FOUR-DIMENSIONAL WALL-CROSSING FROM THREE-DIMENSIONAL FIELD THEORY

WORK DONE WITH

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BASED ON

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OUTLINE

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1. INTRODUCTION

THIS TALK IS ABOUT THE BPS SPECTRUM OF $\mathcal{N}=2$, $D=4$ FIELD THEORIES.

THE BPS SPECTRUM OF THE THEORY ON $\mathbb{R}^4$ IS A "PIECEWISE CONSTANT" FUNCTION OF THE BOUNDARY CONDITIONS AT ∞.

RECENTLY THERE HAS BEEN SOME PROGRESS IN UNDERSTANDING PRECISELY <u>HOW</u> THE SPECTRUM DEPENDS ON BOUNDARY CONDITIONS.

THESE ARE CALLED WALL-CROSSING FORMULAE (WCF). THIS TALK WILL GIVE A PHYSICAL INTERPRETATION AND PROOF OF A FAMOUS WCF OF KONTSEVICH + SOIBELMAN.

2. REVIEW $N=2, D=4$ WALL CROSSING

CONSIDER A THEORY ON $\mathbb{R}^4$ WITH $\mathcal{N}=2$ SUPERPOINCARÉ SYMMETRY $\mathcal{S}$

LET $\mathcal{H}$ BE THE ONE-PARTICLE HILBERT SPACE.

AS A REPRESENTATION OF $\mathcal{S}$, $\mathcal{H}$ DEPENDS ON THE BOUNDARY CONDITIONS OF FIELDS AT ∞.

THESE BOUNDARY CONDITIONS ARE VALUED IN THE MODULI SPACE OF VACUA: $\mathcal{M}_V$.

FOR $u \in \mathcal{M}_V$, WRITE $\mathcal{H}_u$.

FOR ALL $u \in \mathcal{M}_v$ THERE IS AN UNBROKEN ABELIAN GAUGE SYMMETRY OF RANK r, SO $\mathcal{H}$ IS GRADED BY THE SYMPLECTIC LATTICE Γ OF ELEC.+MAG. CHARGES. (OF RANK $2r$).

$$\mathcal{H}_u = \bigoplus_{\gamma \in \Gamma} \mathcal{H}_{\gamma,u}$$

ON EACH SUBSPACE $\mathcal{H}_{\gamma,u}$ THE CENTRAL CHARGE OPERATOR $Z \in \Delta$ IS A SCALAR.

DENOTE THE VALUE $Z_\gamma(u)$

RECALL THE $\mathcal{N}=2$, $D=4$ SUSY ALGEBRA

$$\mathcal{S} = \mathcal{S}_0 \oplus \mathcal{S}_1$$

$$\mathcal{S}_0 = \left(\underset{M_{\mu\nu}}{\mathrm{spin}(1,3)} \ltimes \underset{P_\mu}{\mathbb{R}^4}\right) \oplus \mathfrak{u}(2) \oplus \underset{Z}{\mathbb{R}}$$

$$\mathcal{S}_1 = \left[\mathrm{Spinor} \otimes \mathbb{C}^2\right]_{\mathbb{R}}$$

$$Q_{\alpha I},\ \bar{Q}^{\dot{\alpha} I}$$

$$\{Q_{\alpha I}, \bar{Q}_{\dot{\beta}}{}^{J}\} = 2 P_\mu \sigma^\mu_{\alpha\dot{\beta}}\, \delta_I{}^J$$

$$\{Q_{\alpha I}, Q_{\beta J}\} = 2 Z\, \epsilon_{\alpha\beta}\, \epsilon_{IJ}$$

UNITARY IRREPS SATISFY BPS BOUND:

$$M \geq |Z|$$

<u>DEF.</u> $\mathcal{H}^{BPS}_{\gamma,u}$ = SUBSPACE SATURATING THE BPS BOUND.

ON THIS SUBSPACE $E = |Z_\gamma(u)|$

SOME BPS PARTICLES CAN BE VIEWED AS BOUNDSTATES OF OTHERS: DECAY WHEN BOUNDSTATE ENERGY $E(u) \longrightarrow 0$.

[Cecotti et.al. ; Seiberg & Witten]

$Z_\gamma(u)$ IS LINEAR IN $\gamma = \gamma_1 + \gamma_2$ SO

$$E(u) = |Z_\gamma(u)| - \left(|Z_{\gamma_1}(u)| + |Z_{\gamma_2}(u)| \right) \leq 0$$

$\Longrightarrow$ DECAY ONLY HAPPENS ALONG WALLS OF MARGINAL STABILITY:

$$MS(\gamma_1, \gamma_2) := \{ u \mid Z_{\gamma_1}(u) / Z_{\gamma_2}(u) \in \mathbb{R}_+ \}$$

WHEN γ_1, γ_2 ARE PRIMITIVE

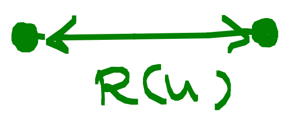
DENEF & MOORE DESCRIBED HOW $\mathcal{H}^{BPS}_{\gamma,u}$

CHANGES ACROSS A WALL:

$$\boxed{\begin{gathered}\mathcal{H}'^{BPS}_{\gamma,u_+} - \mathcal{H}'^{BPS}_{\gamma,u_-} = (J_{12}) \otimes \mathcal{H}'_{\gamma_1,u} \otimes \mathcal{H}'_{\gamma_2,u} \\ J_{12} = \tfrac{1}{2}\left(|\langle \gamma_1, \gamma_2 \rangle| - 1 \right)\end{gathered}}$$

$\langle \gamma_1, \gamma_2 \rangle$ = SYMPLECTIC PRODUCT

BASED ON DENEF'S SUGRA CONSTRUCTION OF BOUNDSTATE SOLUTIONS:



THERE IS A GENERALIZATION FOR DECAYS OF THE FORM

$$\gamma = \gamma_1 + N\gamma_2 \quad , \quad N > 1$$

BUT FOR DECAYS OF THE FORM

$$\gamma = N_1 \gamma_1 + N_2 \gamma_2 \qquad N_1, N_2 > 1$$

THE METHODS OF DM ARE DIFFICULT TO USE.

KONTSEVICH & SOIBELMAN PROPOSED A REMARKABLE FORMULA FOR THE CHANGE OF THE INDEX:

$$\Omega(\gamma; u) := -\frac{1}{2} \mathrm{Tr}_{\mathcal{H}'^{BPS}_{\gamma,u}} (2J_3)^2 (-1)^{2J_3}$$

WHICH INCLUDES ALL CASES.

[IN SUPERGRAVITIES ARISING FROM CY COMPACTIFICATION $\Omega(\gamma; u) =$ "GENERALIZED DONALDSON-THOMAS INVT."]

