Universal Aspects of Many-body Localization transition & Eigenstate Thermalization

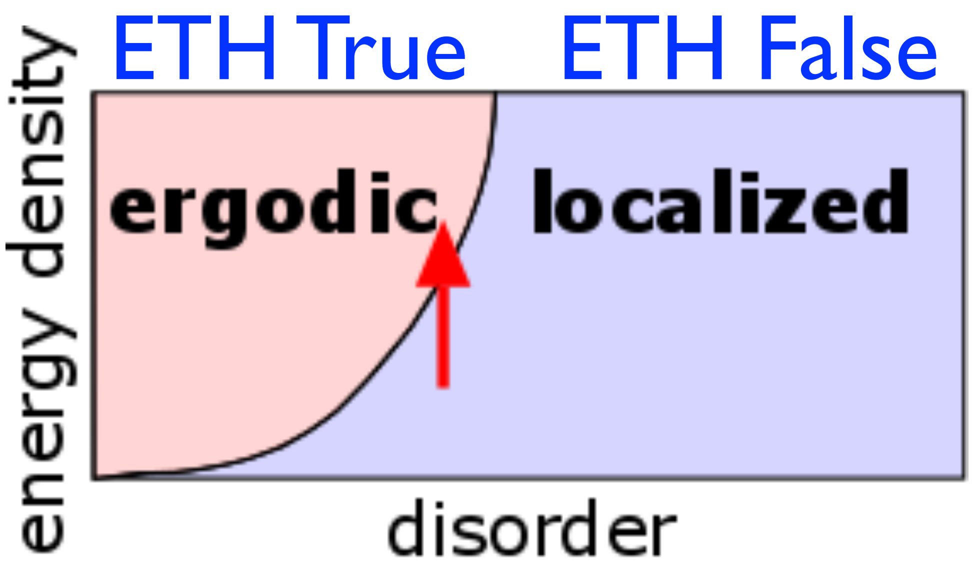
Tarun Grover (KITP, Santa Barbara)

TG: arXiv:1405.1471
TG, Jim Garrison: arXiv:1503.00729
TG, Jim Garrison: arXiv:1512.XXXX

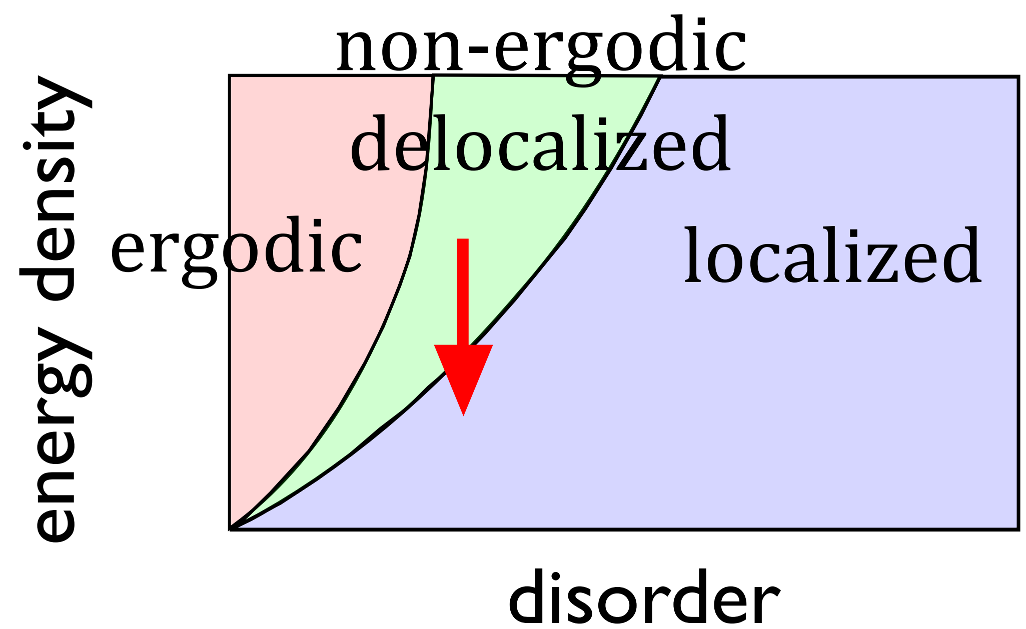
The Kavli Institute for Theoretical Physics
University of California, Santa Barbara

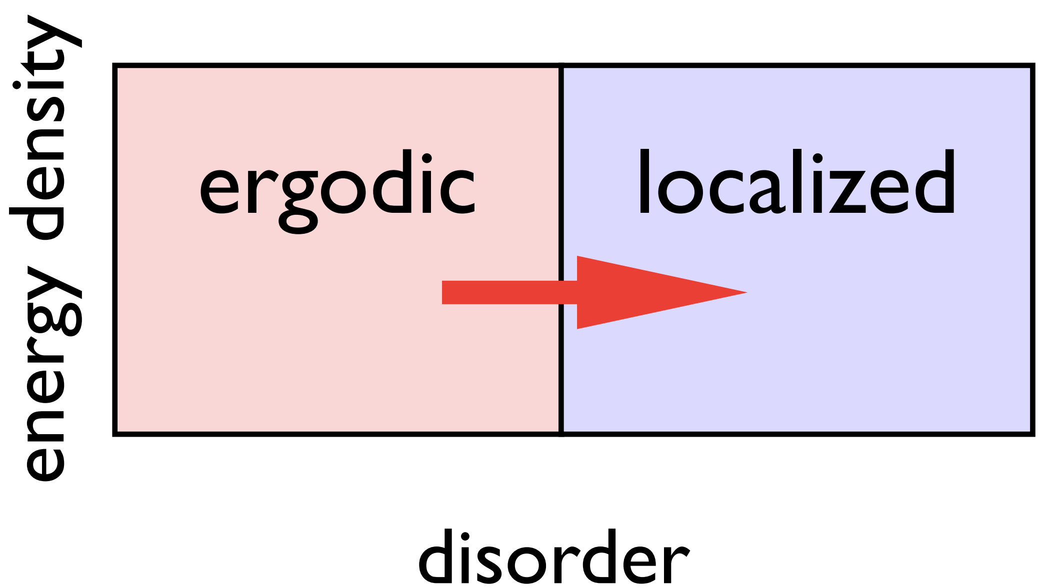
Gordon and Betty MOORE FOUNDATION

MBL Phase diagram?



$= \mathrm{var}(\mu_i)$

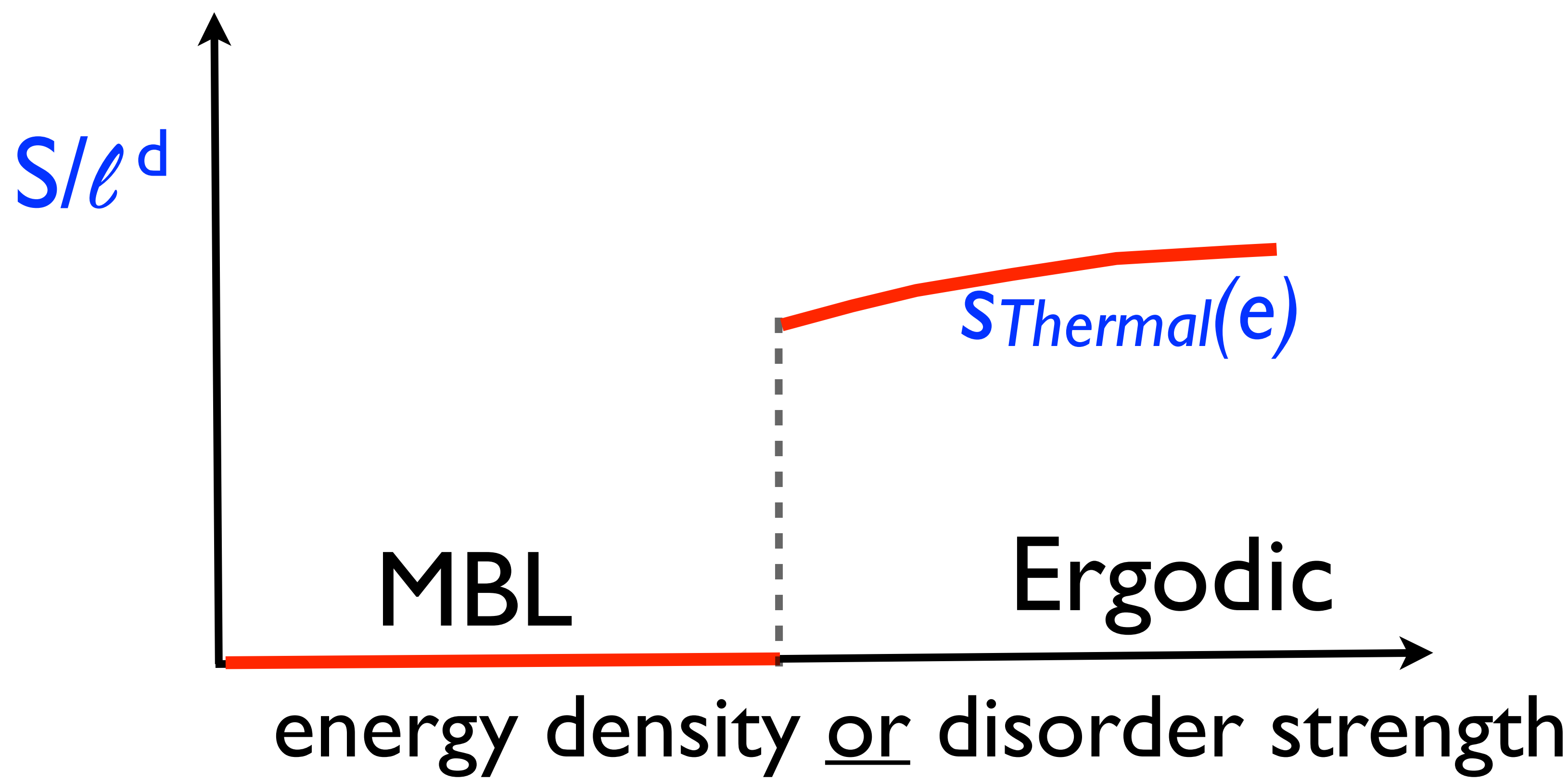




de Roeck, Huveneers, Muller, Schiulaz 2015

Basko, Aleiner, Altshuler, 2005; Oganesyan, Huse 2006; Pal, Huse 2010; Reichman et al 2010; Vosk et al 2014; Potter et al 2015; Serbyn et al 2015; de Luca, Scardicchio 2013; Demler et al 2014; Santos et al 2015, and many others.

Can transition be continuous?



Volume law coefficient "jumps" across the transition.

Pal, Huse 2010; Bauer, Nayak 2013; Imbrie 2014.

