Lattice Field Theory at Non-zero Chemical Potential

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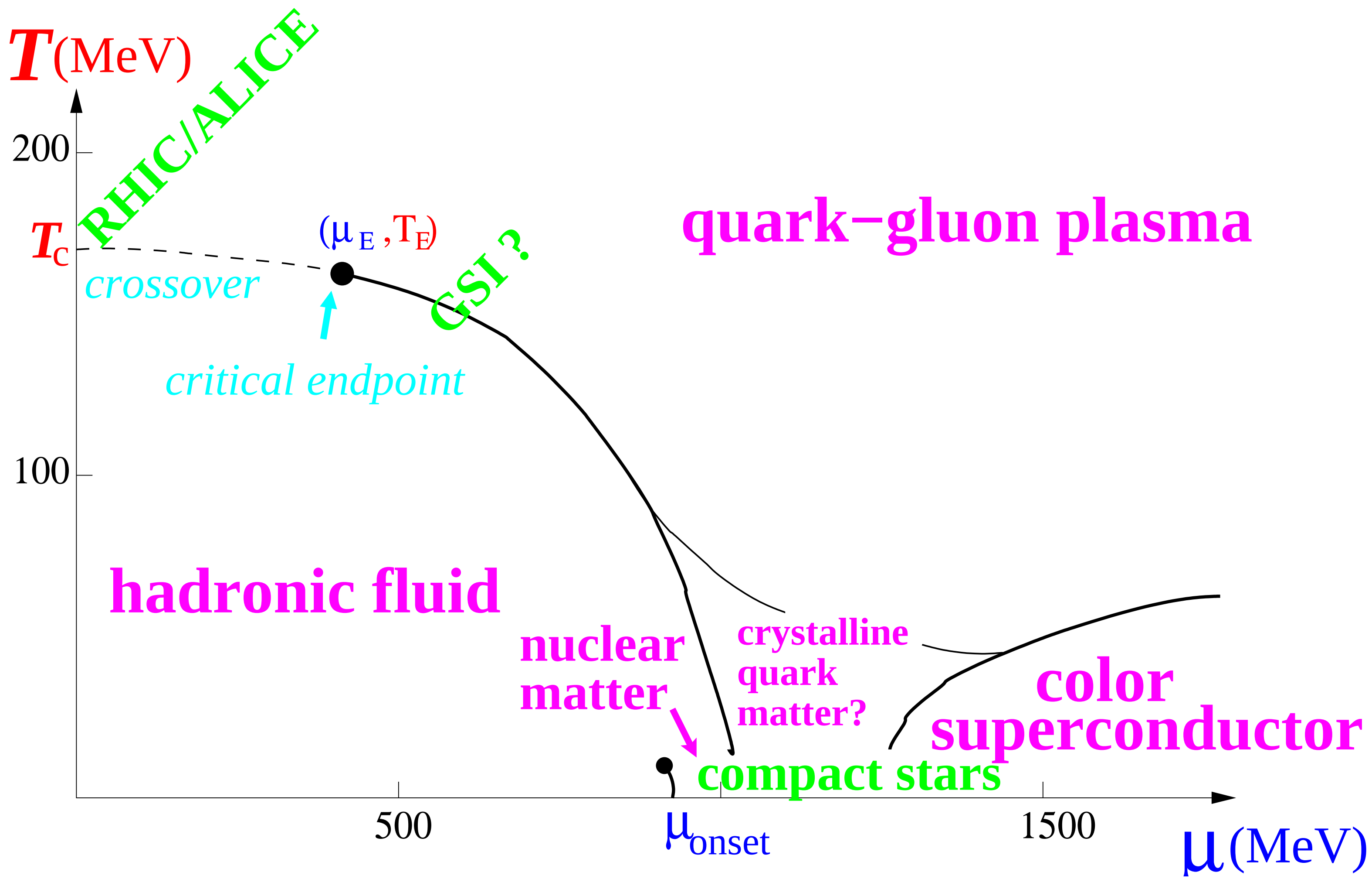
- The Sign Problem and the Silver Blaze Problem
- Progress at small μ/T
- Taylor Expansion of the Free Energy
- NJL model: Fermi Surface and Superfluidity
- Speculations

KITP Santa Barbara 31^{st} March

The QCD Phase Diagram



The Sign Problem for $\mu \neq 0$

In Euclidean metric the QCD Lagrangian reads

$$\mathcal{L}_{QCD} = \bar{\psi}(M + m)\psi + \frac{1}{4}F_{\mu\nu}F_{\mu\nu}$$

with $M(\mu) = \not{D}[A] + \mu\gamma_0$

Straightforward to show $\gamma_5 M(\mu)\gamma_5 \equiv M^\dagger(-\mu) \Rightarrow$

$$\det M(\mu) = (\det M(-\mu))^*$$

ie. Path integral measure is not positive definite for $\mu \neq 0$

Fundamental reason is explicit breaking of time reversal symmetry

Monte Carlo importance sampling, the mainstay of lattice QCD, is ineffective

A formal solution to the Sign Problem is *reweighting* ie. to include the phase of the determinant in the observable:

$$\langle \mathcal{O} \rangle \equiv \frac{\langle\langle \mathcal{O}\,\arg(\det M)\rangle\rangle}{\langle\langle \arg(\det M)\rangle\rangle}$$

with $\langle\langle \ldots \rangle\rangle$ defined with a positive measure $|\det M| e^{-S_{boson}}$

Unfortunately both denominator and numerator are exponentially suppressed:

$$\langle\langle \arg(\det M)\rangle\rangle = \frac{\langle 1 \rangle}{\langle\langle 1 \rangle\rangle} = \frac{Z_{true}}{Z_{fake}} = \exp(-\Delta F) \sim \exp(-\# V)$$

Expect signal to be overwhelmed by noise in thermodynamic limit $V \to \infty$

What goes wrong with the usual positive HMC measure?

$$\det M^\dagger M \begin{cases} M & \text{describes} \qquad\qquad\qquad \text{quarks } q \in 3 \\ M^\dagger & \text{describes conjugate quarks } q^c \in \bar{3} \end{cases}$$

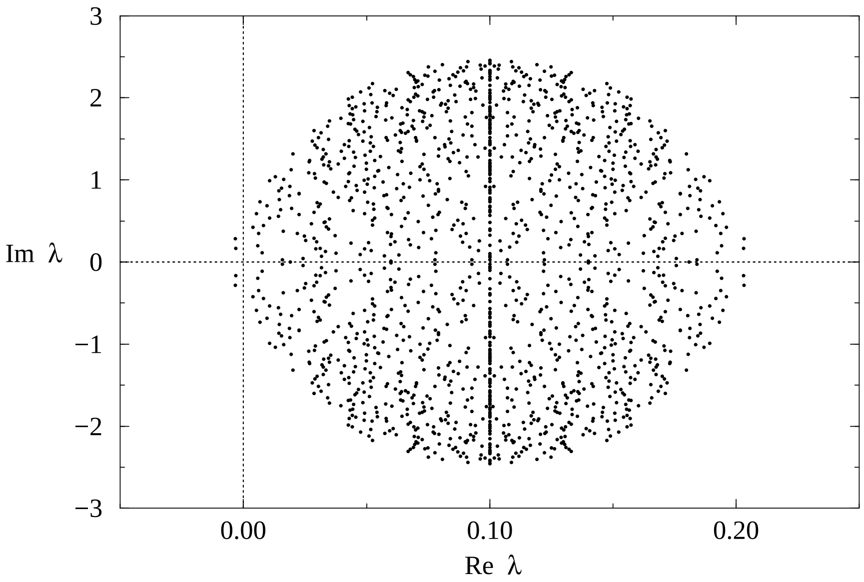
In general $\exists\ qq^c$ gauge singlet bound states with $B > 0$

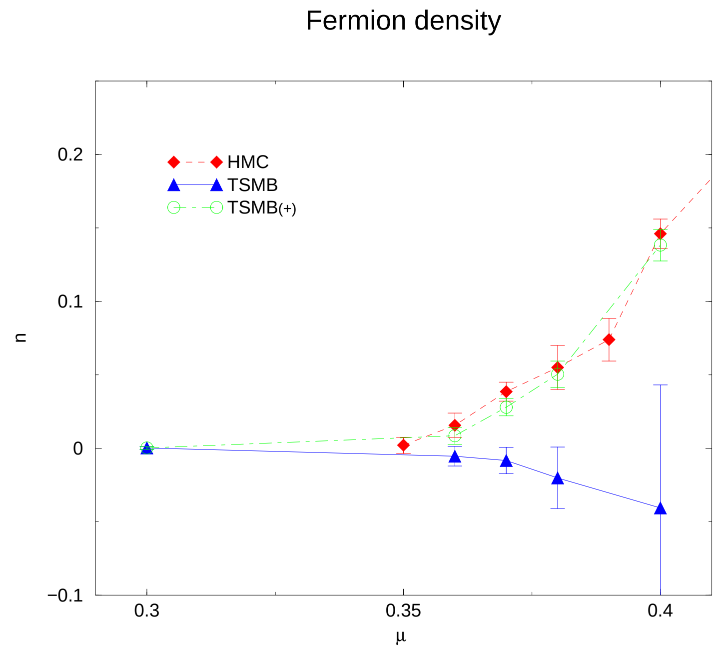
In QCD some qq^c states degenerate with the pion
$\Rightarrow$ unphysical onset of "nuclear matter" at $\mu_o \simeq \frac{1}{2} m_\pi$.

bug for QCD, feature for Two Color QCD. . .

Calculations with the true complex measure $\det^2 M$ nullify effects of qq^c states for the vacuum with $T = 0$, $\frac{1}{2} m_\pi < \mu \lesssim \frac{1}{3} m_N$ by cancellations among configurations with different signs/phases

The *Silver Blaze* Problem. . .





