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Quantum geometric Langlands, $SL(2,\mathbb{R})$-WZNW, and Liouville theory

${}^LG$-loc. system $\longrightarrow$ $\mathcal{D}$-mod on Bun_G

B-branes

A-branes

Observe r.h.s. has "obvious" def $k \neq h^\vee$

$\mathcal{W}({}^L\mathfrak{g})$-conf. bl. $\longrightarrow$ $\hat{\mathfrak{g}}_k$-conf. bl.

Parametrized by local system

(BPZ et al)

$\mathcal{W}(sl_2) = Vir$

② Virasoro $\longrightarrow$ sl_2-WZNW

Set-up: $\hat{sl}_{2,k}$ generators J^a_n, $a = 0, +, -$

Reps: $\mathcal{V}_{j\omega}$ generated from $v_{j\omega}$ such that

$$J^+_{n+w} v_{j,w} = 0 \qquad J^0_n v_{j,w} = 0, \ n > 0$$

$$J^+_{n-w-1} v_{j,w} = 0$$

$$J^0_0 v_{j,w} = (j + \tfrac{k}{2} w) v_{j,w}$$

b) $\hat{\mathfrak{g}}$ - conf. bl. Inv. lin. fct. on $\bigotimes_{r=1}^{n} V_r = V^{(n)}$

$$V_r = V_{\hat{J}_r, w_r}$$

$$f(\eta \cdot v) = 0 \qquad \forall v \in V^{(n)}$$
$$\forall \eta \in \Gamma(X|_{\{z_1, \dots, z_n\}}, \mathfrak{g}_{\mathcal{B}})$$

$$\mathfrak{g}_{\mathcal{B}} = \mathcal{B} \times_{G(1)} \mathfrak{g}$$

Varying $\mathcal{B}$ $\rightsquigarrow$ sheaf of conf. bl. on Bun_G ($\times \mathcal{M}_{g,n}$)

"Poor physicists sheaf-functions corresp"

$$f \longmapsto f\left(\bigotimes_{r=1}^{n} v_{J_r, w_r}\right) \equiv f(v_0)$$

$$\equiv \left\langle \Phi^{\hat{J}_1}_{w_1}(z_1) \cdots \Phi^{\hat{J}_n}_{w_n}(z_n) \right\rangle_{X, \mathcal{B}}$$

funct. on subset $U \subset \mathrm{Bun}_G \times \mathcal{M}_{g,n}$

$\mathcal{D}$-module

$$f(x\cdot v) =: \delta_x f(v_0)$$

$$x \in \bigotimes_{r=1}^{n} \Pi^{(r)}_{j_r, w_r}(\hat{g})$$

or $\langle J^a(z) \Phi^{j_1}_{w_1}(z_1) \cdots \rangle$

$$= \mathcal{D}^a_{\mathcal{B}}(z) \langle \Phi^{j_1}_{w_1}(z) \cdots \rangle_{\mathcal{S}, \mathcal{O}_\mathcal{S}}$$

$\Rightarrow$ PDE for fct. $f(v_0)$,

e.g. for $g=0$, $w_r = 0$

bundles with parabolic struct. (param by x_r)

(gauge group $\begin{pmatrix} 1 & 0 \\ x_r & 1 \end{pmatrix} \begin{pmatrix} a & u \\ 0 & b \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -x_r & 1 \end{pmatrix}$)

$\Rightarrow$ (KZ) $\quad (k+2) \dfrac{\partial}{\partial z_r} g = \displaystyle\sum_{s \neq r} \frac{\mathcal{D}^{(2)}_{rs,x}}{z_r - z_s} g,$

$g(x_1, \ldots, x_n, z_1, \ldots, z_n) = f(v_0).$

2^{nd} order DO

c) <u>Left hand side</u> $W({}^L g) \equiv \mathrm{Vir}$, gen L_n

repr. W_α, $\quad L_n w_\alpha = 0 \quad n > 0$

$L_0 w_\alpha = \alpha(Q - \alpha) w_\alpha$

Conf. bl. lin. fct $\rho : \bigotimes_{r=1}^{m} W_{\alpha_r} \to \mathbb{C}$

$$\rho(\chi \cdot w) = 0 \quad \forall \eta \in \Gamma(X \setminus \{z_1, \dots, z_n\}, \Theta_X)$$

$$\rho(v_0) = \langle \Phi_{\alpha_1}(z_1) \cdots \Phi_{\alpha_m}(z_n) \rangle_X$$

If $\alpha_r = -1/2b \Rightarrow \rho(v_0)$ satisfies PDE $\mathcal{D}_r^{BPZ} \rho = 0$

$$\mathcal{D}_r^{BPZ} = \frac{\partial^2}{\partial z_r^2} - \sum_{s \neq r} \left(\frac{1}{z_r - z_s} \frac{\partial}{\partial z_s} + \frac{\Delta_{\alpha_s}}{(z_r - z_s)^2} \right)$$

d) Correspondence (for $g = 0$, $w_r = 0$)

Key observation (Feigin, Frenkel, Stoyanovsky)