3. KS FORMULA

- INTRODUCE THE SYMPLECTIC TORUS

$$T = \Gamma^* \otimes_{\mathbb{Z}} \mathbb{C}^*$$

$$\gamma \in \Gamma \Rightarrow \text{FUNCTION } X_\gamma : T \to \mathbb{C}^*$$

- CHOOSING A BASIS γ_i, DEFINE:

$$\varpi = \frac{1}{2} \epsilon^{ij} \frac{dX_i}{X_i} \wedge \frac{dX_j}{X_j}, \qquad \epsilon_{ij} = \langle \gamma_i, \gamma_j \rangle$$

- CHOOSE A QUADRATIC REFINEMENT

$$\frac{\sigma(\gamma_1+\gamma_2)}{\sigma(\gamma_1)\sigma(\gamma_2)} = (-1)^{\langle \gamma_1, \gamma_2 \rangle}, \qquad \sigma = \pm 1$$

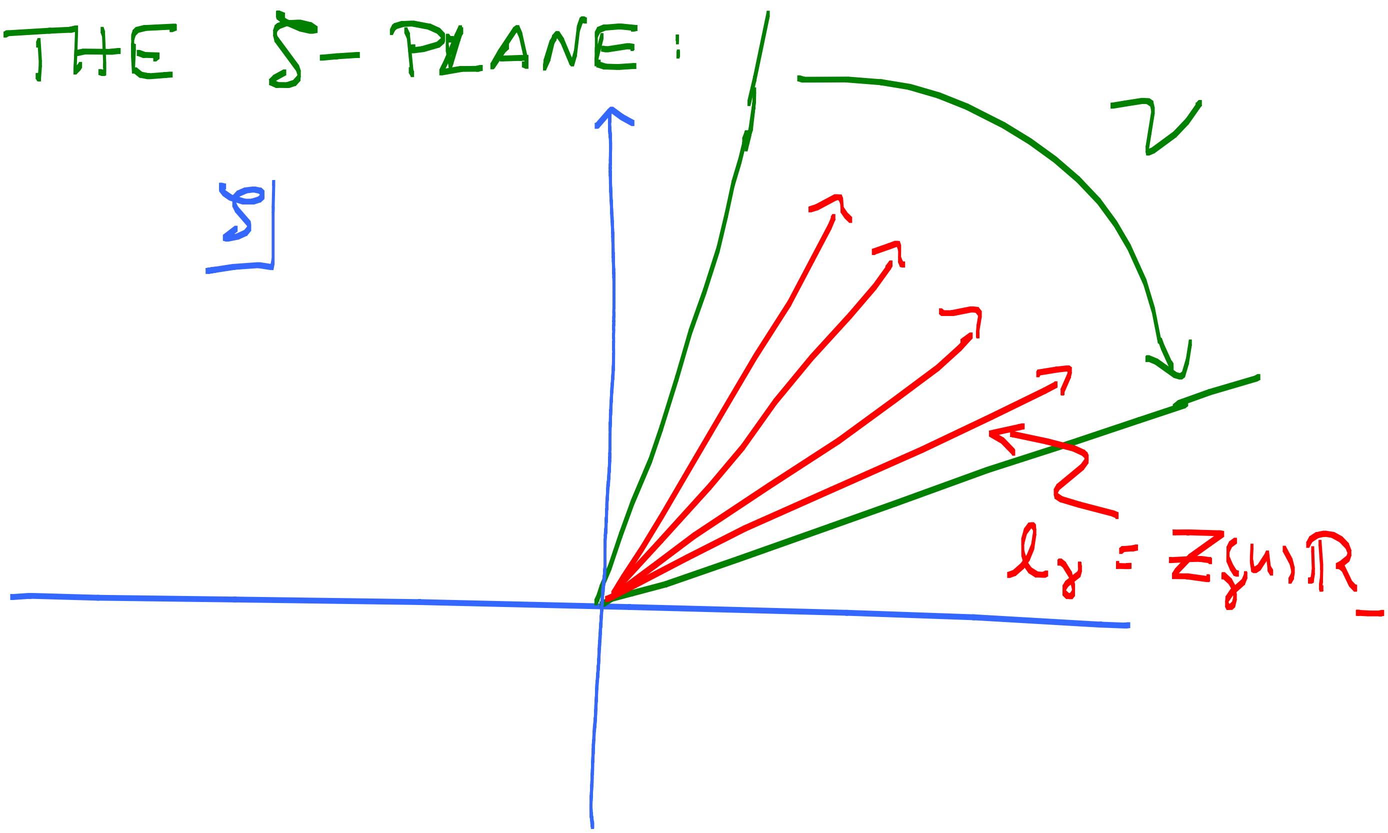
- DEFINE "KS TRANSFORMATIONS"

$$U_\gamma : X_{\gamma'} \to X_{\gamma'} \left(1 - \sigma(\gamma) X_\gamma\right)^{\langle \gamma', \gamma \rangle}$$

TO EACH $\gamma \in \Gamma$ ASSOCIATE THE "BPS RAY" IN THE COMPLEX PLANE

$$l_\gamma := \{ \zeta \mid \zeta \in Z_\gamma(u) \cdot \mathbb{R}_- \}$$

CHOOSE A CONVEX CONE ν IN THE ζ-PLANE:



ASSOCIATE THE SYMPLECTIC TMN:

$$A_\nu = \prod_{l_\gamma \subset \nu}^{\rightarrow} U_\gamma^{\Omega(\gamma;u)}$$

FOR LATER CONVENIENCE

NOTE THAT IF $\gamma_1 = N \gamma_2$, $N>0$

THEN $\ell_{\gamma_1} = \ell_{\gamma_2}$, SO DEFINE

$$S_\gamma := \prod_{\ell_{\gamma'} = \ell_\gamma} U_{\gamma'}^{\Omega(\gamma';u)}$$

AND WRITE :

$$\mathcal{A}_\nu = \overrightarrow{\prod_{\substack{\gamma \text{ PRIM} \\ \ell_\gamma \subset \nu}}} S_\gamma$$

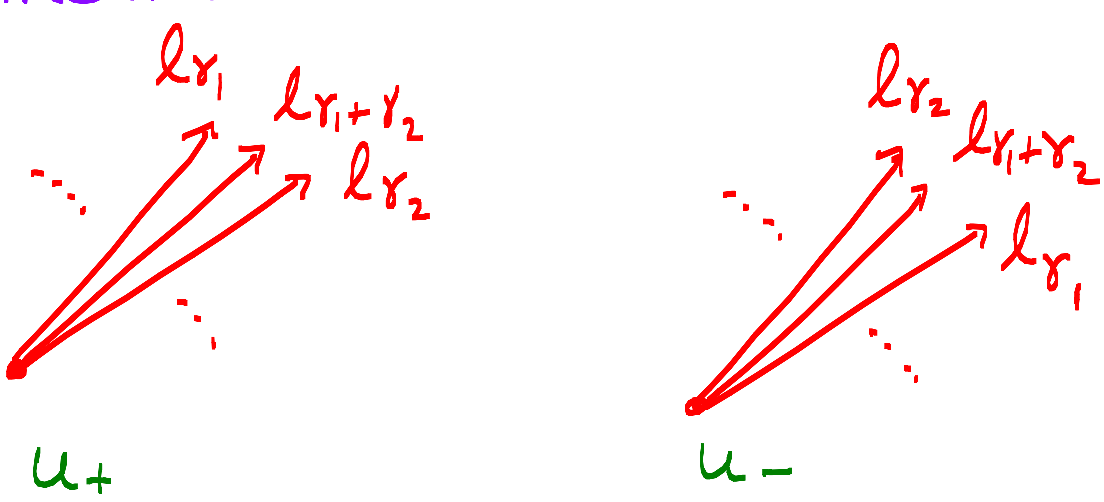
THE $\Omega(\gamma;u)$ DEPEND ON u ...