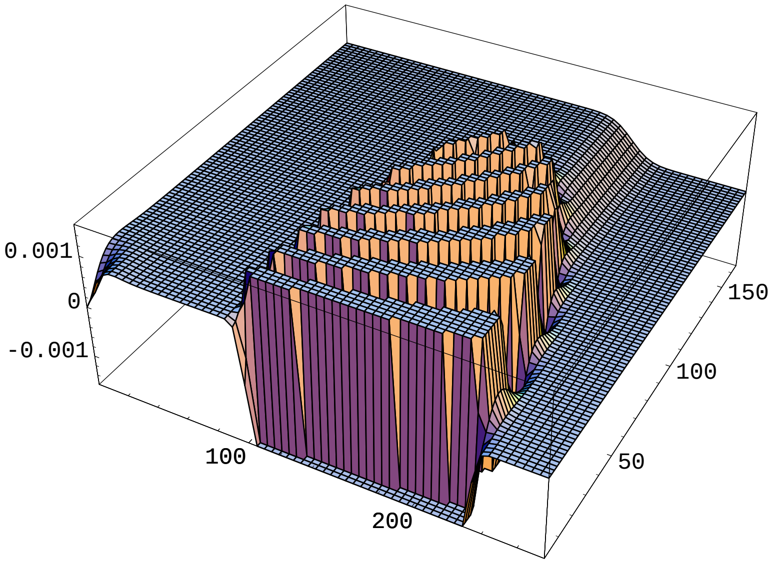
This has been numerically verified, eg. in TSMB simulations of Two Color QCD with $N = 1$ adjoint staggered quarks.

SJH,Montvay,Scorzato,Skullerud, EurPJ C22 (2001) 451

The fake transition to a superfluid phase, forbidden by the Pauli Principle, at $\mu_o a \simeq 0.35$ disappears once configurations with det$M < 0$ are included with the correct weight.

Analytic solution for Random Matrix model in the mesoscopic limit $V \to \infty$ with $m_\pi^2 f_\pi^2 V$ fixed

Akemann, Osborn, Splittorff & Verbaarschot, hep-th/0411030

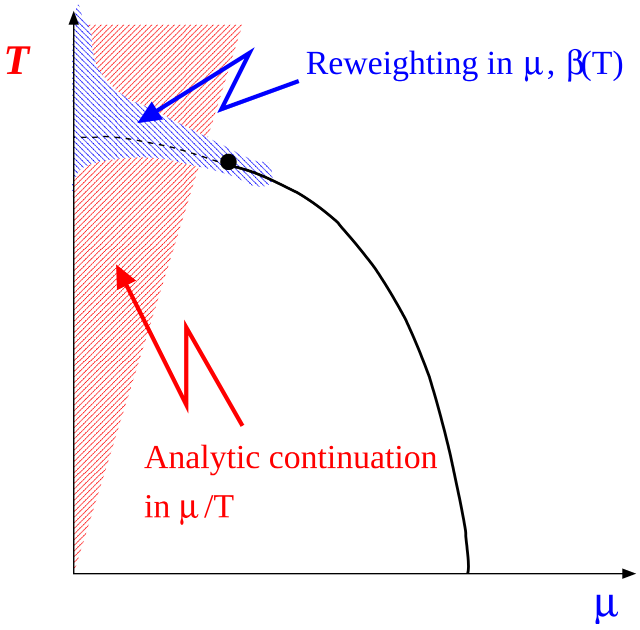


$$\langle \bar{\psi}\psi \rangle = \lim_{m\to 0} \lim_{V\to\infty} V^{-1} \int dxdy \frac{\rho(x,y,m;\mu)}{x+iy+m}$$

For $\mu = 0$ or $N_f = 0$ ρ is real, but in general it is a *complex-valued* spectral density.

In region $x > m$ ρ develops oscillatory structure with wavelength $\sim V^{-1}$, amplitude $\sim e^V \Rightarrow$
here is where the Silver Blaze cancellations take place

Two Routes into the Plane



(I) Analytic continuation in μ/T by either

Taylor expansion @ $\mu = 0$

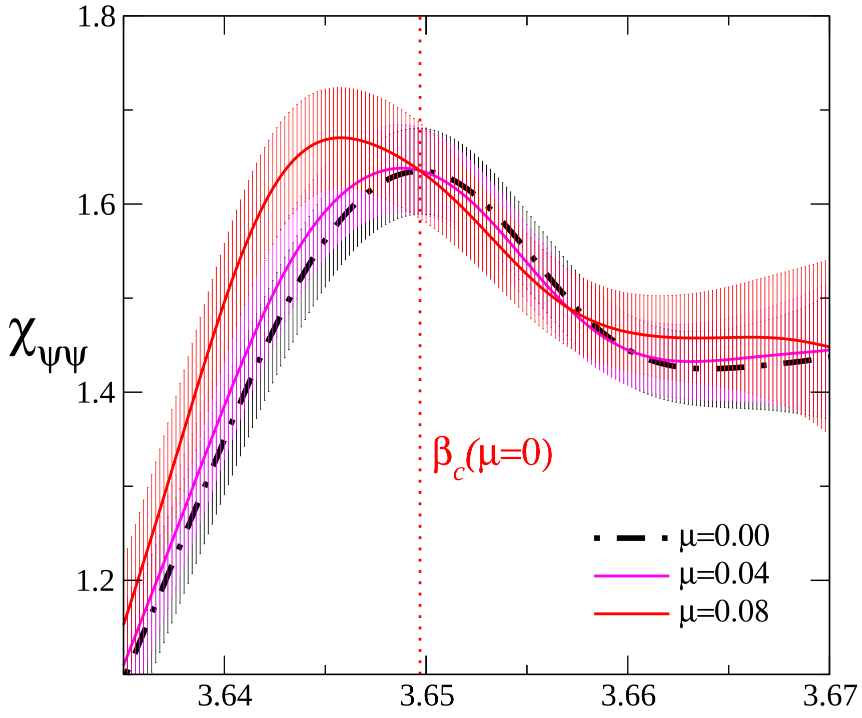
Gavai & Gupta; QCDTARO

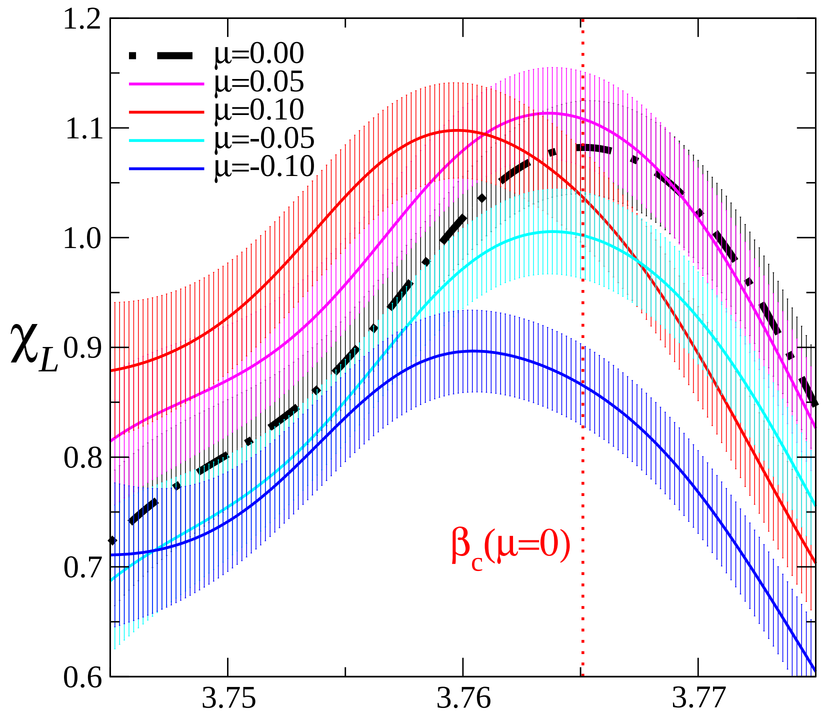
Simulation with imaginary $\tilde{\mu} = i\mu$ de Forcrand & Philipsen; d'Elia & Lombardo

effective for $\frac{\mu}{T} <$ min $\left(\frac{\mu_E}{T_E}, \frac{\pi}{3}\right)$

(II) Reweighting along transition line $T_c(\mu)$ Fodor & Katz

Overlap between (μ, T) and $(\mu + \Delta\mu, T + \Delta T)$ remains large, so multi-parameter reweighting unusually effective





The Bielefeld/Swansea group used a hybrid approach; ie. reweight using a Taylor expansion of the weight:

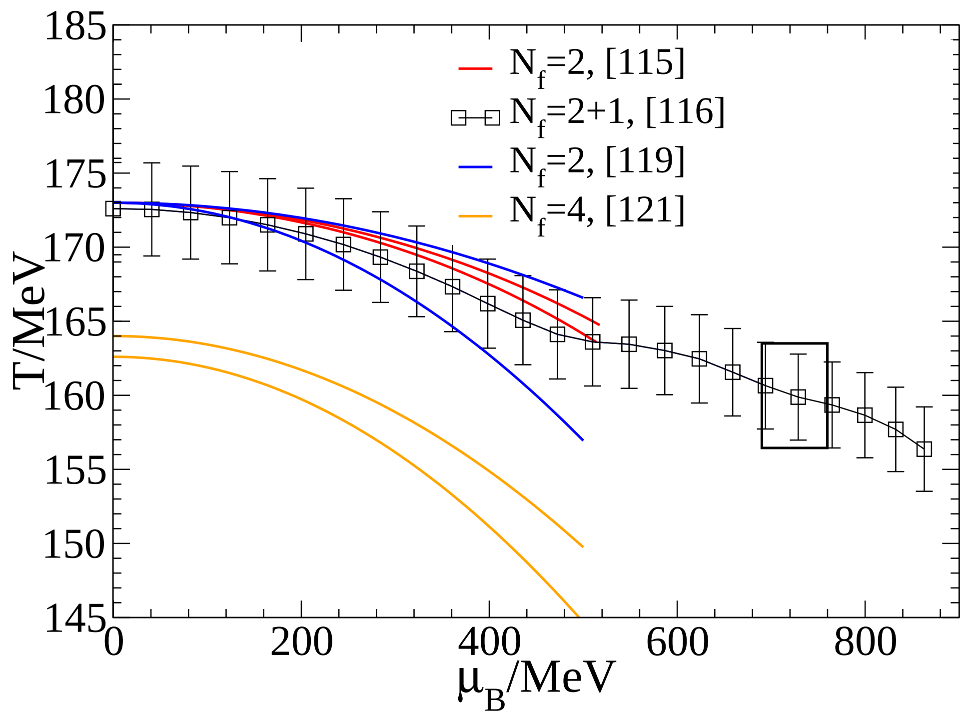
Allton *et al*, NSF-ITP-02-26

$$\ln\left(\frac{\det M(\mu)}{\det M(0)}\right) = \sum_n \frac{\mu^n}{n!} \frac{\partial^n \ln \det M}{\partial \mu^n}\bigg|_{\mu=0}$$

This is relatively cheap and enables the use of large spatial volumes ($16^3 \times 4$ using $N_f = 2$ flavors of p4-improved staggered fermion).

