Adiabatic Statistics and Hall Viscosity of Topological Phases

Nicholas Read

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Topological (gapped) phases in 2 + 1 are related to modular tensor categories

Conformal blocks as trial wavefunctions give *same* MTC as their RCFT, or else gapless

---non-unitary cases

Hall viscosity: *new fundamental physics*

adiabatic transport approach

connection with orbital spin

effective field theory (w. W. Goldberger)

numerical calculations (w. E. Rezayi)

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Gapped (topological) phase in 2+1

$\rightarrow$ 2+1 Topological Quantum Field Theory (TQFT)

or modular tensor category

Moore + Seiberg, Witten, Reshitikhin + Turaev (1988 – 90)

--Finite set of quasiparticle types α

--Fusion: $\phi_\alpha \times \phi_\beta = \sum_\gamma N_{\alpha\beta}^\gamma \phi_\gamma,$ $N_{\alpha\beta}^\gamma \geq 0$ integers

$\rightarrow$ degeneracy of *n* well-separated qptcles all of type α

$$\dim \sim (\text{largest eval of } N_\alpha \text{ matrix})^n$$

as $n \to \infty$.

Operations (linear maps) on these spaces:

--Braiding (exchange of qptcles)
Braid gp rep, YB eq

--Creation/destruction of qptcle/antiptcle pair

--Twist (rotation of a qptcle by 2π), produces phase factor

$$\theta_\alpha = e^{2\pi i s_\alpha}$$

+ further conditions. This structure is called a ribbon tensor category.

Use to define quantum dimension d_α, each α.

Also $\widetilde{S}$ matrix; modular TC (MTC) if $\widetilde{S}$ invertible.

Hermitian structure: *inner product* on the spaces. Turaev (1994)

"Unitary" MTC if *positive definite* . This captures $\boxed{\text{positivity of QM}}$.

In particular, $d_\alpha > 0$ and largest eval of N_α is d_α .

Rational conformal field theories (RCFTs) also produce an MTC.

Moore and Seiberg (1988)

Non-unitary RCFTs (in 2D sense) contain some conf weight h<0 and some d_α is negative, in every known case.

$\boxed{\text{What is relation between "unitary" in 2D RCFT and in 3D TQFT?}}$

Conformal blocks as trial wavefunctions

Conformal blocks come from RCFTs: Moore and NR (1991)

$$\langle\psi(z_1,\bar{z}_1)\cdots\psi(z_N,\bar{z}_N)\tau(w_1,\bar{w}_1)\cdots\tau(w_1,\bar{w}_1)\rangle_{\mathrm{CFT}} = \sum_a |\Psi_a(w_1,\ldots;z_1,\ldots)|^2$$

Blocks Ψ_a are analytic but multivalued functions in w_l
---“monodromy” under e.g. braiding $\Psi_a \to \sum_b \Psi_b M_{ba}$

Many QH trial wavefunctions are conformal blocks, e.g.

$$\Psi_{\mathrm{MR}}(z_1,\ldots,z_N) = \mathcal{A}\left(\frac{1}{z_1-z_2}\frac{1}{z_3-z_4}\cdots\right)\prod_{i<j}(z_i-z_j)^Q\cdot e^{-\frac{1}{4}\sum_i |z_i|^2}$$

and with quasiholes at w_l also.

Usual inner product

For braiding and twist in a top phase, must calculate them by *adiabatic transport* (Berry phase/matrix): for Ψ_a orthonormal

$$|\Psi_b(w_{(0)})\rangle \to \sum_a |\Psi_a(w_{(0)})\rangle B_{ab}$$

where

$$B = M\mathcal{P}\exp i\oint_C (A_w \cdot dw + A_{\overline{w}} \cdot d\overline{w}),$$

$$A_{w,l,ab}(w) = i\left\langle \Psi_a(w) \middle| \frac{\partial \Psi_b(w)}{\partial w_l} \right\rangle$$

If also Ψ_a are holomorphic in *w*, then

$$\boxed{B = M}$$

as desired in MR (1991).

