RESUMMATION FOR PRECISION

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Stress-testing the Standard Model at the LHC
KITP Santa Barbara
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PERTURBATIVE QCD CALCULATIONS

- High-precision theoretical predictions needed for the LHC
  - NLO calculations in QCD are now standard
  - NNLO exists for an increasing number of processes
  - NNNLO has been completed for Higgs

$$\sigma(x,Q^2) = \sigma_{\rm LO}(x,Q^2) + \alpha_s \sigma_{\rm NLO}(x,Q^2) + \alpha_s^2 \sigma_{\rm NNLO}(x,Q^2) + \alpha_s^3 \sigma_{\rm NNNLO}(x,Q^2) + \ldots$$

- Many observables at LHC characterized by multiple scales Q_i
- Multi-scale problems are affected by perturbative logarithmic corrections $\alpha_s^n \log^m(Q_i/Q_j)$
- When $\alpha_s^n \log^m(Q_i/Q_j) \sim 1$ fixed order PT is no longer justified

WHERE DO LOGARITHMS COME FROM ?

- Real emissions diagrams are singular for soft/collinear emissions
- These singularities are cancelled by virtual counterparts
- Finite logarithmic pieces are left over, e.g.

$$V = \kappa$$

$$\alpha_s \int \frac{d\theta}{\theta}\frac{dz}{z}\Theta\left(\kappa - z\theta\right) - \alpha_s \int \frac{d\theta}{\theta}\frac{dz}{z} = -\alpha_s \int \frac{d\theta}{\theta}\frac{dz}{z}\left[\Theta\left(z\theta - \kappa\right)\right]$$

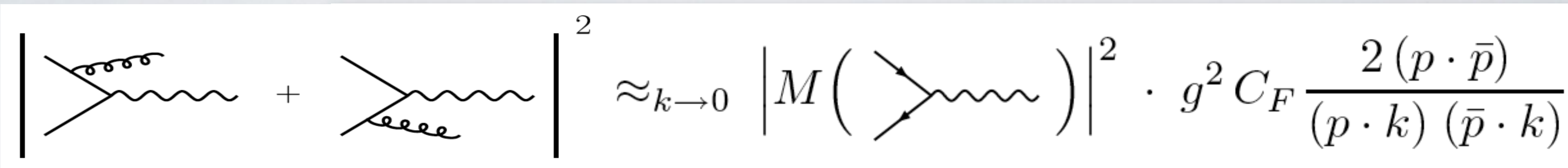
real emission **virtual correction**

$$= -\frac{1}{2}\alpha_s \ln^2 \kappa$$

- These corrections are important for observables *V* that insist on only small deviations from lowest order kinematics (V~0)
- Real radiation is constrained to a small corner of phase space and the logarithms are large
  - event (jet) shapes, e.g. thrust (jet mass): $V=1-T$ ($V= m_{jet}/p_T$)
  - production at threshold: $V=1-M^2/s$
  - transverse momentum: $V= p_T/M$ ...

RESUMMATION: A SKETCH

- All-order calculations are based on factorization
  - Matrix element factorization in soft/collinear limit



Born dipole

  - this can be generalized to the multi-gluon case
  - phase space factorization usually in a conjugate space, e.g.

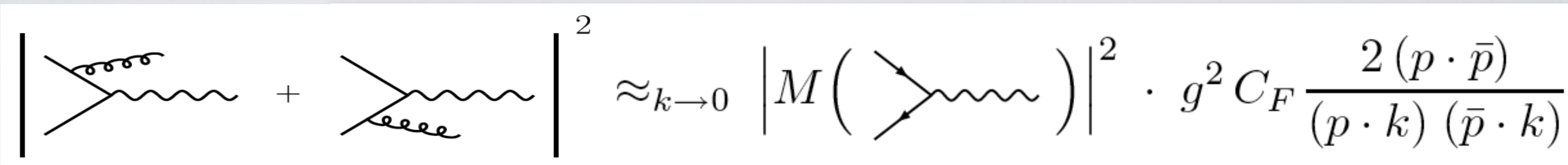
$$\delta^{(2)}\left(\sum_{i=1}^{n} \underline{k}_{Ti} + \underline{Q}_T\right) = \frac{1}{(2\pi)^2}\int d^2\underline{b}\, e^{i\underline{b}\cdot\underline{Q}_T} \prod_{i=1}^{n} e^{i\underline{b}\cdot\underline{k}_{Ti}}$$

  - although other approaches are possible (e.g. CAESAR framework)
- factorization then leads to exponentiation

$$\sigma_{res} = g_0 \exp[L g_1(\alpha_s L) + g_2(\alpha_s L) + \alpha_s g_3(\alpha_s L) + \dots]$$

RESUMMATION: A SKETCH

- All-order calculations are based on factorization
  - Matrix element factorization in soft/collinear limit



Born dipole

- this can be g
- phase sp ce, e.g.

resummation is a systematic re-arrangement of perturbation theory

$\delta^{(2)}\left(\sum \underline{k}_{Ti} \ldots \underline{q}_T\right) \ldots \underline{q}_T \prod e^{i\underline{b}\cdot\underline{k}_{Ti}}$

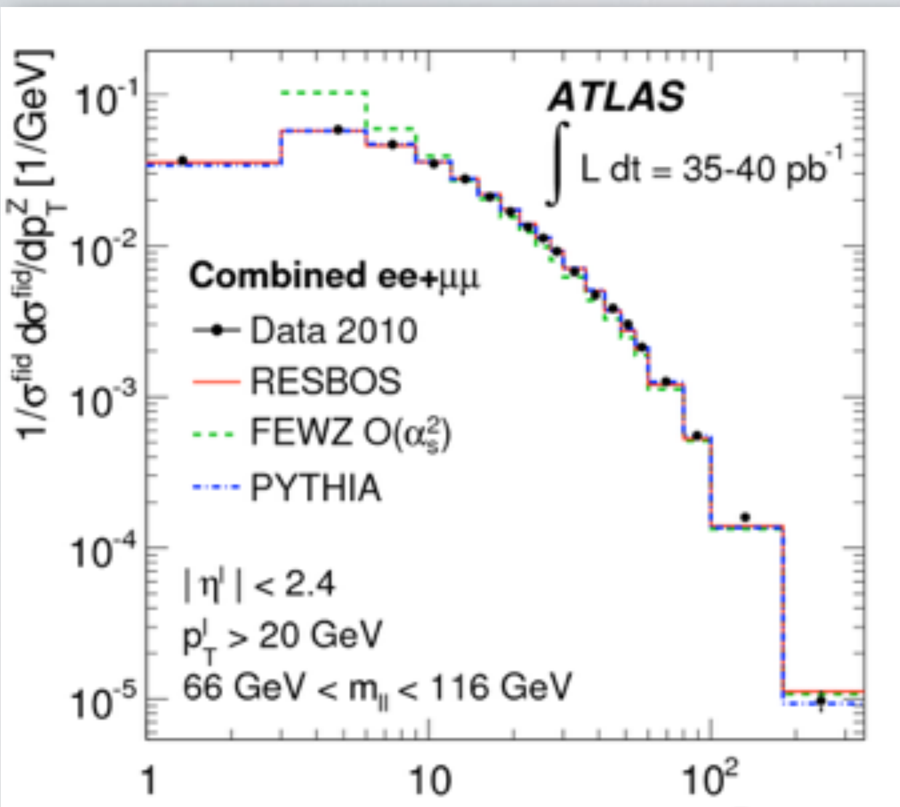
LL NLL NNLL

- although ot proaches a le (e.g. C ramework)

- factorization then leads to exponentiation

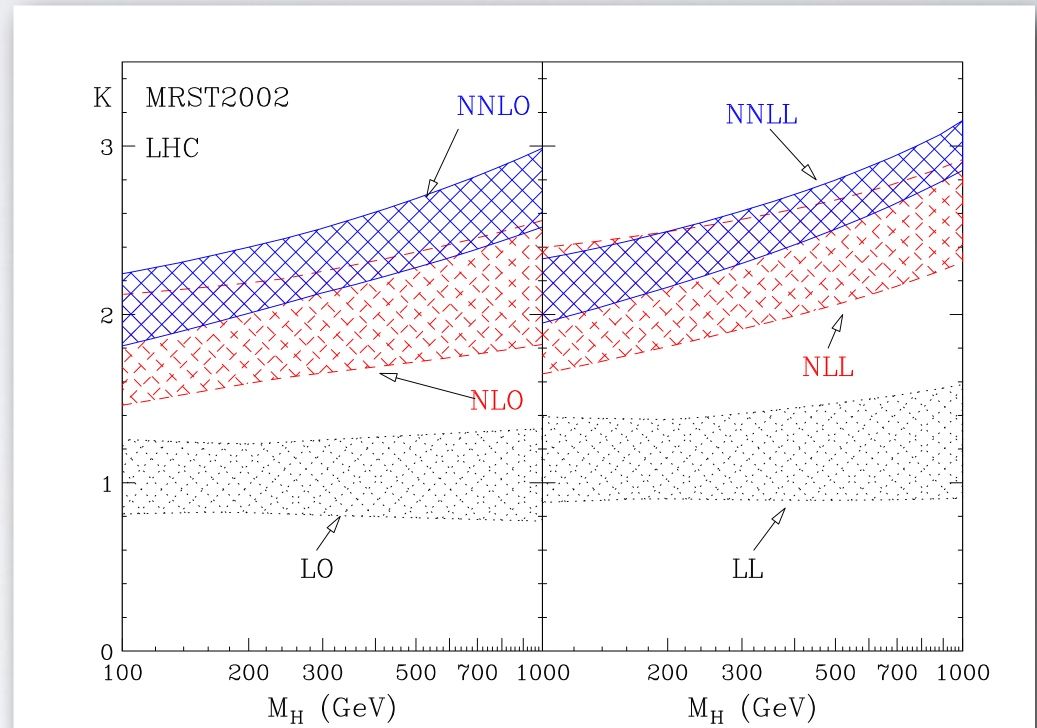
$\sigma_{res} = g_0 \exp[L g_1(\alpha_s L) + g_2(\alpha_s L) + \alpha_s g_3(\alpha_s L) + \ldots]$

RESUMMATION IN ACTION



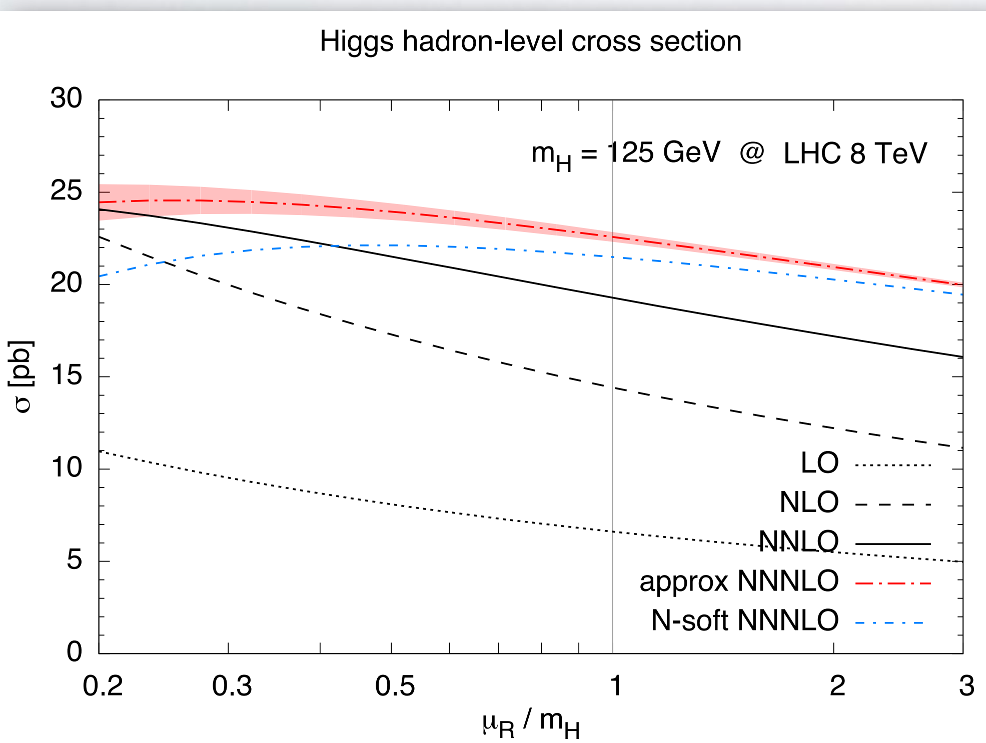
Phys.Lett. B705 (2011) 415-434

1) it's necessary for describing data in particular kinematic limits



Catani, de Florian, Grazzini, Nason (2003)