Note with $L_t = 4$ the lattice is coarse: $a^{-1}(T_c) \simeq 700$MeV

The (Pseudo)-Critical Line

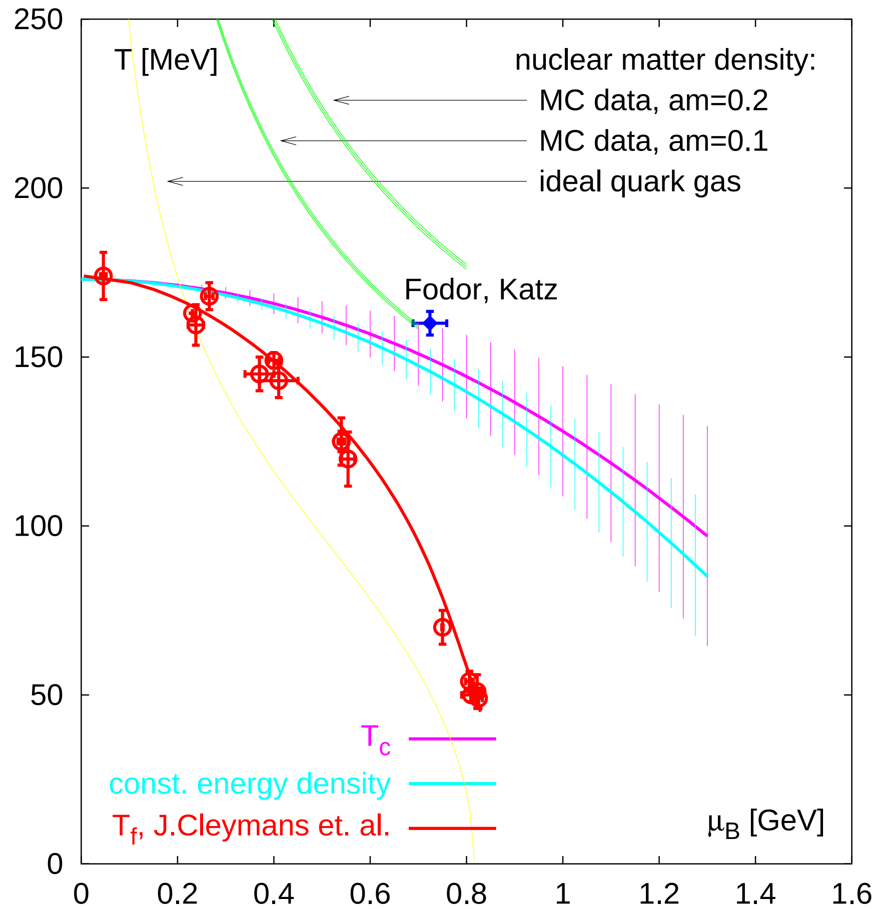


E. Laermann & O. Philipsen, Ann.Rev.Nucl.Part.Sci.53:163,2003

Remarkable consensus on the curvature. . .

Same curvature also seen in direct HMC simulations with $\mu_I = \frac{1}{2}(\mu_u - \mu_d) \neq 0$

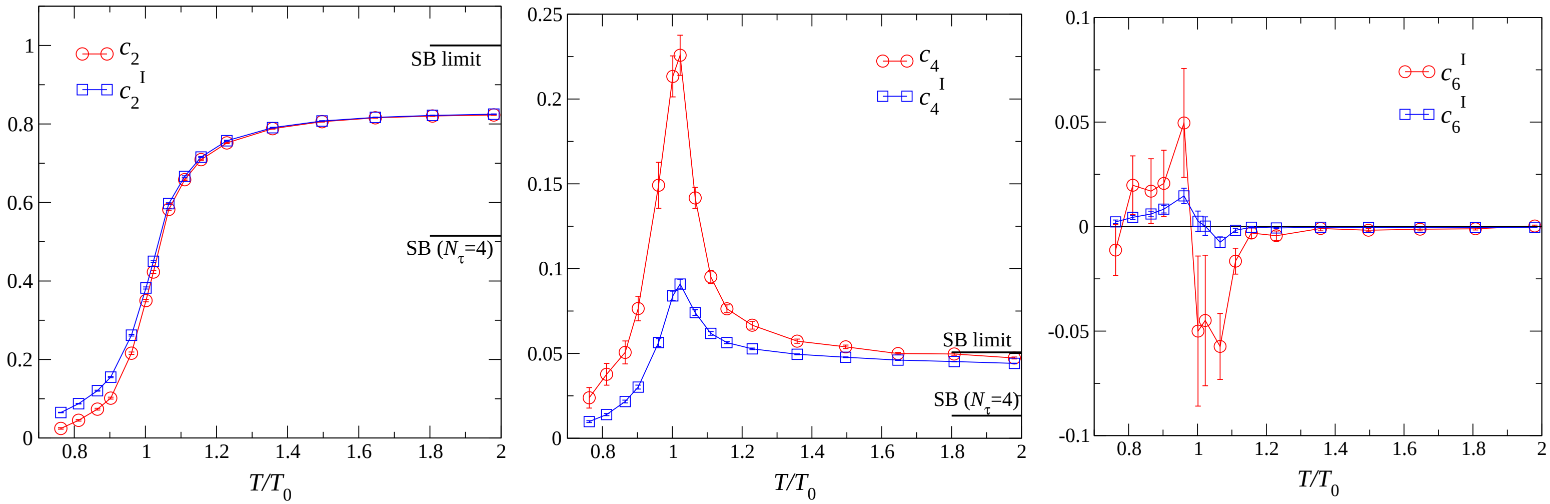
J. Kogut & D. Sinclair, PRD70:094501,2004



The pseudocritical line found lies well above the (μ_B, T) trajectory marking chemical freezeout in RHIC collisions

$\Rightarrow$ is there a region of the phase diagram where *hadrons* interact very strongly (ie. inelastically)? <u>So what?</u>

Taylor Expansion



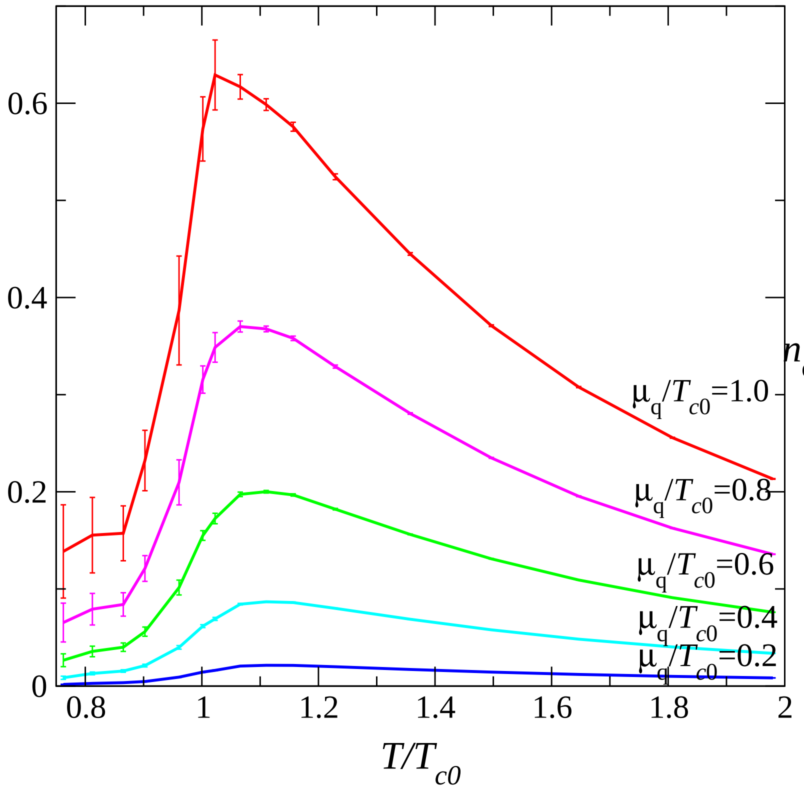
In our most recent work we develop the Taylor expansion of the free energy to $O((\mu_q/T)^6)$ (recall $c_6^{SB}=0$):