As Ψ_a is holomorphic in w_l, issue is orthonormality

$$\langle \Psi_a(w_1,\dots,w_n) | \Psi_b(w_1,\dots,w_n)\rangle_{\mathrm{CFT}} = \mathcal{Z}_{ab}(w_1,\dots,w_n)$$

Integrals over z and definition of conformal blocks $\rightarrow$
(go grand canonical)

$$\sum_a \mathcal{Z}_{aa} = \langle e^{\lambda \int d^2 z \psi(z,\bar{z})} \prod_k \tau(w_k,\bar{w}_k)\rangle_{\mathrm{CFT}}$$

---CFT perturbed by ψ. What is long-distance behavior of perturbed thy?

---- 1) massive
---- 2) massless
---- 3) other?

1) Massive 2D phase

Correlators $\sum_a \mathcal{Z}_{aa}$ generically go to constants as $|w_k - w_l| \to \infty$

But $\mathcal{Z}_{ab}$ cannot all be non-zero because of monodromy:

$$\mathcal{Z}_{ab} \to \sum_{c,d} (M^{\dagger})_{ac} \mathcal{Z}_{cd} M_{db}$$

These imply that

$$\mathcal{Z}_{ab} = \sum_{c,d} (M^{\dagger})_{ac} \mathcal{Z}_{cd} M_{db}$$

Schur's lemma implies $\mathcal{Z}_{ab} \propto \delta_{ab}$

Like order/disorder operators
in stat.mech/field thy

$B = M$ follows!

2) Massless 2D phase

$$\mathcal{Z}_{ab}(w_1.w_2) \sim \frac{1}{|w_1 - w_2|^{\#}} \left[1 + \mathcal{O}\left(\frac{1}{|w_1 - w_2|^2} \right) \right]$$

$\rightarrow$get power-law corrections to holonomy

---no good in a gapped phase (and other problems)

---probably gapless

3) Other ?

Worse!

Can do quasihole spin (twist) similarly NR (2008)

Then either MTC obtained is that of the RCFT, or system is gapless.

But use of non-unitary RCFT will produce some negative d_α , not acceptable in QM top. phase. E.g. Bernevig and Haldane (2008)

Hence these states must be gapless.

Hall Viscosity

Avron, Seiler, and Zograf (1995)

Hall viscosity is the viscosity analog of Hall conductivity:

$\sigma_{ij} =$ stress $\qquad$ $u_{ij} =$ strain

$$\sigma_{ij} = \sum_{k,l} \left[\lambda_{ijkl} u_{kl} - \eta_{ijkl} \frac{d}{dt} u_{kl} + \mathcal{O}\left(\frac{d^2}{dt^2} \right) \right], \qquad (i,j,k,l = 1,\ldots,d)$$

$\lambda_{ijkl} =$ pressure/elasticity $\qquad$ $\eta_{ijkl} =$ viscosity

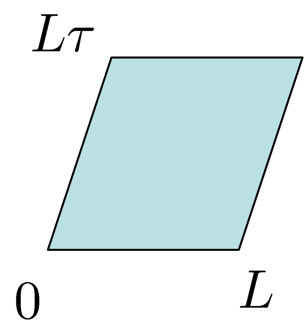
$u_{ij} = u_{ji}$ so $\eta_{ijkl} = \eta_{jikl} = \eta_{ijlk}$

Symmetric part $\eta^{(S)}_{ijkl} = \eta^{(S)}_{klij}$ gives dissipation

Antisymmetric (Hall) part $\eta^{(A)}_{ijkl} = -\eta^{(A)}_{klij}$ non-dissipative

---odd under time-reversal symmetry
---in d=2 isotropic system, only one ind comp: $\eta^{(A)}$

In a top phase, comes from response to varying metric instead of strain, for fixed coordinate system (no independent velocity)
---for torus, equivalent to changing (complex) aspect ratio τ at fixed area:



Hall viscosity is equal to the adiabatic curvature (curl of Berry vector potential) in τ space $(\mathrm{Im}\,\tau > 0)$, divided by area $L^2\mathrm{Im}\,\tau$

Avron, Seiler, and Zograf (1995)

For $\nu \geq 1$ filled Landau levels:

$$\eta^{(A)} = \frac{1}{4}\nu\bar{n}\hbar$$

Avron, Seiler, and Zograf (1995)
(factor of 2: Vignale and Tokatly, 2008)
Levay (1995)

($\bar{n}$ is the particle density) --- ind of τ !

