Modern Challenges for Lattice Field Theory KITP, March 2005

Simulations of QCD at finite baryon density, via imaginary chemical potential

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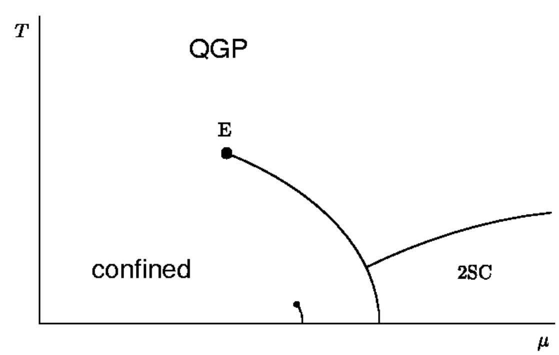
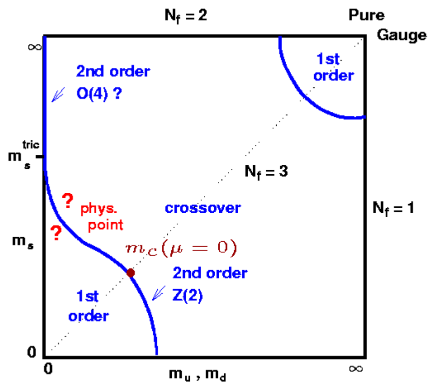
- Present: Analytic continuation, with Owe Philipsen (U. Münster)
- Future: Canonical approach, with Slavo Kratochvila (ETH)

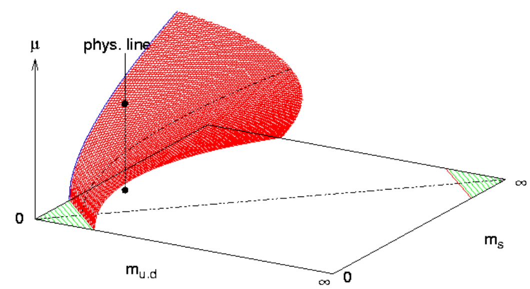
- **Confinement loses meaning when matter is heated and/or compressed**
- **rich phase structure**



- **non-perturbative ⟶ lattice simulations**

The QCD phase transition at finite $T, \mu = 0$





Finite density, $\mu \neq 0$:

$\not{D}(\mu)^\dagger = \gamma_5 \not{D}(-\mu^*)\gamma_5 \Rightarrow \det(\not{D}) \notin R$, unless $\mu = i\mu_I$ (e.g. 0)

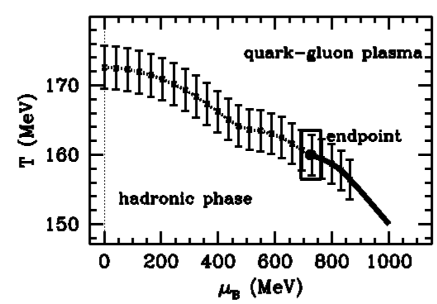
Sign problem

Circumventing the sign problem:

I. Multi-dimensional reweighting in (μ, β)

Fodor & Katz

$$Z(\mu,\beta) = \left\langle \frac{e^{-S_g(\beta)} \det(\mu)}{e^{-S_g(\beta_0)} \det(\mu = 0)} \right\rangle_{\mu=0,\beta_0} Z(\mu = 0, \beta_0)$$



idea: simulate at $\beta_0 = \beta_c(0)$, better overlap by sampling both phases; errors? ovlp.?

II. Taylor expansion of I

Bielefeld/Swansea
Gavai & Gupta

idea: for small μ/T, compute coeffs. of Taylor series $\Rightarrow$local ops. $\Rightarrow$gain V

convergence?