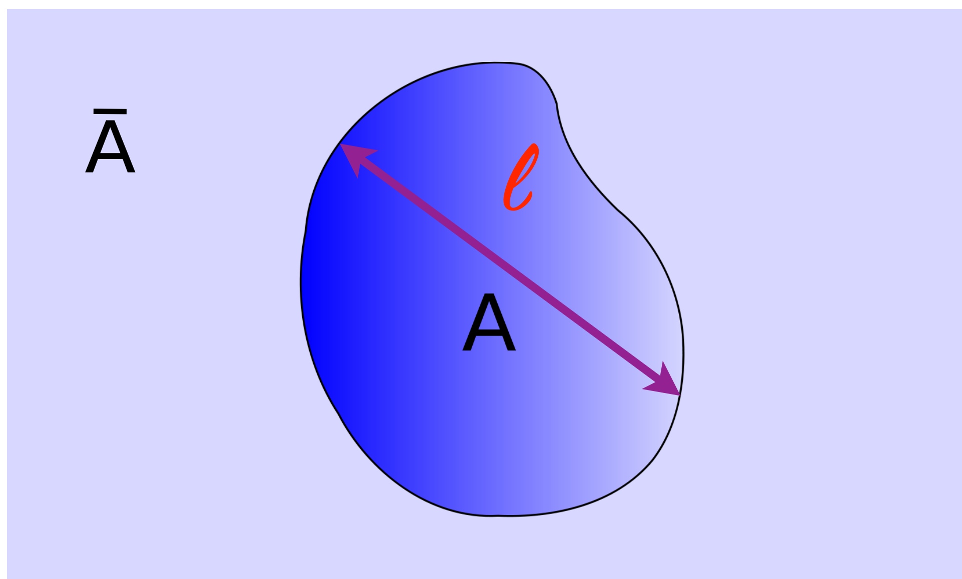
Nature of Eigenstates <u>at</u> the transition?

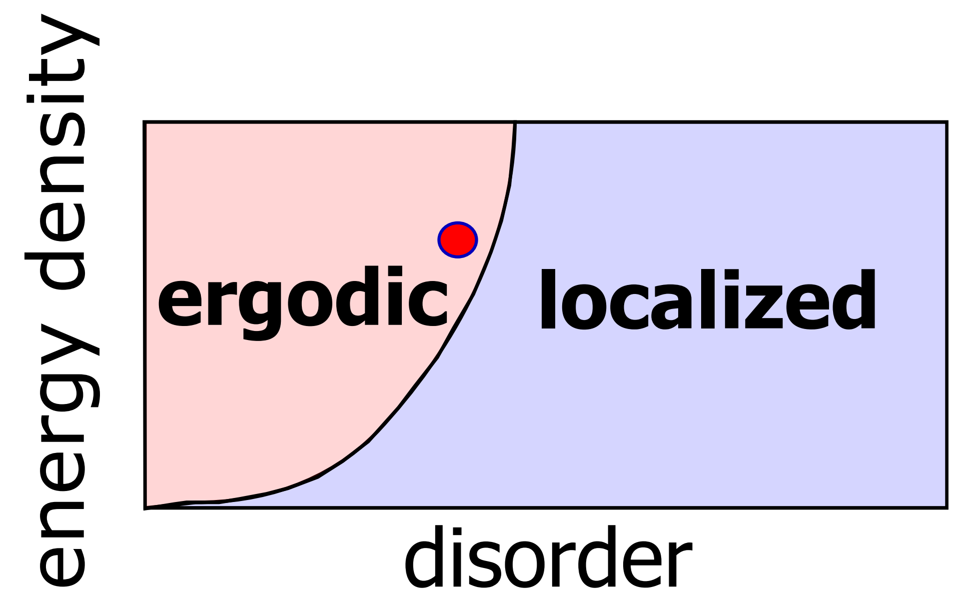
Are they ergodic?

Strategy: *<u>Analyze scaling of entanglement entropy close to the transition.</u>*

<u>Assumption</u>: A <u>diverging length scale</u> on approaching the transition from <u>either</u> side.

Entanglement Scaling Close to MBL Transition





ξ = delocalization length subsystem size ℓ

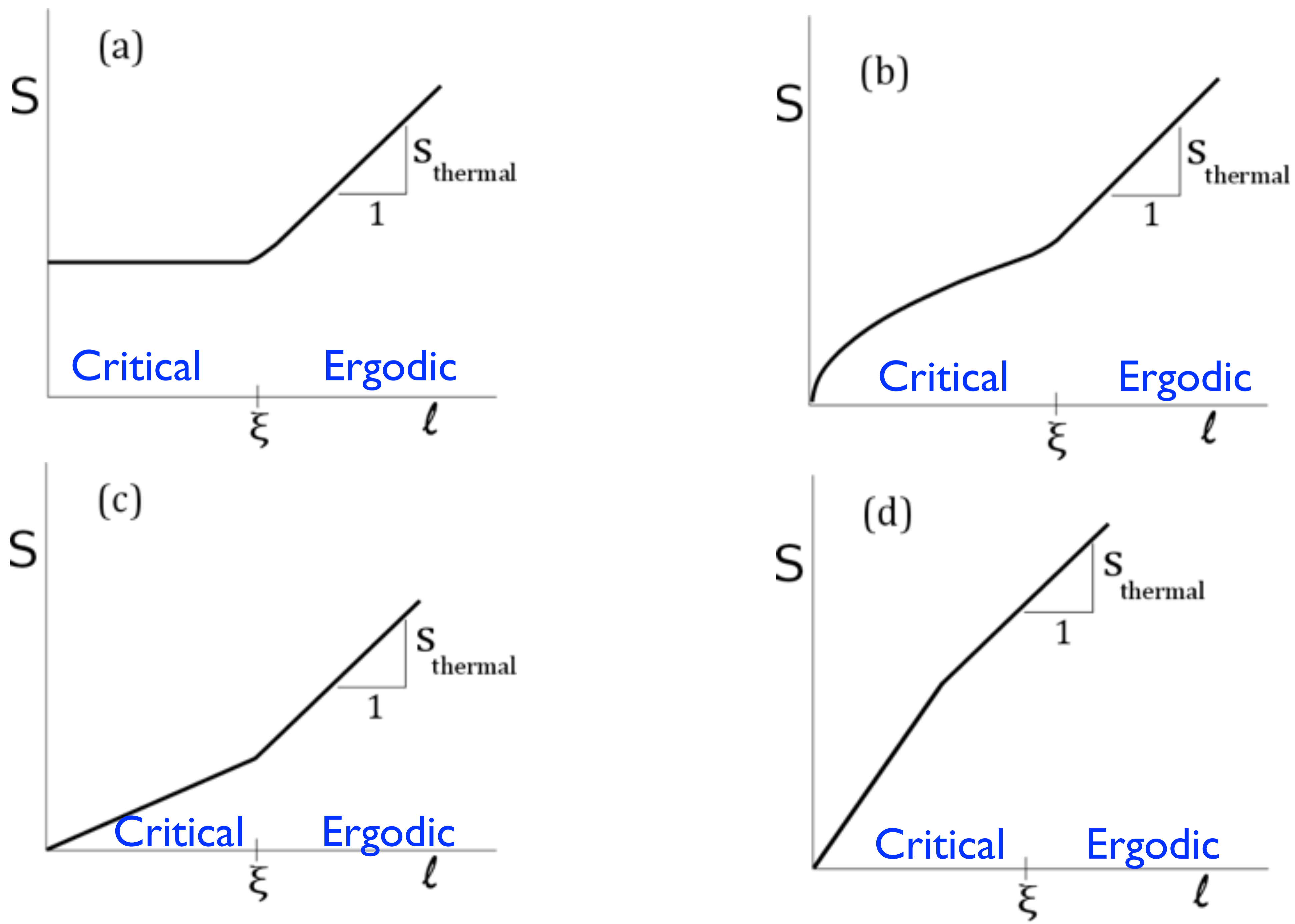
"universal entanglement" of critical point

Unknown

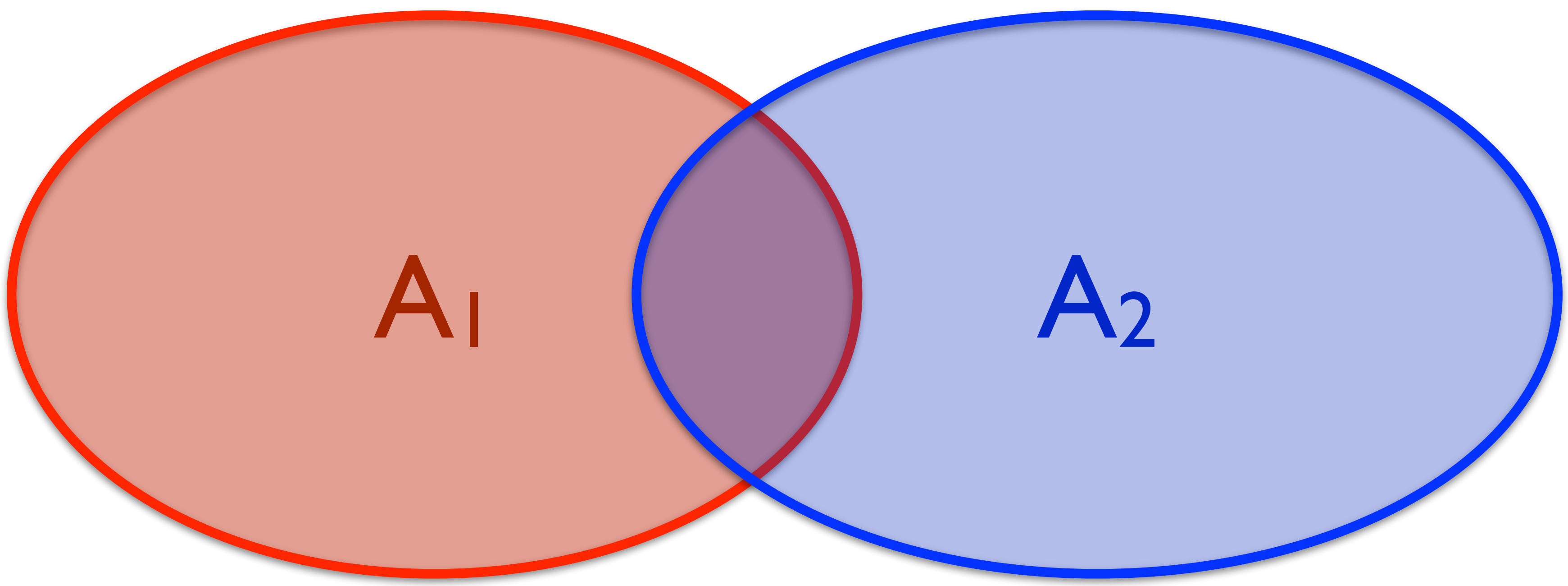
Entanglement of ergodic phase

Known

Critical Entanglement:
A Catalog of Scaling Behaviors.