2) it reduces theoretical uncertainty

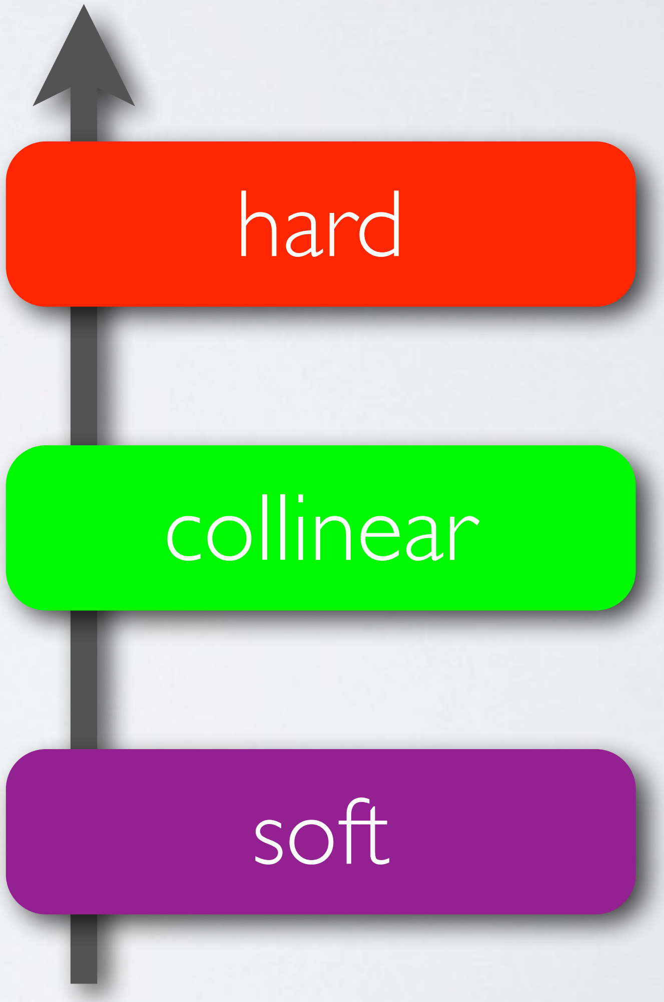


Ball, Bonvini, Forte, SM, Ridolfi (2013)

3) it can be used to approximate higher orders

SOFT COLLINEAR EFFECTIVE THEORY

- Direct QCD (dQCD) resummation based on factorization of QCD matrix elements and phase space in the soft/collinear limit
- An alternative framework for resumming large logs is SCET
- In SCET
    - hard modes are integrated out
    - effective Lagrangian for soft & collinear fields
    - separation of scales leads to factorisation
    - resummation is achieved by RG evolution



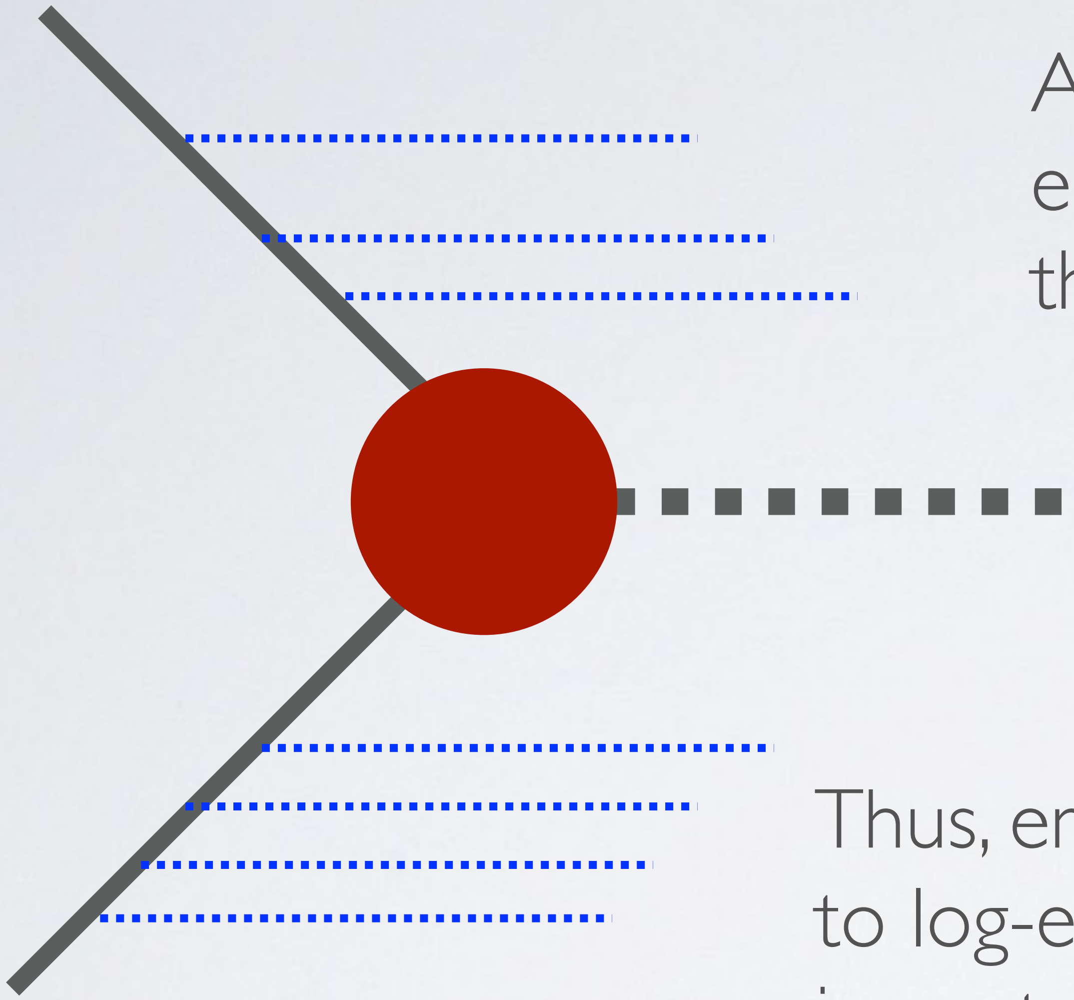
OUTLINE

- Hadronic final-state (jets) resummation discussed in J. Thaler's and A. Larkoski's talks
- Here, I will concentrate on other aspects, trying to underlying recent directions in
  - large-*x* resummation
  - small-*x* resummation
  - Q_T resummation
- A few words on combining resummation

Topic is vast, so a lot of personal taste went into selecting the material
(*excusatio non petita, accusatio manifesta*)

LARGE-X RESUMMATION

PRODUCTION AT THRESHOLD



Absolute threshold: the initial-state energy is just enough to produce the final state with invariant mass Q

$$x = \frac{Q^2}{s} \to 1$$

Thus, emissions are forced to be soft, leading to log-enhanced contributions order-by-order in perturbation theory

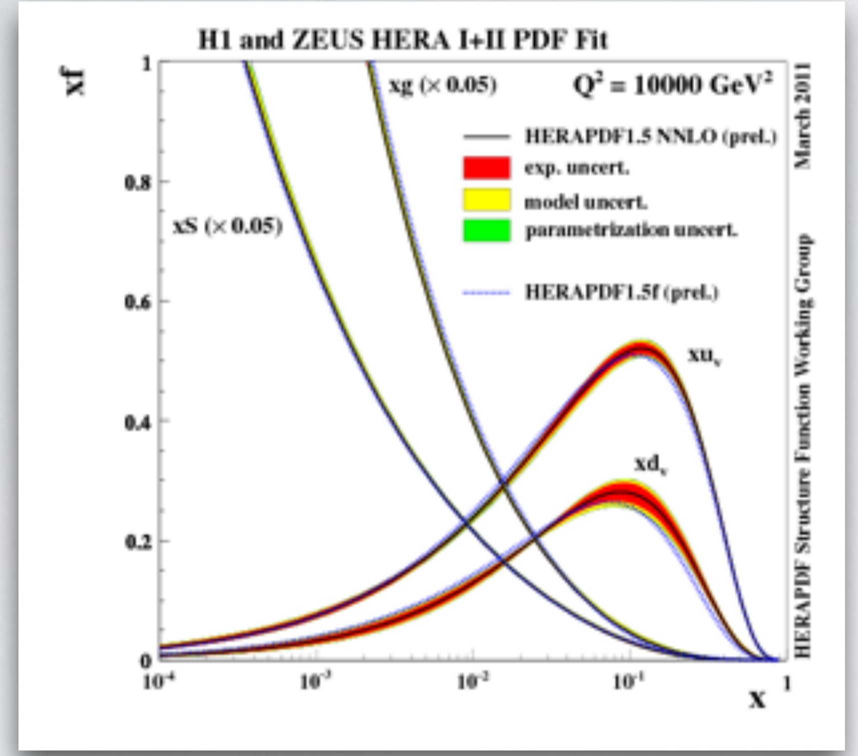
$$\mathrm{LO} : Q^2 = \hat{s}$$

$$\text{beyond LO} : Q^2 = z\hat{s}$$

$$C(z, \alpha_s) \sim \sigma_0 \sum_{n=1} \sum_{k=-1}^{2n-1} \alpha_s^n \left[\frac{\ln^k(1-z)}{1-z} \right]_+$$

$k=-1$: $\delta(1-z)$

WHY THRESHOLD AT THE LHC ?