III. Imaginary μ + analytic continuation

de Forcrand & Philipsen
D'Elia & Lombardo; Chen & Luo; Azcoiti et al.

fermion determinant positive $\Rightarrow$no sign problem

idea: for small μ/T, fit **full** simulation results of imag. μ by Taylor series $\Rightarrow$continuation

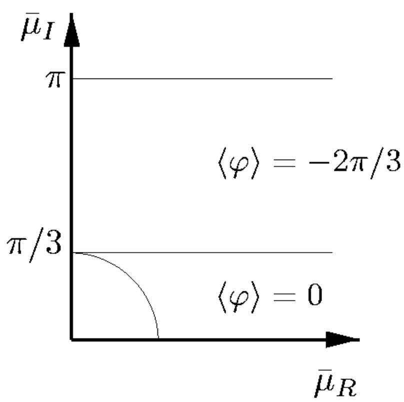
- vary **two** parameters $(\mu, T)\Rightarrow$uncorrelated results
- control over systematics

QCD at complex μ: general properties

$$Z(V,\mu,T) = \mathrm{Tr}\left(\mathrm{e}^{-(\hat{H}-\mu\hat{Q})/T}\right); \quad \mu = \mu_R + \mathrm{i}\mu_I; \quad \bar{\mu} = \mu/T$$

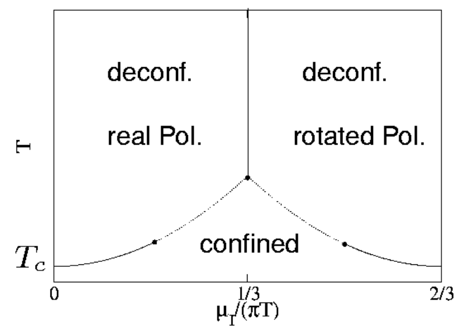
exact symmetries: μ-reflection and μ_I-periodicity — Roberge & Weiss

$$\boxed{Z(\bar{\mu}) = Z(-\bar{\mu}), \qquad Z(\bar{\mu}_R,\bar{\mu}_I) = Z(\bar{\mu}_R,\bar{\mu}_I + 2\pi/N_c)}$$



Z(3)-transitions:
$\bar{\mu}_I^c = \frac{2\pi}{3}\left(n+\frac{1}{2}\right)$

pert./strong coupling:
1rst order for high T
crossover for low T



analytic continuation within arc, $\mu_B \lesssim 500\mathrm{MeV}$: $\langle O\rangle = \sum_n^N c_n \bar{\mu}_I^{2n} \Rightarrow \mu_I \longrightarrow \mathrm{i}\mu_I$

Ia. Analyticity of the (pseudo-) critical line in finite V

location of phase transition from maximum of susceptibilites:

$$\chi(\beta, a\mu, V) = V N_t \left\langle (\mathcal{O} - \langle\mathcal{O}\rangle)^2 \right\rangle, \quad \mathcal{O} \in \{\mathrm{plaq}, \bar{\psi}\psi, |P(x)|\}$$

- finite volume: suscept. always finite and analytic

Critical line $\beta_c(a\mu)$ defined by peak $\chi_{max} \equiv \chi(a\mu_c, \beta_c)$

- implicit function theorem: $\chi(\beta, a\mu)$ analytic $\Rightarrow \beta_c(a\mu)$ analytic!

symmetries: $\Rightarrow \boxed{\beta_c(a\mu) = \sum_n c_n (a\mu)^{2n}}$

What to expect in physical units?

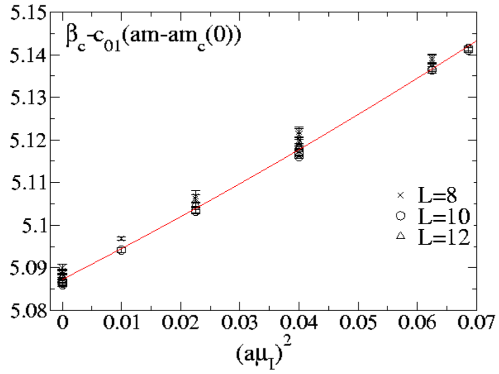
Natural expansion parameter is $\frac{\mu}{\pi T}$:

- thermal perturbation theory
- $\mu_I \longrightarrow$ shift in Matsubara frequencies $(2n+1)\pi T$

$$\Rightarrow \frac{T_c(\mu)}{T_c(\mu=0)} = 1 - O(1)\left(\frac{\mu}{\pi T_c(0,m)}\right)^2 + O(1)\left(\frac{\mu}{\pi T_c(0,m)}\right)^4 + \ldots$$

$N_f = 3$ results, quark mass dependence

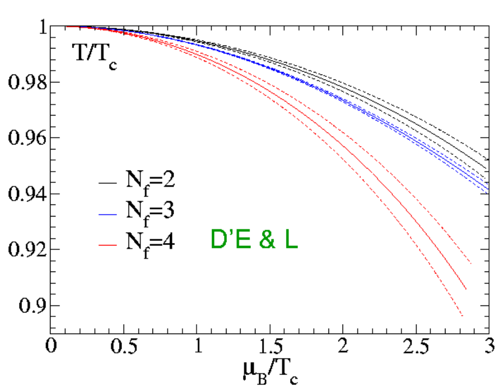
$$\beta_c(a\mu, am) = \sum_{k,l=0} c_{kl}\,(a\mu)^{2k}\,(am - am_c(0))^l$$

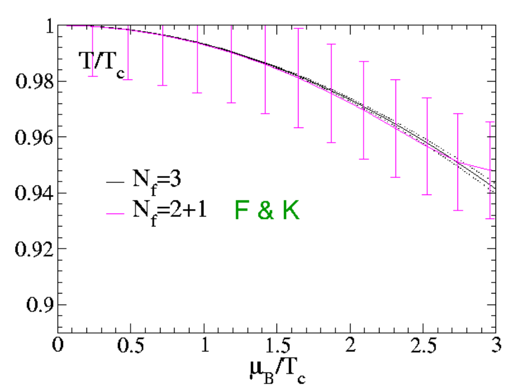


$\Rightarrow$sensitive to μ^4, but data very well described by μ^2 fit !

$$\frac{T_c(\mu, m)}{T_c(\mu = 0, m_c(0))} = 1 + 1.937(17)\frac{m - m_c(0)}{\pi T_c} + 0.602(9)\left(\frac{\mu}{\pi T_c(0,m)}\right)^2 + 0.23(9)\left(\frac{\mu}{\pi T_c(0,m)}\right)^4$$

N_f-dependence





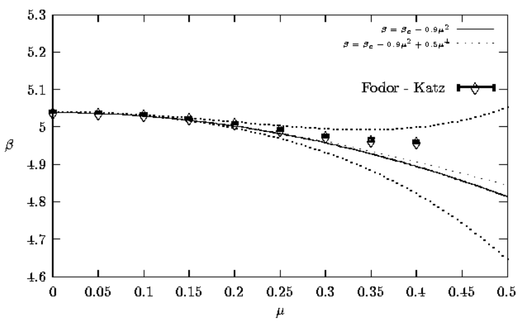
- curvature $\propto N_f$ when $N_c \to \infty$ Toublan
- $N_f = 2 + 1$ same as $N_f = 3$ as long as $m_s \ll \pi T$

Comparison:

$N_f = 4$

imag. μ vs. reweighting

D'Elia, Lombardo

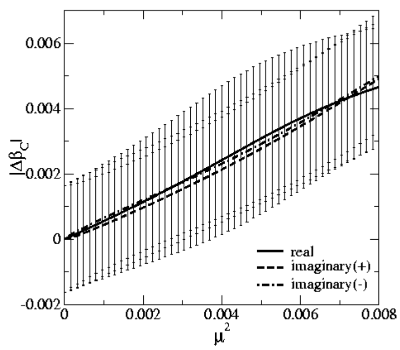


Comparison:

real μ vs. imag. μ

to leading order

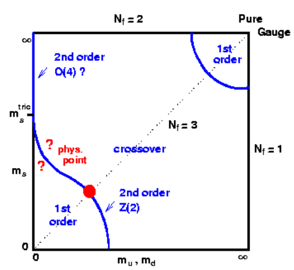
Bielefeld/Swansea

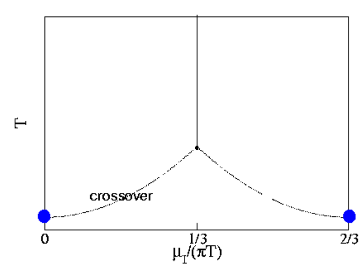


Comparison: analyt. continuation ok in SU(2) Giudice, Papa

Ib. Critical endpoint $(\mu, m_c(\mu))$: numerical strategy

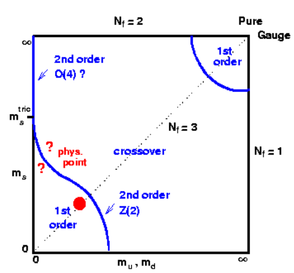
$m = m_c(0)$

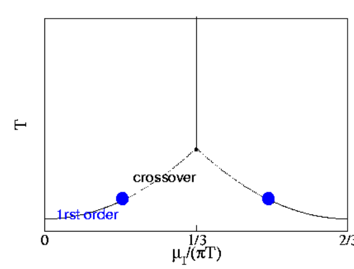




$\Longleftrightarrow$

$m < m_c(0)$





Expect: $\frac{m_c(\mu)}{m_c(\mu{=}0)} = 1 + c_1 \left(\frac{\mu}{\pi T}\right)^2 + \ldots$

$m = 0 \Rightarrow$ true chiral phase transition $\Rightarrow$ $\boxed{c_1 \leq 9}$

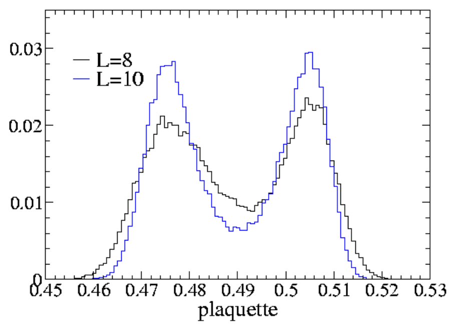
Detecting criticality: Binder cumulant $B_4 \equiv \frac{\langle(\delta\bar{\psi}\psi)^4\rangle}{\langle(\delta\bar{\psi}\psi)^2\rangle^2}$

3d Ising universality : $B_4(m_c,\mu_c) \to 1.604, \quad V \to \infty$

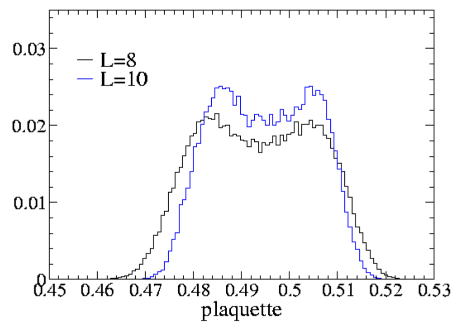
($B_4 = 1$ (first-order), 3 (crossover) for $V = \infty$)

need VERY LONG MC runs for sufficient tunneling statistics

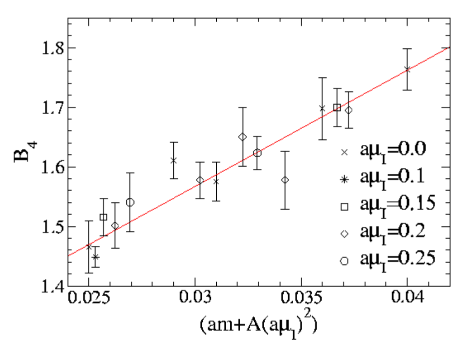
first-order



crossover



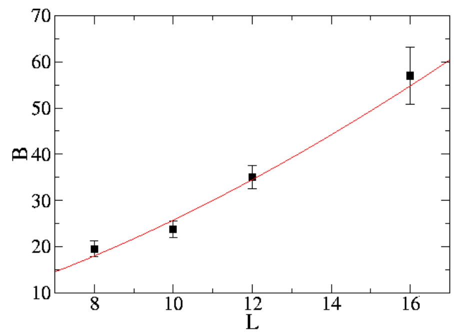
Taylor series $B_4(am, a\mu) = 1.604 + B\left(am - am_c(0) + A(a\mu)^2\right) + \dots$