THE KS FORMULA STATES THAT

$$A_\nu = \prod_{\ell_\gamma \subset \nu}^{\rightarrow} U_\gamma^{\Omega(\gamma;u)}$$

IS CONSTANT IN u AS LONG AS NO BPS RAY ENTERS OR LEAVES THE SECTOR ν.

IN PARTICULAR, ACROSS WALLS OF MARGINAL STABILITY



$\Longrightarrow$ WALL CROSSING FORMULA

4. COMPACTIFICATION OF $N=2, D=4$ FIELD THEORIES

A. SEIBERG-WITTEN SOLUTION

G - COMPACT S.S. GAUGE GROUP, RANK $= r$
R - FIN. DIML. REP.

$\Longrightarrow$ $D=4, N=2$ FIELD THEORY

$$\mathcal{M}_V = (\mathfrak{g}_{\mathbb{C}})^G \cong \mathbb{C}^r \quad \left(u_2 = \mathrm{Tr}\,\Phi^2,\ u_3 = \mathrm{Tr}\,\Phi^3, \dots \right)$$

S & W GAVE FORMULAE FOR

- $Z_\gamma(u)$
- LOW ENERGY ABELIAN GAUGE THRY.

IN TERMS OF

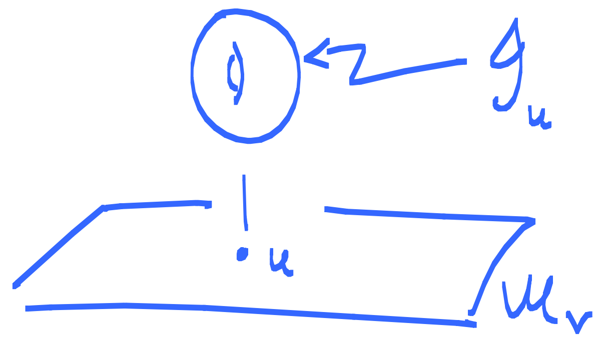
SPECIAL KÄHLER GEOMETRY

REVIEW SPECIAL KÄHLER GEOM:

c.f. D. FREED, hep-th/9712042

VIEW Γ AS A LOCAL SYSTEM OVER $\mathcal{M}_V$

$$\begin{tikzcd} \mathcal{J}_u \arrow[r] \arrow[d] & \mathcal{M} \arrow[d] \\ u \arrow[r, hook] & \mathcal{M}_V \end{tikzcd}$$



$$\mathcal{J} = \Gamma^* \otimes_{\mathbb{Z}} \left(\mathbb{R}/2\pi\mathbb{Z}\right) \cong U(1)^{2r}$$

FIBERS = ABELIAN VARIETIES

IN REGIONS OF $\mathcal{M}_V$ CHOOSE A DUALITY FRAME:

$$\Gamma = \Gamma_{el} \oplus \Gamma_{mag}, \quad \Gamma_{mag} = \Gamma_{el}^*$$

$$= \mathrm{Span}\{\alpha_I\} \oplus \mathrm{Span}\{\beta^I\}$$

$$\langle \alpha_I, \alpha_J \rangle = \langle \beta^I, \beta^J \rangle = 0 \qquad \langle \alpha^I, \beta^J \rangle = \delta_I^{\ J}$$

CHOOSING A DUALITY FRAME,

$\mathcal{J}_u$ HAS PERIOD MATRIX τ_{IJ}

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1. LOW ENERGY LAGRANGIAN:

$$\mathcal{L} = \frac{-1}{4\pi} \operatorname{Im}\tau_{IJ} \left(da^I * d\bar{a}^J + F^I * F^J \right)$$

$$+ \frac{1}{4\pi} \operatorname{Re}\tau_{IJ} \; F^I \wedge F^J$$

$$a^I = Z_{\alpha_I}(u) \qquad I = 1, \ldots, r$$

LOCAL COORD'S ON $\mathcal{M}_V$

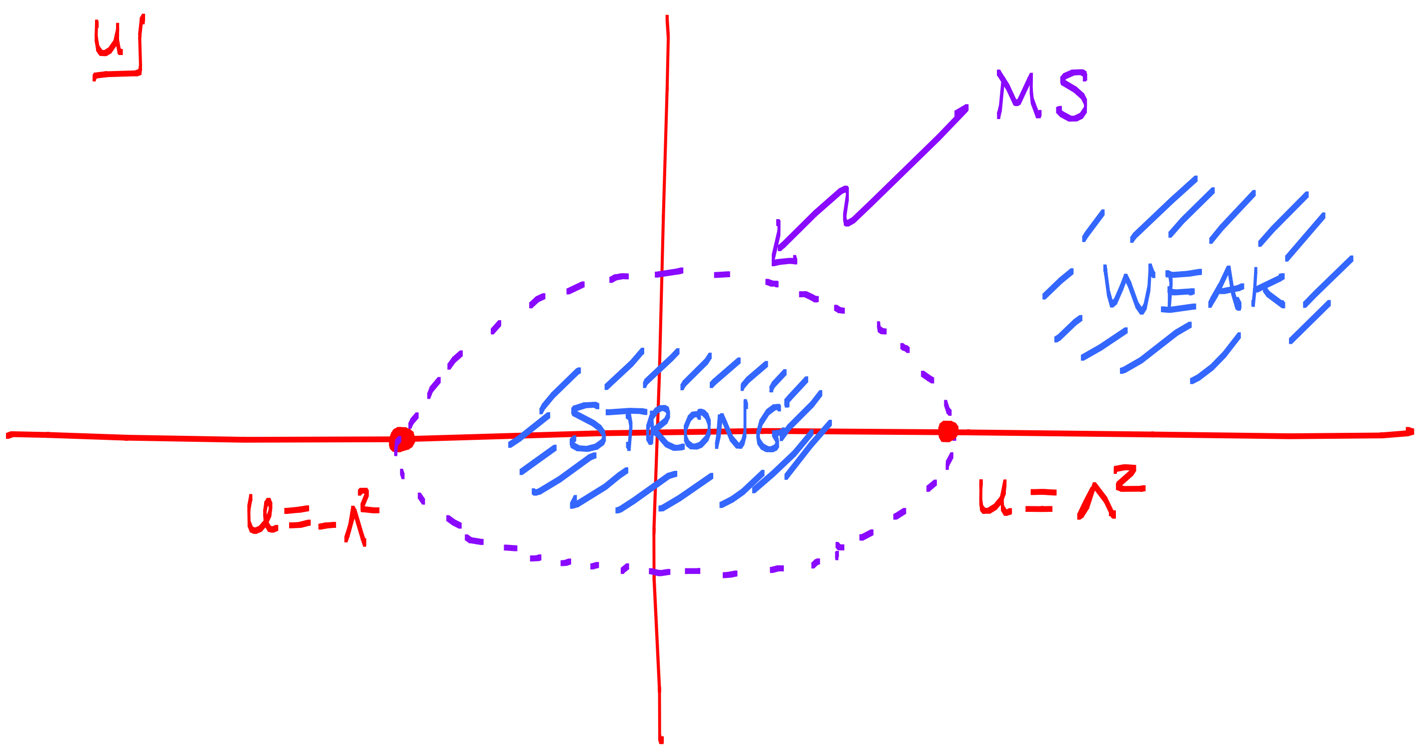
2. CENTRAL CHARGE FUNCTION

$$\boxed{\begin{aligned} Z_\gamma(u) &= a \cdot \gamma_{el} + a_D \cdot \gamma_{mg} \\ \tau_{IJ} &= \frac{\partial a_{DI}}{\partial a^J} = \frac{\partial^2 \mathcal{F}}{\partial a^I \partial a^J} \end{aligned}}$$