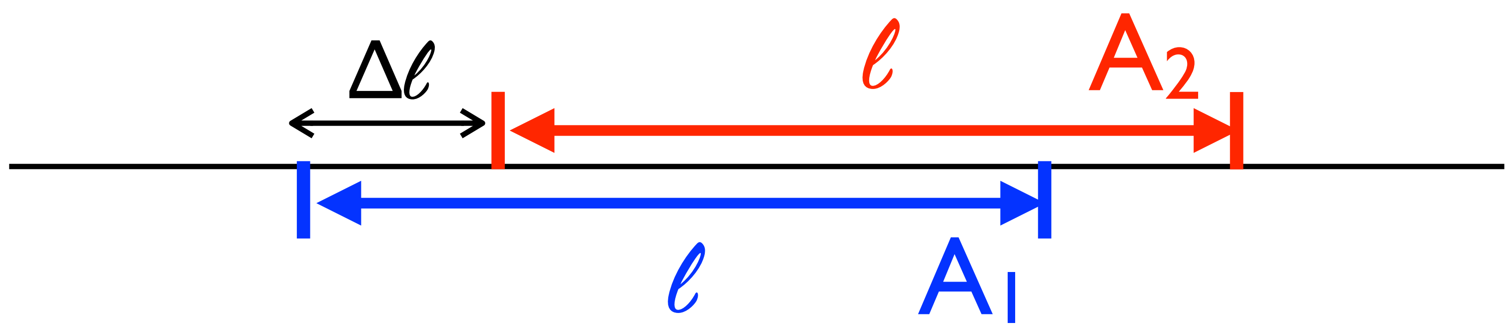
Our toolbox: Strong Subadditivity constraint on Entanglement



$$S(A_1) + S(A_2) - S(A_1 \cup A_2) - S(A_1 \cap A_2) \;\geq\; 0$$

Lieb, Ruskai 1973

Strong Subadditivity & Entanglement Scaling



$$S(A_1) + S(A_2) - S(A_1 \cup A_2) - S(A_1 \cap A_2) \ \geq \ 0$$

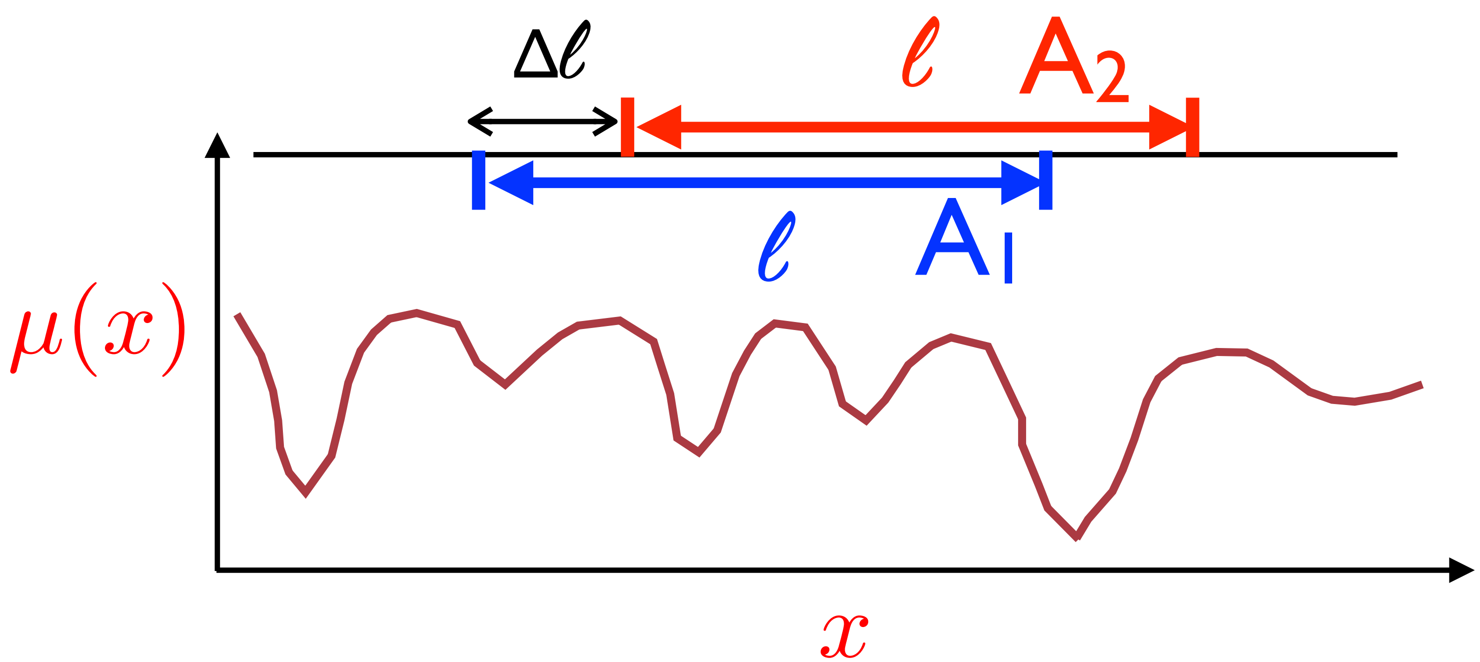
$$\Rightarrow \ S(\ell) + S(\ell) - S(\ell + \Delta\ell) - S(\ell - \Delta\ell) \geq 0$$

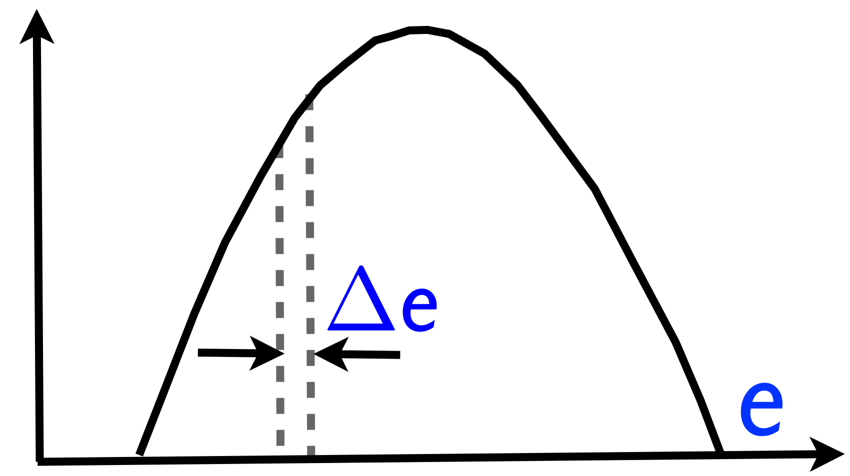
$$\Rightarrow \qquad \boxed{\frac{\partial^2 S(\ell)}{\partial \ell^2} \leq 0}$$

Hirata, Takayanagi 2007

cf: Casini, Huerta's proof of c-theorem 2004

Inequality with Disorder?





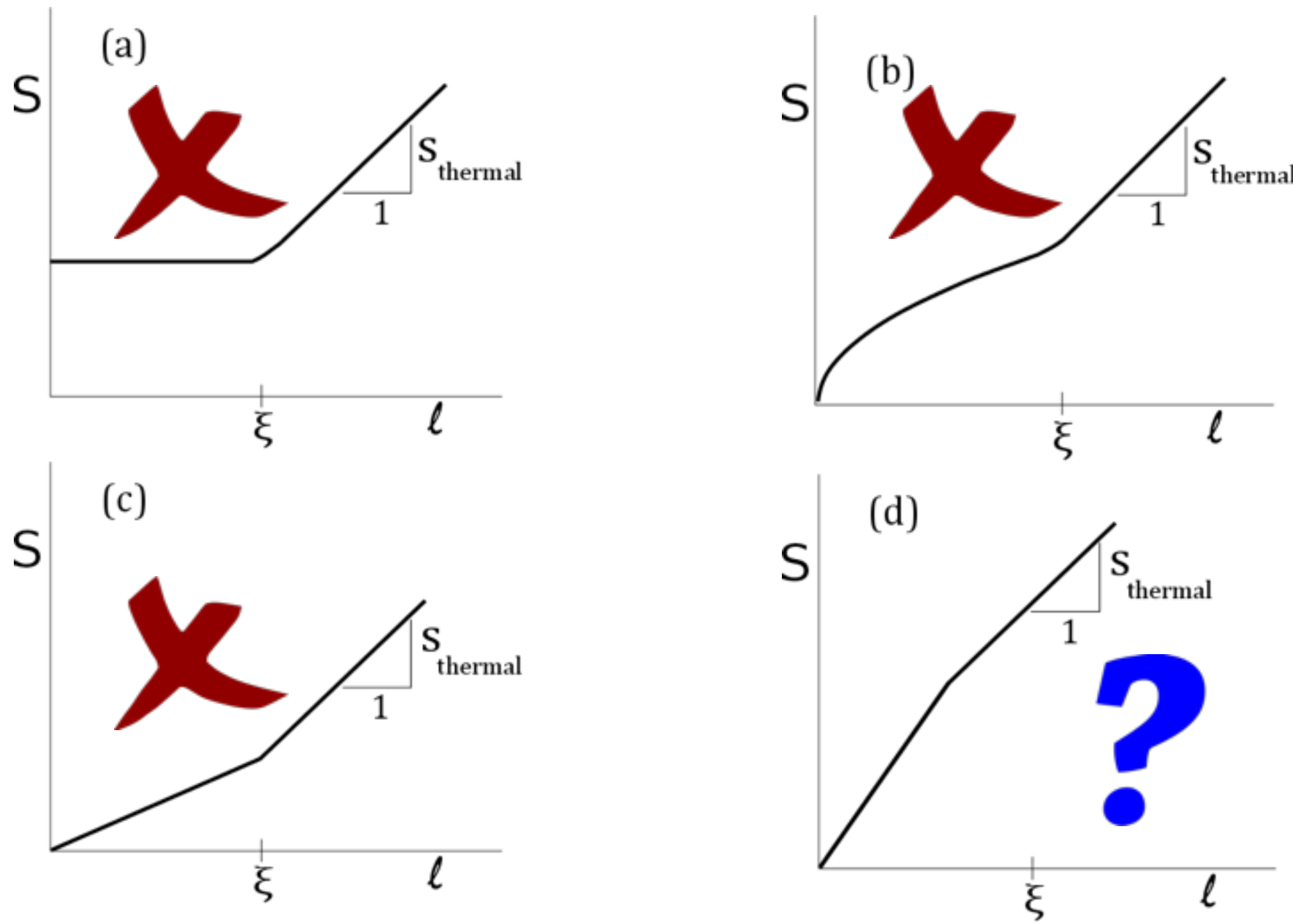
$$S(\ell,e) = \lim_{\Delta e\to 0}\lim_{V\to+\infty}\sum_{\mathcal{D}} P(\mathcal{D})\left(\frac{\sum_{e'=e-\Delta e/2}^{e'=e+\Delta e/2} S_{\mathcal{D}}(\ell,e')}{\mathcal{N}_{\mathcal{D}}}\right)$$