$\Rightarrow \frac{m_c(\mu)}{m_c(\mu=0)} = 1+0.84(36)\left(\frac{\mu}{\pi T}\right)^2 + \dots$ **high quark mass sensitivity of μ_c!**

$m_c(0)$ in agreement with Bielefeld, Columbia; c_1(Bielefeld,impr.) $\sim 600(300)$ **?**

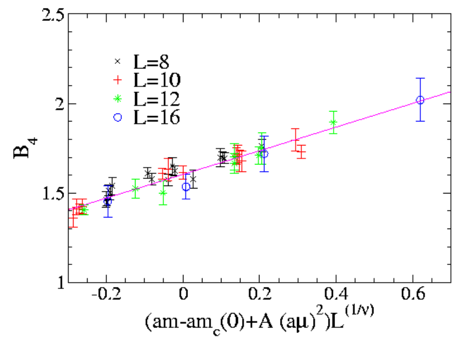
Finite size scaling



FSS:

$\nu = 0.62(3)$

$\nu(Ising) = 0.63$



Can one expect the critical point to be at "small" μ?

If $\mu_c \lesssim 120$ MeV (F & K), then

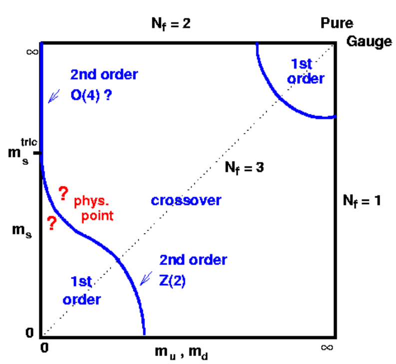
$$1 < \frac{m}{m_c(\mu = 0)} \lesssim 1.05$$

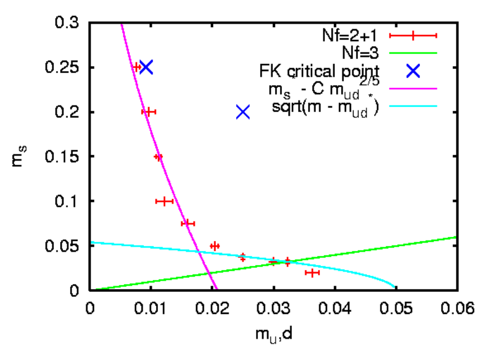
fine tuning of quark masses!

Outlook: $N_f = 2 + 1$ Phase diag. 4d: $(T, \mu, m_{u,d}, m_s)$

Two-step procedure:

I. $(m_s, m_{u,d})$ phase-diagram at $\mu = 0$ $\Rightarrow m_s^c(m_{u,d})$





$\Rightarrow$ strong non-linearities, no linear extrapolations from $N_f = 3$!

$\Rightarrow$ new Fodor & Katz qualitatively consistent with our results $(m_{u,d}/m_{u,d}^c \sim 1.1)$

$\Rightarrow$ consistent with $O(4)$, tri-critical point, $m_s^{\rm tric}/T \sim 2.8$

II. repeat calculation for $\mu \neq 0$

II. Canonical ensemble

with Slavo Kratochvila (hep-lat/0409072)

Fix baryon number B: $\delta(3B - \int d^3x\, \bar\psi\gamma_0\psi)$

$\Rightarrow Z_C(B) = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\left(\frac{\mu_I}{T}\right)\; e^{-i3B\frac{\mu_I}{T}} Z_{GC}(\mu = i\mu_I)$

- Sample $Z_{MC}(\mu = i\mu_{MC})$

  $\Rightarrow \frac{Z_C(B)}{Z_{MC}(i\mu_{MC})} = \langle \frac{1}{\det(i\mu_{MC})} \int d\mu_I \exp(i3B\frac{\mu_I}{T}) \det(i\mu_I)\rangle$