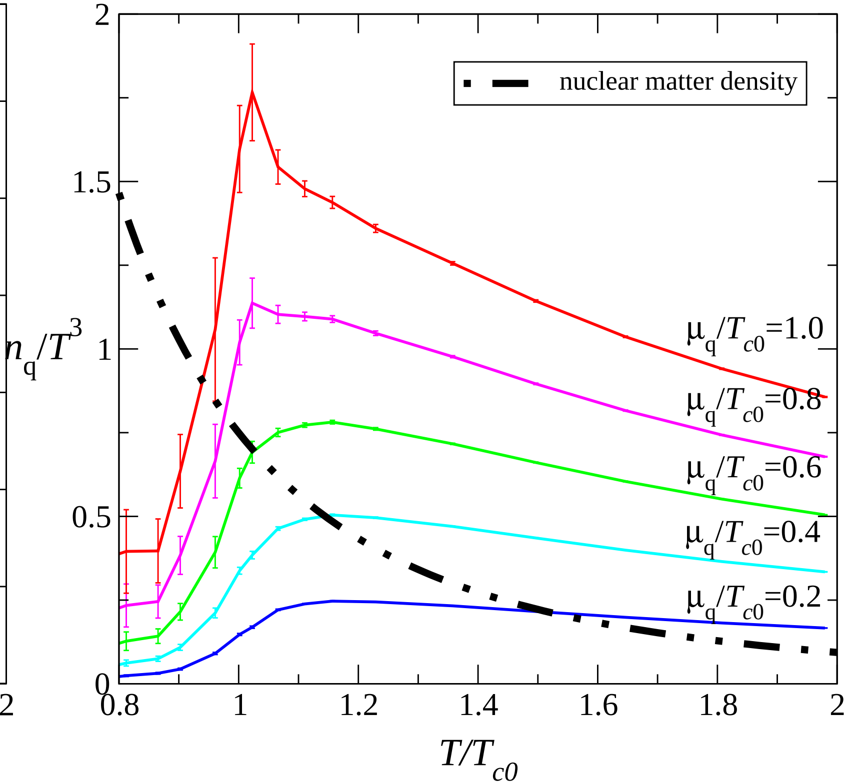
$$\frac{p}{T^4} = \sum_{n=0}^{\infty} c_n(T)\left(\frac{\mu_q}{T}\right)^n \quad \text{with} \quad c_n(T) = \frac{1}{n!}\frac{\partial^n(p/T^4)}{\partial(\mu_q/T)^n}\bigg|_{\mu_q=0}$$

Similarly we define expansion coefficients

$$c_n^I(T) = \frac{1}{n!}\frac{\partial^n(p/T^4)}{\partial(\mu_I/T)^2\partial(\mu_q/T)^{n-2}}\bigg|_{\mu_q=0,\mu_I=0}$$

Equation of State Allton *et al* PRD68(2003)014507, `hep-lat/0501030`

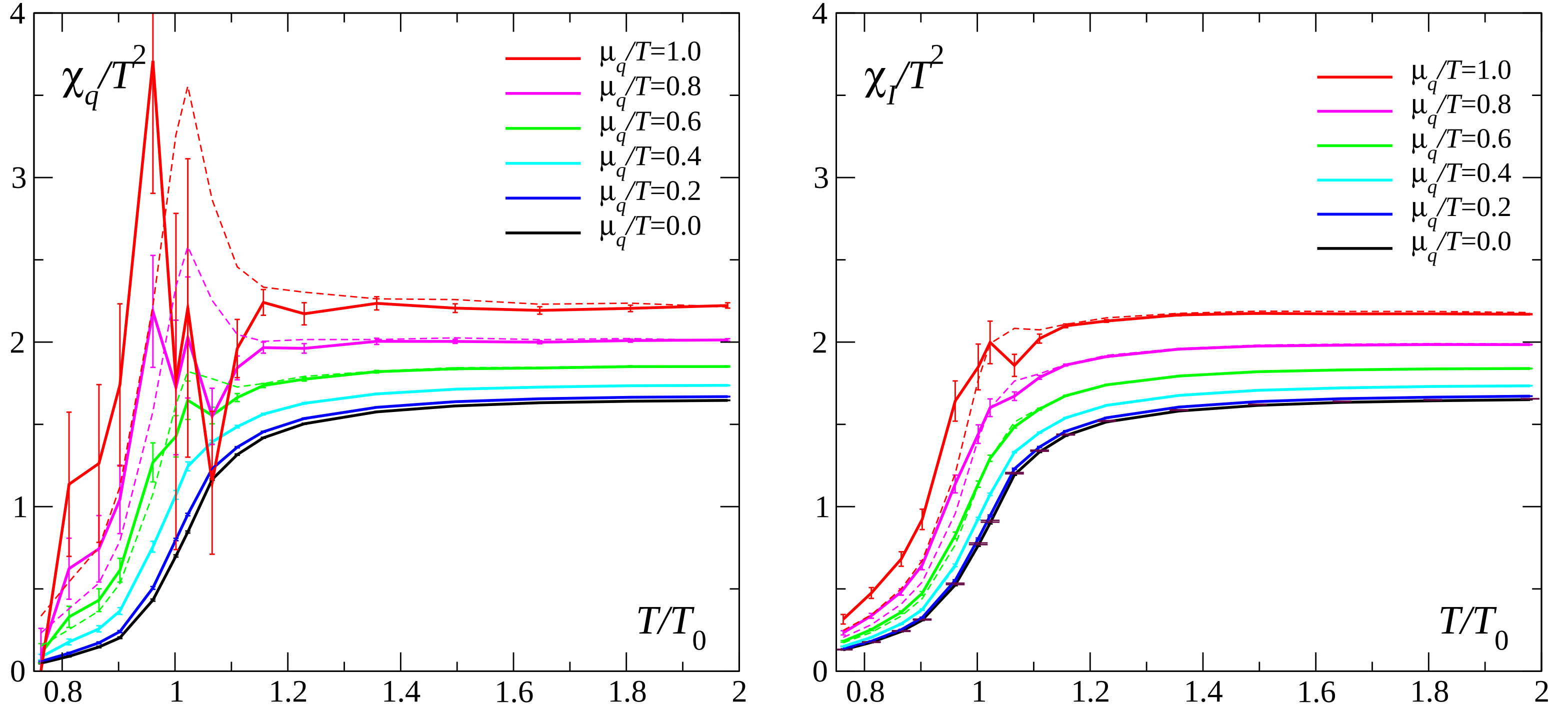




Pressure change $\Delta p/T^4$ Quark density n_q/T^3

$$\Delta \frac{p(\mu,T)}{T^4} = \frac{p(\mu,T)-p(0,T)}{T^4} = \sum_{n=1}^{n_{max}} c_n(T)\left(\frac{\mu}{T}\right)^n \quad ; \quad n_q = \frac{\partial p}{\partial \mu}$$

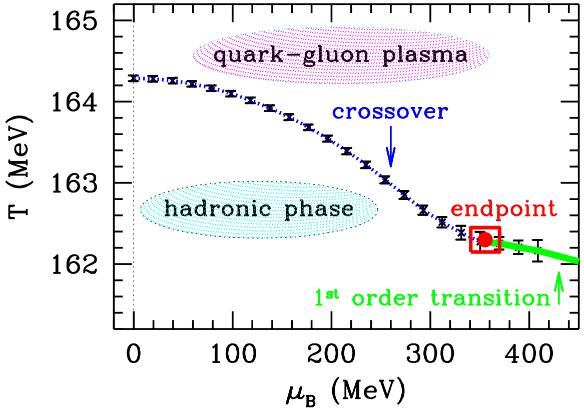
Growth of Baryonic Fluctuations



Quark number susceptibility $\chi_q = \frac{\partial^2 \ln Z}{\partial \mu_q^2}$ appears singular near $\mu_q/T \sim 1$; isospin susceptibility $\chi_I = \frac{\partial^2 \ln Z}{\partial \mu_I^2}$ does not

Massless field at critical point a combination of the Galilean scalar isoscalars $\bar{\psi}\psi$ and $\bar{\psi}\gamma_0\psi$?

The Critical Endpoint μ_E/T_E



Reweighting estimate
via Lee-Yang zeroes
$\mu_E/T_E = 2.2(2)$

Z. Fodor & S.D. Katz JHEP0404(2004)050

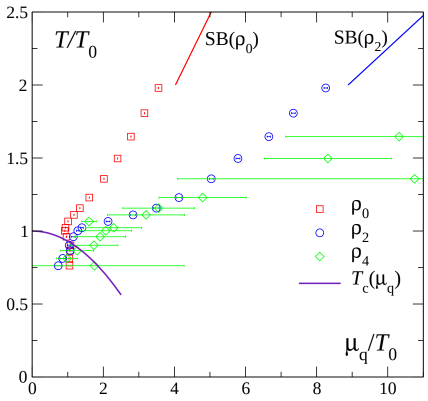
Taylor expansion estimate from apparent radius of convergence