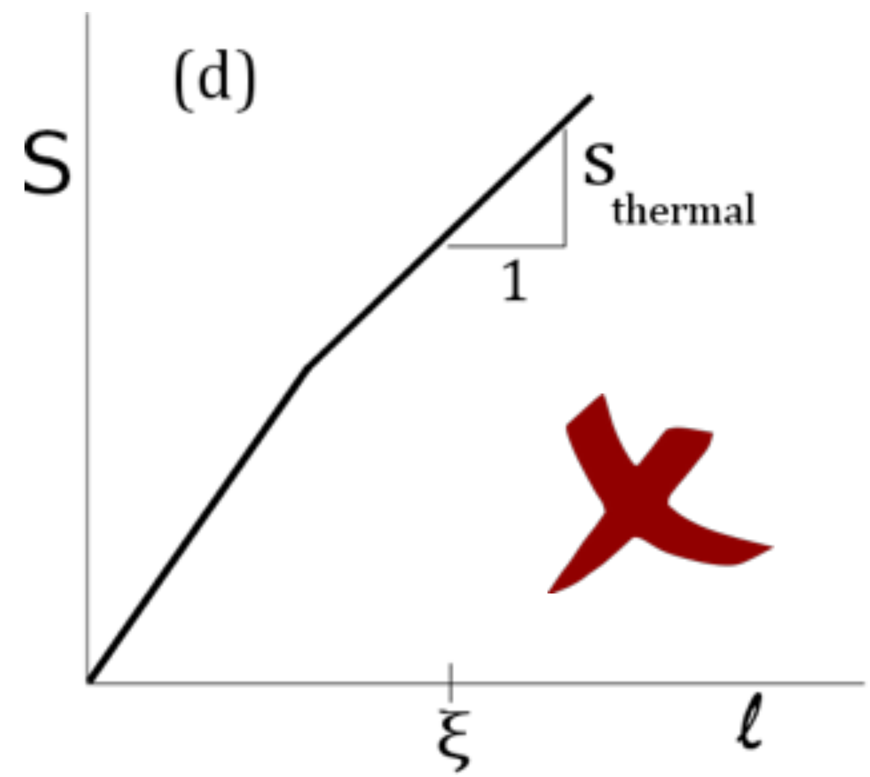
$$\boxed{\frac{\partial^2 S(\ell,e)}{\partial \ell^2} \le 0}$$

TG 2014

Ruling out Possibilities via

$$\frac{\partial^2 S(\ell)}{\partial \ell^2} \leq 0$$





Not allowed because $s \leq s_{\text{Thermal}}$

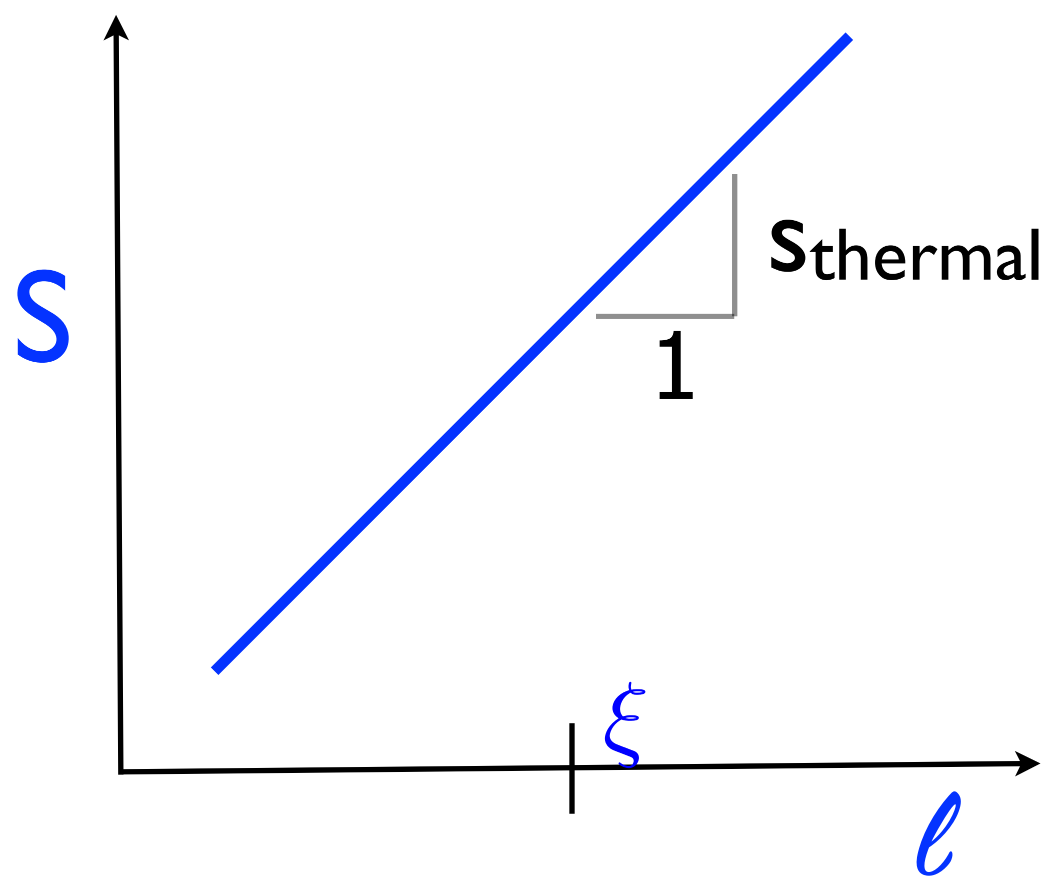
Follows from positivity of "Relative Entropy"

$$\mathrm{tr}\,(\rho_1 \log \rho_1) - \mathrm{tr}\,(\rho_1 \log \rho_2) \geq 0 \qquad \forall \rho_1, \rho_2$$

TG 2014

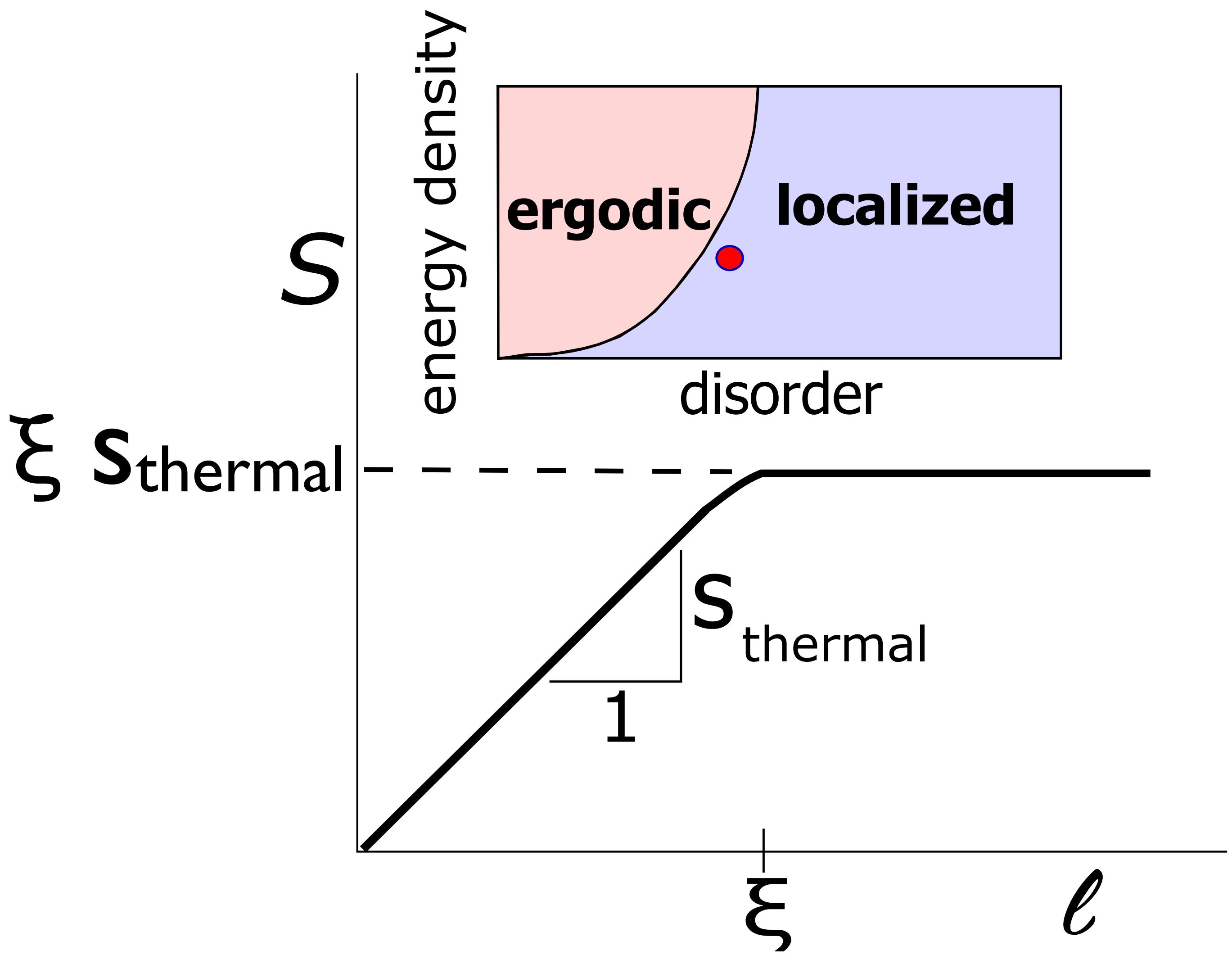
Answer

Critical Entanglement $S = s_{\text{Thermal}}\, \ell \Rightarrow$ Ergodic entanglement!



Note: $\ell/L << 1$

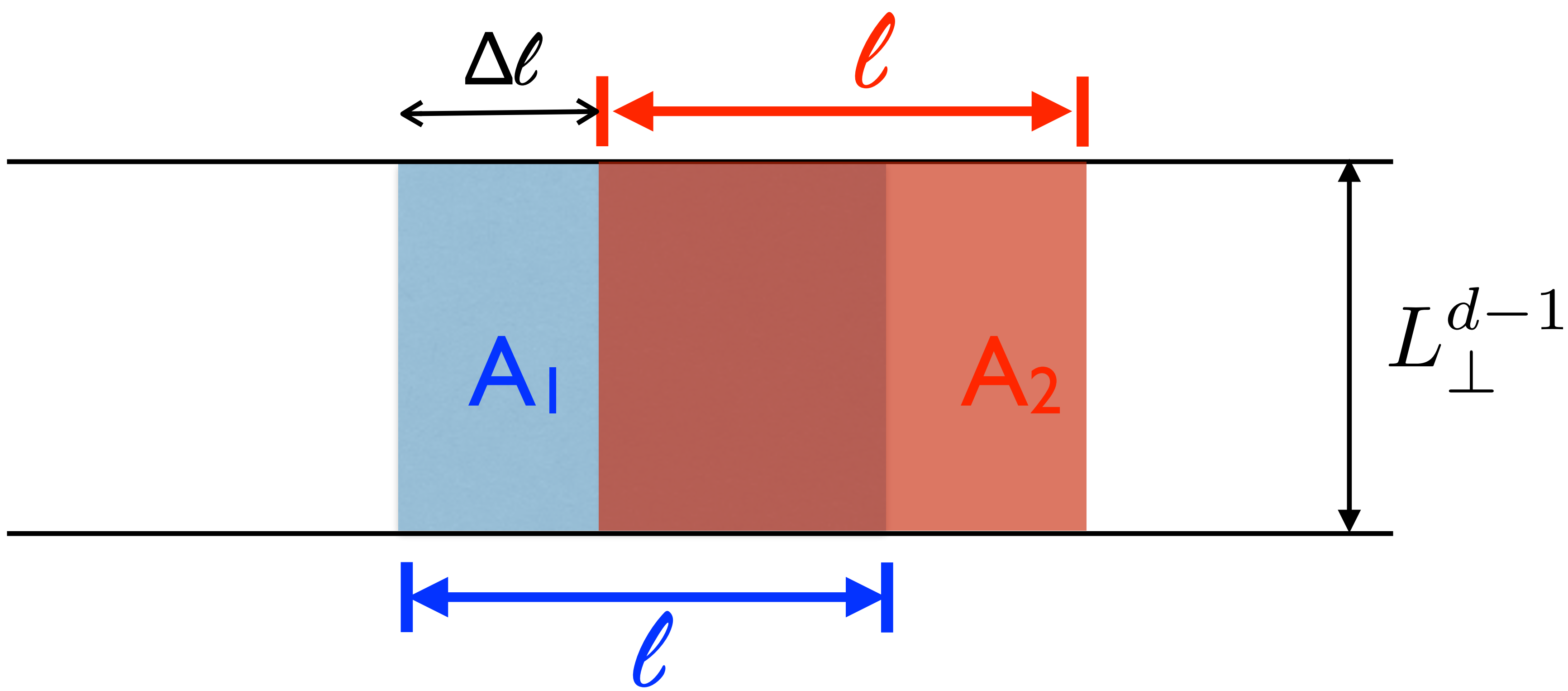
Approaching Transition from the MBL Side



$$\lim_{\ell\to\infty}\lim_{\xi\to\infty}\frac{S}{\ell} \neq \lim_{\xi\to\infty}\lim_{\ell\to\infty}\frac{S}{\ell}$$

