29 July 2008
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A-branes and D-modules

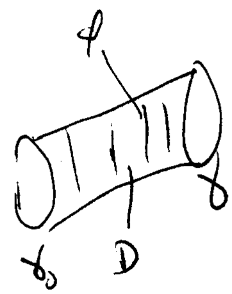
SQM $\quad X \quad$ $f: X \to \mathbb{R}$ $\quad (C_{Morse}, d_{Morse})$

$(\Omega^\cdot, d_{DR})$ $\quad V = |df|^2$

WKB/tunneling limit

SQM on $\mathcal{L}M$ on TM, M symplectic

$$f(\gamma) = \int_\gamma p\,dq = \int_D \varphi^* \omega$$



$df = 0$, $\gamma = $ const

Gradient flow lines, $\bar\partial_J \varphi = 0$

b.c.'s $\qquad \cancel{\varphi}: \partial\Sigma \to L$, L cn Lag.

$U(1)$: $F = 0$ & Lag (else f depends on D)

K-0 : $F \neq 0$ $\quad \partial_n \varphi = F \cdot \partial_t \varphi$ $\quad$ mixed type ND

{ mag field
geom q^n
NC

<u>Ex</u> M H-K target space-filling submanifold

$$(\omega^{-1}F)^2 = -\mathbb{1}$$

I, J, K $\quad F = J \quad$ Note: $(I^{-1}J)^2 = (K^{-1}J)^2 = -1$.

(A,B,A) "new A-brane"

$M = T^*X$, "canonical coisotropic" C

If C is involved in hom's $\Rightarrow \varphi|_C = \text{const} \overset{?}{\Rightarrow} \varphi$ const

No inst means $\hom(C,C)$ has <u>local</u> definition in target

$\underline{\hom}(C,C)$

$\underline{\hom}(C,L)$

φ hol

$\partial_t\varphi = I\,\partial_n\varphi$

$IJ\,\partial_t\varphi$

$\partial_t\varphi = 0$

$\partial_n\varphi = 0$

K-W: $\underline{\hom}(C,C) = D$

$(X = \mathrm{Bun}_G(\Sigma))$

Abstract nonsense

$\mathcal{C}$ $\mathbb{C}$-linear category (A-branes)

$C \in \mathrm{Ob}(\mathcal{C})$

$\hom(C,C)$ alg $A \Rightarrow \mathcal{C} \xrightarrow{J} A\text{-mod} \circlearrowleft \hom(C,C)$

$L \mapsto \hom(C,L)$

Sometimes [illegible] iso

e.g. $\mathcal{C} = D Coh(\mathbb{P}^n)$

$C = \bigoplus_{i=0}^{n-1} \mathcal{O}(i)$ $\qquad$ C "generator"

A 

Here we have a sheafified version

A-branes $\to$ D-modules $\quad (M = T^*X)$

(Equivalence?)

<u>D-modules</u> $\sim$ systems of differential eqs

$(z\partial_z - \alpha)$

soln z^α

Eqs $\underset{\text{R-H}}{\rightsquigarrow}$ solns (w/ monodromies)

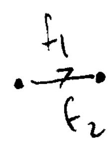
$\binom{\text{reg.}}{\text{hol}}$ D-mod $\cong D_c(X)$

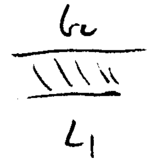
SQM $(\Omega^\bullet, d)$ $\qquad H^\bullet(X) = H^\bullet(X, \mathbb{C})$

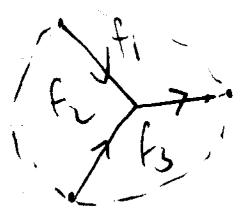
$f = 0$, $f = \log m$, $m = 1$.

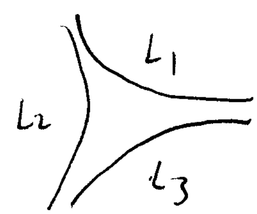
SQM categories

	X	$\mathcal{PM}^{(\text{hard})}$	$\mathcal{P}\overbrace{T^*X}^{M}$
ob	$f: X \to \mathbb{R}$	L g-Lag	Γ_{df}
hom	$\hom(f_1, f_2) = \bigoplus_{p \in \mathrm{Crit}(f_2 - f_1)} \mathbb{C}_p[-\mu_p]$	$\hom(L_1, L_2) = \bigoplus_{q \in L_1 \cap L_2} \mathbb{C}_q[-\deg q]$	$\hom(\Gamma_{df_1}, \Gamma_{df_2}) = \bigoplus_{p \in \mathrm{Crit}(f_2 - f_1)} \mathbb{C}_{df(p)}[-\mu_p]$









Count $\#\mathcal{M}_{\mathrm{Morse}}(\vec{f}, \vec{p})$ $\qquad$ $\#\mathcal{M}_{\mathrm{Fuk}}(\vec{L}, \vec{q})$