$$\mu_E/T_E \gtrsim |c_4/c_6| \quad \sim \quad 3.3(6)$$

Allton *et al* PRD68(2003)014507

$$\mu_E/T_E \gtrsim 1.1(2)$$

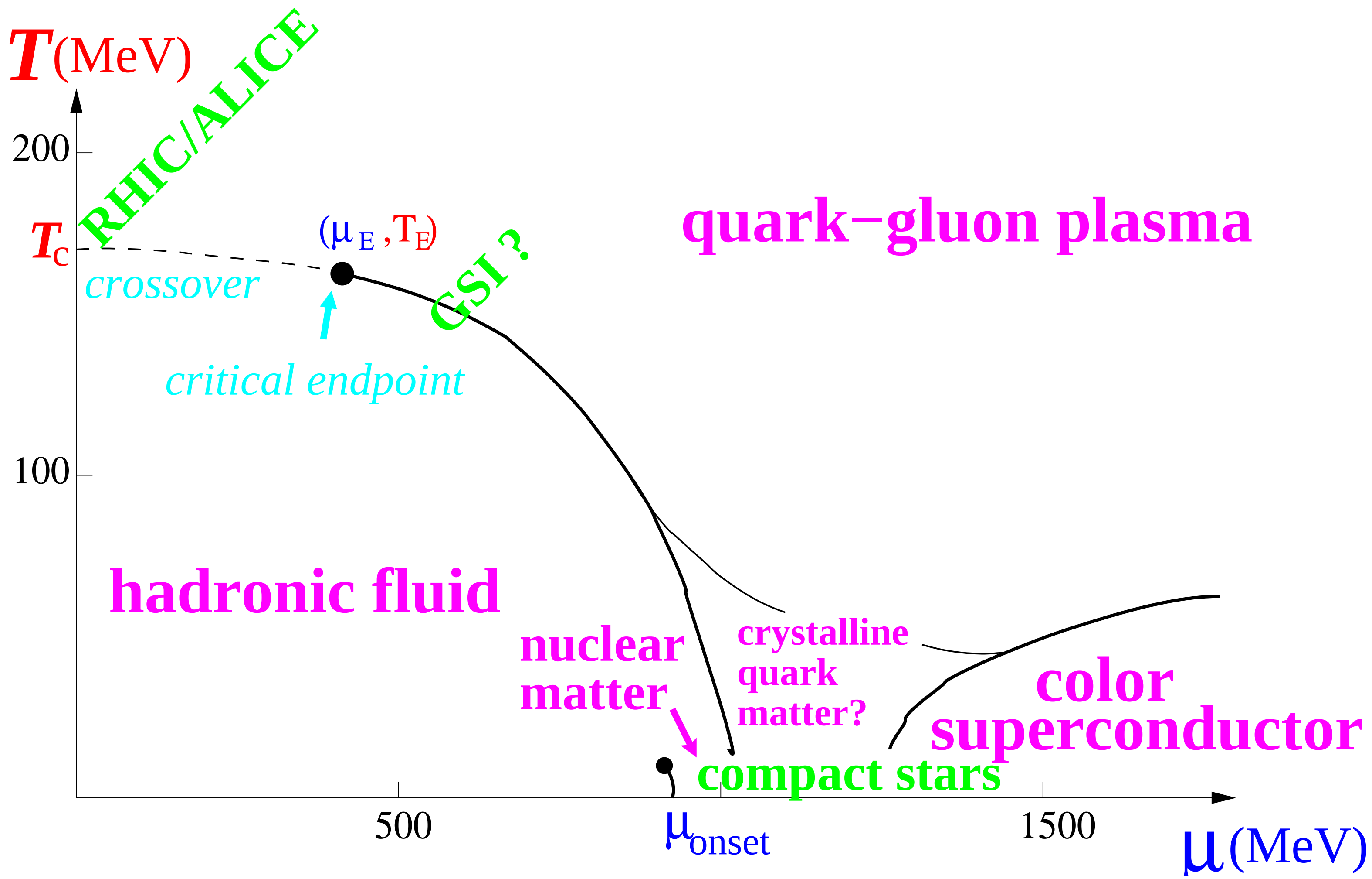
Gavai & Gupta `hep-lat/0412035`



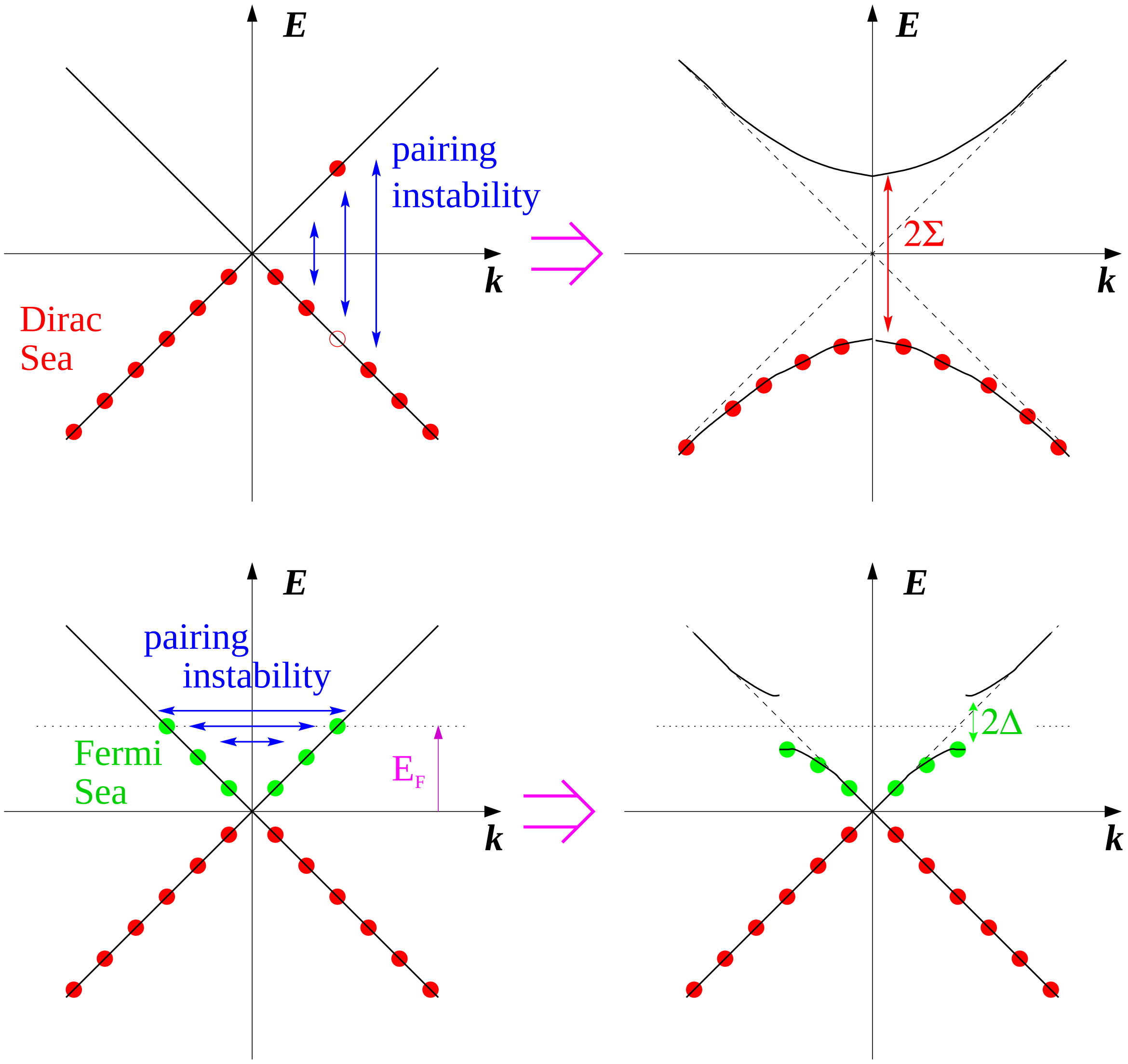
Analytic estimate via Binder cumulant $\langle(\delta\mathcal{O})^4\rangle/\langle(\delta\mathcal{O})^2\rangle^2$ evaluated at imaginary $\mu \Rightarrow \mu_E/T_E \sim O(20)$!

P. de Forcrand & O. Philipsen NPB673(2003)170

The QCD Phase Diagram



χSB vs. Cooper Pairing



Color Superconductivity

In the asymptotic limit $\mu \to \infty$, $g(\mu) \to 0$, the ground state of QCD is the *color-flavor locked* (CFL) state characterised by a BCS instability, [D. Bailin and A. Love, Phys.Rep. 107(1984)325]
ie. diquark pairs at the Fermi surface condense via

$$\langle q_i^\alpha(p) C\gamma_5 q_j^\beta(-p)\rangle \sim \varepsilon^{A\alpha\beta}\varepsilon_{Aij} \times \text{const.}$$

breaking $\text{SU(3)}_c \otimes \text{SU(3)}_L \otimes \text{SU(3)}_R \otimes \text{U(1)}_B \otimes \text{U(1)}_Q$
$\longrightarrow \text{SU(3)}_\Delta \otimes \text{U(1)}_{\tilde{Q}}$

The ground state is simultaneously
superconducting (8 gapped gluons, ie. get mass $O(\Delta)$),

superfluid (1 Goldstone),

and *transparent* (all quasiparticles with $\tilde{Q} \neq 0$ gapped).
[M.G. Alford, K. Rajagopal and F. Wilczek, Nucl.Phys.B537(1999)443]

At smaller densities such that $\mu/3 \sim k_F \lesssim m_s$, expect pairing between u and d only $\Rightarrow$ "2SC" phase

$$\langle q_i^\alpha(p) C\gamma_5 q_j^\beta(-p)\rangle \sim \varepsilon^{\alpha\beta 3}\varepsilon_{ij} \times \text{const.}$$

SU(3)$_c \longrightarrow$ SU(2)$_c \Rightarrow$ 5/8 gluons get gapped
Global SU(2)$_L\otimes$SU(2)$_R\otimes$U(1)$_B$ unbroken

In the electrically-neutral matter expected in compact stars, $k_F^d - k_F^u = \mu_e = -2\mu_I \sim 100$MeV $\Rightarrow \langle qq\rangle$ condensate can have $\vec{k} \neq 0$ breaking translational invariance $\Rightarrow$ *crystallisation*

Other ideas: a 2SC/normal mixed phase (plates? rods?) or a gapless 2SC where $\langle qq\rangle \neq 0$ but $\Delta = 0$?

The most urgent issue of all – whether quark matter exists in our universe – requires quantitative knowledge of the EOS $p(\mu), \epsilon(\mu)$ for all $\mu > \mu_o$

Four Fermi Models with $\mu \neq 0$

Effective description of soft pions interacting with constituent quarks

$$\begin{aligned}\mathcal{L}_{NJL} &= \bar{\psi}(\partial\!\!\!/ + m + \mu\gamma_0)\psi - \frac{g^2}{2}[(\bar{\psi}\psi)^2 - (\bar{\psi}\gamma_5\vec{\tau}\psi)^2] \\ &\sim \bar{\psi}(\partial\!\!\!/ + m + \mu\gamma_0 + \sigma + i\gamma_5\vec{\pi}.\vec{\tau})\psi + \frac{2}{g^2}(\sigma^2 + \vec{\pi}.\vec{\pi})\end{aligned}$$

Full global symmetry is SU(2)$_L \otimes$SU(2)$_R \otimes$U(1)$_B$

Dynamical χSB for $g^2 > g_c^2 \Rightarrow$
isotriplet Goldstone $\vec{\pi}$, constituent quark mass $\Sigma \gg m$

Scalar isoscalar diquark $\psi^{tr}C\gamma_5 \otimes \tau_2 \otimes A^{color}\psi$ breaks U(1)$_B$
$\Rightarrow$ diquark condensation signals superfluidty