For (i) paired superfluids (e.g. p+ip), and (ii) conformal blocks used as trial QH wavefunctions

$$\boxed{\eta^{(A)} = \frac{1}{2}\bar{s}\bar{n}\hbar}$$

NR (2009)

where $\bar{s}$ is (minus) the mean orbital spin per particle:

("real" spin neglected here)

$$\bar{s} = \begin{cases} 1/2 & \text{for p-ip} \\ Q/2 & \text{for Laughlin } \nu = 1/Q \text{ state} \\ \nu^{-1}/2 + h_\psi & \text{for general conformal block states} \end{cases}$$

$\bar{s} = \mathcal{S}/2$ where $\mathcal{S}$ is the shift on the sphere: $N_\phi = \nu^{-1}N - \mathcal{S}$

Should be:

---quantized within trans/rot invariant topological phase

---general result for all such phases (Other fluids?)

For classical plasma $\eta^{(A)} = \dfrac{\bar{n} k_B T}{2\omega_c}$ Lifshitz and Pitaevskii, Physical Kinetics

--- electron in $\mathcal{N}$th LL has orbital spin $\mathcal{N} + 1/2$, cf. Levay (1995)
due to cyclotron motion. This equals kinetic energy / $\hbar\omega_c$.

Hence thermal average at high T (using equipartition) $\Rightarrow$

$$\bar{s} = \frac{k_B T}{\hbar \omega_c}$$

and

$$\eta^{(A)} = \frac{1}{2} \bar{s} \bar{n} \hbar = \frac{\bar{n} k_B T}{2\omega_c}$$

NR (2009)

Relation to spin

Fix coordinates $x^1 = x,\ x^2 = y,\ 0 \leq x, y \leq 1,$ metric is

$$ds^2 = \sum_{i,j} g_{ij} dx^i dx^j = \frac{A}{\mathrm{Im}\,\tau}(dx^2 + 2\mathrm{Re}\,\tau\, dx dy + |\tau|^2 dy^2).$$

Under an "active" coordinate transformation in

$$\mathrm{SL}(2,\mathbf{R}) = \left\{ \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) : \quad a,b,c,d \in \mathbf{R},\ ad - bc = 1 \right\},$$

the change in g_{ij} at fixed area *A* is described by $\tau \to \dfrac{a\tau + b}{c\tau + d}.$

E.g. under small transformations $\left(\begin{array}{cc} 1 & \varepsilon \\ \varepsilon & 1 \end{array} \right), \left(\begin{array}{cc} 1+\varepsilon' & 0 \\ 0 & 1-\varepsilon' \end{array} \right),\ \varepsilon, \varepsilon'$ real,

$\tau = i$ (square) undergoes $i \to i + 2\varepsilon, \quad i \to i + 2\varepsilon' i$ --- two distinct shears.

Commutator is $= I + 2\varepsilon\varepsilon' \left(\begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right)$ --- an SO(2) rotation! cf. Levay (1995)

Leaves g_{ij} invariant --- rotation of space.

Adiabatic curvature (curl of Berry connection) is given by the commutator of transformations, evaluated on a state, so picks up expectation of generator of $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. But as this is equivalent to rotation in real space,

we identify eigenvalue as "orbital spin". Note the upper half-plane (τ space) is

$$\cong \mathrm{SL}(2,\mathbf{R})/\mathrm{SO}(2).$$

[Cf. adiabatic rotations of coherent state for SU(2) spin, for spin in *z* direction, rotations about *x* and *y* commute to give S_z, and pick up expectation value. Note the space is $S^2 = \mathrm{SU}(2)/\mathrm{U}(1)$.]

