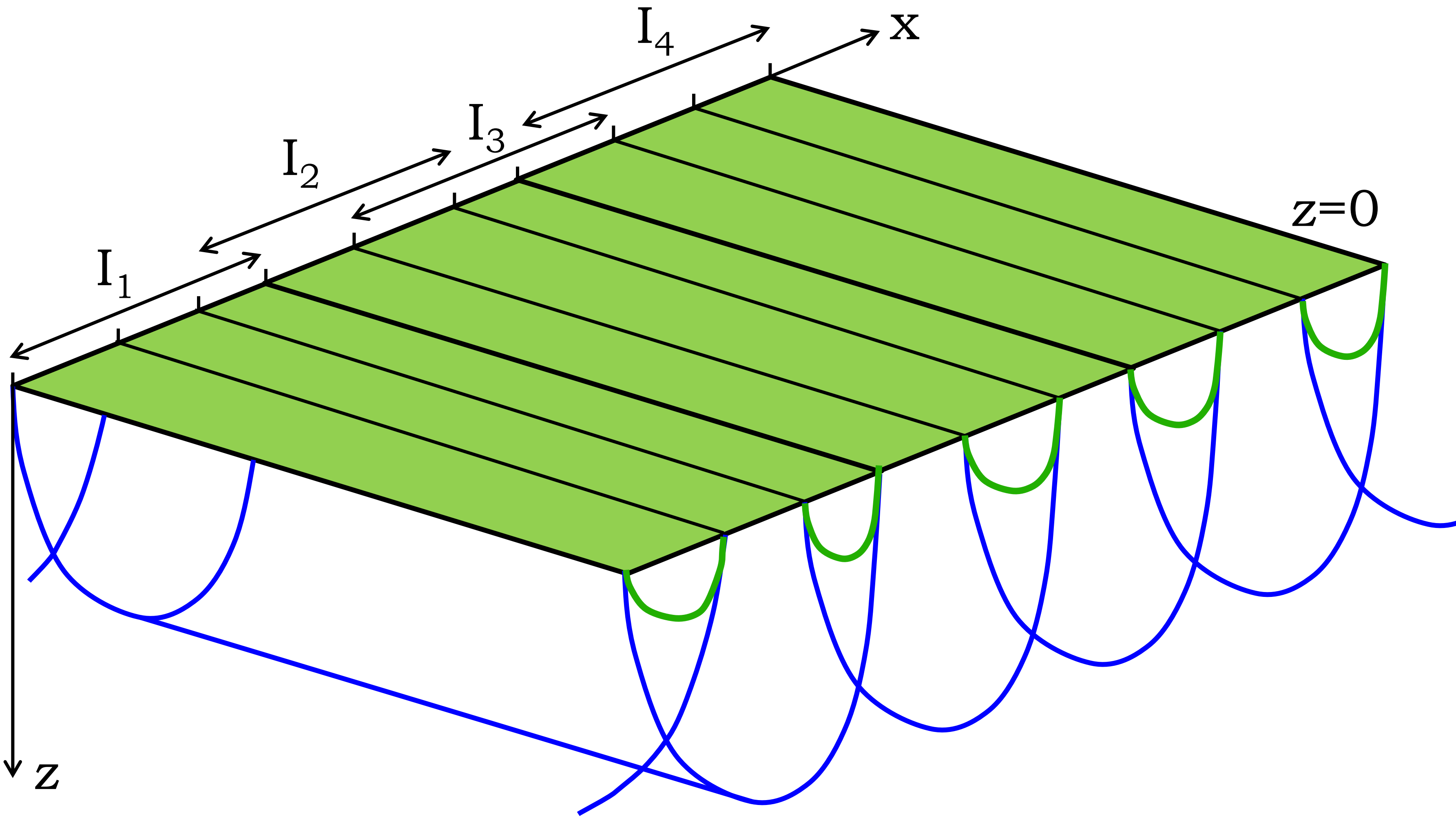
in “continuum limit”:

S_{BH} in bulk = differential entropy on boundary

geometric inequality: $\hat{S}(\{I_k\}) \leq \Sigma_{k=1}^{\infty} S(I_k) - \Sigma_{k=1}^{\infty} S(I_k \cap I_{k+1})$

Higher dimensional “holes”:

- “outer envelope” readily extends to higher dimensions
- extend to AdS_{d+1} in Poincare coordinates and tile t=0 surface with strips/slabs of constant width