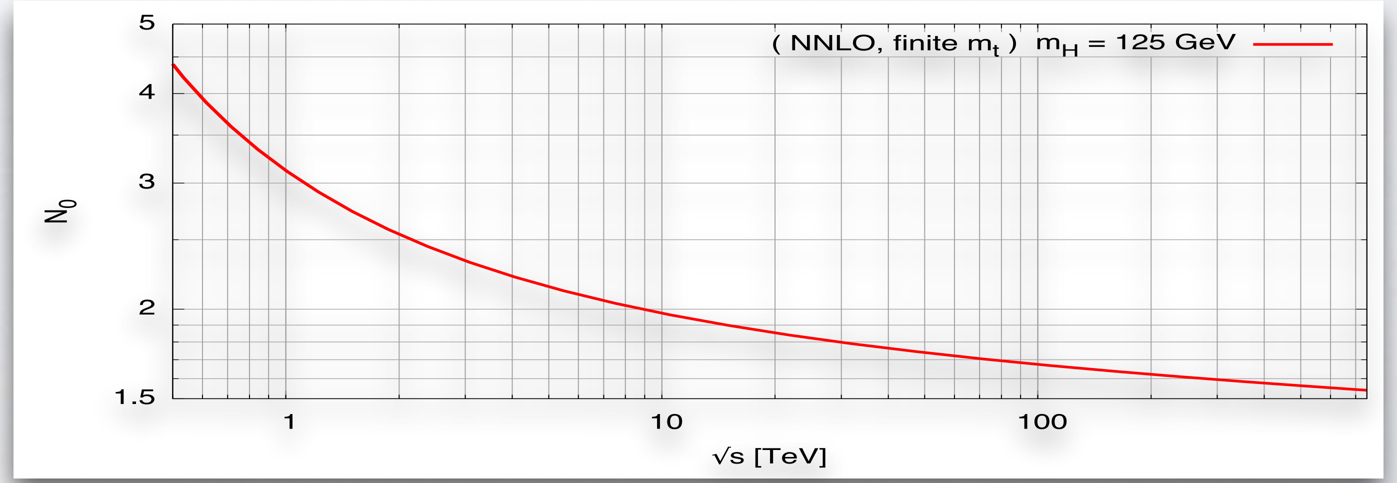
Gluon PDF shows a steep increase at low *x*

$$\hat{s} = x_1 x_2 s$$

Region of partonic threshold is enhanced in the convolution

- More precise argument in Mellin space: a saddle-point approximation indicates the region that gives the bulk of the contribution to the inverse Mellin integral.
- This region turns out to be fairly narrow around the (real) saddle-point.

Bonvini, Forte Ridolfi (2012)



THRESHOLD RESUMMATION

Momentum space: singular and distributional terms for $z \to 1$
Mellin space: terms that do not vanish at large N

$$C_{\rm res}(N,\alpha_s) = \bar{g}_0\big(\alpha_s,\mu_{\rm F}^2\big)\,\exp\bar{\mathcal{S}}(\alpha_s,N),$$

$$\bar{\mathcal{S}}(\alpha_s,N) = \int_0^1 dz\,\frac{z^{N-1}-1}{1-z}\left(\int_{\mu_{\rm F}^2}^{m_{\rm H}^2\frac{(1-z)^2}{z}}\frac{d\mu^2}{\mu^2}2A\big(\alpha_s(\mu^2)\big) + D\big(\alpha_s([1-z]^2 m_{\rm H}^2)\big)\right),$$

$$\bar{g}_0(\alpha_s,\mu_{\rm F}^2) = 1 + \sum_{k=1}^{\infty}\bar{g}_{0,k}(\mu_{\rm F}^2)\alpha_s^k,$$

Anastasiou *et al.* (2014)

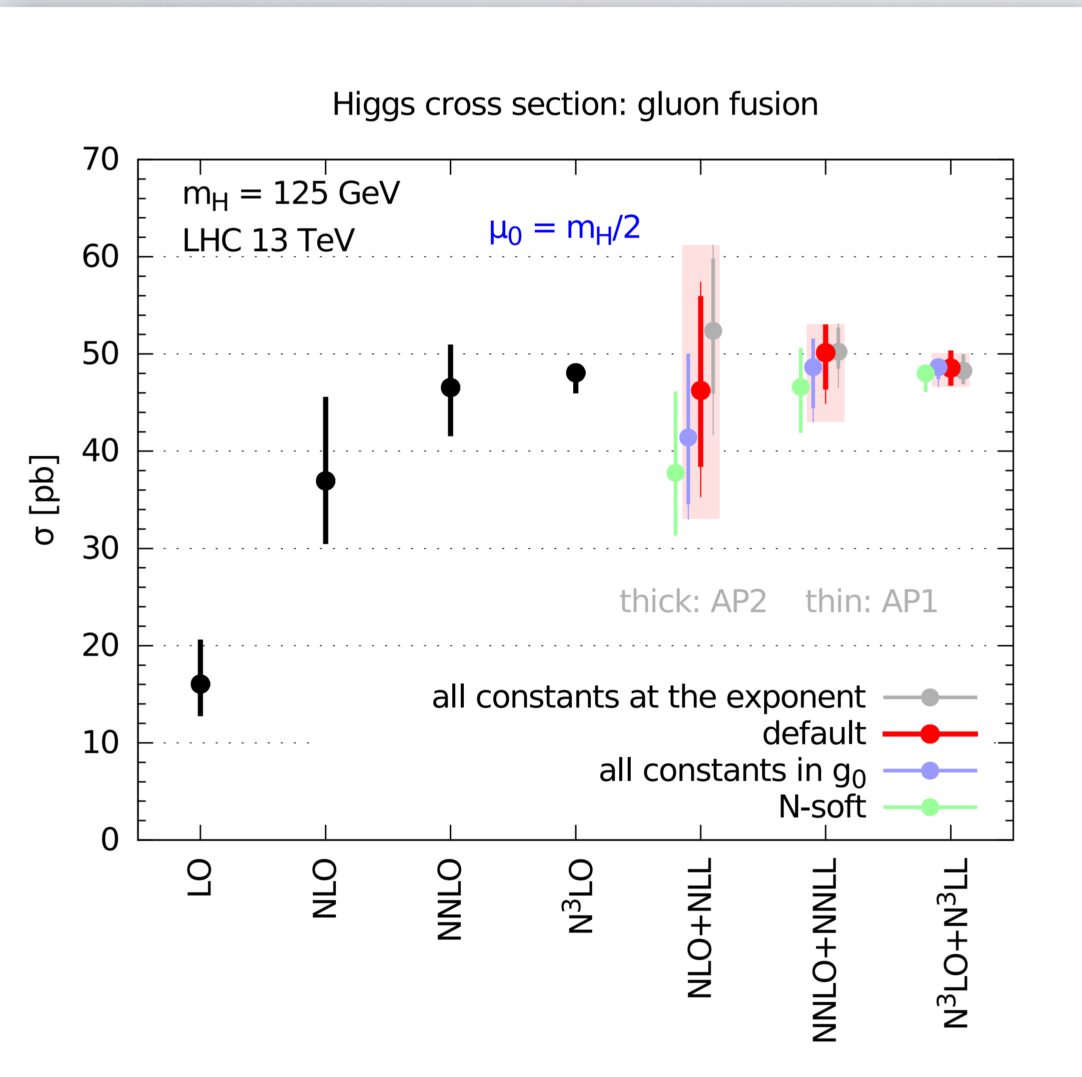
$$A(\alpha_s) = \sum_{k=1}^{\infty}A_k\alpha_s^k, \qquad D(\alpha_s) = \sum_{k=1}^{\infty}D_k\alpha_s^k,$$

Catani *et al.* (2002)
Moch, Vogt (2005)
Laenen, Magnea (2005) [...]

- constants can go in the exponent of in front of it
- state of the art N^{3}LL (but the 4-loop cusp)
- next-to-eikonal can be important (e.g. $(1-z)^2/z$)
- systematic studies underway

Laenen *et al.* (2015),
Larkoski, Neill, Stewart (2015)

AN EXAMPLE: HIGGS IN GLUON FUSION



N^{3}LO: Anastasiou *et al.* (2015)

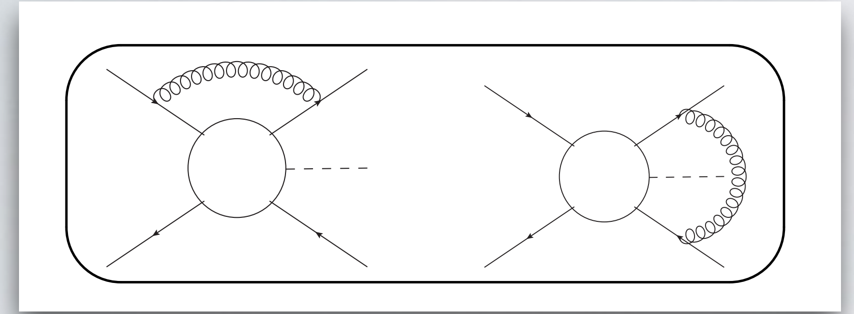
- resummed (and matched) converges faster than pure FO
- resummation is perturbative, i.e. captures the effect of the first few orders, so that N^{3}LO+N^{3}LL ~ N^{3}LO
- they provide further handles to estimate uncertainty from missing higher orders (e.g. subleading logs)

order	σ [pb]	
N^{3}LO	$48.1^{+0.1}_{-1.8}$	scale variation
N^{3}LO	48.1 ± 2.0	$\overline{\text{CH}}$ at 95% DoB
N^{3}LO+N^{3}LL	48.5 ± 1.9	scale+resummation variations
all-order estimate	48.7	from accelerated fixed-order series
all-order estimate	48.9	from accelerated resummed series

Bonvini, SM (2014),
Bonvini, SM, Muselli, Rottoli (2016)

see also Catani *et al.* (2014), Ahmed *et al.* (2014/2015), Schmidt and Spira (2015), …
also Becher *et al.* in SCET