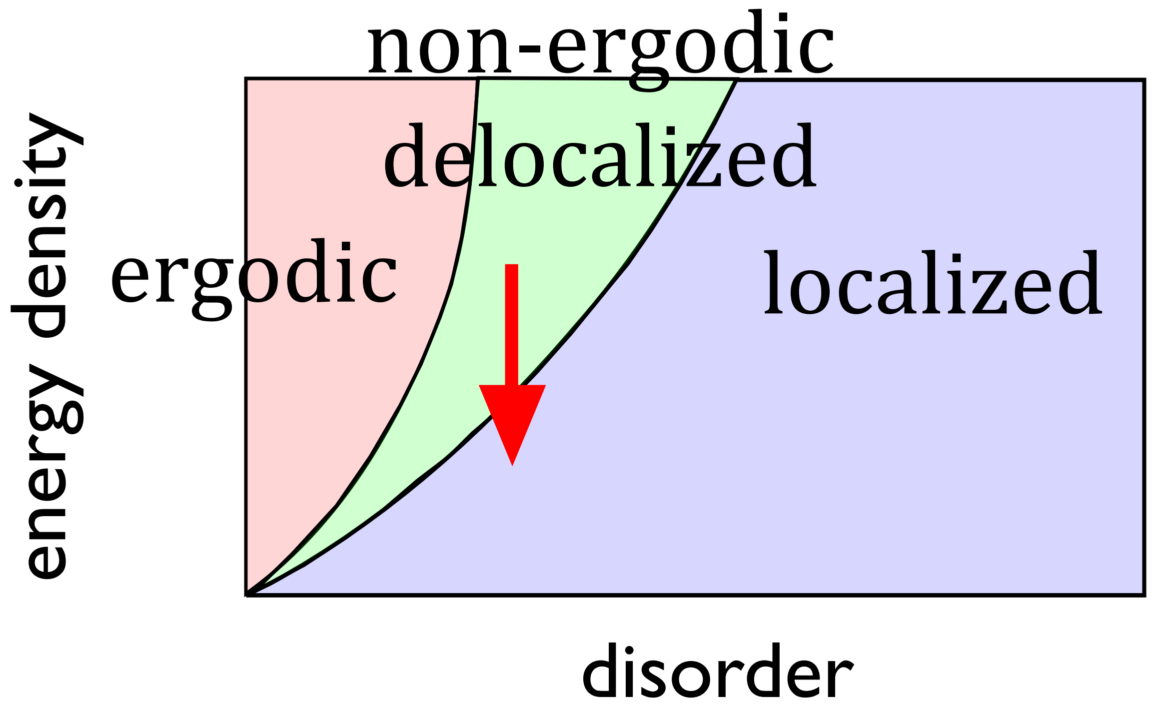
Transition continuous because area-law coefficient diverges from the MBL side!

Generalization to Higher Dimensions



$$\frac{\partial^2 S(\ell, L_\perp)}{\partial \ell^2} \leq 0$$

Same conclusion. Critical point ergodic.



(i) Can a non-ergodic delocalized phase be connected to an ergodic one without a phase transition?

Answer: No

(ii) Restrictions on entanglement scaling at the critical point between localized and the non-ergodic delocalized phase?

Answer: EE at transition can't be area law.

Consequences

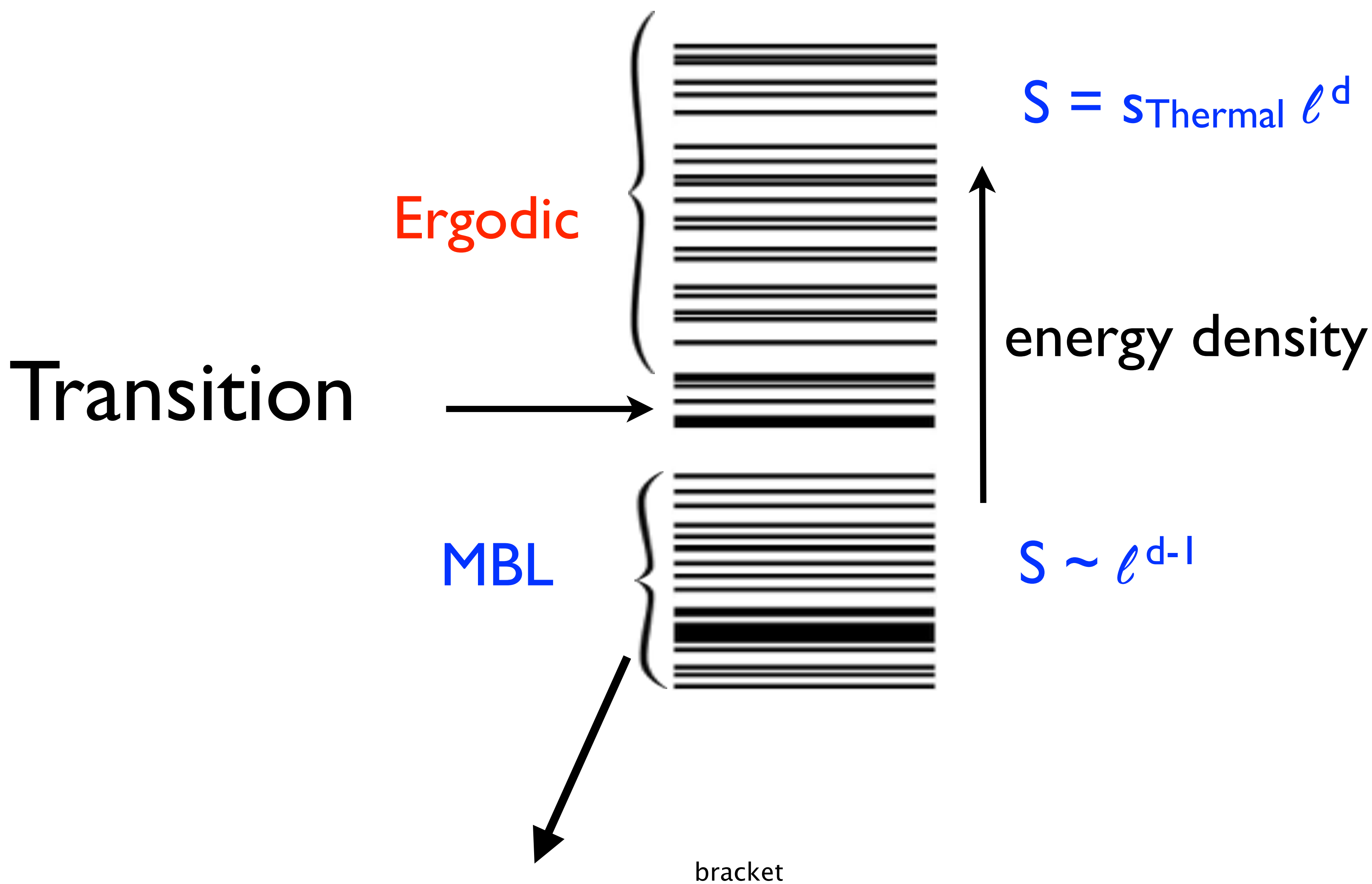
Eigenstates AT the transition satisfy:

$$\langle\psi|O|\psi\rangle = \frac{\mathrm{tr}\left(Oe^{-\beta H}\right)}{\mathrm{tr}\left(e^{-\beta H}\right)}$$

Long time evolution (how long?) of a state with critical energy density should yield a thermal state.

$$\mathrm{tr}_{\overline{A}} \lim_{t\to\infty} e^{iHt}|\psi_0\rangle\ \langle\psi_0|e^{-iHt} \propto e^{-\beta H_A}$$

Scenario discussed so far...



$\langle S \rangle$ progressively exhibits ergodicity on approaching transition.

An Alternative "Rare Region" Scenario:

$\log\langle e^S\rangle$ a correct scaling function and not $\langle S\rangle$.

Can $\langle S\rangle_{\text{critical}}$ be non-ergodic while the transition remains continuous?

Motivated by work of Ehud Altman and collaborators.

Strong subadditivity again constrains $\log\langle e^S\rangle$:

$$\frac{d^2\ \log\langle e^S\rangle}{d\ell^2} - \text{Variance'}\left(\frac{dS}{d\ell}\right) \leq 0$$

A toy model for this scenario

$$\frac{e^{a_L \ell^d}}{e^{a_E \ell^d} + e^{a_L \ell^d}}$$ = probability of finding localized region in length ℓ

$$\frac{e^{a_E \ell^d}}{e^{a_E \ell^d} + e^{a_L \ell^d}}$$ = probability of finding ergodic region in length ℓ

MBL Phase

$\ell \gg \xi$: $a_L > a_E$

$\langle S \rangle \sim \ell^{d-1}$

$\log \langle e^S \rangle = c\,\ell^d$

$(c < s_{\text{thermal}})$

$\ell \ll \xi$: $a_L = a_E$

$\langle S \rangle = s_{\text{thermal}} \ell^d / 2$

$\log \langle e^S \rangle \sim (s_{\text{thermal}})\, \ell^d$

+ non ergodic subleading terms

Ergodic Phase

$\ell \gg \xi$: $a_E > a_L$

$\langle S \rangle = \log \langle e^S \rangle = s_{\text{thermal}} \ell^d$

$\ell \ll \xi$: $a_L = a_E$

$\langle S \rangle = s_{\text{thermal}} \ell^d / 2$

$\log \langle e^S \rangle \sim (s_{\text{thermal}})\, \ell^d$

+ non ergodic subleading terms

Not allowed because $\langle S \rangle_{\text{critical}} < s_{\text{thermal}} \ell$ violates SSA.

Part II

Universal Aspects of Eigenstate Thermalization

Work with Jim Garrison (UCSB).