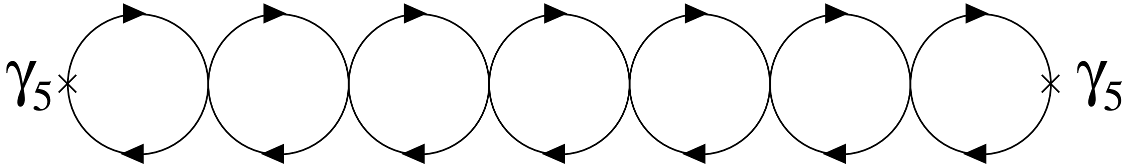
Lattice four-fermi models:

Preserve QCD-like symmetries and have either an interacting continuum limit (2+1d) or can be considered a cutoff effective theory (eg. $a^{-1} \approx 700$MeV for NJL in 3+1d)

Exhibit spontaneous χSB at low μ, but no confinement physics, and no sign of any "nuclear matter" phase where both $\langle\bar{\psi}\psi\rangle > 0$ and $n_B > 0$

$\Rightarrow$ for $\mu > \mu_c \sim \Sigma$ describe "relativistic quark matter"

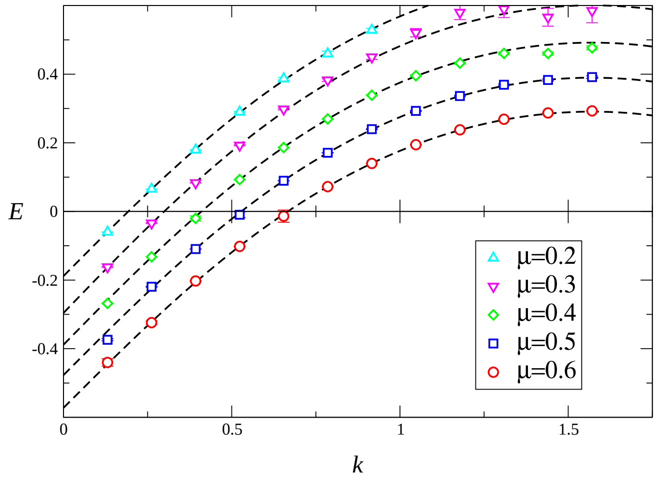
Simulable with $\mu > 0$ because the Goldstone channel dominated by



which are only available to $q\bar{q}$, and not qq^c

Can break U(1)$_B$ with a real measure because excluded from Vafa-Witten theorem

Fermion Dispersion relation



μ	K_F	β_F	$K_F/\mu\beta_F$
0.2	0.190(1)	0.989(1)	0.962(5)
0.3	0.291(1)	1.018(1)	0.952(4)
0.4	0.389(1)	0.999(1)	0.973(1)
0.5	0.485(1)	0.980(1)	0.990(2)
0.6	0.584(3)	0.973(1)	1.001(2)

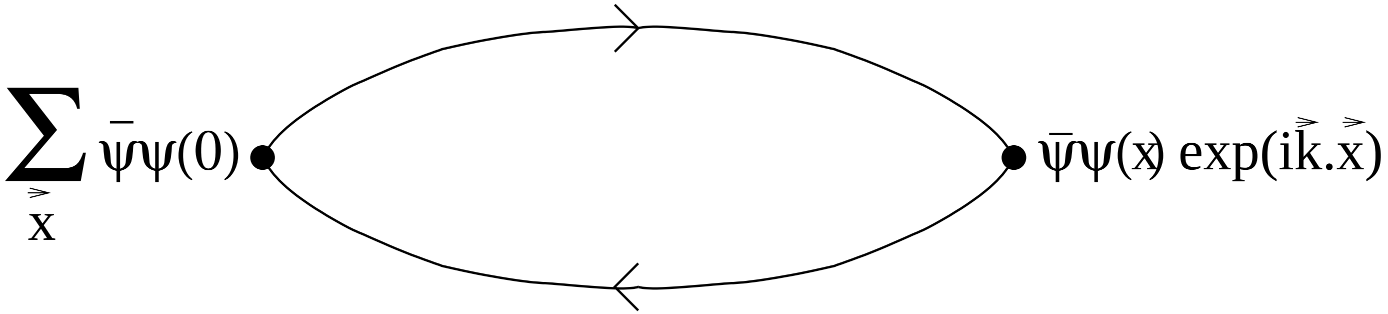
The fermion dispersion relation is fitted with

$$E(|\vec{k}|) = -E_0 + D \sinh^{-1}(\sin|\vec{k}|)$$

yielding the Fermi liquid parameters

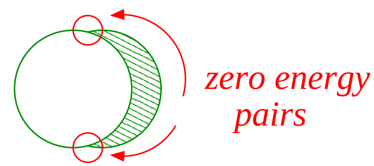
$$K_F = \frac{E_0}{D}; \qquad \beta_F = D\frac{\cosh E_0}{\cosh K_F}$$

Meson Correlation Functions (2+1d)



For $\vec{k} \neq 0$ can always excite a particle-hole pair with almost zero energy $\Rightarrow$ algebraic decay of correlation functions

$|\vec{k}| \ll \mu$

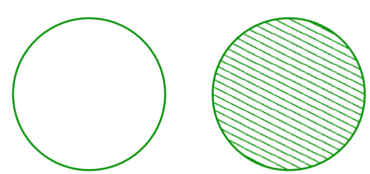


$\Rightarrow C \sim \frac{1}{x_0^2}$

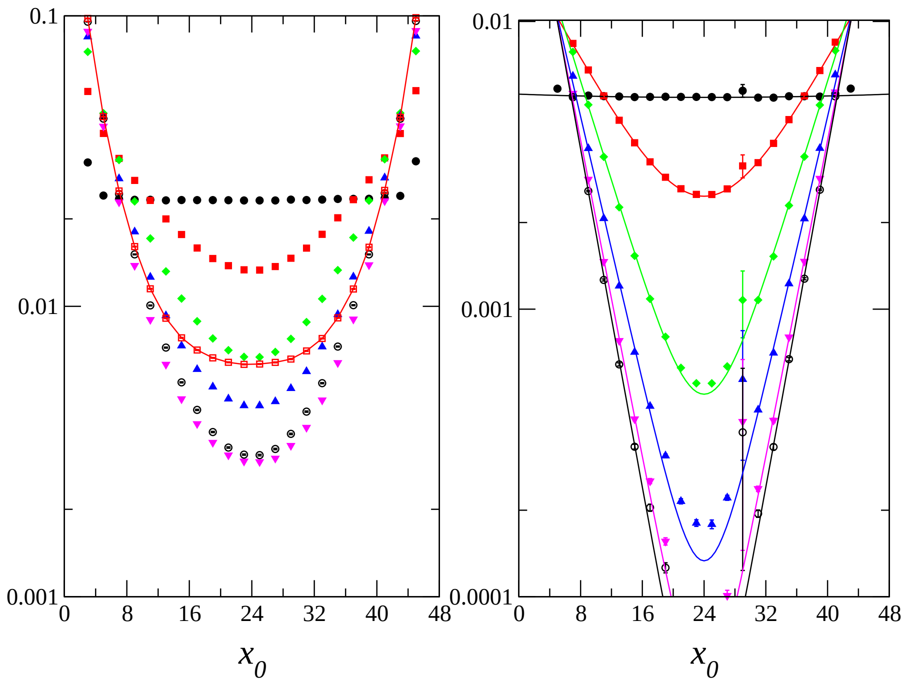
$|\vec{k}| = 2\mu$

$\Rightarrow C \sim \frac{1}{x_0^{3/2}}$

$|\vec{k}| > 2\mu$

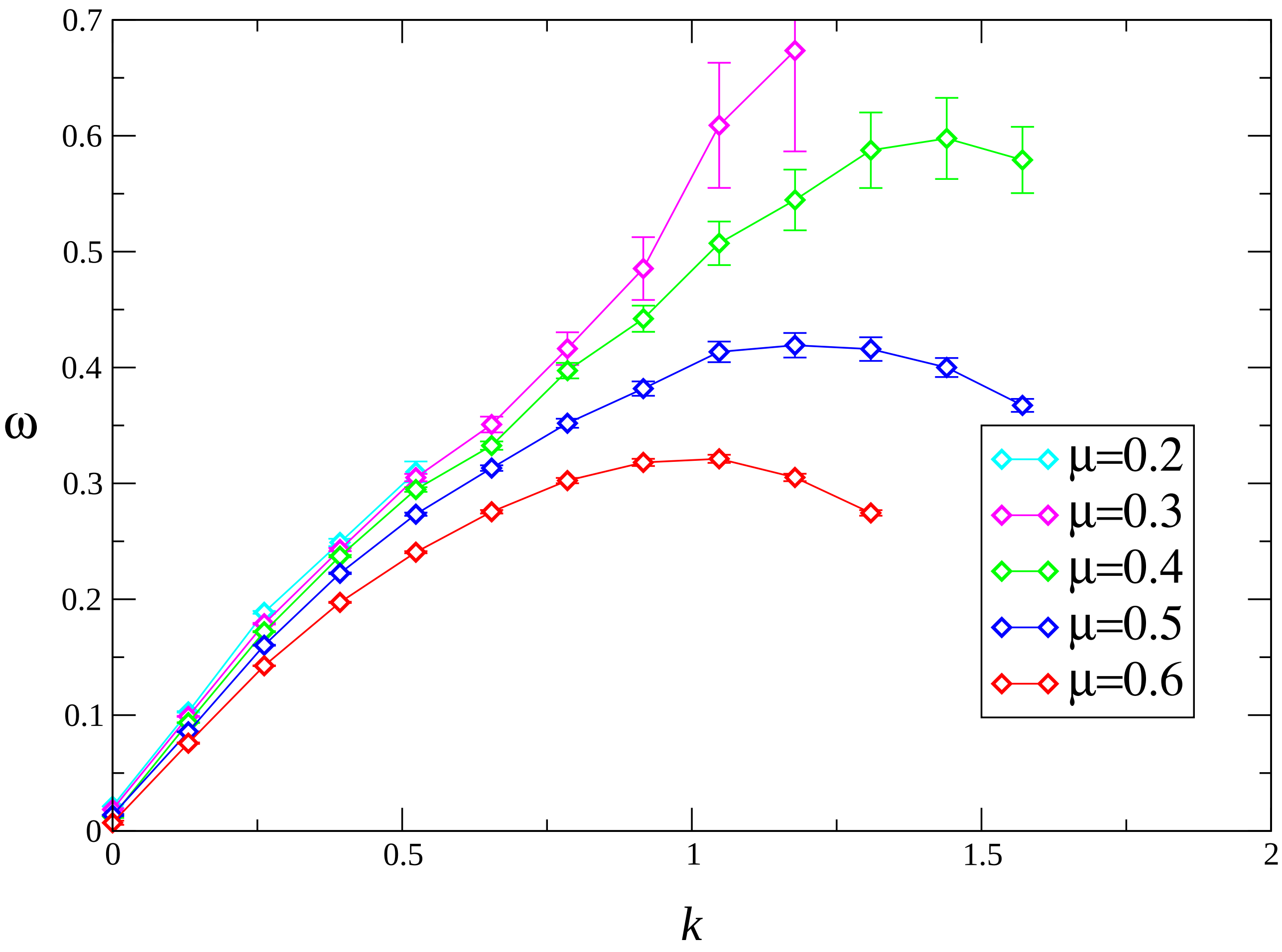


$\Rightarrow C \sim \frac{e^{-(|\vec{k}|-2\mu)x_0}}{x_0^{3/2}}$



eg. in the spin-1 channel at $\mu a = 0.6$, $C_{\gamma_\perp}$ (left) looks algebraic as predicted by free field theory, but $C_{\gamma_{||}}$ (right) decays exponentially.