A SECOND EXAMPLE: *ttH*



$$z = \frac{M^2}{\hat{s}} \to 1$$

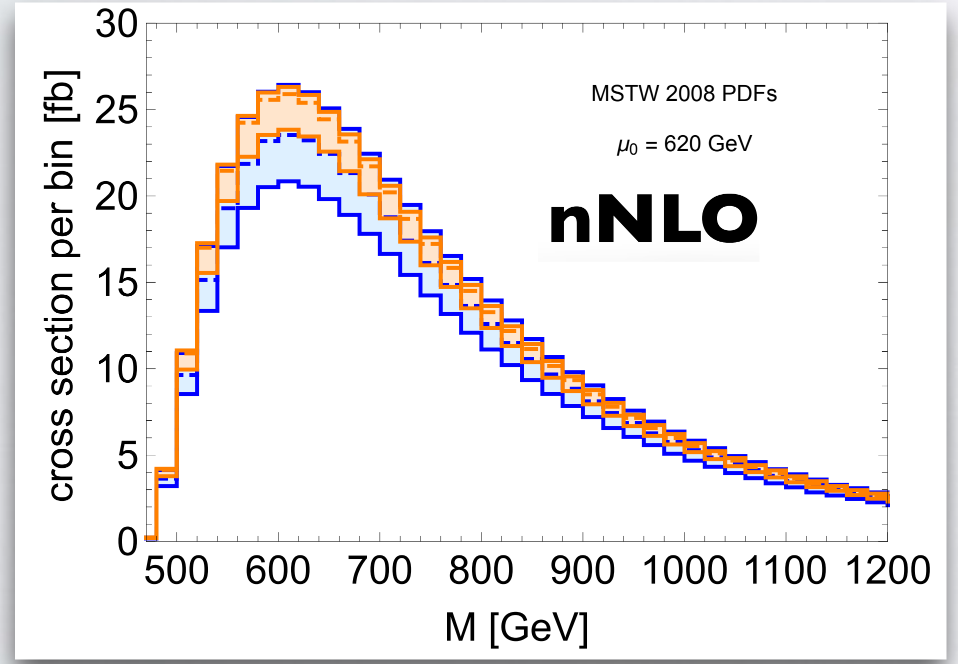
$M^2 = (p_3 + p_4 + p_5)^2$

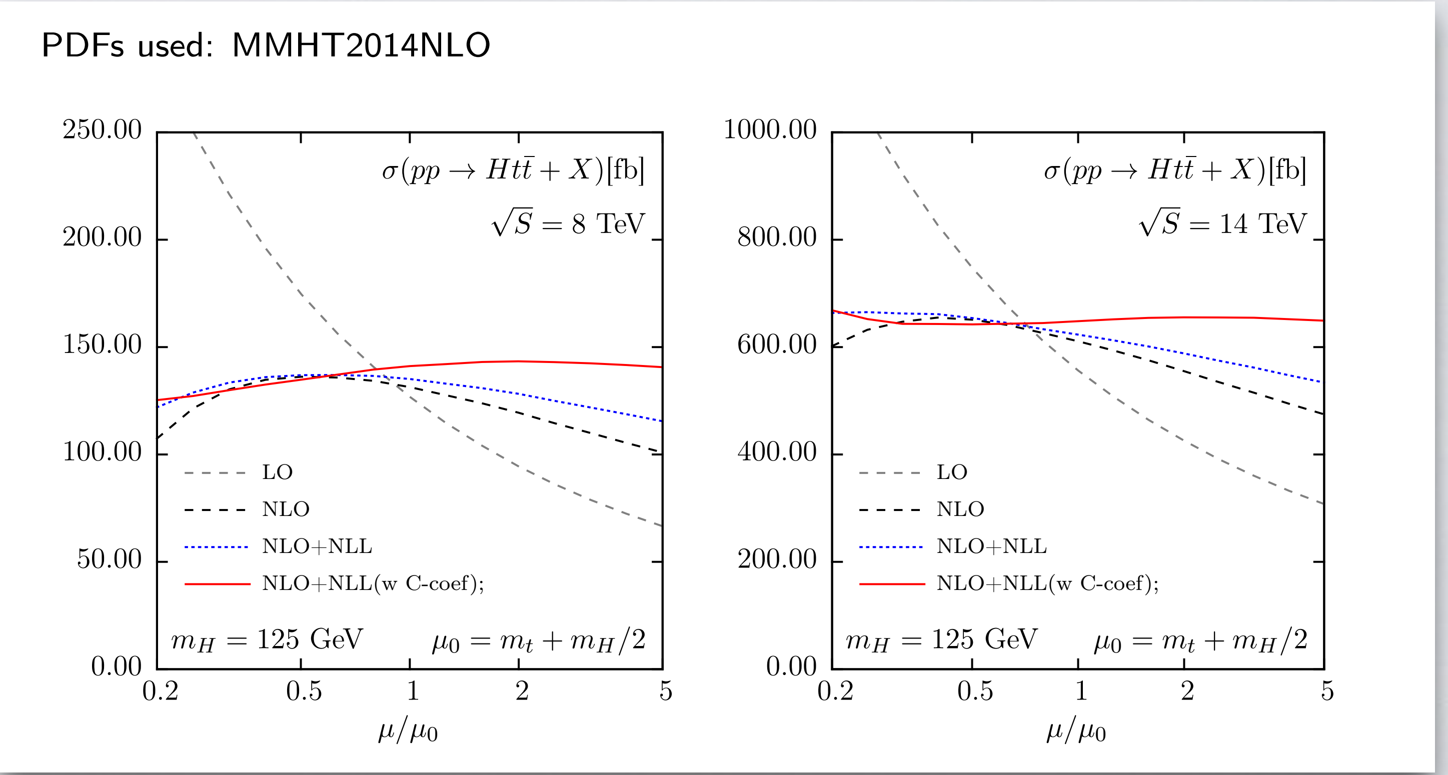
or

$M^2 = (m_H + 2m_t)^2$

color structure (same as *tt**): dealing with many dipoles and matrices

more complicated final state, different choices for threshold variable



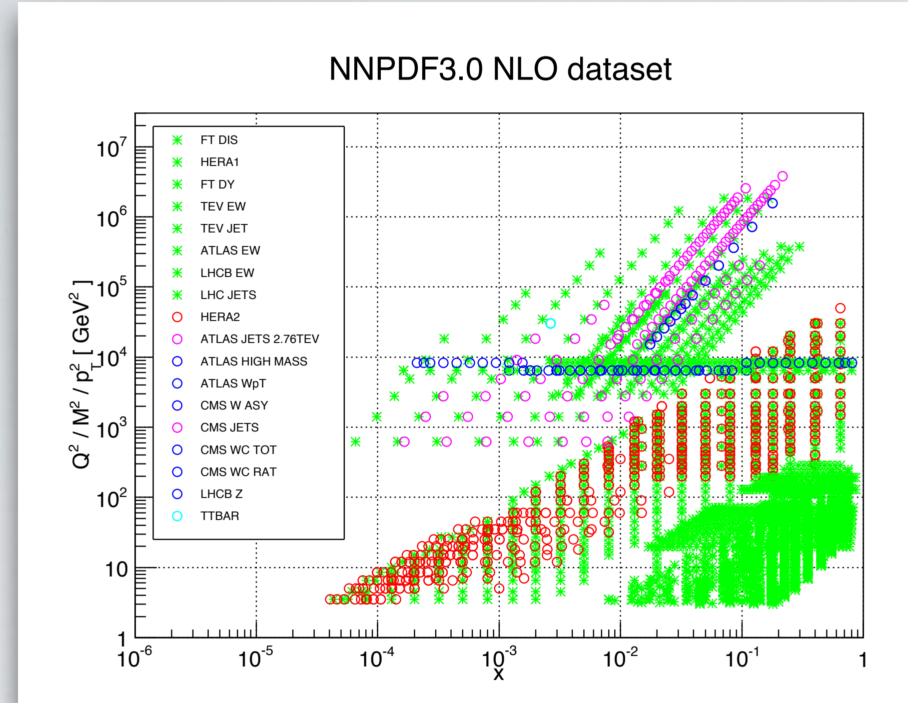


Broggio, Ferroglia, Pecjak, Signer, Yang (2015)

Kulesza, Motyka, Stebel, Theeuwes (2015)

*extensive work on *tt* by eg. Beneke *et al.* and Ferroglia *et al.*

PDFs AT LARGE X



$$\sigma(x,Q) = \sigma_0 C\left(\frac{x}{x_1 x_2}, \alpha_s(\mu)\right) \otimes f_1(x_1,\mu) \otimes f_2(x_2,\mu)$$

- coefficient functions contain large-x logs
- PDF evolution doesn't (in MSbar)

$$P_{gg}(x) \sim \frac{A(\alpha_s)}{(1-x)_+}$$

Process	observable	resummation available
DIS	$d\sigma/dx/dQ^2$ (NC, CC, charm, ...)	YES
DY Z/γ	$d\sigma/dM^2/dY$	YES
DY W	differential in the lepton kinematics	NO
$t\bar{t}$	total σ	YES
jets	inclusive $d\sigma/dp_t/dY$	YES/NO

- performing a resummed fit is relatively straightforward
- data set is restricted: no jets

it should be easy to compute

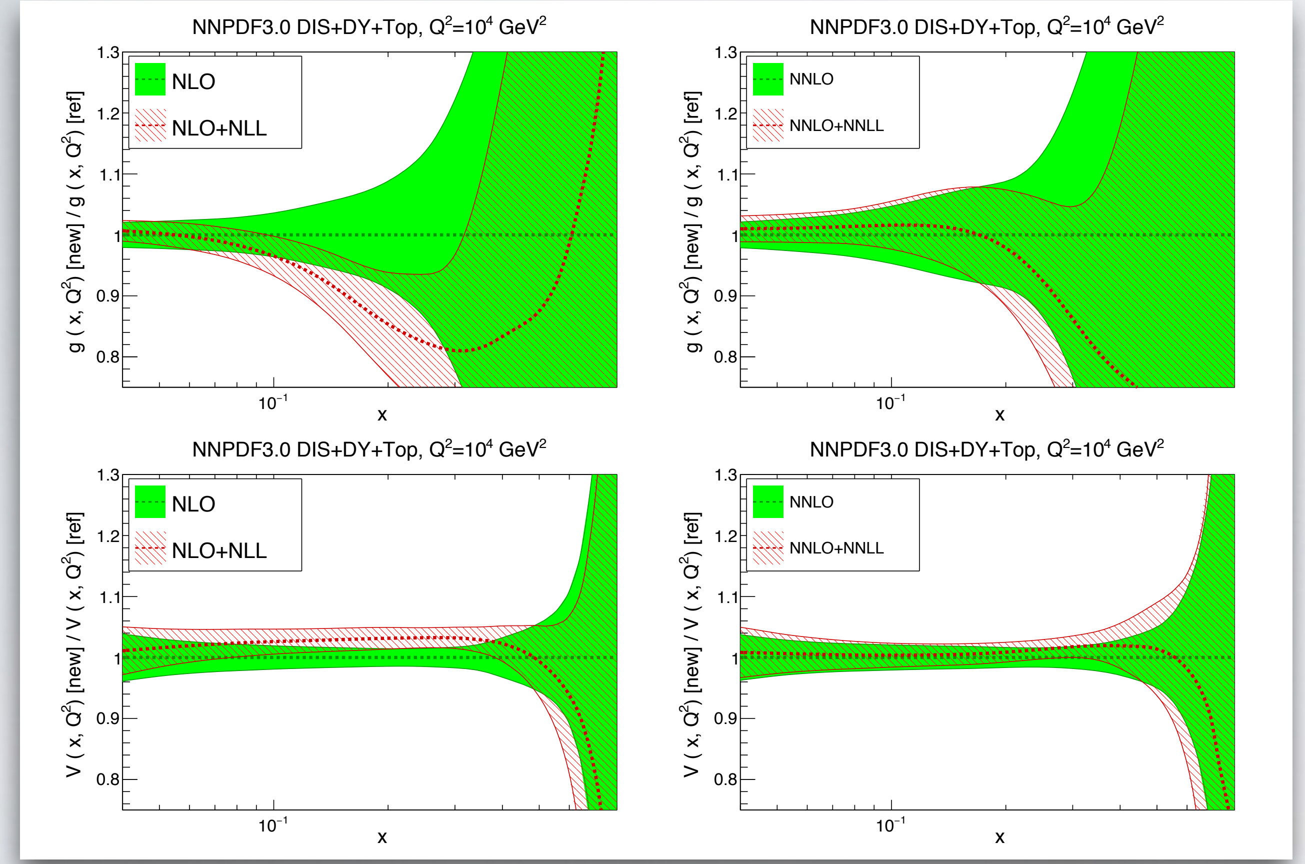
two different calculations exist at NLL but no implementation

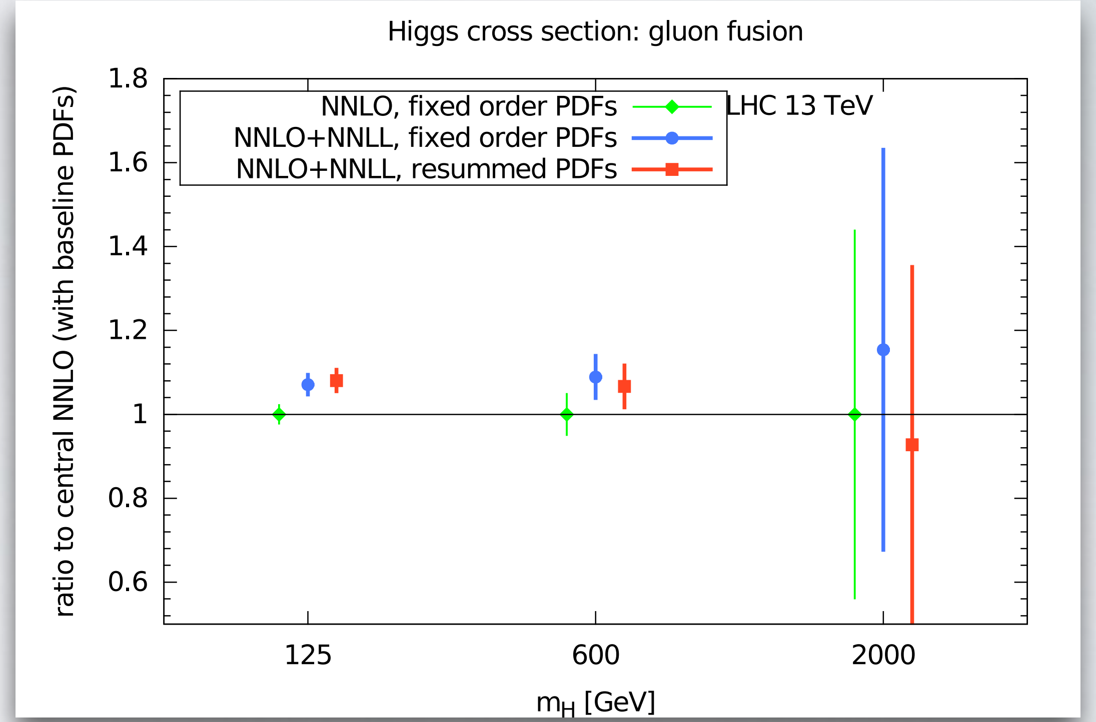
DIS, DY available from TROLL (TROLL Resums Only Large-x Logarithms)
www.ge.infn.it/~bonvini/troll