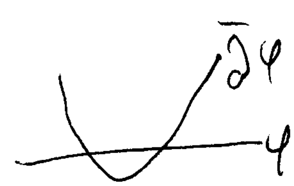
Thm (F-O) $\forall \vec{f}, \vec{p}\ \exists \varepsilon > 0$ s.t.

$$\mathcal{M}_{\mathrm{Morse}}(\vec{f}, \vec{p}) \overset{\text{or'd}}{\underset{\text{diffeo}}{\cong}} \mathcal{M}_{\mathrm{Fuk}}(\Gamma_{\varepsilon \vec{df}}, \vec{q} = d\vec{f}(p))$$

$\Gamma_{df_1(p)} \cap \Gamma_{df_2(p)}$

Prf Via approximation

Proof is local



$$H^\cdot(X, \mathbb{C})$$

<u>Cor</u> $\mathrm{Morse}(X) \cong \mathrm{Fuk}_{\mathrm{Graph}}(T^*X)$

Suggests <u>dilation</u> connects A-branes on T^*X to const sheaves on X

Const sheaves and categorification

$\mathcal{F} = \mathcal{F}^\bullet$ "const" if $H^\bullet(\mathcal{F}^\bullet)$ loc. constant on strata

dg cat $Sh(X)$

microlocalization $(X \rightsquigarrow T^*X)$

$$\begin{array}{cc} \mathcal{F} & \mathcal{G} \\ {\scriptstyle cc}\big\downarrow & {\scriptstyle cc}\big\downarrow \\ cc(\mathcal{F}) & cc(\mathcal{G}) \end{array} \quad \text{on } \mathrm{Lag}(T^*X) \qquad \square$$

$$\chi(\mathcal{F}, \mathcal{G}) = cc(\mathcal{F}^\vee) \cdot cc(\mathcal{G})$$

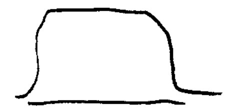
Remark $U \subset X$ open

$cc(\iota_* \mathbb{C}_U) = \cancel{\lim_{\varepsilon \to 0^+}} \Gamma_{\varepsilon df}$, $f = \log m$

m

U

$\varepsilon f = \log m^{\varepsilon}$



$m|_U > 0$, $m|_{X-U} = 0$

Define functor $\iota_* \mathbb{C}_U \to \Gamma_{df}$

$\uparrow$

generate $D_c(X)$

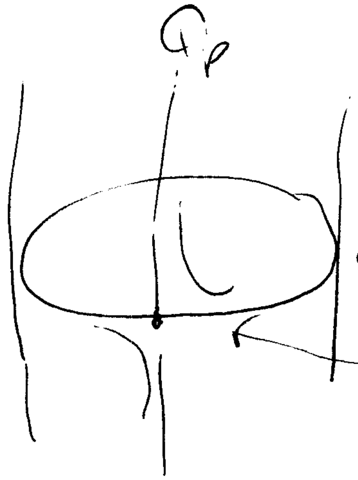
Get $D_c(X) \hookrightarrow Fuk(T^*X)$

Thm (Nadler–Zaslow) This is a quasi-embedding

Thm (Nadler) This is an equivalence.

⑦

$M = T^*S^1$

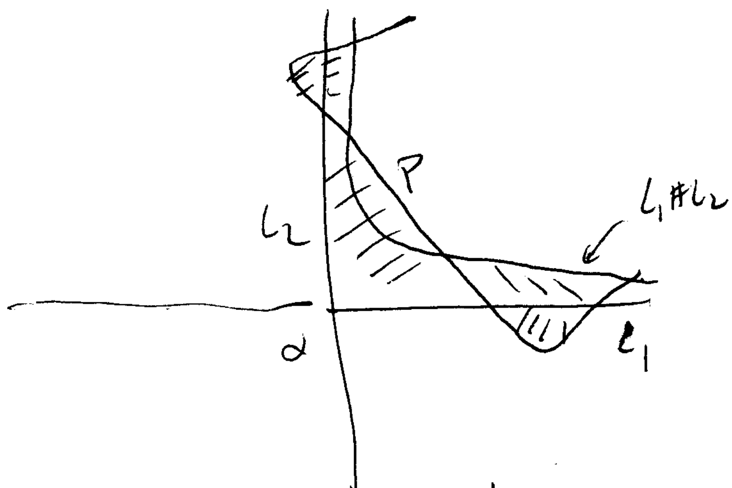


$m = x^2$
$d\log m = 4x$

$S^1,\ p,\quad U = S^1 \setminus p$

LES $\quad H^\bullet(S^1, p) \to H^\bullet(S^1) \to H^\bullet(p) \xrightarrow{[1]}$

$i_{U*}\mathbb{C}_U \to \mathbb{C}_{S^1} \to i_{p*}\mathbb{C}_p$



~~Cone(α)~~ $\quad d_{hom(cone(\alpha), P)} = \begin{pmatrix} d_{L_1} & \\ & d_{L_2} \end{pmatrix}$

"Theorem Z" of F-O-O-O

surgery = cone