29 July 2008
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A-branes and Hitchin fibers

I. Hitchin fibration $\rightsquigarrow$ Question: what are Hitchin fibers?

II. Review of A-branes (Fukaya category) and constr. sheaves $\rightsquigarrow$ Language: constr. sheaves

III. Springer theory $\rightsquigarrow$ Answer: Springer sheaf

IV. Fourier transform $\rightsquigarrow$ Proof

I. $G/\mathbb{C}$ reductive

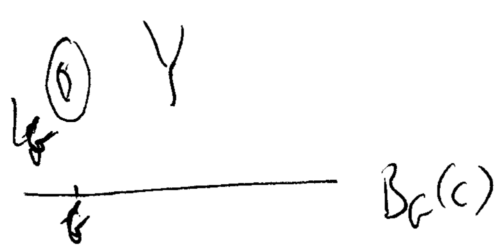
$C/\mathbb{C}$ smooth projective curve

$$\mathcal{H} : T^* Bun_G(C) \longrightarrow B_G(C)$$

L-fibration affine base

$$\begin{array}{c} T^*_P Bun_G(C) = H^0(C, \mathfrak{g}^*_P \otimes \omega_C) \\ \Big\downarrow \mathcal{H} \\ B_G(C) = H^0(C, \mathfrak{g}^*/\!/G \cdot \omega_C) \end{array}$$

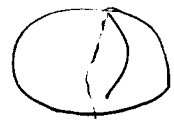
$$\mathfrak{g}^*/\!/G \simeq \mathfrak{t}^*/W_G$$



Special case

$C =$  cuspidal ell. curve

$$\begin{array}{c} C \\ \uparrow \pi \\ \mathbb{P}^1 \end{array}$$

$\mathbb{P}^1 =$ 

Restrict to G-bundles on C s.t. $\pi^*\mathcal{P}$ is trivializable

$\mathrm{Bun}_G^0(C) = \mathfrak{g}/G$ adj. quot

descent data ($\mathfrak{g}$), acts (G)

Hitchin fibration

$$\mathcal{H} : T^*\mathrm{Bun}_G^0(C) \to \mathfrak{h}^*/W$$

$$T^*\mathrm{Bun}_G^0(C) = T^*(\mathfrak{g}/G) = \{[X,Y]=0\}/G, \quad \mathfrak{g} \cong \mathfrak{g}^*$$

$$(X,Y) \mapsto e(Y)$$

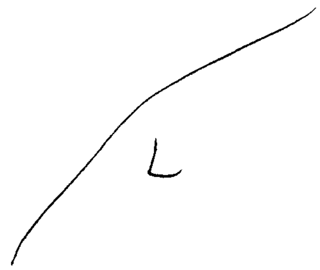
Hitchin fibers

$$\lambda \in \mathfrak{h}^{reg}/\!/W \mapsto \mathcal{L}_\lambda = \mathcal{H}^{-1}(\lambda) = \{[X,Y]=0,\ e(Y)=\lambda\}/G$$

<u>We have</u>: $L_\lambda \subset \mathfrak{g} \times \mathfrak{g}$ sympl. v. sp.

$$= \{[X,Y]=0,\ e(Y)=\lambda\}$$

<u>Ex</u> L_λ <u>exact</u> Lagrangian



$\omega = d\alpha$

$[\alpha|_L] = 0 \in H^1(L)$

II. <u>Thm</u> $Sh_c(X) = F(T^*X)$

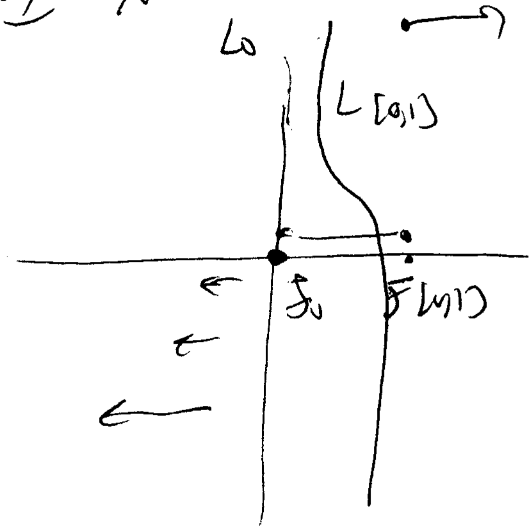
<u>Objs</u> of $Sh_c(X)$: Complexes of sheaves of $\mathbb{C}$-vector spaces on X with constr cohom.