Summary of ETH

Srednicki 1994

$$\langle E_\alpha | O | E_\beta \rangle = O(E)\delta_{\alpha\beta} + e^{-S(E)/2} f_O(E,\omega) R_{\alpha\beta}$$

$$E = \frac{E_\alpha + E_\beta}{2} \qquad \omega = E_\alpha - E_\beta$$

O(E) = microcanonical expectation value of *O*,

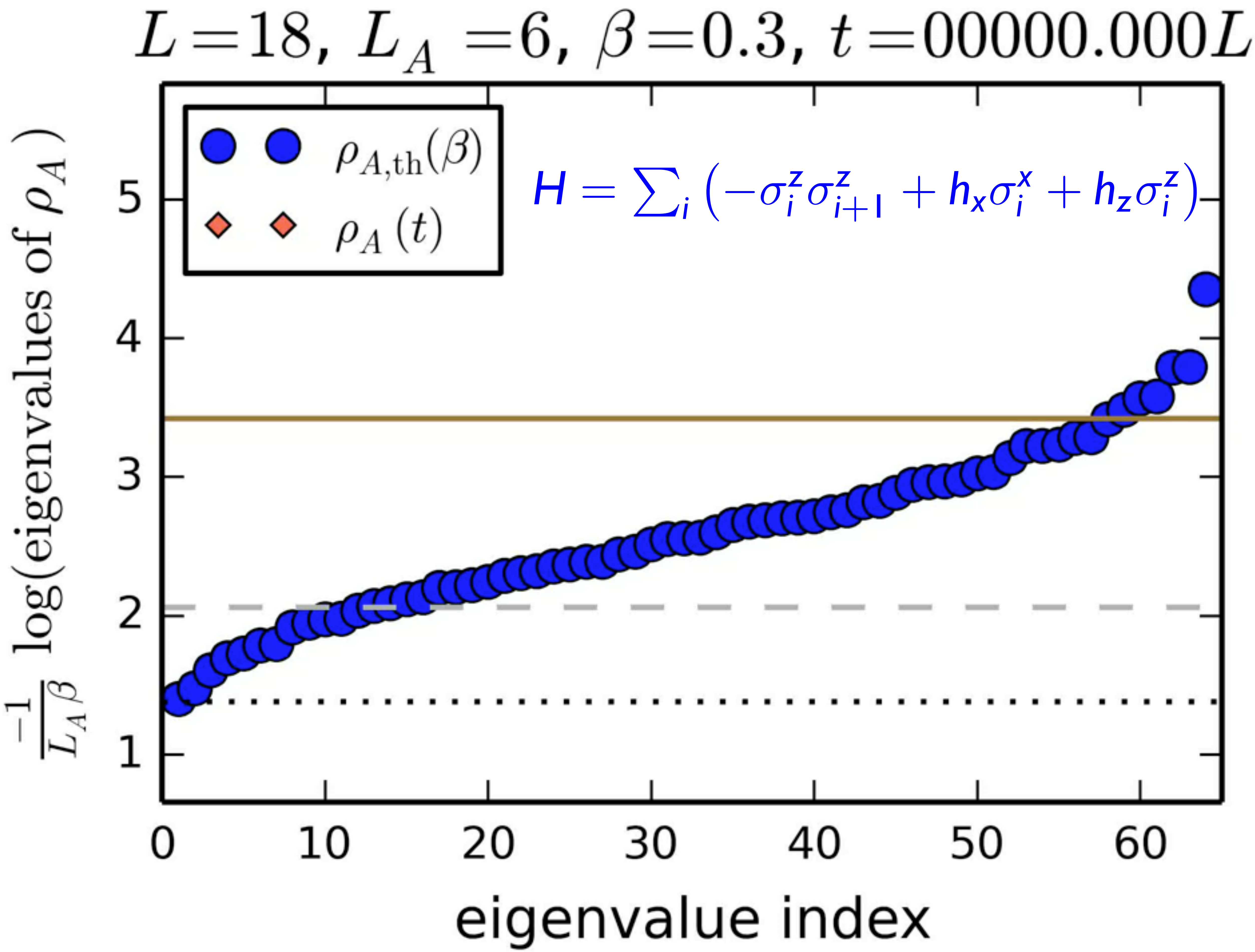
$f_O(E,\omega)$ smooth function,

R random complex variable with zero mean and unit variance.

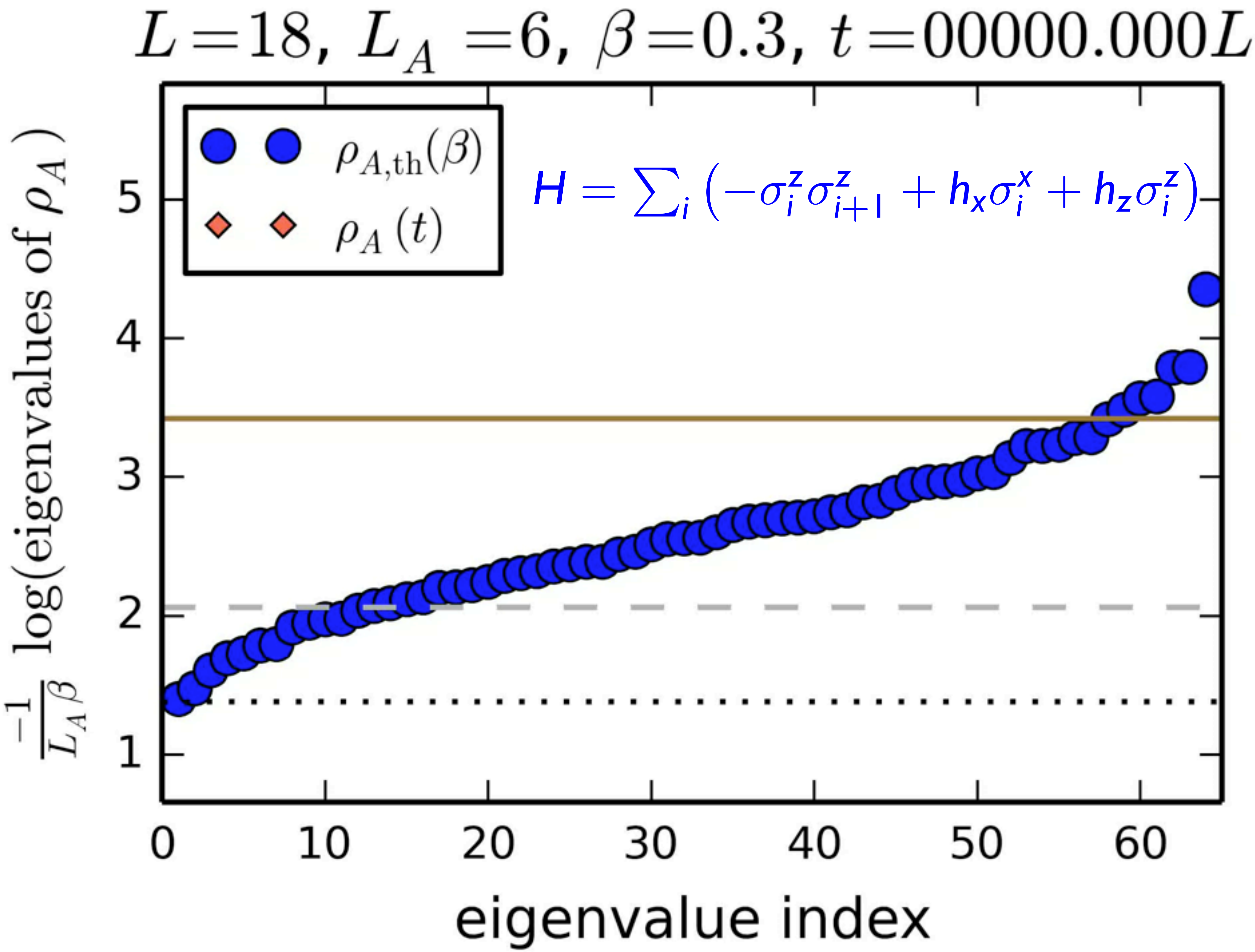
Rigol, Dunjko, Olshanii 2008; Khatami, Pupillo, Srednicki, Rigol (2014).

Questions

- Can one calculate properties of a system at all temperatures using a single eigenstate?
- Does thermalization occurs in a region A even when V_A/V is held fixed i.e. subsytem not much smaller than the total system?
- Is the thermalization time for local Vs non-local operators vastly different?

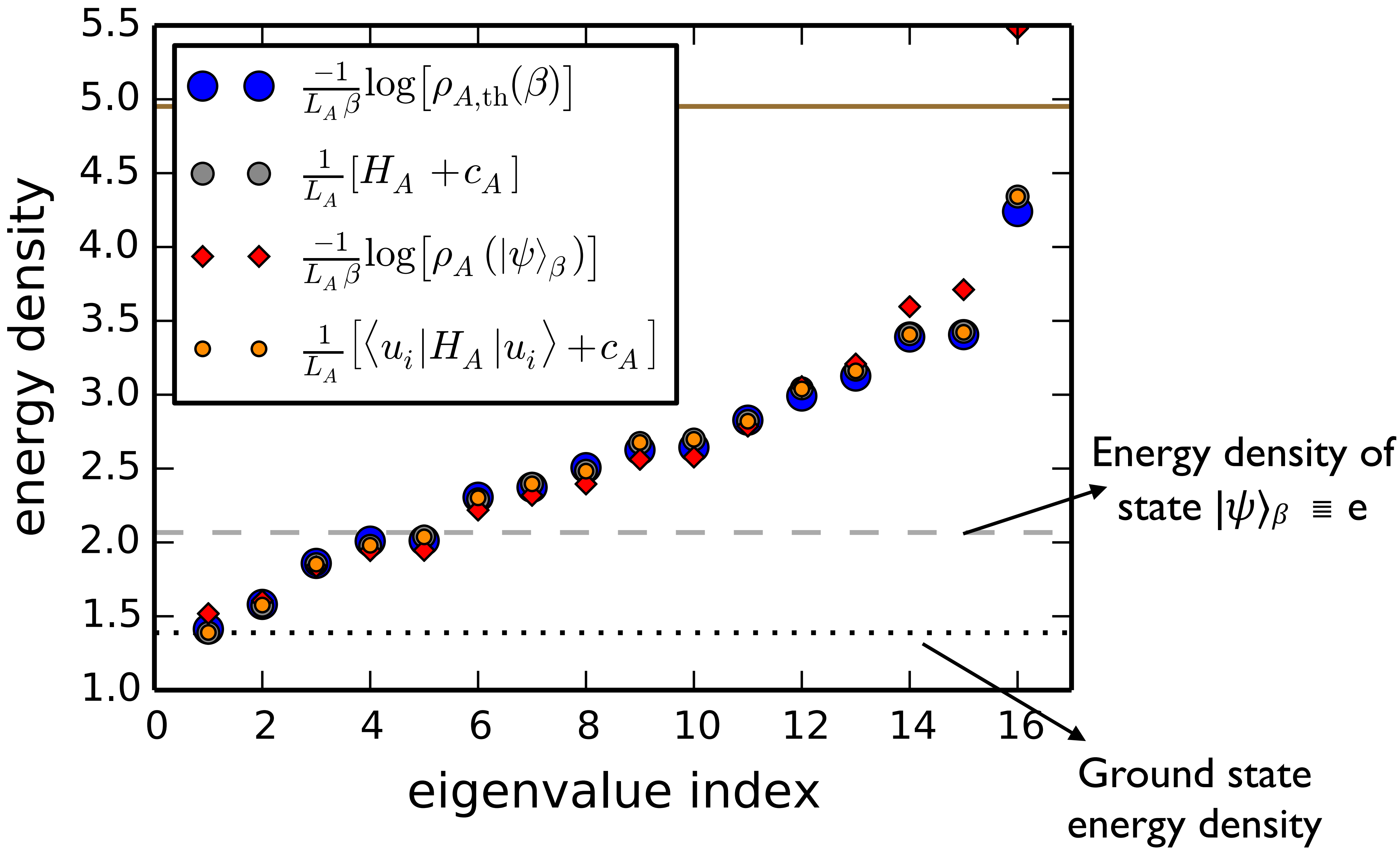


$L = 18,\ L_A = 6,\ \beta = 0.3,\ t = 00000.000L$
$\rho_{A,\mathrm{th}}(\beta)$
$\rho_A(t)$
$H = \sum_i \left(-\sigma_i^z\sigma_{i+1}^z + h_x\sigma_i^x + h_z\sigma_i^z\right)$
$\frac{-1}{L_A\beta}\log(\text{eigenvalues of } \rho_A)$
eigenvalue index