S & W IDENTIFY $\mathcal{J}_u$ AS JACOBIANS OF AN EXPLICIT FAMILY OF RIEMANN SURFACES

BASIC EXAMPLE: $G = SU(2)$

$$\Sigma_u : \quad y + \frac{\Lambda^4}{y} = x^2 - 2u$$



$$a = \oint_\alpha x \frac{dy}{y} \qquad a_D = \oint_\beta x \frac{dy}{y}$$

SPECTRUM:

$$\mathcal{H}^{BPS}_{WEAK} = \bigoplus_{n\in\mathbb{Z}} HM(2n,1) \oplus VM(2,0)$$

$$\oplus \text{ CONJUGATE}$$

$$\mathcal{H}^{BPS}_{STRONG} = HM(2,-1) \oplus HM(0,1)$$

$$\oplus \text{ CONJUGATE}$$

KS IDENTITY:

$$U_{2,-1} U_{0,1} = U_{0,1} U_{2,1} U_{4,1} \cdots U_{2,0}^{-2} \cdots U_{6,-1} U_{4,-1} U_{2,-1}$$

IT IS TRUE !!!

B. COMPACTIFY ON A CIRCLE.

- NOW CONSIDER THE THEORY ON $\mathbb{R}^3 \times S^1_R$.

- LOW ENERGY THEORY IS A 3D σ-MODEL : $\mathbb{R}^3 \longrightarrow \mathcal{M}$

$$a^I(\vec{x}, x^4) \longrightarrow a^I(\vec{x})$$

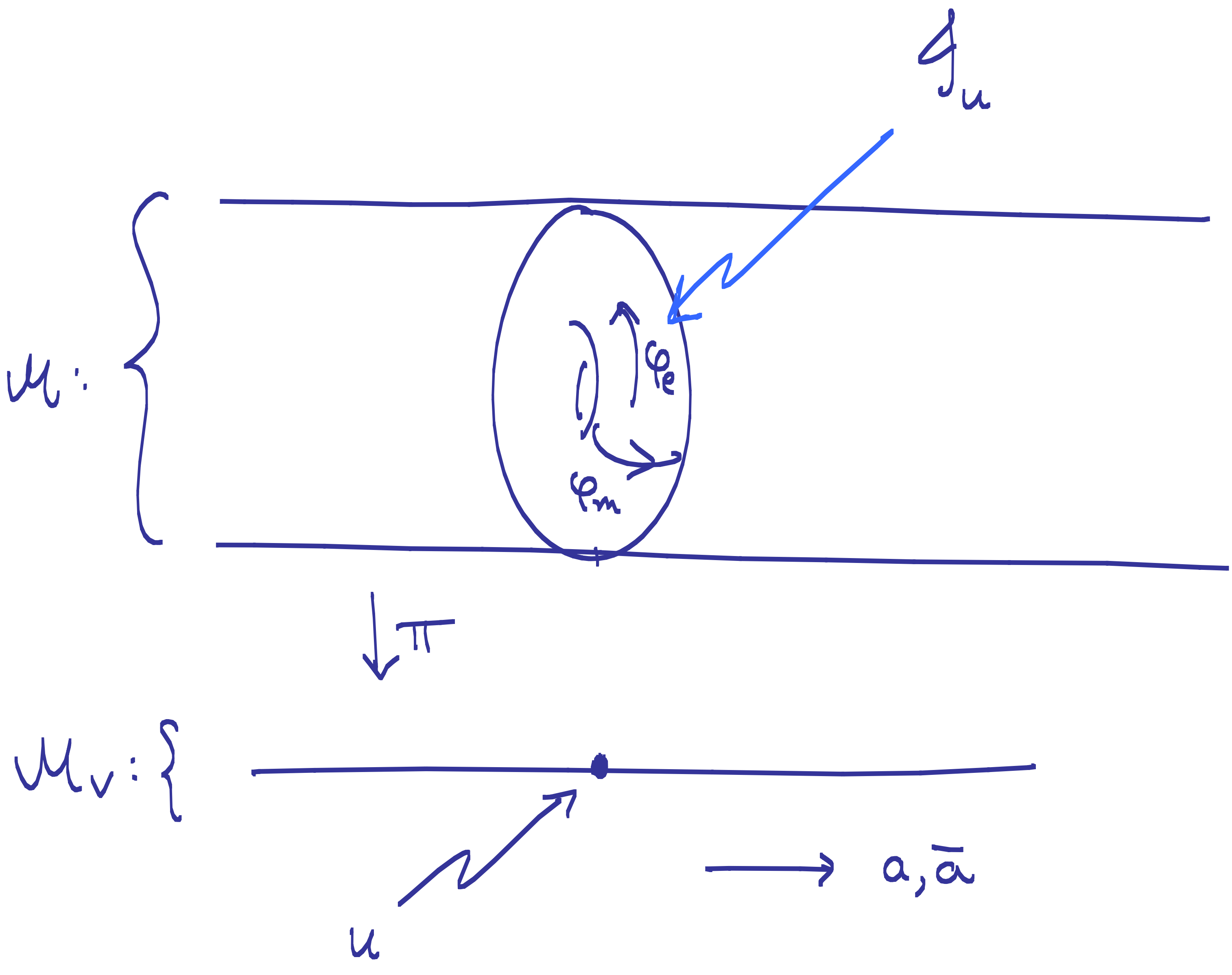
$$\left.\begin{aligned} \varphi_e^I &= \int_{S^1} A_4^I \, dx^4 \\ \varphi_{m,I} &= \int_{S^1} (A_{D,4})_I \, dx^4 \end{aligned}\right\} \text{PERIODIC!}$$

- SUPERSYMMETRY $\Longrightarrow$

$\mathcal{M}$ MUST CARRY A HYPERKÄHLER METRIC

LET US TRY TO DESCRIBE IT

TOPOLOGICALLY $\mathcal{M}$ IS A TORUS FIBRATION OVER $\mathcal{M}_v$:



THE SEMI-FLAT METRIC

LEADING $R \to \infty$ APROXIMATION:

USE DIMENSIONAL REDUCTION

+ DUALIZATION OF 3D GAUGE FIELD:

$$\mathcal{L}^{(3)} = -\frac{R}{2} \operatorname{Im}\tau_{IJ}\, da^I * d\bar{a}^J - \frac{1}{8\pi^2 R} (\operatorname{Im}\tau)^{-1, IJ}\, dz_I * d\bar{z}_J$$

$$dz_I = d\varphi_{m,I} - \tau_{IJ}\, d\varphi_e^J$$

THIS DEFINES THE SEMIFLAT METRIC

$$g^{sf} = R (\operatorname{Im}\tau) |da|^2 + \frac{1}{4\pi^2 R} (\operatorname{Im}\tau)^{-1} |dz|^2$$

C. THE KEY IDEA

- THE METRIC g^{sf} RECEIVES QUANTUM CORRECTIONS FROM BPS PARTICLE WORLD-LINES WRAPPING S^1.