- Fourier transform each determinant exactly (work $\propto L_s^9 L_t$) $\Longleftarrow$ low T

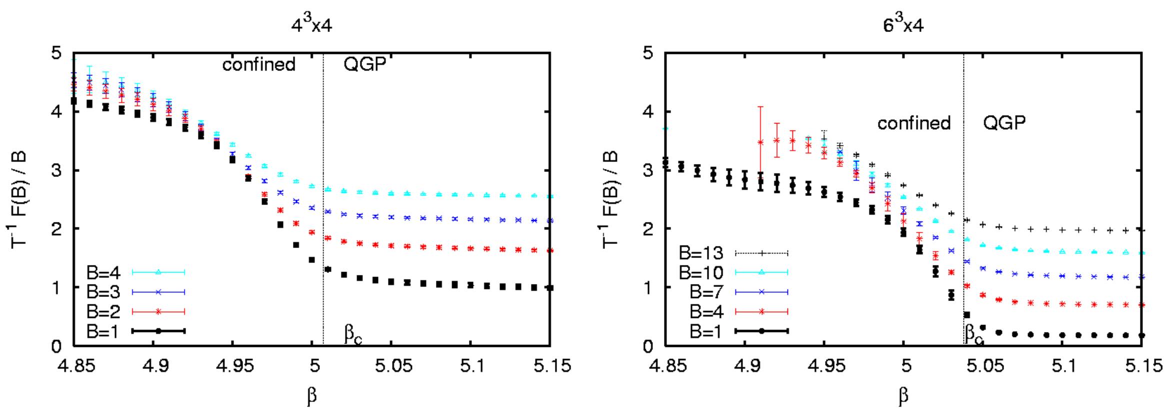
- Sign problem $\leftrightarrow$ noise in $Z_C(B) \Longleftarrow$ governed by B (not V)

- Study few-baryon system at low temperature: Nuclear Physics!

- Overlap: combine several (μ_{MC}, β) ensembles with Ferrenberg-Swendsen

- Here: KS fermions, $N_f = 4$, $am = 0.05$, $a \sim 0.3$ fm (P.T. first order $\forall\mu$)

Free energy versus temperature

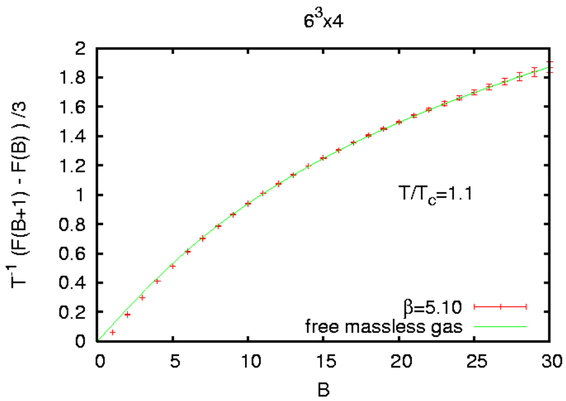


Chemical potential versus density

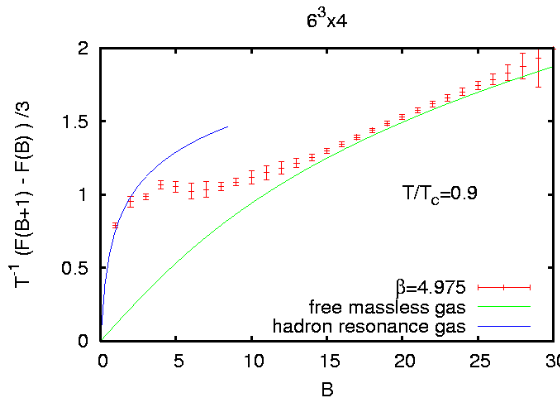
Do NOT use fugacity expansion $Z_{GC}(\mu) = \sum_B \exp(3B\frac{\mu}{T}) Z_C(B)$