In a topological phase, adiabatic curvature is $\mathrm{SL}(2,\mathbf{R})$-invariant on upper-half τ-plane (like that on sphere for spin).

Effective field theory

NR, Goldberger, to appear

Induced action for a 2+1 topological phase in *external* electromagnetic and gravitational fields:

$$S = \frac{\nu}{4\pi}\int d^3x\,\sqrt{g}\varepsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda + \frac{\nu\mathcal{S}}{4\pi}\int d^3x\,\sqrt{g}\varepsilon^{\mu\nu\lambda}A_\mu\partial_\nu\omega_\lambda + \dots$$

Chern-Simons term　　　　Wen-Zee term

A_μ = e.m. vector potential, ω_μ = the spin connection for SO(2) *spatial* rotations only.

Vary wrt A_0 and integrate, use Gauss-Bonnet Theorem $\int d^2x\,R = 2\pi(2-2\mathcal{G})$ on genus $\mathcal{G}$ surface ($\mathcal{G}=0$ is the sphere),

$$N = \nu N_\phi + \nu\mathcal{S}(1-\mathcal{G}) \longrightarrow N_\phi = \nu^{-1}N - \mathcal{S}(1-\mathcal{G})$$

so $\mathcal{S}$ is the shift. Coupling of orbital spin to curvature of space.

Wen and Zee (1992)

Induced action has local Galilean, not Lorentz, invariance.

NR, Goldberger, to appear

Apply to variation of τ for torus: use $\omega_\mu \sim g^{-1}\partial_\mu g$ and

$$\nu \nabla \times \mathbf{A}/2\pi = \bar{n} - \nu \bar{s} R/2\pi$$

R vanishes, reproduces the adiabatic Hall viscosity result.
---explains why the shift and Hall viscosity are related by $\bar{s} = \mathcal{S}/2$.

Like the Chern-Simons term, the Wen-Zee term cannot be renormalized, because it is not the integral of a local *gauge-invariant* combination of fields. (Or because of angular momentum conservation in perturbative corrections.)

Hence the result for $\eta^{(A)}$ from trial wavefunctions will hold throughout a trans/rot inv topological phase.

Varying wrt metric gives (complicated) expression for stress tensor of QH state.

Use of Hall viscosity as a diagnostic tool for numerics:
Measure shift on torus (unbiased) instead of extrapolating energies for different shifts on sphere. Don't need the trial state.