$t\bar{t}$ available from top++ www.alexandermitov.com/software

de Florian, Vogelsang (2007, 2013), Kidonakis Owens (2000)

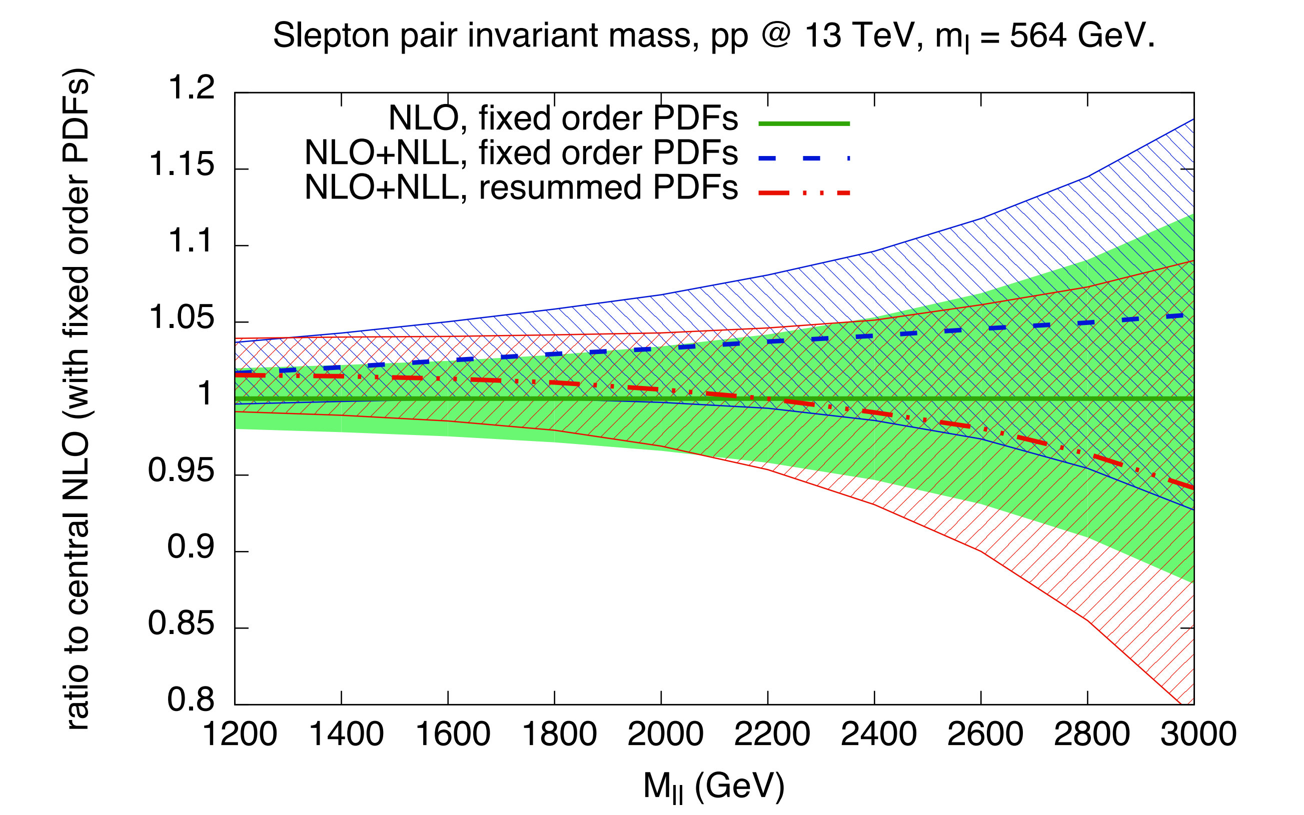
PDFs WITH THRESHOLD RESUMMATION





Bonvini *et al.* (2015), Beenakker *et al.* (2015)

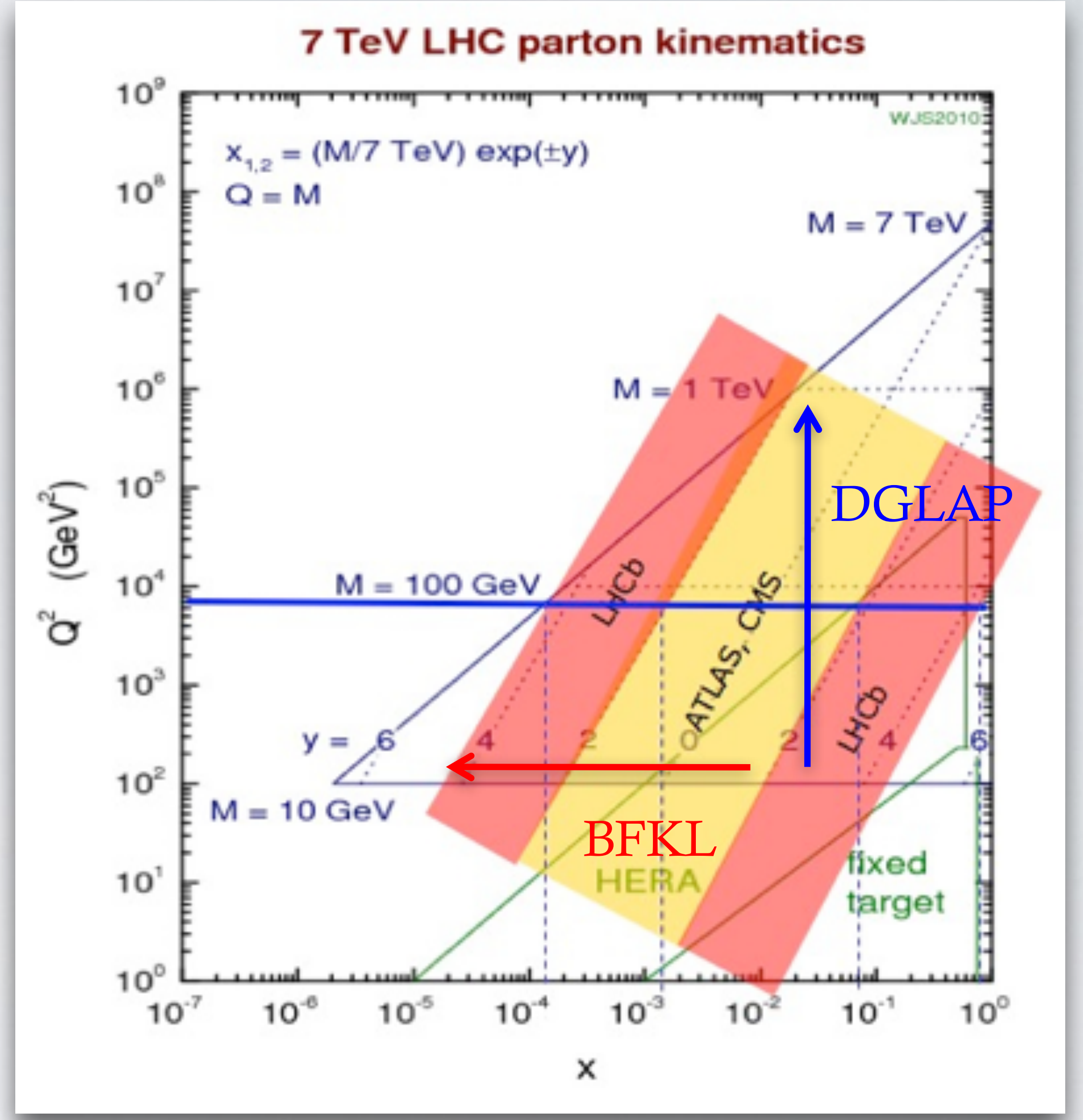
- effects on SM Higgs negligible
- more pronounced at high-mass but still within PDF errors
- Resummed still PDFs not competitive because of missing jet data



SMALL-X RESUMMATION

LHC KINEMATICS

- PDFs are largely unconstrained at low *x*
- LHC does probe this region
- Is DGLAP enough to describe this region?
- Do we need BFKL?



DGLAP: Q^2 evolution for N moments of the parton density

$$\frac{d}{d\ln(Q^2/\mu^2)}G(N,Q^2) = \gamma(N,\alpha_s)G(N,Q^2)$$

BFKL: small-x evolution for M moments of the parton density

$$\frac{d}{d\ln(1/x)}G(x,M) = \chi(M,\alpha_s)G(x,M)$$

Mellin moments:

logs $\leftrightarrow$ poles

$$\ln^k\frac{Q^2}{\mu^2} \leftrightarrow \frac{1}{M^{k+1}}$$

$$\ln^k\frac{1}{x} \leftrightarrow \frac{1}{N^{k+1}}$$

BASICS OF HIGH-ENERGY RESUMMATION

- Small-x logs present both in coefficient functions and in the evolution of the parton densities

- Evolution:
  - one constructs a resummed anomalous dimension from the BFKL kernel at next-to-leading log
  - however naive procedure leads to results not supported by HERA data

Altarelli, Ball, Forte
Ciafaloni, Colferai, Salam, Stasto

- Coefficient functions:
  - the resummation here is known at leading log level
  - first developed for heavy quarks and DIS

Catani, Ciafaloni, Hautmann (1991)
Catani, Hautmann (1994)

  - (subleading) running coupling terms are important

Ball (2008)

- Recent work in SCET and work in progress for DIS applications

Rothstein and Stewart (2016)

Pathak *at al.* in progress

SMALL-X RESUMMATION FROM HELL

- Problem studied by different groups in late '90s /early '00s
- The main interest was DIS at low x

    for a comparative review see HERA-LHC Proc. arXiv:0903.3861

- Key ingredients:
  - stable solution of the running coupling BFKL equation
  - match to standard DGLAP at large N (x)
  - important subleading effects
  - resummed coefficient functions (DIS, DY, HQ…)