The interpolating operator for $C_{\gamma_{||}}$ in terms of continuum fermions is $\bar{q}(\gamma_0 \otimes \tau_2)q$
ie. with same quantum numbers as baryon charge density



Dispersion relation $\omega(|\vec{k}|)$ extracted from meson channel interpolated by an operator $\bar{\psi}(\gamma_0 \otimes \tau_2)\psi$

A massless vector excitation?

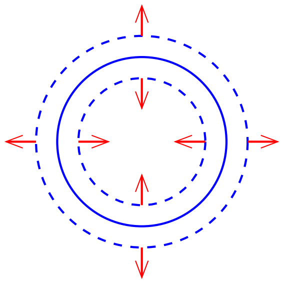
SJH & C.G. Strouthos PRD70(2004)056006

Sounds Unfamiliar?

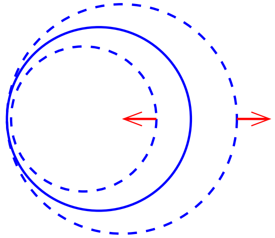
Light vector states in medium are of of great interest:

Brown-Rho scaling, vector condensation. . .

In the Fermi liquid framework a possible explanation is a *collective excitation* thought to become important as $T \to 0$: *Zero Sound*

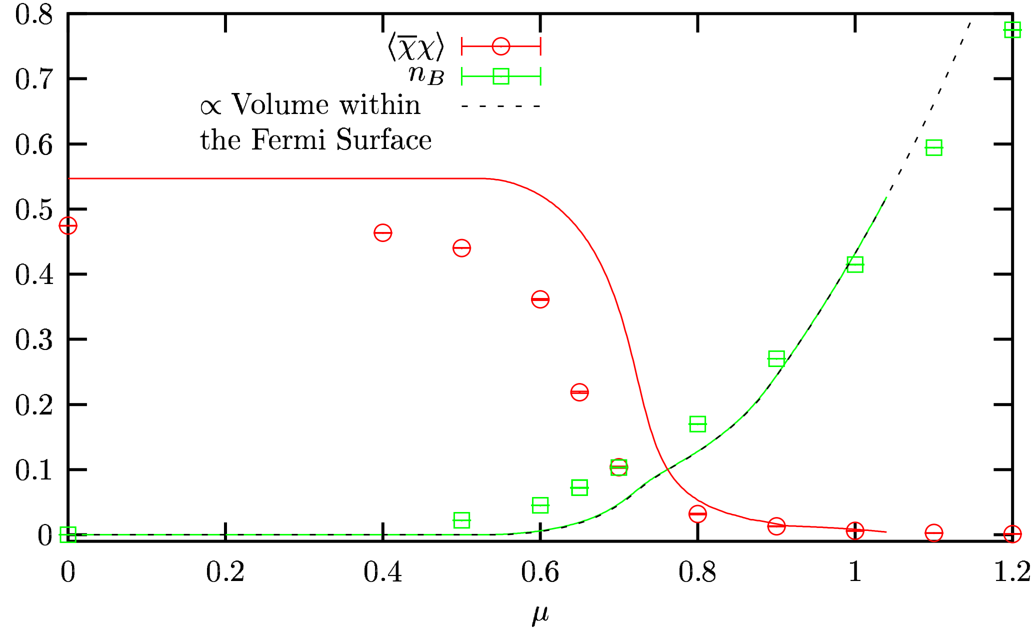


Ordinary FIRST sound is a breathing mode of the Fermi surface: velocity $\beta_1 \simeq \frac{1}{\sqrt{2}} \frac{k_F}{\mu}$

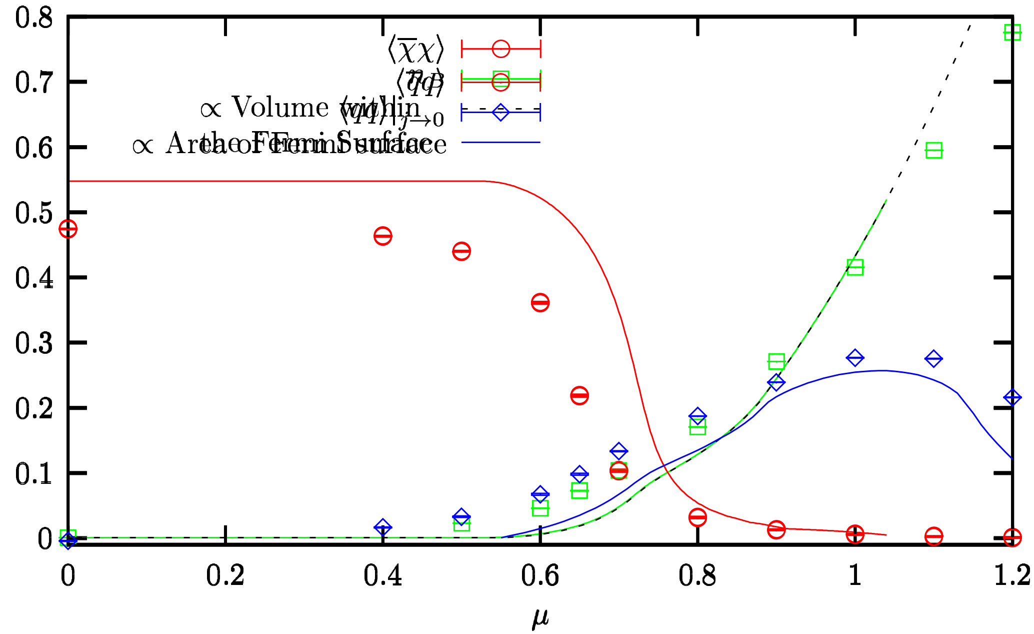


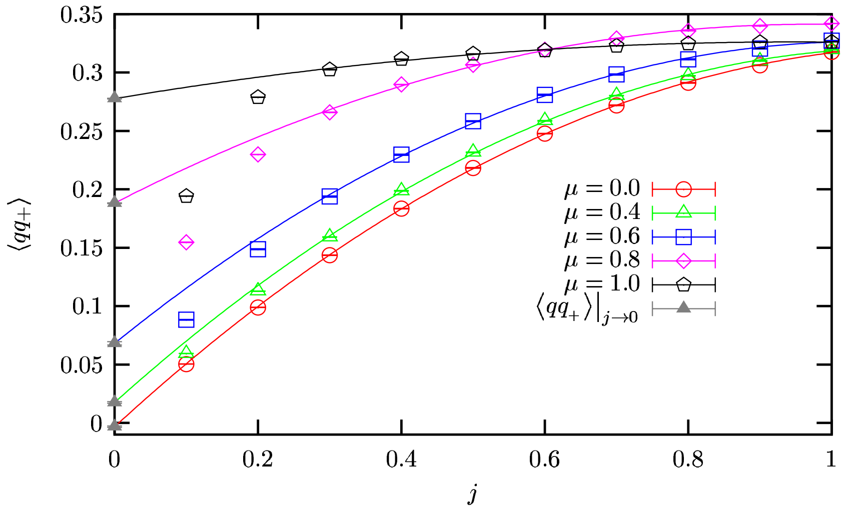
ZERO sound is a propagating distortion of the Fermi surface: velocity $\beta_0 \sim \beta_F$ must be determined self-consistently

Diquark Condensation (3+1d)



Diquark Condensation (3+1d)