$\mathcal{J}^\bullet$

Twisted complexes

<u>Objs</u> of $F(T^*X)$: Smooth, (not necc. compact) exact Lag. submanifolds with flat connection (+ grading, rel pin str) (grade homs) (orient moduli spaces)

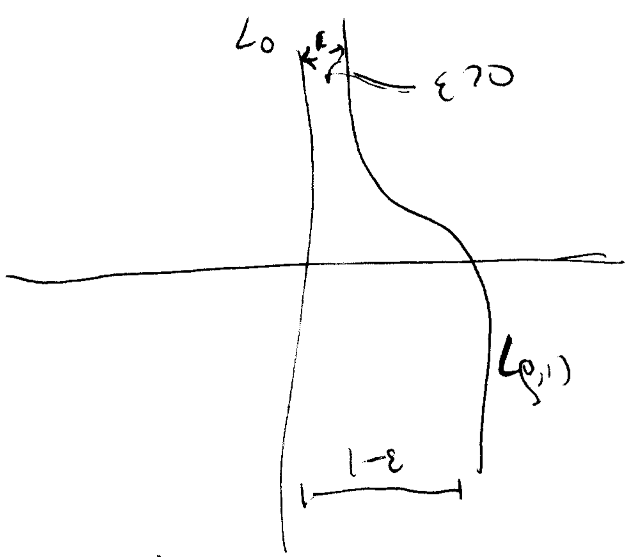
Ex $X = \mathbb{R}$



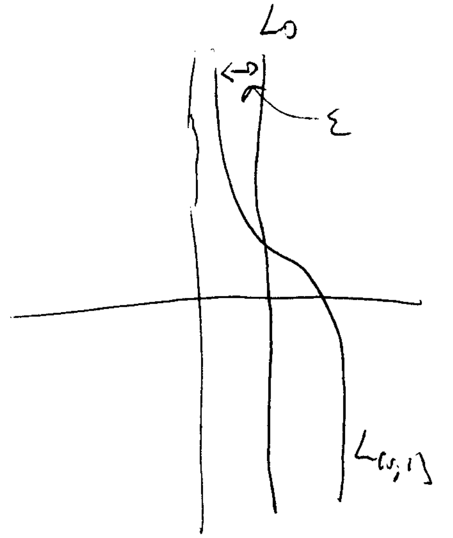
$\hom(L_0, L_{[0,1]})$

$\hom(L_{[0,1]}, L_0)$

Morphisms propagate forward in time



$\hom(L_0, L_{[0,1]}) = 0$



$\hom(L_{[0,1]}, L_0) = \mathbb{C}$

$$\hom_{\mathbb{R}}(\mathcal{F}_0, \mathcal{F}_{[0,1]}) \qquad j_0 : \{0\} \hookrightarrow \mathbb{R}$$

$$\| \qquad\qquad j_{[0,1]} : (0,1) \hookrightarrow \mathbb{R}$$

$$\hom_{\mathbb{R}}(j_{0!}\mathbb{C}_{\{0\}}, j_{(0,1)*}\mathbb{C}_{(0,1)})$$

$$\|$$

$$\hom_{(0,1)}(\underbrace{j_{(0,1)}^* j_{0!}\mathbb{C}_{\{0\}}}_{= 0}, \mathbb{C}_{(0,1)}]$$