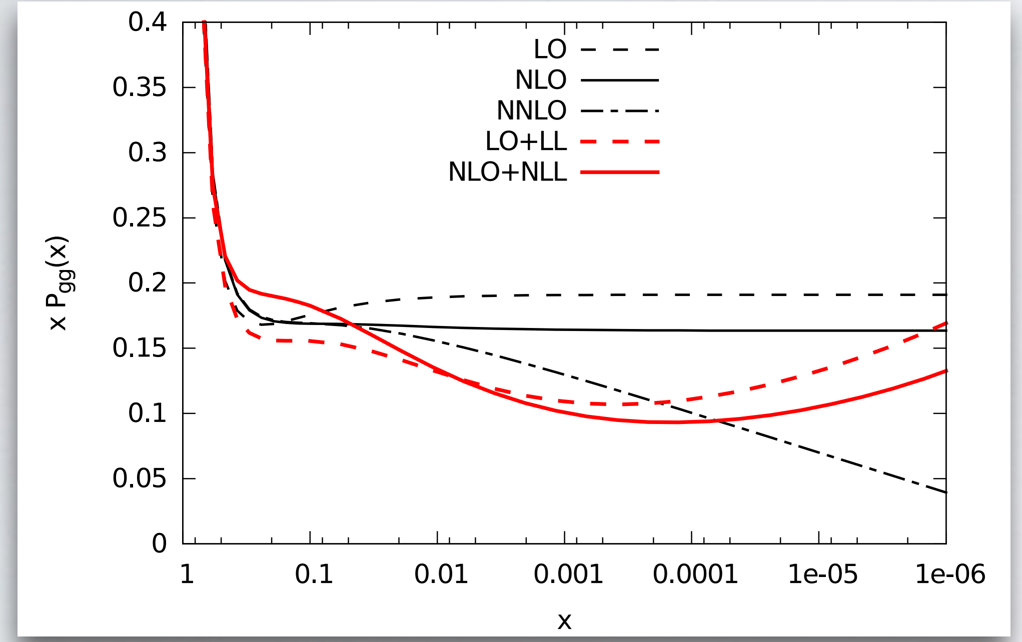
Little phenomenology because a comprehensive code was missing

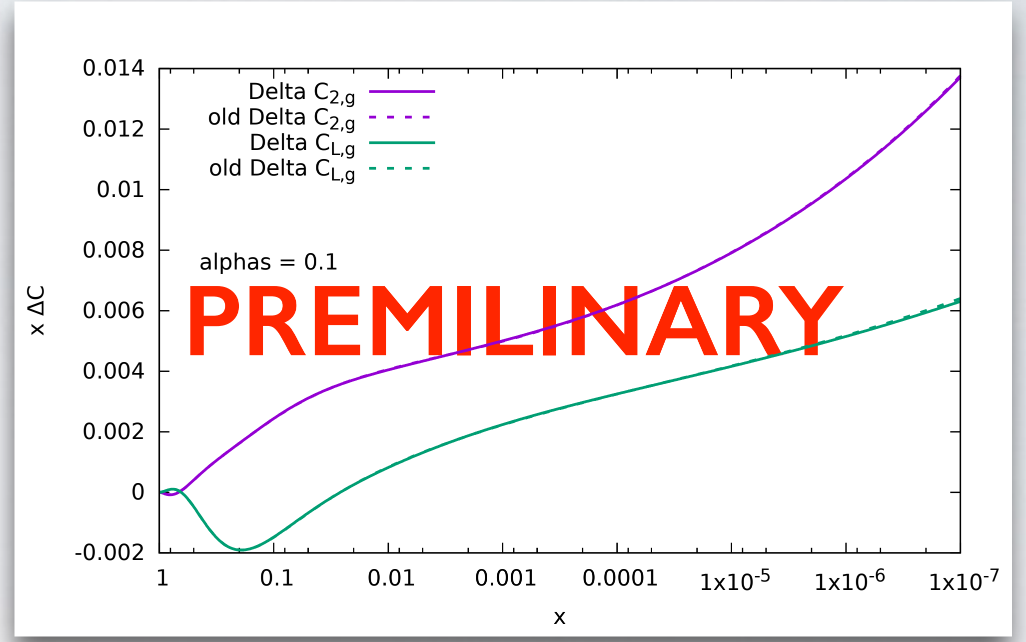
We are preparing a public code
`HELL: High-Energy Large Logarithms`
which will deliver resummed splitting functions and coefficient functions.

Bonvini, SM, Peraro (2016-hopefully!)

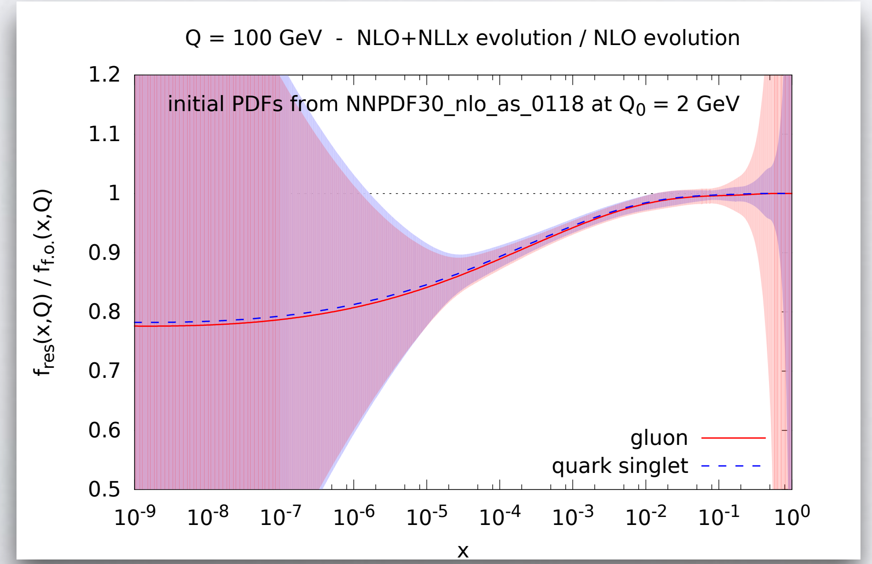
TOWARDS A RESUMMED FIT

theory inputs: splitting functions and coefficient functions

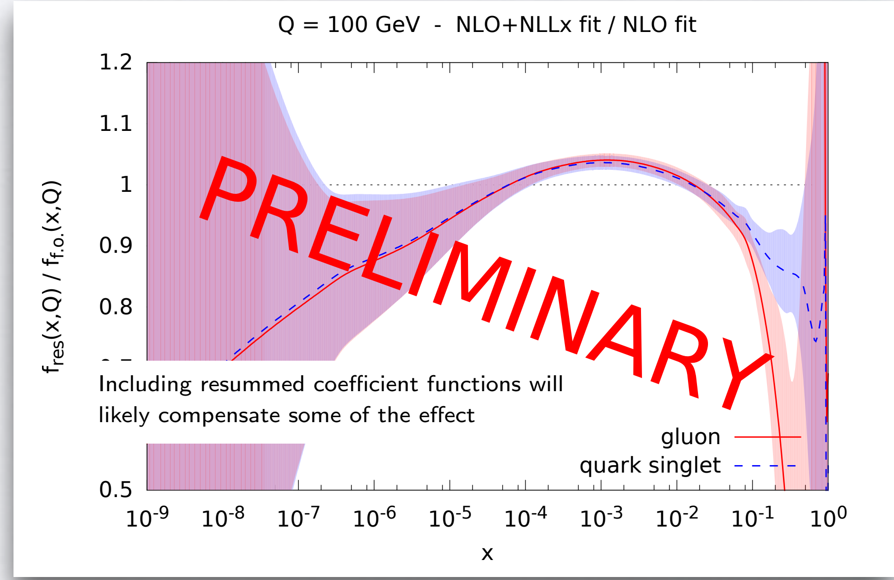




PDFs with resummed evolution



re-fitted PDFs with resummed evolution



Q_T RESUMMATION

TRANSVERSE MOMENTUM RESUMMATION

- One of the most studied distribution both in Higgs and DY
- High-accuracy NNLO+NLO calculations exist both in dQCD and SCET
- Codes available such DYQT, DYRES, Resbos, CuTe, *etc*.

e.g. Collins, Soper, Sterman; Catani *et al.;* Becher, Neubert; Neill, Rothstain Vaidya; Monni, Re.

- resummation often performed in b-space

$$\delta^{(2)}\left(\sum_{i=1}^{n} \underline{k}_{Ti} + \underline{Q}_T\right) = \frac{1}{(2\pi)^2}\int d^2\underline{b} e^{i\underline{b}\cdot\underline{Q}_T}\prod_{i=1}^{n} e^{i\underline{b}\cdot\underline{k}_{Ti}}$$

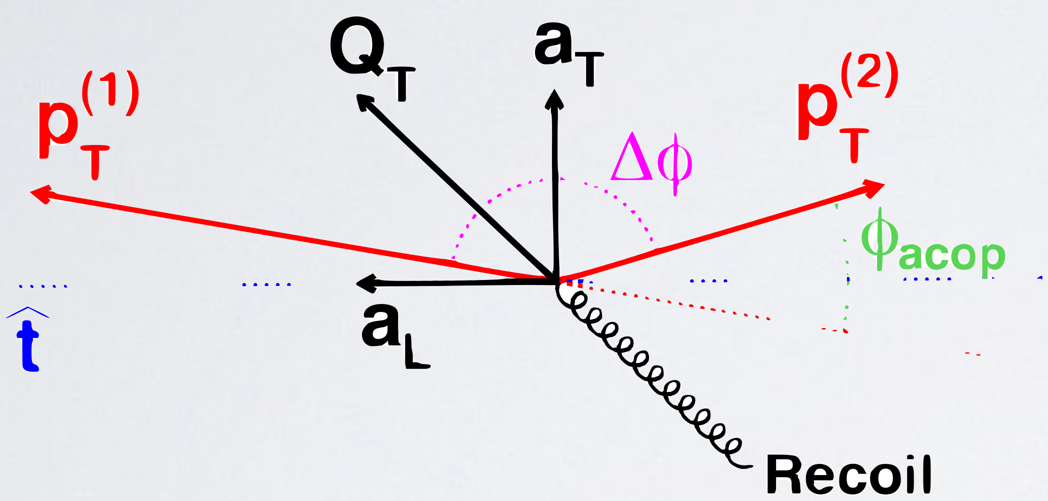
resummed exponent

$$\frac{d\sigma}{dQ_T^2} \simeq \int_0^\infty db\, b\, J_0(bQ_T)e^{-R(b)}\Sigma(x_1, x_2, \cos\theta^*, bM)$$

non-log terms and PDFs

RELATED VARIABLES

- Variables introduced by the DØ collaboration for studying the transverse momentum of the Z boson
- Experimental viewpoint: one wants to measure angles rather than momenta



$$\phi^* = \tan\left(\phi_{\rm acop}/2\right)\sin\theta^*$$

$$a_T = \frac{\underline{Q}_T \times (\underline{p_T}^{(1)} - \underline{p_T}^{(2)})}{|\underline{p_T}^{(1)} - \underline{p_T}^{(2)}|}$$