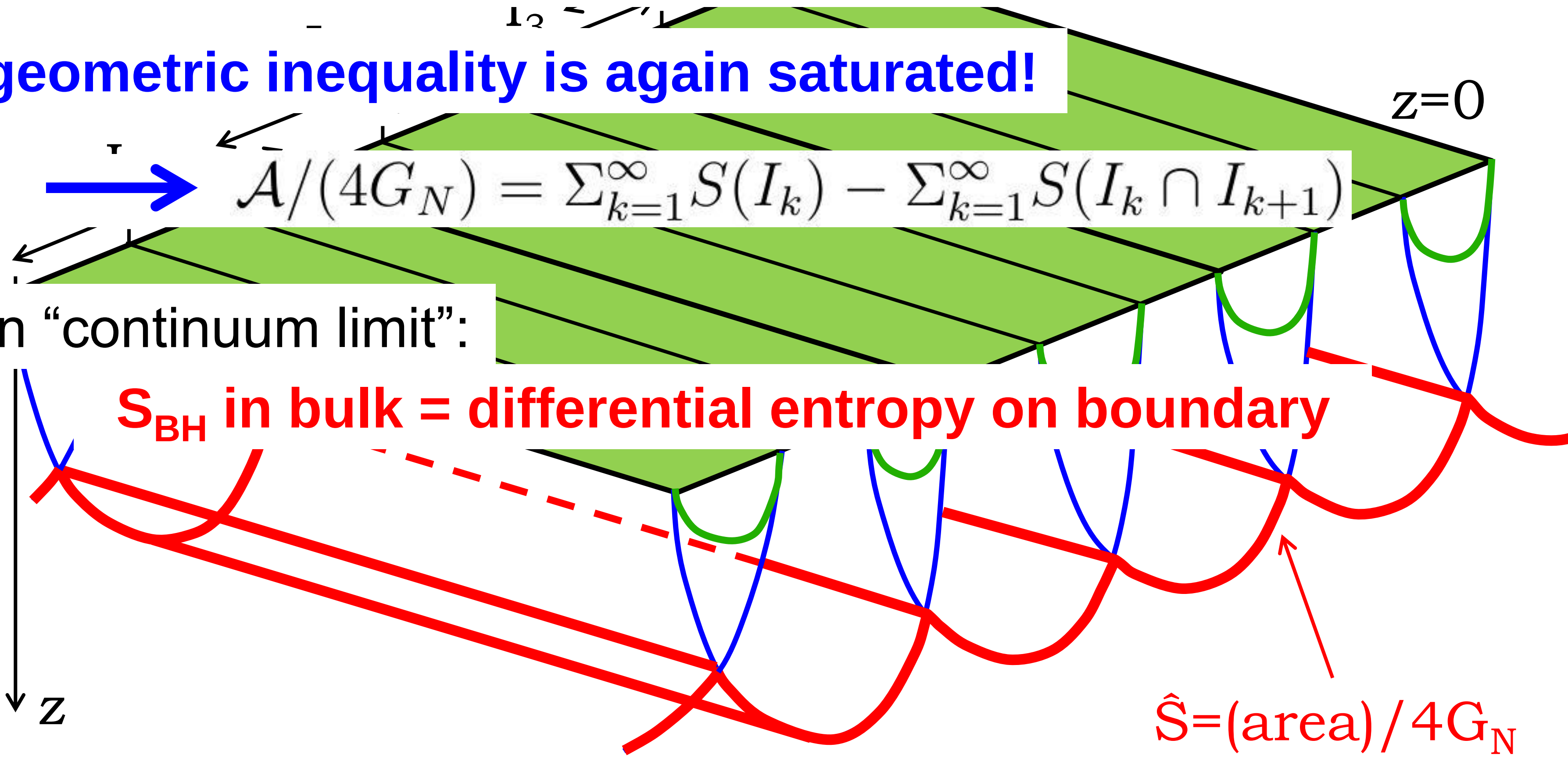
Higher dimensional "holes":

- "outer envelope" readily extends to higher dimensions
- extend to AdS_{d+1} in Poincare coordinates and tile t=0 surface with strips/slabs of constant width
- hole-ographic prescription extends to arbitrary surfaces (with planar symmetry, ie, z=z(x)) in the bulk with $\hat{S}$ ~ area of surface
- **geometric inequality is again saturated!**

$$\rightarrow \quad \mathcal{A}/(4G_N) = \Sigma_{k=1}^{\infty} S(I_k) - \Sigma_{k=1}^{\infty} S(I_k \cap I_{k+1})$$

in "continuum limit":