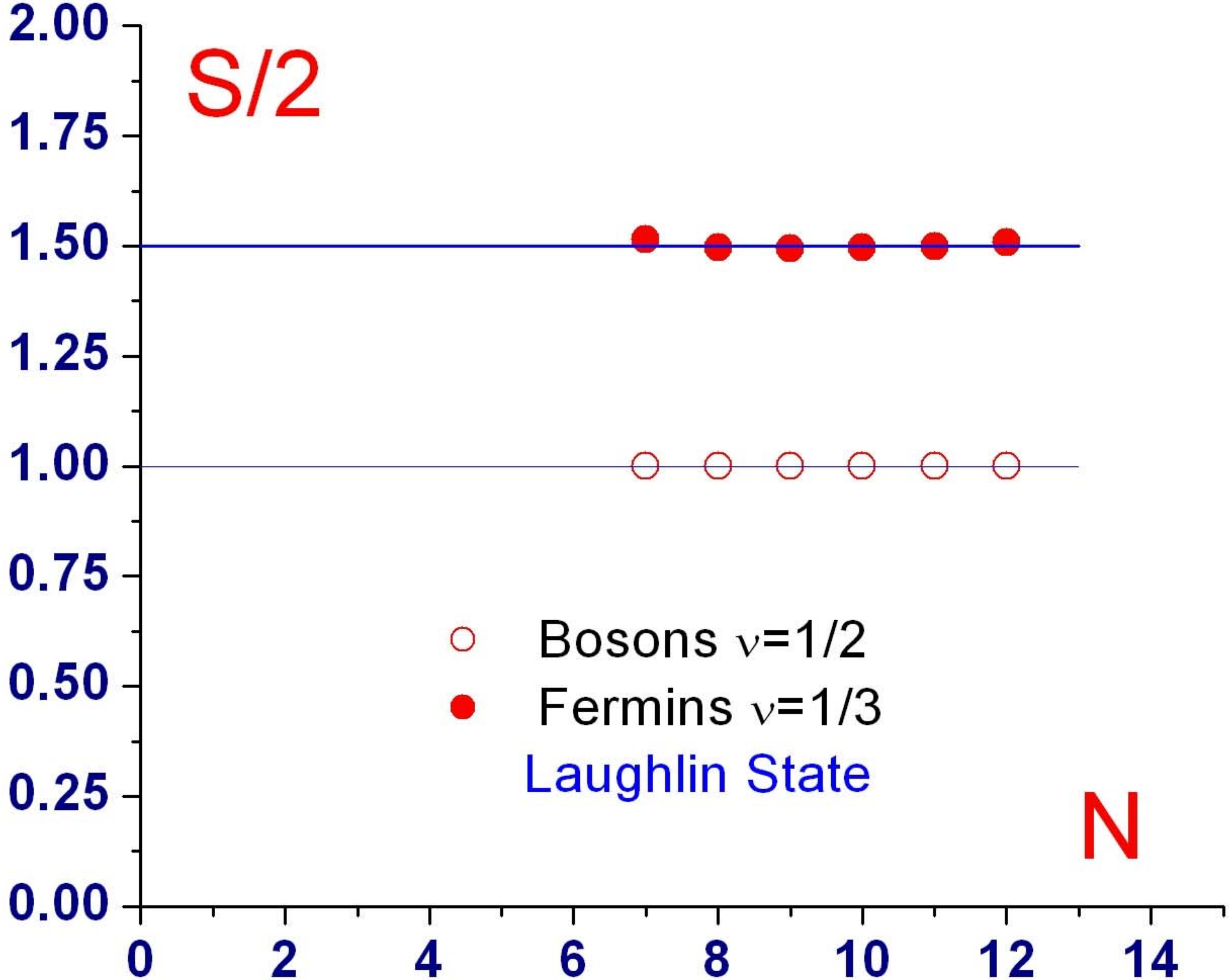
Numerical approach: Rezayi, NR, in progress