- THEREFORE THE QUANTUM CORRECTIONS DEPEND ON THE BPS SPECTRUM.

- THE TRUE METRIC g SHOULD BE A SMOOTH METRIC ON $\mathcal{M}$ AWAY FROM THE LOCUS IN $\mathcal{M}_V$ WHERE BPS PARTICLES BECOME $M=0$.

- SMOOTHNESS OF g ACROSS WALLS OF M.S. IMPLIES A WCF.

CLAIM: IT IS THE KS WCF.

5. TWISTOR SPACE APPROACH

WE WILL USE HITCHIN'S THEOREM: KNOWING $(\mathcal{M}, g)$ IS EQUIVALENT TO KNOWING TWISTOR SPACE $Z := \mathcal{M} \times \mathbb{CP}^1$ AS A HOLOMORPHIC MANIFOLD.

<u>THEOREM</u>: IF $\mathcal{M}$ IS HK OF DIMENSION $4r$ THEN:

1. $\exists$ HOLO. FIBRATION

$$p : Z \longrightarrow \mathbb{CP}^1$$

$M_\zeta = p^{-1}(\zeta) =$ M IN COMPLEX STRUCTURE ζ

2. $\exists$ HOLOMORPHIC SECTION

ϖ OF $\Omega^2_{Z/\mathbb{CP}^1} \otimes \mathcal{O}(2)$

$\varpi|_{M_\zeta} =$ HOLOMORPHIC SYMPLECTIC FORM ON M_ζ

3. $\forall x \in M$, $\exists$ HOLOMORPHIC SECTION

$s_x : \mathbb{CP}^1 \longrightarrow Z$ WITH NORMAL BUNDLE $\mathcal{O}(1)^{\oplus 2r}$

4. $\exists$ ANTI-HOLOMORPHIC $\sigma : Z \to Z$ COVERING $\zeta \to -1/\bar{\zeta}$

GIVEN 1, 2, 3, 4 ONE CAN RECONSTRUCT THE METRIC:

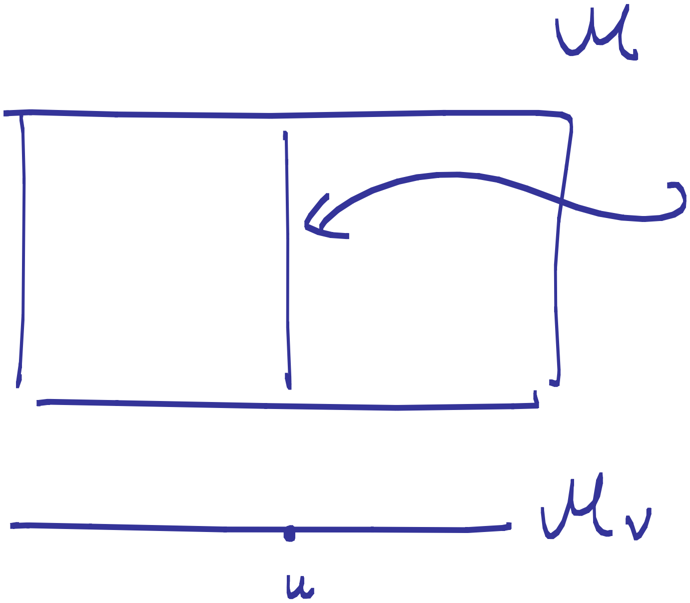
FOR $\zeta \in \mathbb{C}^*$:

$$\varpi = -\frac{i}{2\zeta}\omega_+ + \omega_3 - \frac{i}{2}\zeta\,\omega_-$$

$\uparrow$ KÄHLER FORM

$$\omega_+ = \omega_1 + i\omega_2$$

OUR STRATEGY IS TO CONSTRUCT THE HOLOMORPHIC SECTIONS S_x EXPLICITLY FOR THE 3D σ-MODEL TARGET.



$\mathcal{J}_u \cong \mathbb{R}^{2r}/\Gamma$

- FOR $\zeta = 0$, $\mathcal{J}_u$ IS HOLOMORPHIC
- FOR $\zeta \neq 0, \infty$ $\mathcal{J}_u$ IS NEITHER HOLO. NOR ANTI-HOLO.

- HOLOMORPHIC FUNCTION ON $\mathcal{M}_\zeta$ IS DETERMINED BY RESTRICTION TO SOME (i.e. ANY) FIBER $\mathcal{J}_{u_0}$

- A BASIS OF C^∞ FUNCTIONS ON THE TORUS $\mathcal{J}_{u_0}$ IS LABELLED BY $\gamma \in \Gamma_{u_0}$

- CALL THE HOLOMORPHIC FUNCTIONS IN A NEIGHBORHOOD OF $\mathcal{J}_{u_0}$: $\mathcal{X}_\gamma$

THE LEADING, NO QUANTUM CORRECTIONS, APPROXIMATION:

$$\boxed{\mathcal{X}^{sf}_\gamma = \exp\left[\pi R \zeta^{-1} Z_\gamma + i\,\theta_\gamma + \pi R \zeta \bar{Z}_\gamma\right]}$$

[A. NEITZKE & B. PIOLINE]

$\theta_\gamma : \Gamma^* \otimes \mathbb{R}/2\pi\mathbb{Z} \longrightarrow \mathbb{R}/2\pi\mathbb{Z}$

- CHECK

$$\varpi = \frac{1}{8\pi^2 R} \epsilon_{ij} \frac{d\mathcal{X}_i}{\mathcal{X}_i} \wedge \frac{d\mathcal{X}_j}{\mathcal{X}_j}$$

$$= \frac{1}{8\pi}\left[\frac{i}{\zeta}\langle dZ, d\theta\rangle + \cdots\right]$$

$$\langle dZ, d\theta\rangle = -da^I \wedge dZ_I$$

ETC.

6. SINGLE-PARTICLE CORRECTIONS

NOW WE INCLUDE THE FIRST Q.C.

- FOR SIMPLICITY CONSIDER $r = 1$.
- CONSIDER A POINT $u_* \in \mathcal{M}_V$

WHERE A SINGLE HM BECOMES HAS $M \to 0$

$\Rightarrow$ DOMINANT CONTRIBUTION NEAR u_*.