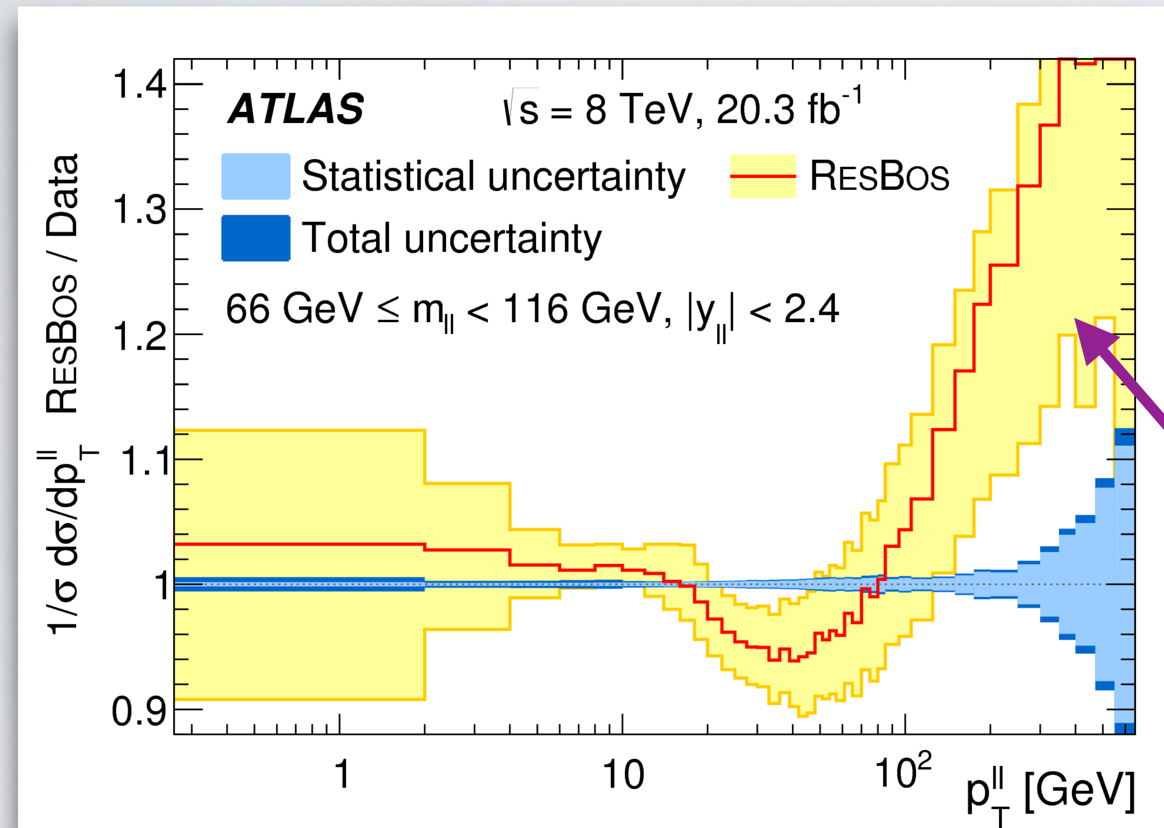
Vesterinen and Wyatt et al. (2008/10)

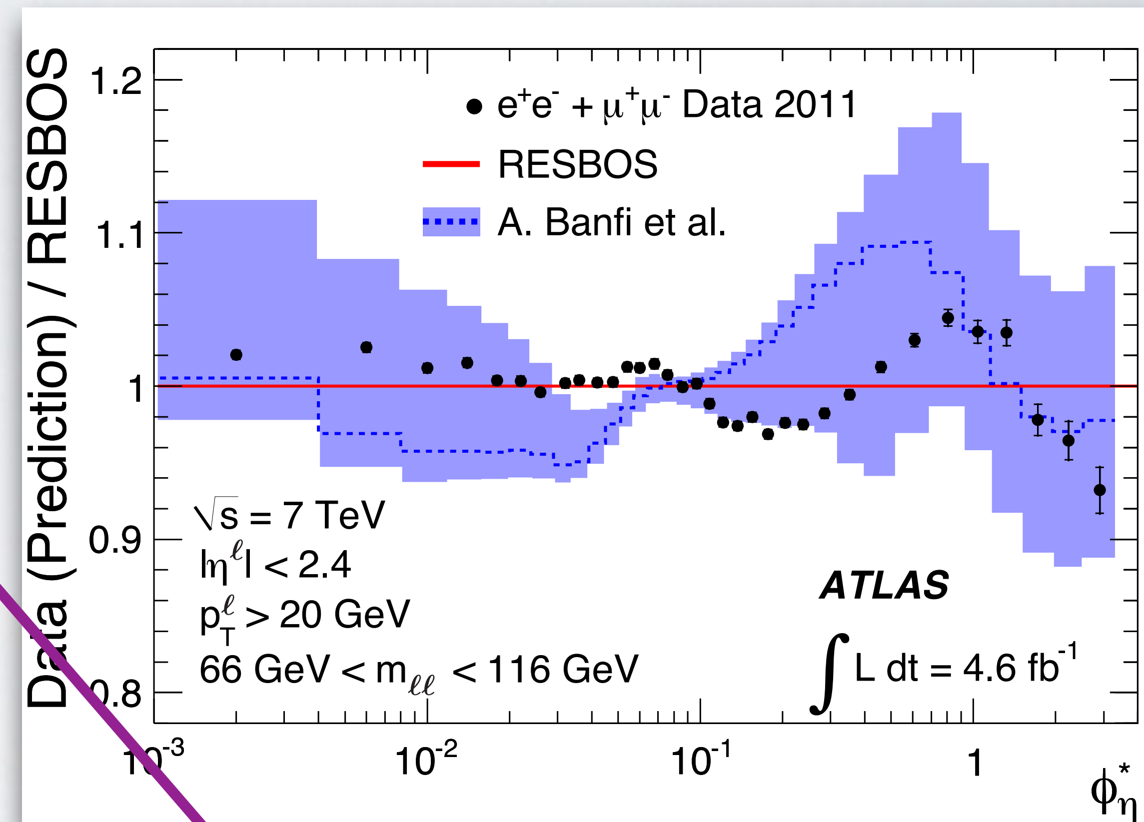
resummation closely related to Q_T

$$\frac{d\sigma}{d\phi^*} = \frac{\pi\alpha^2}{sN_c}\int_0^\infty d(bM)\cos(bM\phi^*)e^{-R(b)}$$
$$\times\Sigma(x_1, x_2, \cos\theta^*, bM)$$

Banfi, Dasgupta, SM, Tomlinson (2010)
Guzzi, Nadolsky, Wang (2014)

COMPARISON TO DATA: NEED FOR BETTER PRECISION





btw: what's going on here? "wrong" scale??

- experimental uncertainty below 1% !
- NNLL+NLO not adeguate (uncertainty 10%)

recent theory developments make N^{3}LL +NNLO possible in the near future!

Include these data in PDF fits ?

Li, Zhu (2016) Boughezal *et al.* (2015), Gehrmann-De Ridder *et al.* (2016)

COMBINING RESUMMATIONS

LARGE X

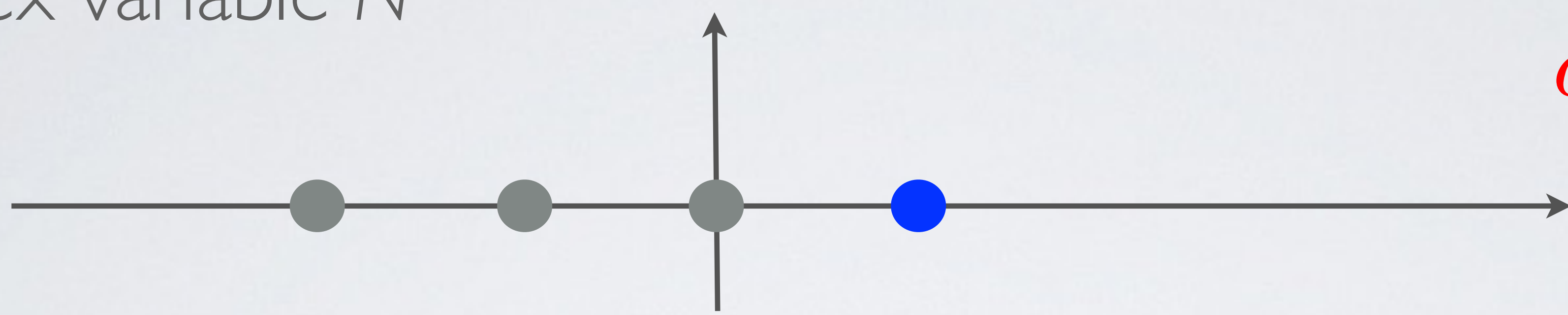
SMALL X

SMALL QT

LARGE X & SMALL X

Coefficient functions in N-space are analytic (meromorphic) function of complex variable N

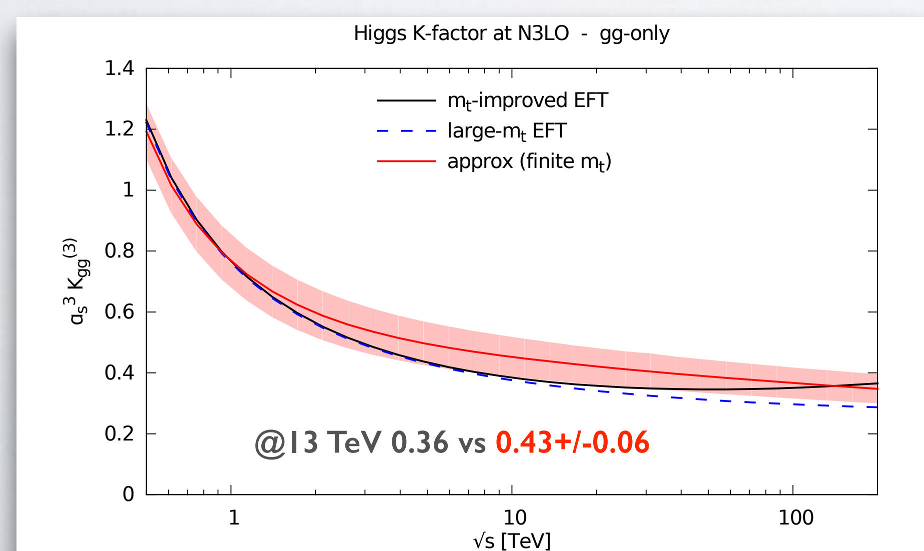
$\alpha_s^n \ln^{2n} N$



poles at integer N BFKL (single) pole

Basic idea from complex calculus: an analytic function can be reconstructed from the knowledge of its singularities

Modify threshold resummation so that it doesn't spoil analyticity



used to construct approx N^{3}LO (Higgs and tt)

(Muselli) Ball, Bonvini, Forte, SM, Ridolfi (2013/14)

work in progress for full resummation

Small Q_T & small x

- Complementary work on this topic:
  - from saturation to Sudakov — Mueller, Xiao, Yuan (2013)
  - double differential in SCET — Procura, Waalewiin, Zeune (2015)
  - from Q_T to low-x — SM (2015) Forte, Muselli (2015)

R left unchanged to LL*x*

$$\frac{d\sigma}{dQ_T^2} \simeq \int_0^\infty db\, b\, J_0(bQ_T) e^{-R(b)} \Sigma(x_1, x_2, \cos\theta^*, bM)$$

PDFs and coefficient functions are small-*x* resummed

- Low-*x* DY with k_t-cut advocated for PDF studies

Oliveira, Martin, Ryskin (2012)

- I plan to do phenomenological studies with HELL

LARGE X & SMALL Q_T

- Formalism to perform joint resummation developed a while back

Laenen, Sterman, Vogelsang (2000)

- NLL (in both variables) resummation performed for Higgs, DY and implemented in the code RESUMMINO

Kulesza, Sterman, Vogelsang (2002)
Fuks *et al.* (2013)

- A very recent come back

Joint transverse momentum and threshold resummation beyond NLL

Gillian Lustermans,[1,2] Wouter J. Waalewijn,[1,2] and Lisa Zeune[1]
[1] *Nikhef, Theory Group, Science Park 105, 1098 XG, Amsterdam, The Netherlands*
[2] *ITFA, University of Amsterdam, Science Park 904, 1018 XE, Amsterdam, The Netherlands*

To describe the transverse momentum spectrum of heavy color-singlet production, the joint resummation of threshold and transverse momentum logarithms is investigated. We obtain factorization theorems for various kinematic regimes valid to all orders in the strong coupling, using Soft-Collinear Effective Theory. We discuss how these enable resummation and how to combine regimes. The new ingredients in the factorization theorems are calculated at next-to-leading order, and a range of consistency checks is performed. Our framework goes beyond the current next-to-leading logarithmic accuracy (NLL).