---evaluate Berry phase γ for a loop C in τ of M discrete steps as

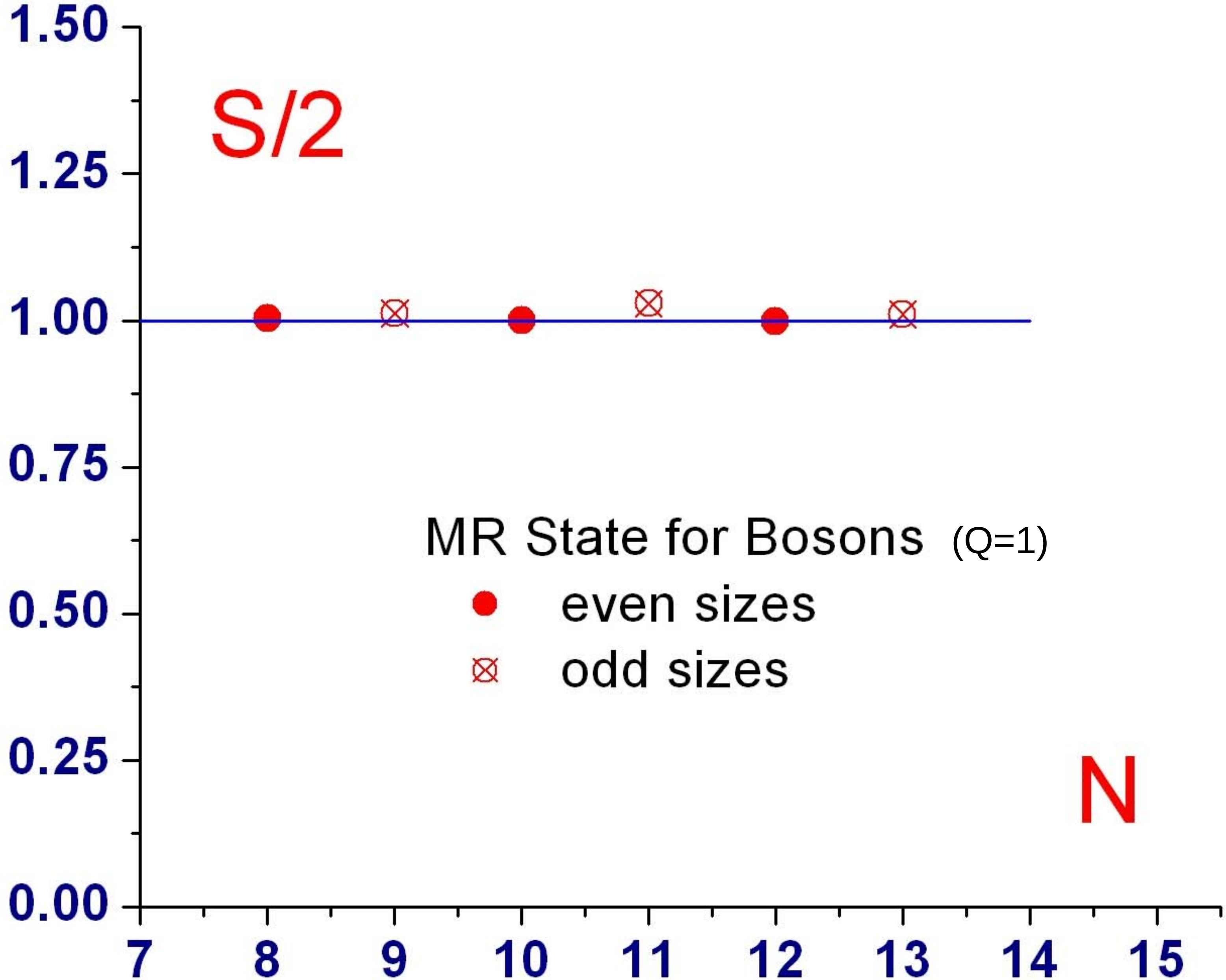
$$e^{i\gamma} = \prod_{l=1}^{M} \langle \Psi(\tau_{l+1}) | \Psi(\tau_l) \rangle$$

Use C small, M large. Divide γ by area of C , and by N to get $\bar{s}$.