S_{BH} in bulk = differential entropy on boundary

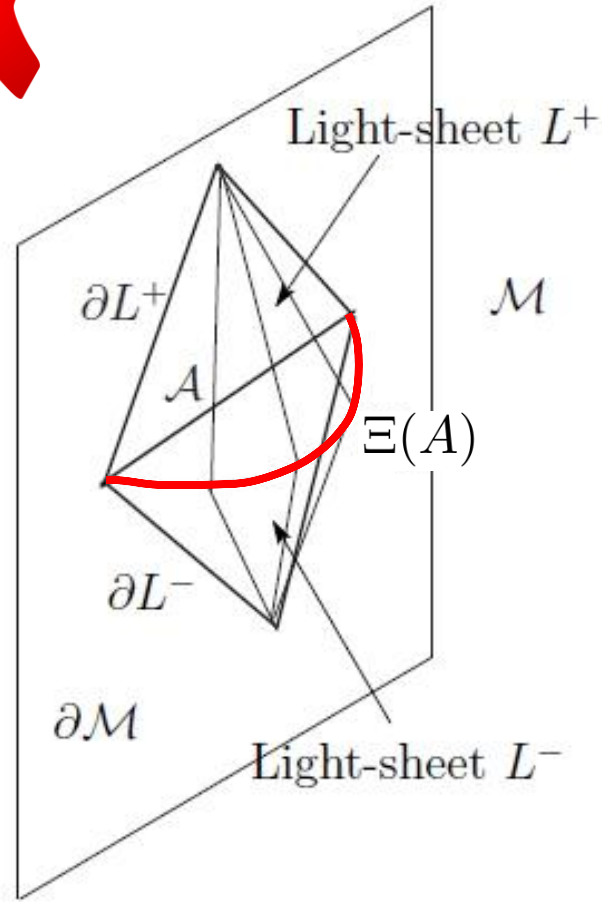


(Hubeny & Rangamani)

Causal Holographic Information:

- evaluate S_{BH} for extremal surface on boundary of bulk causal wedge

$$\chi(A) = \frac{\Xi(A)}{4\,G_N}$$



- natural extension of “observer story” to higher dimensions

- applying hole-graphic prescription to strip decomposition in higher dimensions, find: “geometric inequality” is **NOT** saturated!

$$\longrightarrow \sum_{k=1}^{\infty} \chi(I_k) - \sum_{k=1}^{\infty} \chi(I_k \cap I_{k+1}) \sim \left(\frac{z_{max}}{\delta}\right)^{d-4}$$

- **sub-leading divergences are nonlocal!!** (in contrast to S_{EE})

(Freivogel & Mosk)

Gauss-Bonnet Gravity:

- holographic entanglement entropy: $S(A) \; = \; \min_{\partial V \,=\, \Sigma} S_{JM}$

where $S_{JM} = \frac{1}{4G_N} \int d^{d-1}x \, \sqrt{h} \left[1 + \frac{2\,\lambda L^2}{(d-2)(d-3)} \mathcal{R} \right]$

(Hung, RM & Smolkin; de Boer, Kulaxizi & Parnachev)

$$S_{JM} = \sum_{k=1}^{\infty} S(I_k) - \sum_{k=1}^{\infty} S(I_k \cap I_{k+1})$$

General Backgrounds:

$$ds^2 = -g_0(z)\, dt^2 + g_1(z)\, dx^2 + \sum_{i=2}^{d-1} g_i(z)\, (dx^i)^2 + f(z)\, dz^2$$

$$\longrightarrow \quad S_{BH} = \sum_{k=1}^{\infty} S(I_k) - \sum_{k=1}^{\infty} S(I_k \cap I_{k+1})$$

Time-dependent bulk surfaces:

- salient lessons:
  - boundary data: two “independent” surfaces defining family of intervals: $\vec{\gamma}_L(\lambda) = \{t_L(\lambda), x_R(\lambda)\}$; $\vec{\gamma}_R(\lambda) = \{t_R(\lambda), x_R(\lambda)\}$
  - define intervals by finding extremal HEE surface which is tangent to bulk surface at each point ;

Classical mechanics lemma:

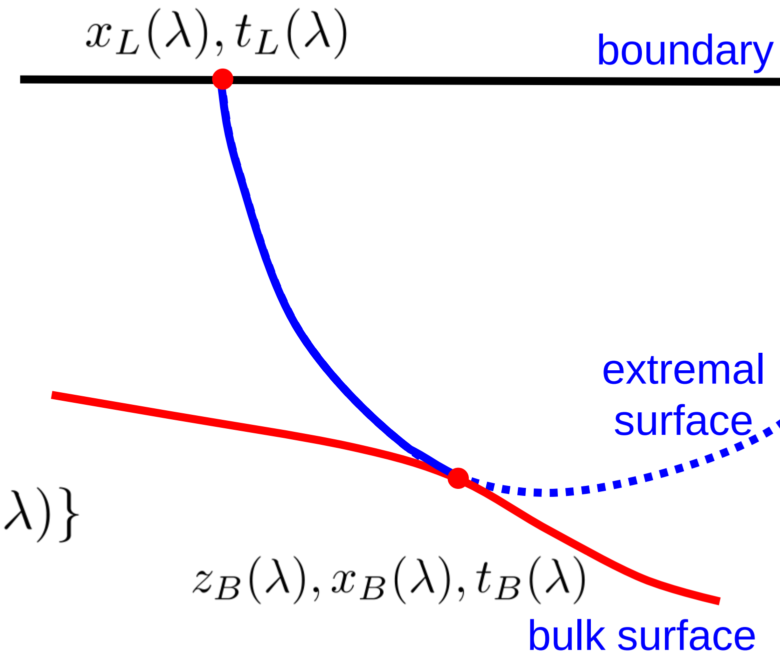
- consider on-shell action: $S_{on} = \int_{s_i, q_i^a}^{s_f, q_f^a} ds \, \mathcal{L}(q^a, \partial_s q^a)$
- varying boundary conditions:

$$\delta S_{on} = p_f^a \, \delta q_f^a - E_f \, \delta s_f - p_i^a \, \delta q_i^a + E_i \, \delta s_i + \int ds [\, eom \cdot \delta q \,]$$

- consider family of boundary conditions: $\{s_i(\lambda), q_i^a(\lambda)\}, \ \{s_f(\lambda), q_f^a(\lambda)\}$

$$0 = \int_0^1 d\lambda \left[p_f^a \, \partial_\lambda q_f^a - E_f \, \partial_\lambda s_f - p_i^a \, \partial_\lambda q_i^a + E_i \, \partial_\lambda s_i \right]$$

Classical mechanics lemma:



- apply lemma to entropy problem
  with end-point data:

$$\{s_i(\lambda), q_i^a(\lambda)\} = \{s=0, z=0, x_L(\lambda), t_L(\lambda)\}$$

$$\{s_f(\lambda), q_f^a(\lambda)\} = \{s_{tang}(\lambda), z_B(\lambda), x_B(\lambda), t_B(\lambda)\}$$

- also use reparametrization invariance of entropy functional

$$\longrightarrow \underbrace{\int_0^1 d\lambda\, \mathcal{L}(q_B^a, \partial_\lambda q_B^a)}_{\text{Gravitational Entropy}} = \underbrace{\int_0^1 d\lambda\, \frac{dq_L^a}{d\lambda}\frac{dS_{on}}{dq_L^a}}_{\text{Differential Entropy}}$$

for general surfaces in general backgrounds (planar symmetry)

Questions:

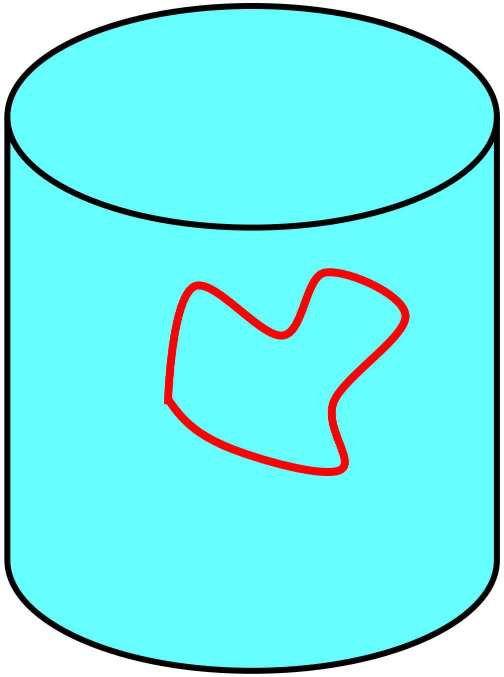
- why entanglement entropy?

evaluating many holographic probes at leading order in large N involves “extremizing bulk area”

→ same construction applies

eg, reconstruct length of curve in bulk from two-point correlator of high dimension operator

$$length = \int_0^1 d\lambda\, \frac{dq_L^a}{d\lambda}\, \frac{\partial_{q_L^a} \langle \mathcal{O}(q_L^a)\, \mathcal{O}(q_R^a)\rangle}{\langle \mathcal{O}(q_L^a)\, \mathcal{O}(q_R^a)\rangle}$$



Questions:

- why entanglement entropy? other differential observables

- what are consistency conditions for boundary data, ie, $\vec{\gamma}_L(\lambda)$ and $\vec{\gamma}_R(\lambda)$, to define bulk surface in AdS? in general bkgd's?

- if consistency conditions not met, what does differential entropy evaluate?

- beyond planar symmetry: tiling boundary with finite regions? extension of differential entropy?

Conclusions:

- holographic S_{EE} suggests new perspectives

→ quantum information & entanglement are keys to fundamental issues in quantum gravity

- spacetime entanglement: S_{BH} applies for generic large regions
- "hole-ography" points to a precise definition in AdS/CFT context

Lots to explore!

Happy 60th, Joe!!

“Canada hands U.S. crushing OT loss, wins women's hockey gold”

CAN 3 / 2 USA

“Canada destroys Sweden 3-0 for men's hockey gold”

SWE 0 / 3 CAN

“US men's hockey falls to Canada”

USA 0 / 1 CAN