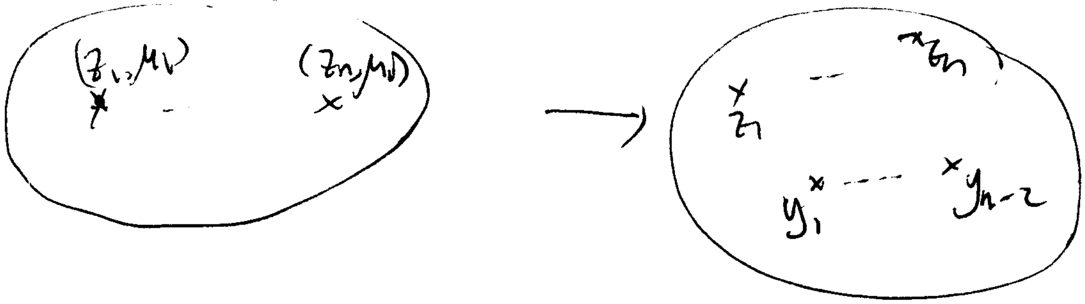
There ex map $G : Sol_{BPZ} \to Sol_{KZ'}$,

where (KZ') obtained by Radon-trsf.

$$x_r \to \frac{\partial}{\partial \mu_r}, \quad \frac{\partial}{\partial x_r} \to -\mu_r.$$

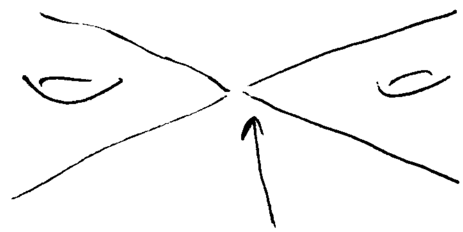
Key ingredient: Ch. of var. $\underline{\mu} = (\mu_1, \dots, \mu_n)$ to (y, u)

(SOV) $$\sum_{r=1}^{n} \frac{\mu_r}{t - z_r} = u \, \frac{\prod_{a}^{n-2} (t - y_a)}{\prod_{r=1}^{n} (t - z_r)}$$



e) <u>What does it help us?</u>

There ex. constr. of conf. bl. (J.T.) with param. in terms of behavior on $\overline{M}_{g,n}$



$\mathcal{W}_\alpha$ varying $\alpha \in \mathbb{C}$.

$\rightsquigarrow$ moduli sp $\mathcal{MC}^{Vir}_{X_{g,n}}$ of conf bl

$$\dim \mathcal{MC}^{Vir}_{X_{g,n}} = \dim \mathcal{O}_{\mathcal{B}^{Vir}_{g}}(X)$$

<u>Alternative POV</u>: parametrize conf. bl. by KZ-loc. syst.

(3) Hecke op

Recall geom. origin: el. bundle modif.

~~Sects of~~ $(\mathcal{B}, \mathcal{B}', z, \beta : \mathcal{B} \hookrightarrow \mathcal{B}')$

Sects of $\mathcal{B}'$ = allowed to have poles along certain lines in fiber at z

Q: Can we define a reasonable op. mapping (anti 6).

$$f_{\mathcal{B}} \to (\mathcal{H}f)_{\mathcal{B}'} \; ?$$

Note: Defining inv $f(\eta v)$ $\forall \eta \in \mathfrak{g}^{\mathcal{B}'}_{out}$

now involves η with poles at z!

$\leadsto$ Represent $\mathcal{H}f$ by extra insertion

$$\mathcal{V}^{(n)} \to \mathcal{V}^{(n+1)} = \bigotimes_{r=0}^{n} \mathcal{V}_r$$

$$\mathcal{V}_0 = \mathcal{V}_{j_0, \nu_0}$$

Claim: There ex. a distinguished choice for $\mathcal{V}_0$, $\mathcal{V}_0 = \mathcal{V}_{k/2, 0}$

s.t.

$$\mathcal{C}_{\mathcal{B}'} \simeq \mathbb{C}^2 \otimes \mathcal{C}_{\mathcal{B}}$$

and variation of z described by certain loc. syst (monodromy)

In terms of fcts on Bun_G:

$$\mathcal{H}\left(\langle \Phi^{j_1}_{\nu_1}(z_1) \cdots \rangle_{x,\mathcal{B}} \right)$$

$$= \lim_{\mu \to 0} \mu^{2+k} \int dx \, e^{i\mu x} \langle \Phi^{k/2}_0(x|z) \, \Phi^{j_1}_{\nu_1}(x_1|z_1) \cdots \rangle_{x,\mathcal{B}}$$

becomes "prefactored" for $k=-2$.

How to see this?

Key observ. V_0 deg. $(J^+_{-1} v_0 = 0)$

$\Rightarrow \langle \ldots \rangle^{\hat{sl}_2}_{\mathcal{B}'}$ represented by

$$\langle \Phi_{-1/2b}(z)\, \Phi_{\alpha_1}(z_1) \cdots \Phi_{\alpha_n}(z_n) \rangle^{Vir}_X$$

and indeed $\quad \mathcal{C}^{Vir}_X (W_{-1/2b} \otimes W_{\alpha_1} \otimes \cdots \otimes W_{\alpha_n})$

$$= \mathbb{C}^2 \otimes \mathcal{C}^{Vir}_X (W_{\alpha_1} \otimes \cdots \otimes W_{\alpha_n})$$

Variation of $z \rightsquigarrow$ BPZ lin. syst.