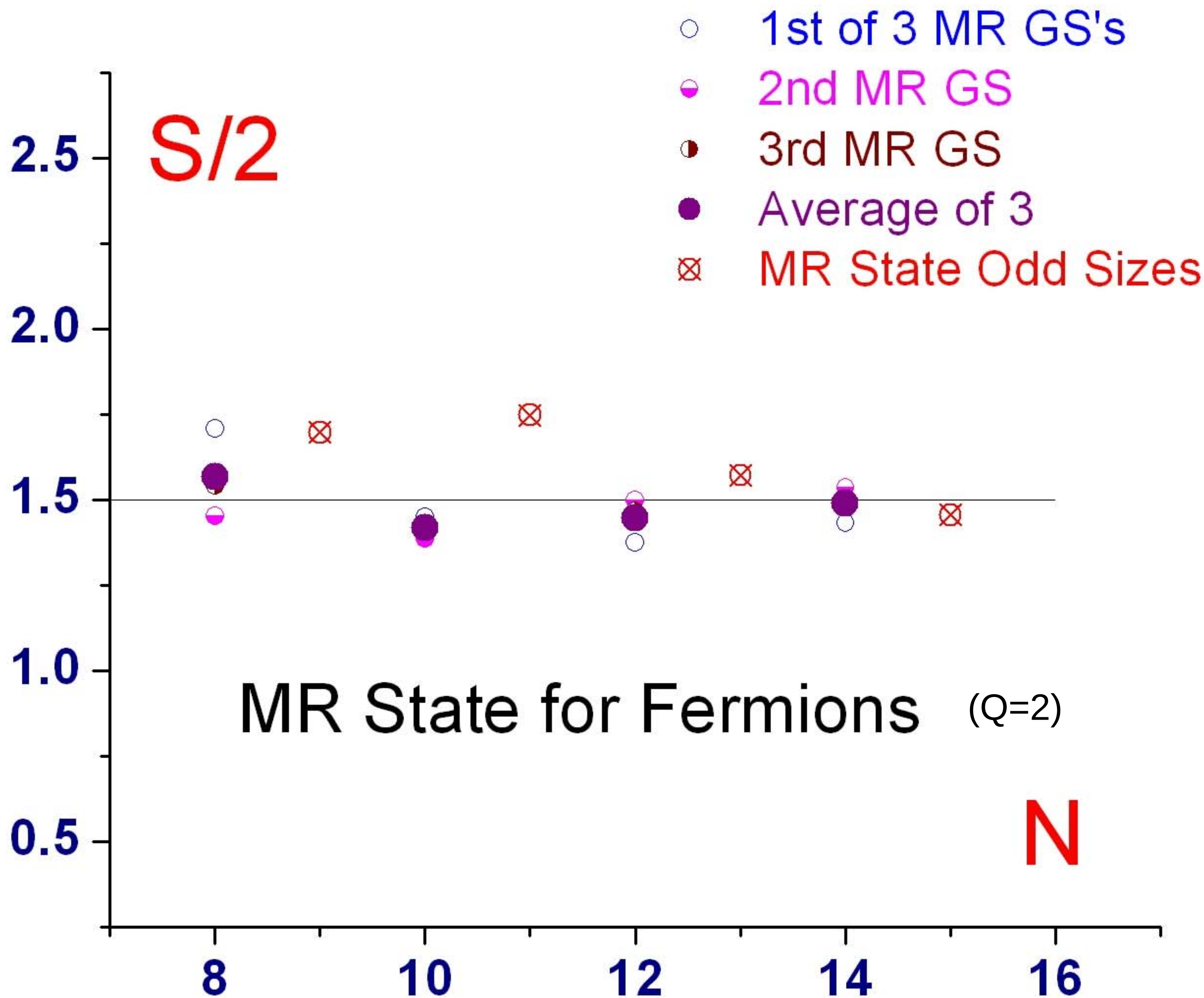
For trial states (Laughlin, Moore-Read), confirm expected values when N sufficiently large.