Add source $j[\psi^{tr}\Gamma_A\psi + \bar{\psi}\Gamma_A\bar{\psi}^{tr}]$

Diquark condensate estimated by taking $j \to 0$

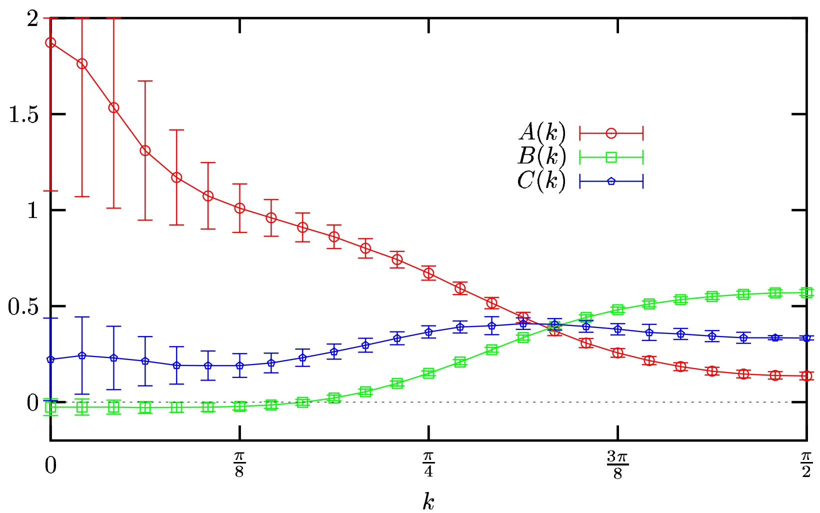
Our fits exclude $j \leq 0.2$

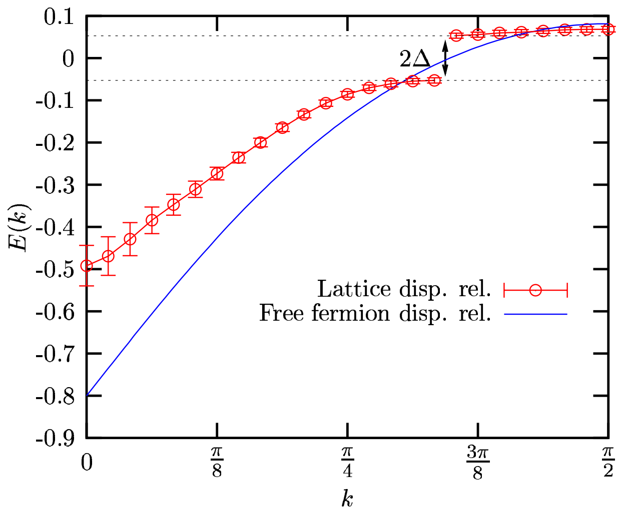
The Superfluid Gap

Quasiparticle propagator:

$$\langle\psi_u(0)\bar{\psi}_u(t)\rangle = Ae^{-Et} + Be^{-E(L_t-t)}$$
$$\langle\psi_u(0)\psi_d(t)\rangle = C(e^{-Et} - e^{-E(L_t-t)})$$

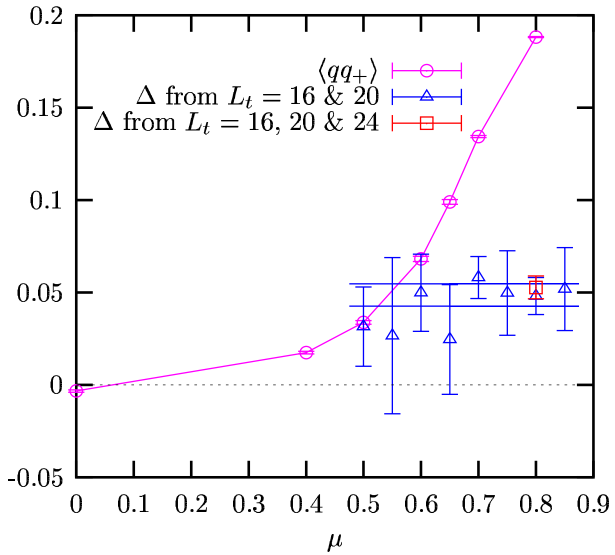
Results from $96 \times 12^2 \times L_t$, $\mu a = 0.8$ extrapolated to $L_t \to \infty$ (ie. $T \to 0$) then $j \to 0$





The gap at the Fermi surface signals superfluidity

SJH & D.N. Walters PLB548(2002)196 PRD69(2004)076011



- $\Delta/\Sigma_0 \simeq 0.15 \Rightarrow \Delta \simeq 60$MeV
in agreement with self-consistent approaches

- Similar formalism to study non-relativistic model for
EITHER nuclear matter (with or without pions)
$\Rightarrow$ calculation of E/A D. Lee & T. Schäfer `nucl-th/0412002`
OR Cold atoms with tunable scattering length
$\Rightarrow$ study of BEC/BCS crossover M. Wingate `cond-mat/0502372`

In either case non-perturbative due to large dimensionless parameter $k_F|a| \gg 1$, with a the s-wave scattering length.

N.B. $\mu_I \neq 0$ or $m_\pi < \infty$ reintroduces sign problem!

Cold Quarks, Hot Glue...

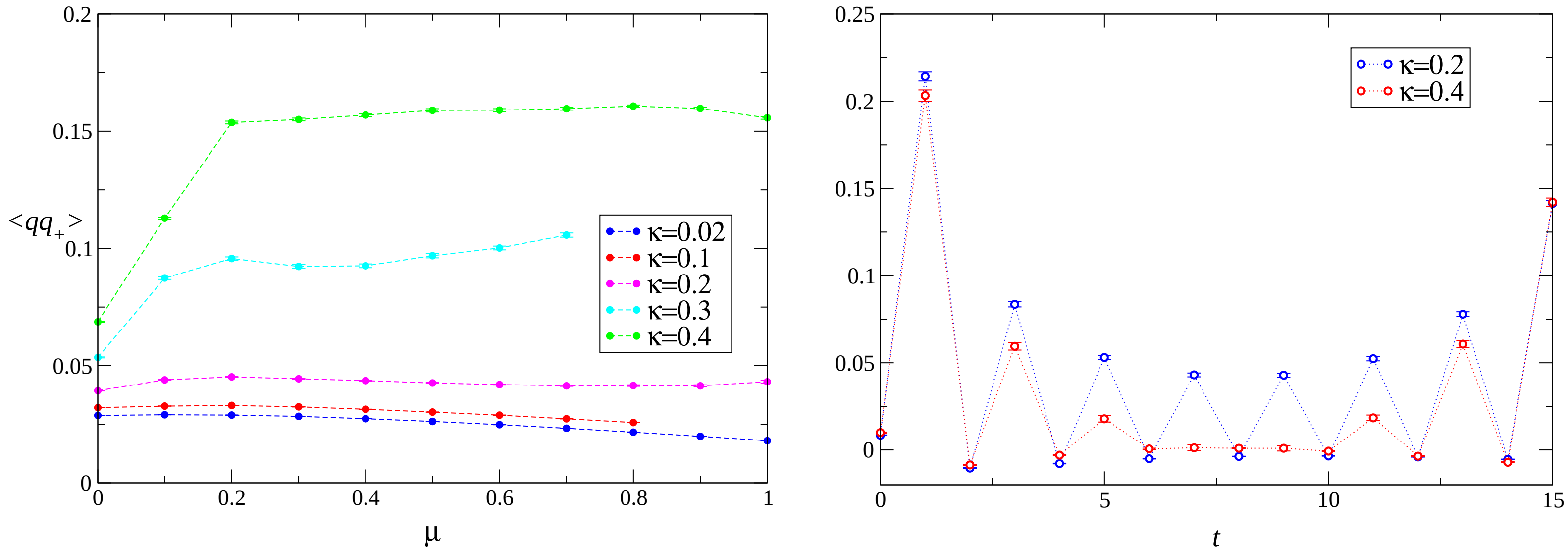
NJL models permit study of Fermi surface – Zero Sound, perhaps the most interesting effect, is expected in systems with *short-ranged* interactions, but not in gauge theories.

Conjugate quarks supposedly invalidate the quenched approximation (Gocksch, Stephanov) – arguments assume tightly bound qq^c states resulting from confinement.

What if we could generate cold, non-confining gluon configurations?

We have tried generating deconfining 3d configurations using the Dimensional Reduction approach to hot QCD, then "reconstructing" the timelike direction to permit quenched inversions of $(M(\mu)+m)$

Resulting model contains only static modes, so NOT a systematic effective description of high densities.



The DR model is $3d$ SU(2) gauge-Higgs with parameters β, κ, λ

Hart & Philipsen, NPB 572 (2000) 243

Large κ yields results similar to Two Color QCD

Small κ qualitatively different – no variation of $\langle qq_+\rangle$ with μ

Gauge-fixed quark propagator $\mathcal{G}(\vec{k},t)$ indicates restored chiral symmetry (sawtooth) and exponential decay $(\Delta > 0)$

Intriguingly, $\mathcal{G}$ has no significant $\vec{k}$-dependence $\Rightarrow$ no means of identifying k_F or the Fermi surface

What is the gauge invariant signal for a Fermi surface?

Are we using the right basis?

Large cancellations between diagrams/configurations hint at low calculation efficiency. Maybe gauge covariant quarks and gluons not natural degrees of freedom at high density?

Intriguing 3d example: approximate duality between scalar QED and complex scalar field theory

Kajantie, Laine, Neuhaus, Rajantie & Rummukainen NPB 699 (2004) 632

$$\mathcal{L}_{SQED} = \frac{1}{4}F^2 + |D\phi|^2 + m^2|\phi|^2 + \lambda|\phi|^4 - \frac{1}{2}\varepsilon_{ijk}H_i F_{jk}$$

H is a real source term coupled to a real B-field

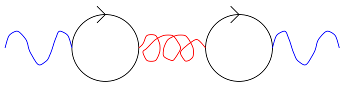
$$\mathcal{L}_{SFT} = [(\partial - \tilde{e}H)_k\tilde{\phi}^*][(\partial + \tilde{e}H)_k\tilde{\phi}] + \tilde{m}^2|\tilde{\phi}|^2 + \tilde{\lambda}|\tilde{\phi}|^4 + \cdots$$

$\tilde{e}H_3$ with $\tilde{e} = 2\pi/e$ is a real chemical potential for the conserved charge density $2\mathrm{Im}(\tilde{\phi}^*\partial_3\tilde{\phi})$

Duality exact at Coulomb/Higgs $\Leftrightarrow$ broken/symmetric phase transition

What is the physical origin of the sign problem? eg:

- Two Color QCD with $N=1$ adjoint staggered quarks – superconductor at large μ?
- Repulsive Hubbard model away from half-filling – model of cuprate superconductivity?
- Technicolor – chiral fermions in complex representations
- "τ_3-QED" describing planar superconductivity by giving the photon a mass via a mixed Chern-Simons term



$\mathrm{det}M \neq \mathrm{det}M^* \text{ since } \{\gamma_5, \not{D}\} \neq 0$

Dorey & Mavromatos NPB 386 (1992) 614

- QCD itself?

Conjecture: *any* system exhibiting spontaneous breaking of a local symmetry by a pairing mechanism has a sign problem when formulated in terms of local gauge covariant degrees of freedom

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And finally. . .