CHOOSE DUALITY FRAME SO IT HAS CHARGE $(q, 0)$, $q > 0$

KK REDUCTION $\Rightarrow$ TARGET SPACE METRIC IS A GIBBONS-HAWKING ANSATZ: [Seiberg Witten; Ooguri Vafa; Seiberg Shenker]

$$g = V^{-1}(\vec{x}) \left(\frac{d\varphi_m}{2\pi} + A \right)^2 + V(\vec{x})\, d\vec{x}^2$$

$$F = * dV \qquad \vec{x} \in \mathbb{R}^3$$

HERE:

$$a = x^1 + i x^2$$

$$\varphi_e = 2\pi R\, x^3 \qquad \text{PERIODIC}$$

$$V(\vec{x}) = \frac{g^2 R}{4\pi} \sum_{n\in\mathbb{Z}} \frac{1}{\sqrt{g^2 R^2 |a|^2 + \left(q\frac{\varphi_e}{2\pi} + n\right)^2}}$$

$$= V^{sf} + V^{inst}$$

$$V^{sf} = -\frac{g^2 R}{4\pi}\left(\log\frac{a}{\Lambda} + \log\frac{\bar{a}}{\bar{\Lambda}}\right)$$

$$V^{inst} = \frac{g^2 R}{2\pi}\sum_{n\neq 0} e^{inq\varphi_e} K_0\left(2\pi R |nqa|\right)$$

$$\sim e^{-2\pi R|nqa|}$$: INSTANTON CONTRIBUTION

NOW, WHAT ARE THE HOLO. FUNCTIONS ON TWISTOR SPACE?

ALGEBRA OF HOLO FUNCTIONS $\{\mathcal{X}_\gamma\}$ ON TWISTOR SPACE IS GENERATED BY:

$$\mathcal{X}_e := \mathcal{X}_{(1,0)} = \exp\{i\varphi_e + \cdots\}$$

$$\mathcal{X}_m := \mathcal{X}_{(0,1)} = \exp\{i\varphi_m + \cdots\}$$

$$\mathcal{X}_{(k_1,k_2)} = (\mathcal{X}_e)^{k_1}(\mathcal{X}_m)^{k_2}$$

$$(k_1, k_2) \in \mathbb{Z}^2 \cong \Gamma_u$$

DETERMINE χ_e AND χ_m

FROM A DIFFERENTIAL EQUATION

HK STRUCTURE: $\alpha = 1, 2, 3$:

$$\omega^\alpha = dx^\alpha \wedge \left(\frac{d\varphi_m}{2\pi} + A \right) + \frac{1}{2} V \epsilon^{\alpha\beta\gamma} dx^\beta dx^\gamma$$

$\Rightarrow$ COMPUTE $\varpi = -\frac{i}{2\zeta}\omega_+ + \omega_3 - \frac{i}{2}\zeta\omega_-$

$$\boxed{\varpi = -\frac{1}{4\pi^2 R} \frac{d\chi_e}{\chi_e} \wedge \frac{d\chi_m}{\chi_m}}$$

WE FIND:

$$\mathcal{X}_e = \mathcal{X}_e^{sf} = \exp\left[\frac{\pi R}{\zeta} a + i\varphi_e + \pi R \zeta \bar{a} \right]$$

BUT

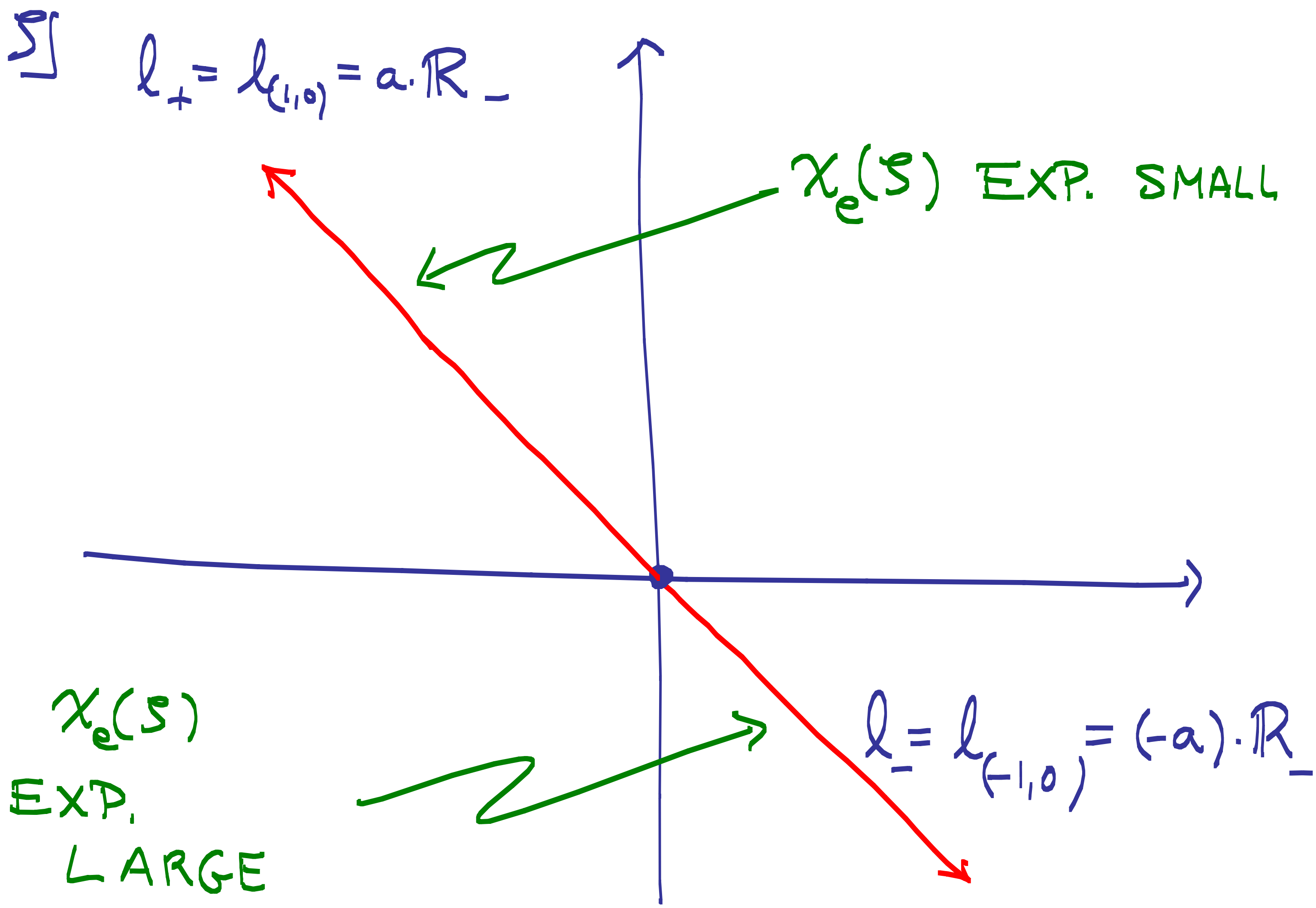
$$\mathcal{X}_m = \mathcal{X}_m^{s.f.} \cdot \mathcal{X}_m^{inst.}$$

$$\mathcal{X}_m^{sf} = \exp\left[\frac{\pi R}{\zeta} \cdot a_D + i\varphi_m + \pi R \zeta \overline{a_D} \right]$$

$$a_D = \frac{q^2}{2\pi i}\left(a \log \frac{a}{e\Lambda} \right)$$

$$\mathcal{X}_m^{inst} = \text{INSTANTON CONTRIBUTION}$$

$$\boxed{\mathcal{X}^{\text{inst}}_m(\zeta) = \exp\left\{ \frac{iq}{4\pi}\int_{\ell_+} \frac{d\zeta'}{\zeta'}\,\frac{\zeta'+\zeta}{\zeta'-\zeta}\,\log\left(1-\mathcal{X}_e(\zeta')^{q}\right) - \frac{iq}{4\pi}\int_{\ell_-} \frac{d\zeta'}{\zeta'}\,\frac{\zeta'+\zeta}{\zeta'-\zeta}\,\log\left(1-\mathcal{X}_e(\zeta')^{-q}\right) \right\}}$$