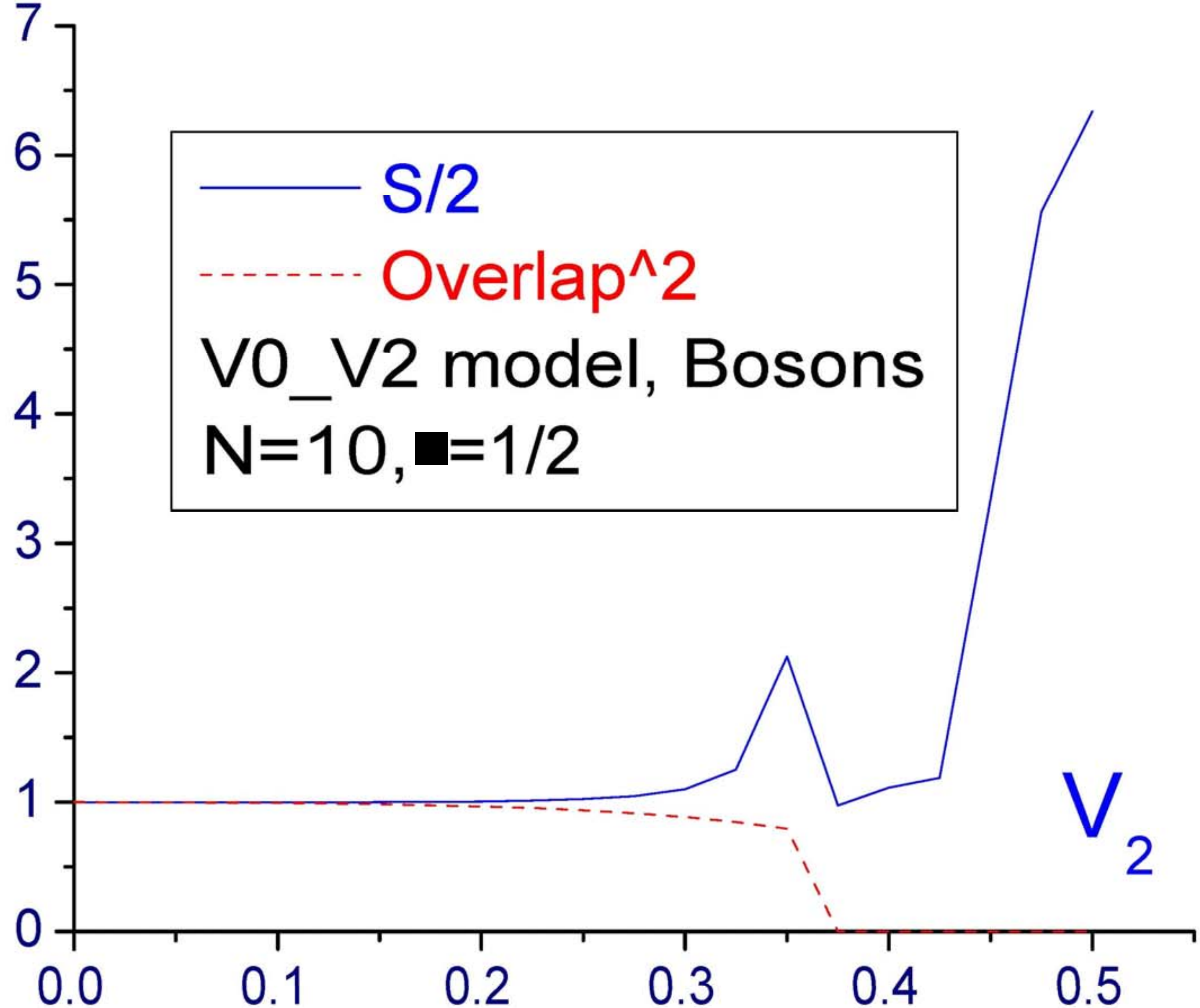
2.00
1.75
1.50
1.25
1.00
0.75
0.50
0.25
0.00
S/2
Bosons ν=1/2
Fermins ν=1/3
Laughlin State
N
0
2
4
6
8
10
12
14



S/2
MR State for Bosons (Q=1)
even sizes
odd sizes
N
1.50
1.25
1.00
0.75
0.50
0.25
0.00
7
8
9
10
11
12
13
14
15



1st of 3 MR GS's
2nd MR GS
3rd MR GS
Average of 3
MR State Odd Sizes
S/2
2.5
2.0
1.5
1.0
0.5
MR State for Fermions (Q=2)
N
8
10
12
14
16



S/2
Overlap^2
V0_V2 model, Bosons
N=10, ■=1/2
V
2
0
1
2
3
4
5
6
7
0.0
0.1
0.2
0.3
0.4
0.5

Conclusion

1) Adiabatic statistics: either given by the monodromy of blocks, or state not gapped

2) Trial functions from non-unitary RCFTs don't give a top. phase

3) Hall viscosity: new bit of basic physics
potential use in numerical diagnostics