Saddle point approximation:

$$Z_{GC}(\mu) = \int d\rho \exp(-\frac{V}{T}(f(\rho)) + \mu\rho)) \Longrightarrow \mu \approx f'(\rho) \approx \frac{F(B+1)-F(B)}{3}$$



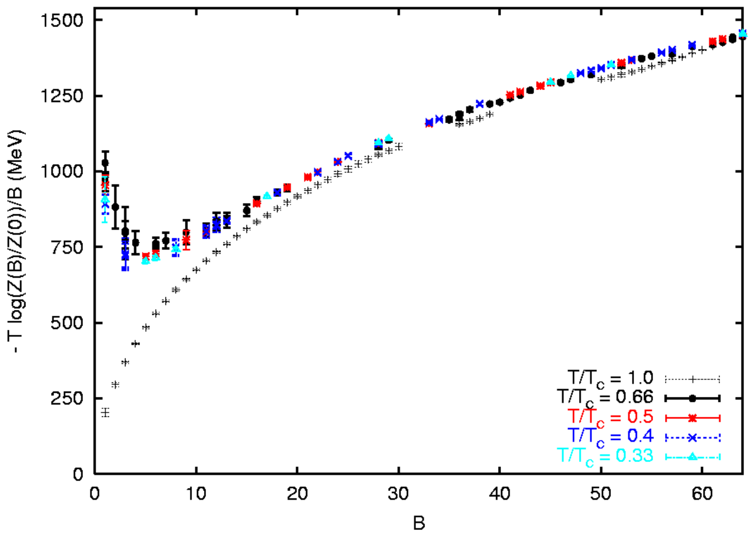
QGP



phase coexistence

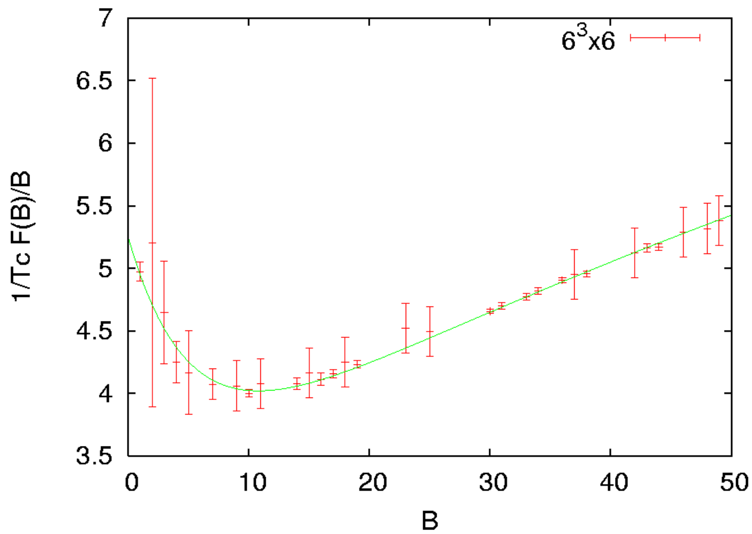
Lower temperatures $\longrightarrow \frac{1}{3}T_c$ nuclear attraction

$$4^3 \times 4, 6, 8, 10, 12, \ \beta = 5.04$$



free energy per baryon

$6^3 \times 6,\ \beta = 5.04 \to \frac{2}{3}T_c$



free energy per baryon

Conclusions

- Imaginary chemical potential: no sign pb., uncorrelated results (μ_I, β)
- Transition line accessible to simulations for $\mu_q/T \lesssim 1$

- consistent among all approaches!

- very flat $\approx \mu^2$, small quark mass dependence

- precision calculation?

- Critical endpoint (T_E, μ_E) still exploratory (esp. $a \to 0$)
- Critical endpoint extremely quark mass sensitive

$\Rightarrow \mu_c \lesssim 400$ MeV requires nature to fine tune m_q's

- Canonical approach promising: low T, nuclear physics