May 2016

An Exponential Regulator for Rapidity Divergences

Ye Li,[1] Duff Neill,[2] and Hua Xing Zhu[2]
[1] *Fermilab, PO Box 500, Batavia, IL 60510, USA*
[2] *Center for Theoretical Physics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA*

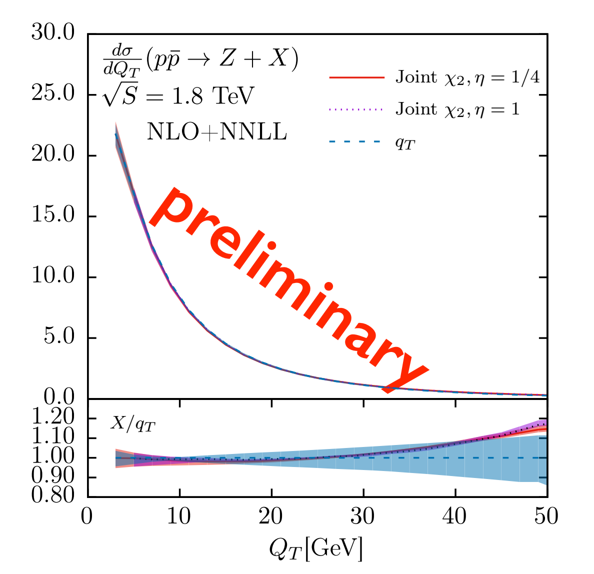
[…] We show how this factorization acts as a mother theory to both traditional threshold and transverse momentum resummation, recovering the classical results for both resummations […]

April 2016

sometime this summer

NNLL in both variables

Ferrera, SM, Theeuwes (work in progress)



CONCLUSIONS

- I have discussed a personal selection of topics in resummation
  - large-x resummation: high-precision & PDFs
  - small-x resummation: vast kinematic plane, towards low-x PDFs
  - Q_T resummation: data challenge theory
- A recent renaissance for joint resummation(s)

THANK YOU VERY MUCH
FOR YOU ATTENTION

BACKUP

soft-gluon resummation

The standard resummed result

$$C_{\rm res} = g_0(\alpha_s) \exp\left[\frac{1}{\alpha_s} g_1(\alpha_s \ln N) + g_2(\alpha_s \ln N) + \alpha_s g_3(\alpha_s \ln N) + \ldots\right]$$

$$g_0(\alpha_s) = 1 + \alpha_s g_{0,1} + \alpha_s^2 g_{0,2} + \alpha_s^3 g_{0,3} + \ldots$$

Higgs in gg fusion

- g_1, g_2 and g_3 are known, g_4 partly. They don't depend on m_t.
- All the m_t dependence is in the constant g_0
  - $g_{0,1}$ known exactly
  - $g_{0,2}$ known as a power expansion in m_H/m_t
  - $g_{0,3}$ recent landmark calculation (Anastasiou *et al.*)

- (Almost) all ingredients for N^{3}LL′ resummation
- All non vanishing terms appearing at N^{3}LO level are known (i.e. soft+virtual approximation)

A simple example: NLO

- Let's consider first NLO in the heavy top limit (OK for soft)

$$C^{(1)} = 4A_g(z)\tilde{\mathcal{D}}(z) + \left(\frac{4C_A}{\pi}\zeta_2 + \frac{11}{2\pi}\right)\delta(1-z) - \frac{11}{2\pi}\frac{(1-z)^3}{z}$$

$$\tilde{\mathcal{D}}(z) = \left[\frac{\ln\frac{1-z}{\sqrt{z}}}{1-z}\right]_+, \quad A_g(z) = \frac{C_A}{\pi}\frac{1-2z+3z^2-2z^3+z^4}{z}$$

Upon Mellin transform:

logarithmic terms

constant contribution

highly suppressed at large N

$$\tilde{\mathcal{D}}_1(N) = \frac{1}{2}\left[\psi_0^2(N) + 2\gamma_E\psi_0(N) + \gamma_E^2\right]$$

$$\psi_0(N) = \ln N + \mathcal{O}\left(\frac{1}{N}\right)$$

How well does the soft contribution approximate the full NLO ?

NLO: soft approximation

- Let's expand the resummation to first non-trivial order

$$C_{\text{res}} = 1 + \alpha_s C_{\text{res}}^{(1)} + \mathcal{O}(\alpha_s^2)$$
$$C_{\text{res}}^{(1)} = g_{1,2} \ln^2 N + g_{2,1} \ln N + g_{0,1}$$

- This captures the correct large-N behavior but has a cut at finite N
- Wrong type of singularity! What does it look like in z space ?
- The inverse Mellin contains

$$\mathcal{D}_k^{\log} = \left(\frac{\ln^k \ln \frac{1}{z}}{\ln \frac{1}{z}} \right)_+$$

with $k=1$ (and a delta contribution)

- This is not what we find at fixed order

Three improvements

The analytic structure at finite *N* can be restored by a more careful study of the kinematics

1. Poles vs cuts:

$$\mathcal{D}_1^{\log}(z) \to \mathcal{D}_1(z) = \left(\frac{\ln(1-z)}{1-z}\right)_+$$

2. Phase-space for soft gluon emission:

$$p_{gg}(z) \int_{\Lambda}^{M\frac{(1-z)}{\sqrt{z}}} \frac{dk_t}{k_t} = \frac{A_g(z)}{1-z}\left(\ln\frac{1-z}{\sqrt{z}} + \ln\frac{M}{\Lambda}\right)$$

Forte, Ridolfi (2002)

which suggests the replacement

$$\mathcal{D}_1(z) \to \tilde{\mathcal{D}}_1(z) = \left(\frac{\ln\frac{1-z}{\sqrt{z}}}{1-z}\right)_+$$

Three improvements

3. Collinear improvement

$$p_{gg}(z)\int_{\Lambda}^{M\frac{(1-z)}{\sqrt{z}}}\frac{dk_t}{k_t}=\frac{A_g(z)}{1-z}\left(\ln\frac{1-z}{\sqrt{z}}+\ln\frac{M}{\Lambda}\right)$$

- resummation of logs only requires $A_g(1)=1$
- the full splitting function accounts for collinear emissions (but at least $1/N$ suppressed)

$$A_g(z)=\frac{C_A}{\pi}\left[1-(1-z)+2(1-z)^2+\dots\right]$$

- We keep first and second order (contribution to our uncertainty)
- We cannot keep the full because of double poles (more later)

$$\text{soft1}:\quad \tilde{\mathcal{D}}_1(N)\to\tilde{\mathcal{D}}_1(N+1)$$

$$\text{soft2}:\quad \tilde{\mathcal{D}}_1(N)\to 2\tilde{\mathcal{D}}_1(N)-3\tilde{\mathcal{D}}_1(N+1)+2\tilde{\mathcal{D}}_1(N+2)$$

Final prescription for soft

- These improvements can be generalized to higher orders
- Write the standard resummed result in the following form

$$C_{\rm res}(N,\alpha_s) = g_0(\alpha_s)\exp\sum_{n=1}^{\infty}\alpha_s^n\sum_{k=0}^{n} b_{n,k}\,\mathcal{D}_k^{\log}(N)$$

- Substitute the fully improved functions to obtain*

$$C_{\rm soft1}(N,\alpha_s) = \bar{g}_0(\alpha_s)\exp\sum_{n=1}^{\infty}\alpha_s^n\sum_{k=0}^{n} b_{n,k}\,\hat{\mathcal{D}}_k(N+1)$$

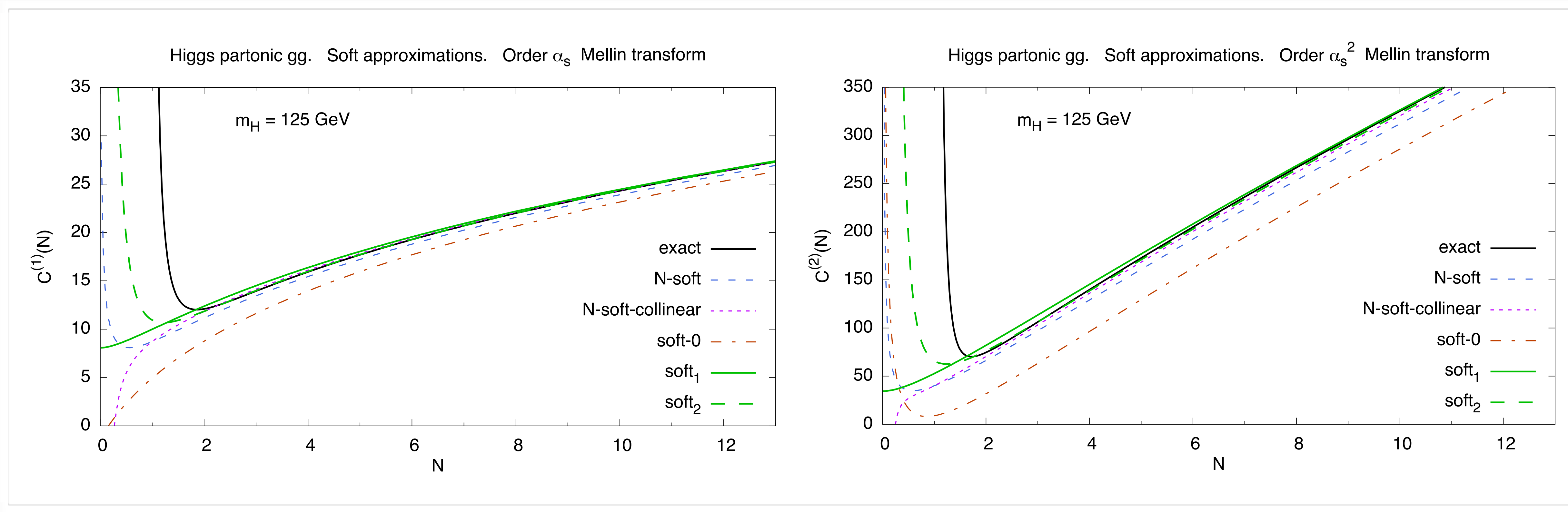
$$C_{\rm soft2}(N,\alpha_s) = \bar{g}_0(\alpha_s)\exp\sum_{n=1}^{\infty}\alpha_s^n\sum_{k=0}^{n} b_{n,k}\left[2\hat{\mathcal{D}}_k(N) - 3\hat{\mathcal{D}}_k(N+1) + 2\hat{\mathcal{D}}_k(N+2)\right]$$

- The various expressions only differs for terms that vanish at large *N*
- Note that there's a reshuffling of the constant terms (always adjusted for in the plots for known orders)

* the difference between "hat" and "tilde" is not crucial

Comparison to exact results

We expand the soft resummation(s) to finite order and compare to known NLO and NNLO



- Our approximations ($soft_1$ and $soft_2$) reproduce the FO down to $N \sim 2$
- Soft+virt in z-space (soft-0) undershoots both NLO and NNLO
- N-soft (no improvements) also good, especially at NLO:

$$\frac{\ln^k \ln \frac{1}{z}}{\ln \frac{1}{z}} = \frac{\sqrt{z}}{1-z} \ln^k \frac{1-z}{\sqrt{z}} \times \left[1 + \mathcal{O}\left[(1-z)^2\right]\right]$$