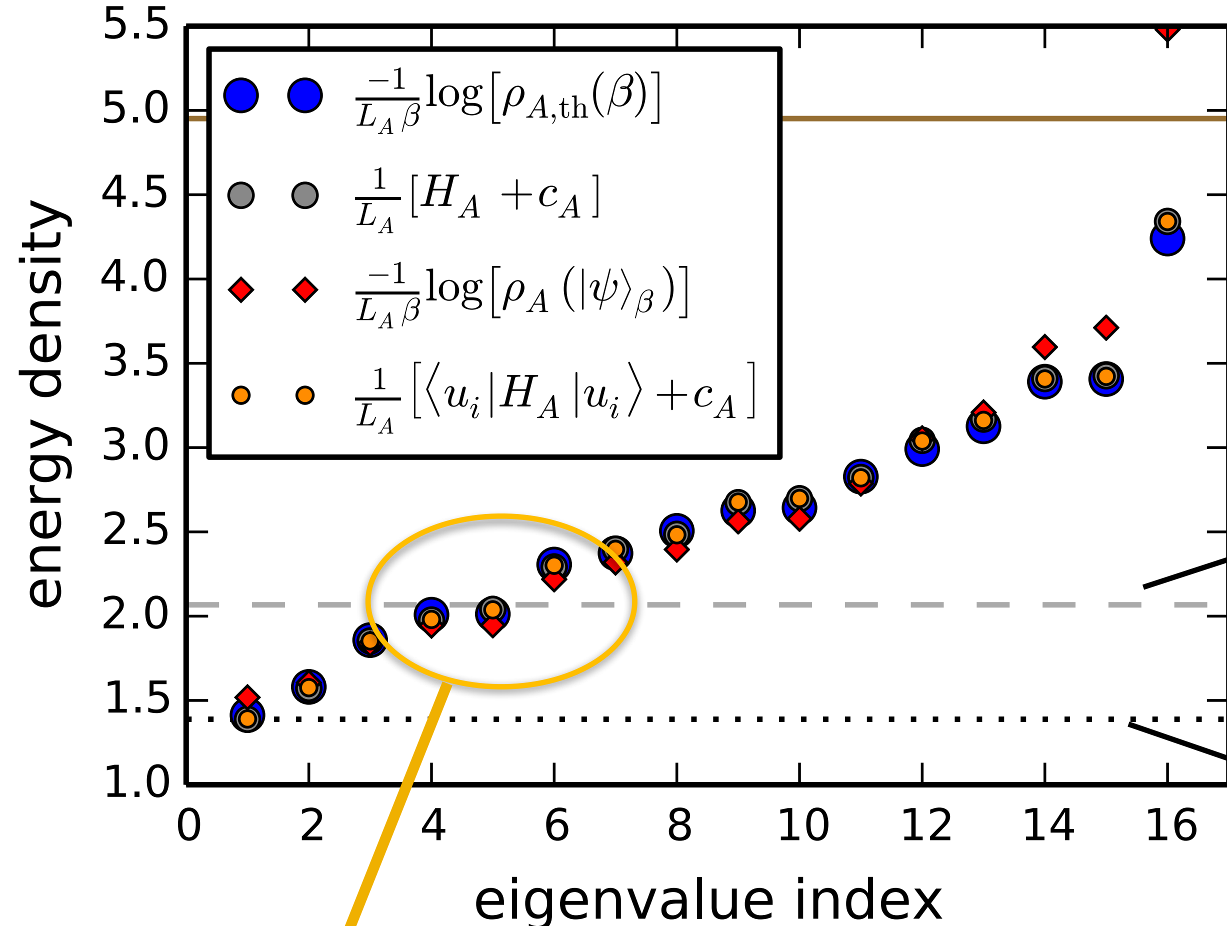
$L=18,\ L_A=6,\ \beta=0.3,\ t=00000.000L$
$\rho_{A,\mathrm{th}}(\beta)$
$\rho_A(t)$
$H=\sum_i\left(-\sigma_i^z\sigma_{i+1}^z+h_x\sigma_i^x+h_z\sigma_i^z\right)$
$\frac{-1}{L_A\beta}\log(\text{eigenvalues of }\rho_A)$
1
2
3
4
5
0
10
20
30
40
50
60
eigenvalue index

Same conclusion with reduced density matrix of an eigenstate (instead of time evolved state)



L=21, L_A = 4

Why is this interesting?



L=21, L_A = 4

Energy density of state $|\psi\rangle_\beta$ ≡ e

Ground state energy density

If ETH was true only for "few-body" operators, in the $V \to \infty$ limit, the spectra need to match <u>**only**</u> at the energy density corresponding to the eigenstate.

Physical Content of $\rho_A(|\psi\rangle_\beta) = \rho_{A,\mathrm{th}}(\beta)$

$$\rho_A(|\psi\rangle_\beta) = \frac{\mathrm{tr}_{\overline{A}}\left(e^{-\beta H}\right)}{\mathrm{tr}\left(e^{-\beta H}\right)} \approx \frac{e^{-\beta H_A}}{\mathrm{tr}_A\left(e^{-\beta H_A}\right)}$$

$$|\psi\rangle_\beta = \sum_i \sqrt{\lambda_i}|u_i\rangle \otimes |v_i\rangle$$

$\lambda_i \propto e^{-\beta E_{A,i}}$ $\quad |u_i\rangle$ = Approximate Eigenstate of H_A

Rare quantum fluctuations of energy in a subsystem allow one to access properties at **<u>all</u>** energy densities as $V_A/V \to 0$.

Consequence #1:

Thermodynamics using a single eigenstate.

$$\rho_A(|\psi\rangle_\beta) = \rho_{A,\mathrm{th}}(\beta)$$

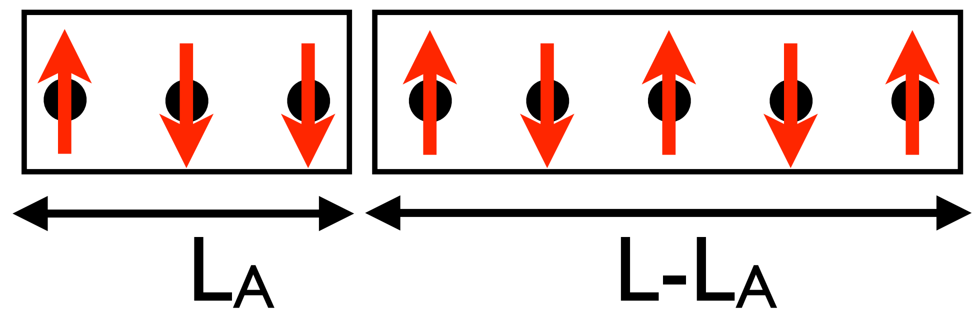
would imply that Renyi entropies encode free energy at various temperatures.

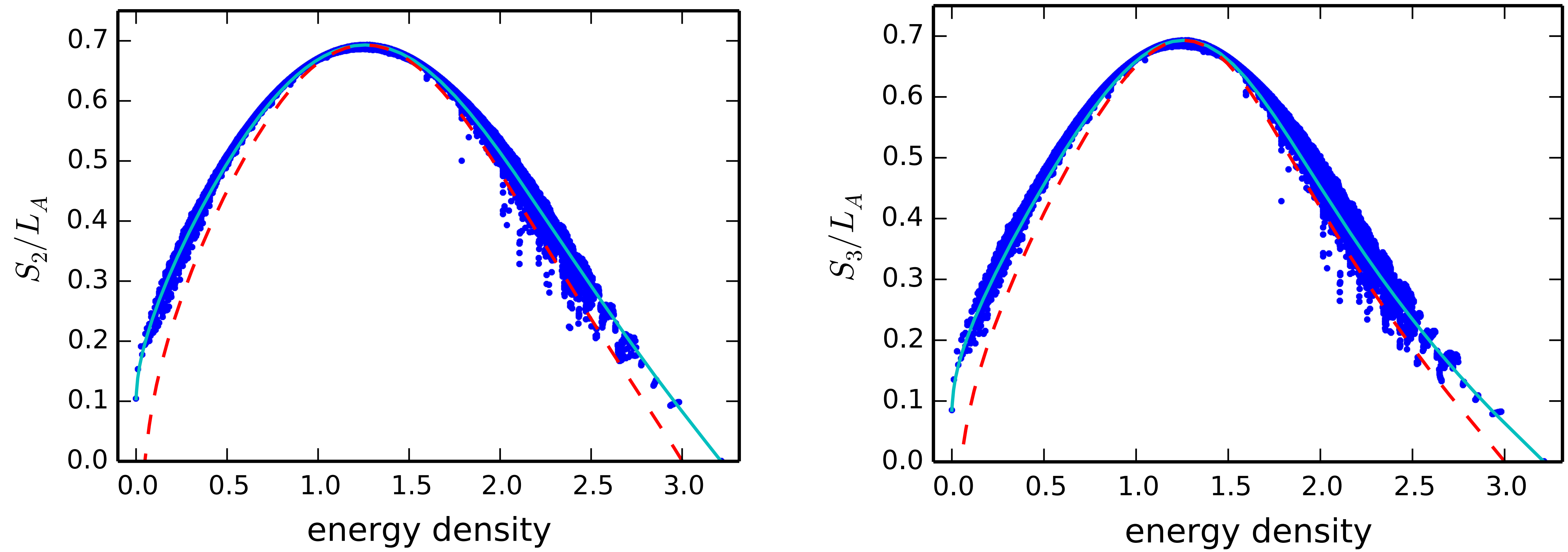
$$S_n(|\psi\rangle_\beta) \ = \ \frac{n}{n-1} V_A \beta \left(f(n\beta) - f(\beta)\right)$$

(holds only to leading order due to conical singularity)

Numerical check on the conjecture

$H = \sum_i \left(-\sigma_i^z \sigma_{i+1}^z + h_x \sigma_i^x + h_z \sigma_i^z\right)$