KEY FEATURES OF THE SOLUTION

1. AS A FUNCTION OF ζ, $\mathcal{X}_m$ IS DISCONTINUOUS ACROSS THE BPS RAY:

$$\ell_\gamma := \left\{ \zeta \;\middle|\; \frac{Z_\gamma}{\zeta} \in \mathbb{R}_- \right\}$$

FOR $\gamma = (\pm 1, 0)$, GENERATORS OF Γ_{el}

2. ACROSS THIS RAY:

$$(\mathcal{X}_e, \mathcal{X}_m)^{cw} = U_\gamma \, (\mathcal{X}_e, \mathcal{X}_m)^{ccw}$$

$$= \left(\mathcal{X}_e, \; \mathcal{X}_m \left(1 - \mathcal{X}_e^{\pm q}\right)^{\mp q} \right)^{ccw}$$

3. $\forall \gamma$, $\mathcal{X}_\gamma \underset{\zeta \to 0, \infty}{\sim} \mathcal{X}_\gamma^{s.f.} \left(1 + O(1)\right)$

OBSERVATION: X_γ ARE THE SOLUTION OF A RIEMANN-HILBERT PROBLEM.

R-H PROBLEM:

FIND A PIECEWISE HOLO. FUNCTION WITH PRESCRIBED SINGULARITIES AND ASYMPTOTICS.

7. MULTI-PARTICLE CONTRIBUTIONS

TO TAKE INTO ACCOUNT ALL BPS PARTICLES WE CANNOT USE A LOW ENERGY EFFECTIVE LAG., BECAUSE THE PARTICLES WILL BE MUTUALLY NONLOCAL.

PROPOSAL: THE HOLOMORPHIC FUNCTIONS ARE CONSTRUCTED FROM A RIEMANN-HILBERT PROBLEM IN THE ζ-PLANE

TO GIVE THE PROBLEM A SIMPLE FORMULATION VIEW THE COLLECTION OF FUNCTIONS

$$\mathcal{X}_\gamma = \exp\left[i\theta_\gamma + \cdots \right]$$

AS A FAMILY OF MAPS

$$\mathcal{X}(\zeta) : \mathcal{M} \longrightarrow T = \Gamma^* \underset{\mathbb{Z}}{\otimes} \mathbb{C}^*$$

PIECEWISE HOLOMORPHIC IN ζ.

- RECALL T_u HAS FUNCTIONS

$$X_\gamma : T_u \longrightarrow \mathbb{C}^*$$

- RECALL T_u IS SYMPLECTIC

$$\varpi^T := \frac{1}{2} \epsilon^{ij} \frac{dX_i}{X_i} \wedge \frac{dX_j}{X_j}$$

RIEMANN-HILBERT PROBLEM:

1.) $\mathcal{X}(\zeta)$ IS DISCONTINUOUS ACROSS BPS RAYS ℓ_γ:

$$\mathcal{X}^{cw} = S_\gamma(\mathcal{X}^{ccw})$$

[RECALL: $S_\gamma = \prod_{\ell_{\gamma'} = \ell_\gamma} U_{\gamma'}^{\Omega(\gamma'; u)}$]

2.) $\mathcal{X}(\zeta)$ HAS ASYMPTOTICS FOR $\zeta \to 0, \infty$ GIVEN BY $\mathcal{X}^{sf}(\zeta)$, UP TO $\mathcal{O}(1)$ CORRECTIONS

$$Y := (\mathcal{X}^{sf})^{-1}\mathcal{X} : \mathfrak{g} \to \mathfrak{g}$$

i.e.

$$Y_0 = \lim_{\zeta \to 0} Y(\zeta) \quad \& \quad Y_\infty = \lim_{\zeta \to \infty} Y(\zeta)$$

EXIST

SOLUTION:

$$\mathcal{X}_\gamma(\zeta) = \mathcal{X}^{sf}_\gamma(\zeta) \cdot \exp\left\{ -\frac{1}{4\pi i} \sum_{\gamma' \in \Gamma} \Omega(\gamma';u) \langle \gamma, \gamma' \rangle \cdot \int_{\ell_{\gamma'}} \frac{d\zeta'}{\zeta'} \frac{\zeta'+\zeta}{\zeta'-\zeta} \log\left[1 - \sigma(\gamma') \mathcal{X}_{\gamma'}(\zeta')\right] \right\}$$

ITERATING THIS EQUATION

(AS A SUM OVER TREES...)

GIVES THE FULL INSTANTON EXPANSION!

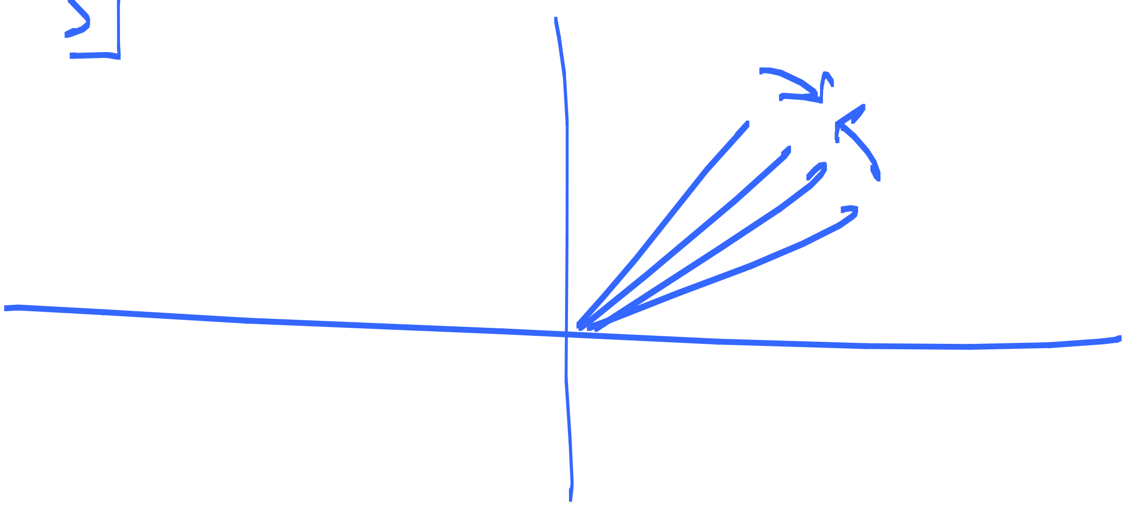
$\Rightarrow$ EXPLICIT CONSTRUCTION OF TWISTOR COORD'S

- WE RECONSTRUCT THE METRIC FROM

$$\varpi = \frac{1}{4\pi^2 R} \mathcal{X}^*(\varpi^T)$$

- AS u CROSSES A WALL OF MS BPS RAYS PILE UP

$\boxed{\zeta}$



BUT THE JUMP OF $\mathcal{X}$ IN THE RH PROBLEM IS CONTINUOUS AS A FUNCTION OF u: THAT IS THE KS FORMULA

THUS: THE KS FORMULA GUARANTEES THE CONTINUITY OF THE H.K. METRIC ACROSS WALLS OF M.S. !