<u>Challenge</u> $Sh_c(\mathfrak{g}) = F(T^*\mathfrak{g})$

$$? \longleftrightarrow L$$

<u>III Springer Theory</u>

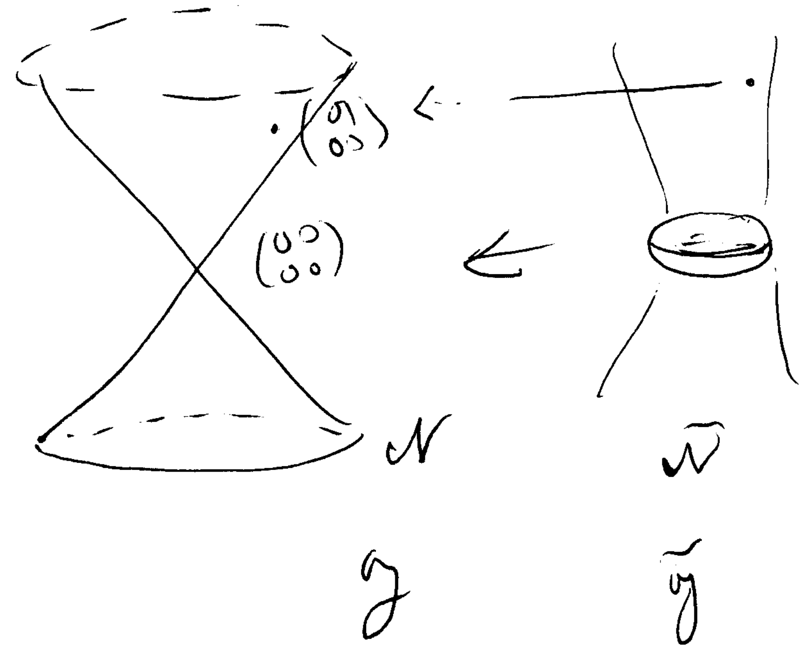
<u>Aim</u> $\begin{pmatrix} \lambda & & \\ & \lambda_2 & \\ & & \lambda \end{pmatrix}$ W permuting $\left(\begin{pmatrix} 0 & 1 \\ & 0 \end{pmatrix}\right)$

$\circlearrowleft W$?

Groth-Springer resolution

$$\begin{array}{ccc} W\text{-bundle} & \tilde{\mathfrak{g}} = (x, \in \mathfrak{b}) & \tilde{N} = T^*\mathcal{B} \\ \downarrow & \downarrow \tilde{\mu} & \text{Borel subalgebra } \mathfrak{b} \in \mathcal{B} \\ \mathfrak{g}^{reg.ss} \hookrightarrow & \mathfrak{g} \ni X & \hookleftarrow N = \text{nilp cone} \end{array}$$

$\mathfrak{sl}_2\mathbb{C}$



Construct W-reps on sheaves

<u>Thm</u> (Lusztig, Borho–MacPherson)

$$R\mu_* \mathbb{C}_{\tilde{\mathcal{N}}} = S_{\mathcal{N}}$$

$$\mathrm{End}(S_{\mathcal{N}}) = \mathbb{C}[W]$$

<u>Thm</u> $\quad F(T^*\mathfrak{g}) = Sh_c(\mathfrak{g})$

$$L_\lambda \longleftrightarrow F.T.(S_{\mathcal{N}})$$

$\circlearrowright$

move $\lambda \in \mathfrak{h}^{reg}$

$\rightsquigarrow$ continuation maps

$\rightsquigarrow$ Braid gp action $\longleftrightarrow$ $\mathbb{C}[W]$-str.

Thm

$$\begin{tikzcd}
 & F(V_x \times V_\xi) \arrow[dl] \arrow[dr] & \\
Sh_c(V_x) \arrow[d] & \circ & Sh_c(V_\xi) \arrow[d] \\
Sh_c^{con}(V_x) \arrow[rr, leftrightarrow, "FT"] & & Sh_c^{con}(V_\xi)
\end{tikzcd}$$

dilate