THE RESULTING METRIC PASSES A NUMBER OF CONSISTENCY TESTS.

BUT... WHY IS OUR PROPOSAL THE RIGHT ONE ?

WHY IS THE METRIC THE RIGHT ONE FOR THE PHYSICAL PROBLEM?

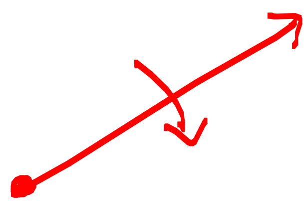
8. PHYSICAL PROOF OF THE KS FORMULA

RH IS EQUIVALENT TO A DIFF. EQ.:

$$A_\zeta = \mathcal{X}^{-1}\zeta\partial_\zeta \mathcal{X}$$

IS CONTINUOUS IN ζ-PLANE:

ACROSS ℓ_γ



$$\mathcal{X}^{-1}\zeta\partial_\zeta \mathcal{X} \to (S\mathcal{X})^{-1}\zeta\partial_\zeta(S\mathcal{X})$$

$$= \mathcal{X}^{-1}\zeta\partial_\zeta \mathcal{X}$$

$\Rightarrow$ A_ζ IS HOLOMORPHIC FOR $\zeta \in \mathbb{C}^*$

$$\Rightarrow \quad \zeta \partial_\zeta X = A_\zeta X \qquad 2$$

STRUCTURE GROUP: SYMPL(Γ)

ASYMPTOTICS $\Rightarrow$

$$A_\zeta = \zeta^{-1} A_\zeta^{(-1)} + A_\zeta^{(0)} + \zeta A_\zeta^{(+1)}$$

SINCE S_γ IS INDPT. OF R, u, Λ ....

SAME ARGUMENT $\Rightarrow$ X SATISFIES A SET OF DIFFERENTIAL EQATIONS:

$$\frac{\partial}{\partial u}\chi = \mathcal{A}_u \cdot \chi$$

$$\frac{\partial}{\partial \bar{u}}\chi = \mathcal{A}_{\bar{u}} \cdot \chi$$

$$\Lambda \frac{\partial}{\partial \Lambda}\chi = \mathcal{A}_\Lambda \cdot \chi$$

$$\bar{\Lambda} \frac{\partial}{\partial \bar{\Lambda}}\chi = \mathcal{A}_{\bar{\Lambda}} \cdot \chi$$

$$R \frac{\partial}{\partial R}\chi = \mathcal{A}_R \cdot \chi$$

$$\zeta \frac{\partial}{\partial \zeta}\chi = \mathcal{A}_\zeta \cdot \chi$$

$$\mathcal{A}_i = \zeta^{-1} \mathcal{A}_i^{(-1)} + \mathcal{A}_i^{(0)} + \zeta \mathcal{A}_i^{(+1)}$$

KEY POINT: THESE EQUATIONS ALL FOLLOW FROM THE PHYSICS OF THE 4D GAUGE THEORY!!

$$\frac{\partial}{\partial u}\chi = \mathcal{A}_u \cdot \chi$$

$$\frac{\partial}{\partial \bar{u}}\chi = \mathcal{A}_{\bar{u}} \cdot \chi$$

HOLOMORPHY ON $\mathcal{M}_g$

ALSO HOLOMORPHY...

$$\Lambda \frac{\partial}{\partial \Lambda}\chi = \mathcal{A}_\Lambda \cdot \chi$$

$$\bar{\Lambda} \frac{\partial}{\partial \bar{\Lambda}}\chi = \mathcal{A}_{\bar{\Lambda}} \cdot \chi$$

VIEW Λ AS BACKGROUND VEV OF A VM.

$$R \frac{\partial}{\partial R}\chi = \mathcal{A}_R \cdot \chi$$

$$\zeta \frac{\partial}{\partial \zeta}\chi = \mathcal{A}_\zeta \cdot \chi$$

ANOMALOUS SCALE AND R-SYMMETRY

STOKES PHENOMENON

THE ζ- DIFF. EQ. HAS AN IRREGULAR SINGULAR POINT AT $\zeta = 0, \infty$; SOLUTIONS EXHIBIT STOKES PHENOM.

IN THIS INTERPRETATION S_γ ARE STOKES FACTORS ASSOCIATED WITH ℓ_γ.

REMAINING EQUATIONS: ISOMONODROMIC DEFORMATION

$\Rightarrow$ STOKES FACTORS ARE INDP'T OF $R, u, \Lambda, \ldots$

$\Rightarrow$ CHECK AT LARGE R IN 1- INSTANTON APPROXIMATION.

9. TAKE-HOME SUMMARY

1. WE CONSTRUCT THE HK METRIC FOR CIRCLE-COMPACTIFICATION OF $\mathcal{N}=2, D=4$ FIELD THEORIES.

2. QUANTUM CORRECTIONS TO THE DIMENSIONAL REDUCTION METRIC COME FROM BPS STATES.

3. CONTINUITY OF THE QUANTUM-CORRECTED METRIC IS EQUIVALENT TO THE KS WCF.

4. USE TWISTOR TRANSFORM AND WRITE HOLOMORPHIC FUNCTIONS AS AN EXPLICIT SUM OVER BPS INSTANTONS.

10. CONCLUSION

– OTHER THINGS WE HAVE DONE –

- MASSIVE HM'S: GENERALIZES KS.

- THERE ARE STRONG CONNECTIONS WITH THE tt^* EQUATIONS OF CECOTTI & VAFA.

- THE FUNCTIONS $\mathcal{X}_\gamma$ ARE 't HOOFT–WILSON–MALDACENA LOOP OPERATOR VEV'S; MOREOVER THERE IS A NICE INTERPRETATION IN TERMS OF A 3D TFT [TO APPEAR]

- THE MODULI SPACE $(\mathcal{M}, g)$ IS A MODULI SPACE OF A HITCHIN SYSTEM [S. CHERKIS & A. KAPUSTIN]. FOR SU(2) WE HAVE CONSTRUCTED IT AS A SPACE OF STOKES MATRICES GLUED BY KS TRANSFORMATIONS.

[TO APPEAR.]

— TO DO —

- UNDERSTAND BETTER THE NEED FOR THE QUADRATIC REFINEMENT.

- SINGULARITIES AT SUPERCONFORMAL POINTS REMAIN TO BE UNDERSTOOD

- RELATIONS TO INTEGRABLE SYSTEMS

- RELATION TO THE WORK OF JOYCE & BRIDGELAND / TOLEDANO LAREDO

- GENERALIZATION TO SUGRA.

- SHOULD GIVE EXPLICIT FORMULATION OF Q.C.'S TO HYPERMULTIPLET MODULI SPACES.