Blue dots • = Renyi Entropy density S_n/L_A of individual eigenstates

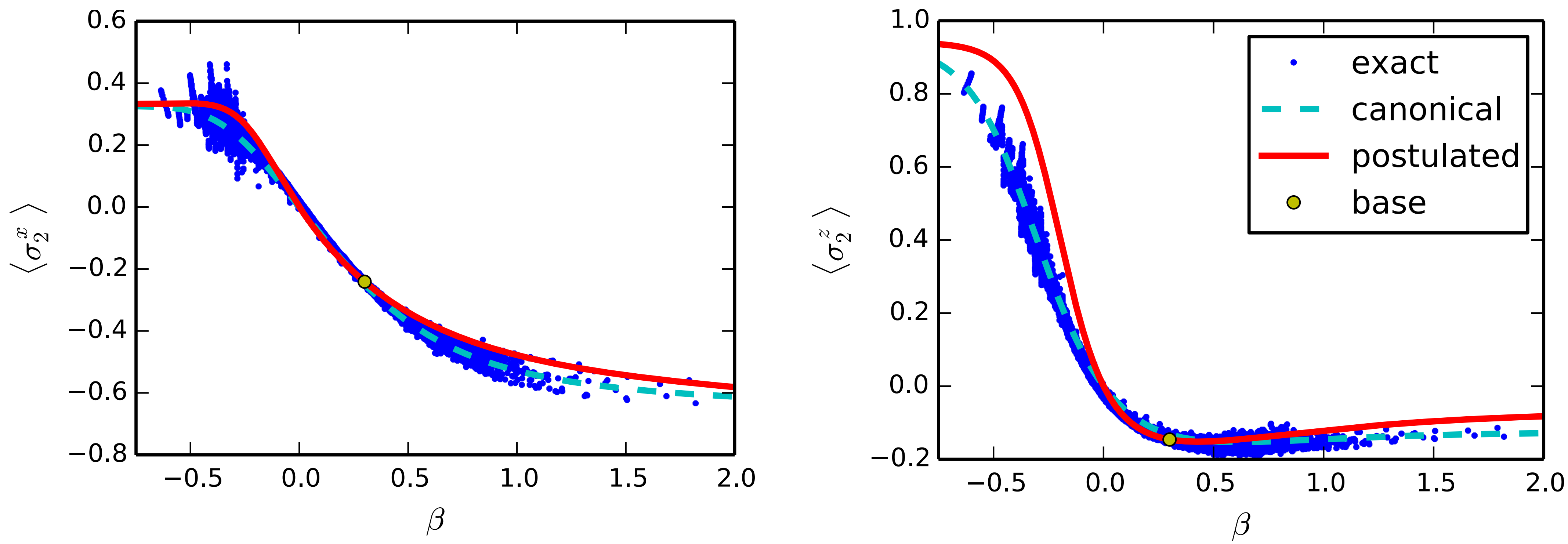
$$= -\frac{1}{n-1}\frac{\log(\mathrm{tr}\rho_{A,th}^n)}{L_A} = \frac{n}{n-1}\beta(f(n\beta) - f(\beta))$$

+ subleading corrections due to conical singularity.

$= \frac{n}{n-1}\beta\left(f(n\beta) - f(\beta)\right)$ in a system with open boundary conditions.

Consequence #2:

Expectation value of observables at all temperatures using a single eigenstate.

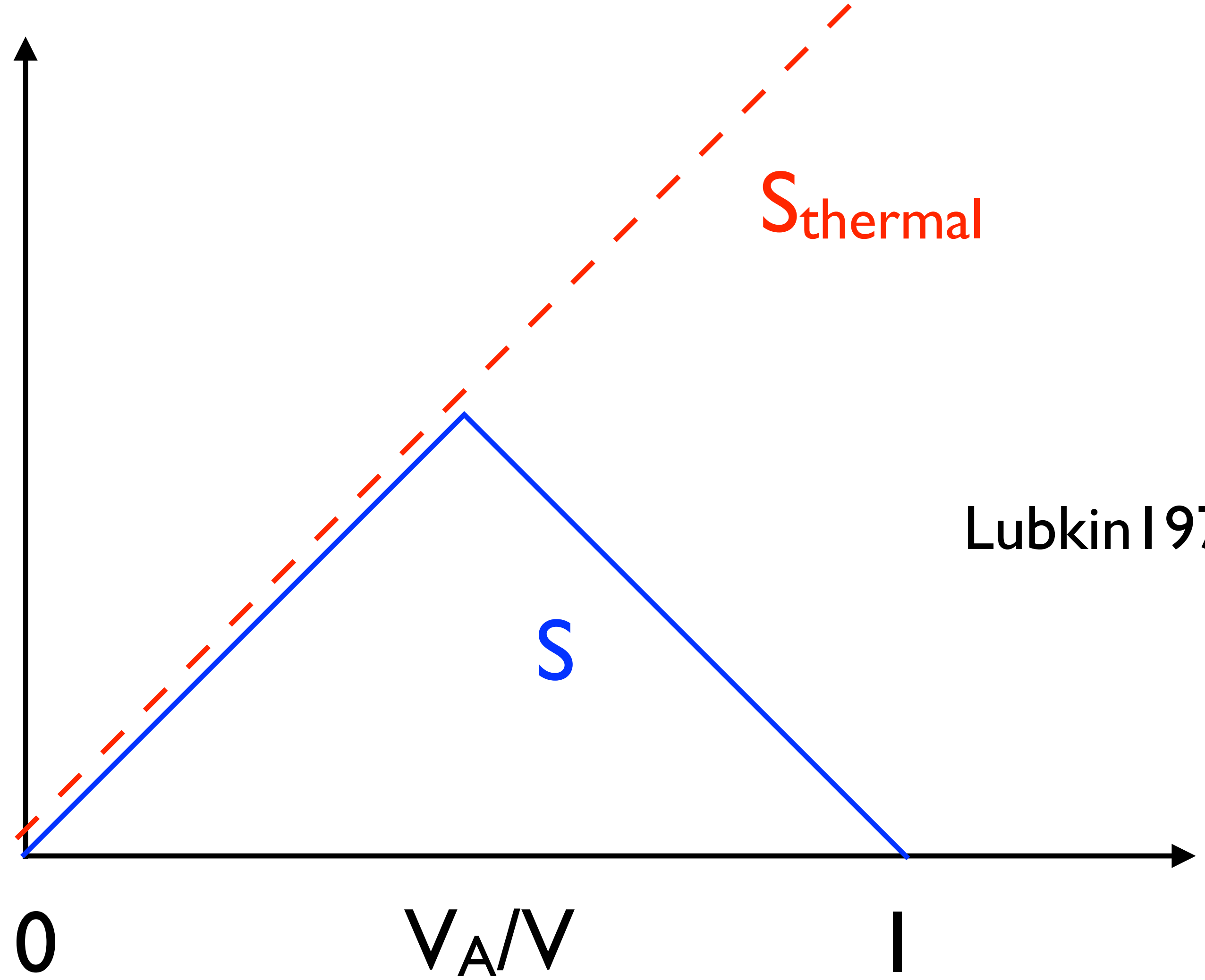


$$\langle O(x)O(y)\rangle_{n\beta} = \frac{\mathrm{tr}_A\left(\rho_A^n(|\psi\rangle_\beta)O(x)O(y)\right)}{\mathrm{tr}_A\left(\rho_A^n(|\psi\rangle_\beta)\right)}$$

+ corrections of order $e^{-x/\xi_T}, e^{-y/\xi_T}$

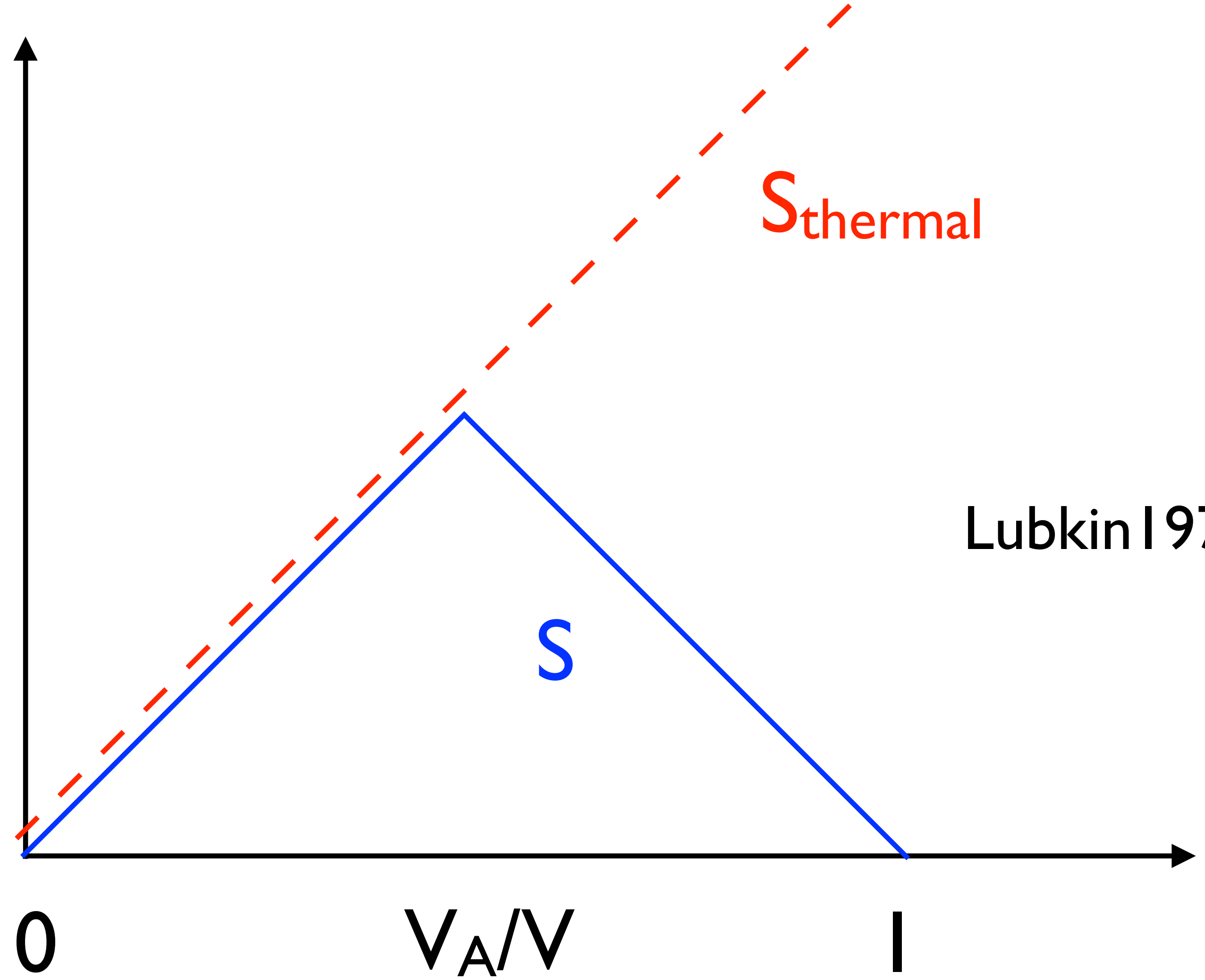
Questions

- Can one calculate properties of a system at all temperatures using a single eigenstate?
- Does thermalization occurs in a region A even when V_A/V is held fixed i.e. subsystem not much smaller than the total system?
- Is the thermalization time for local and non-local operators vastly different?



Lubkin1978; Lloyd, Pagels 1988;
Page 1993

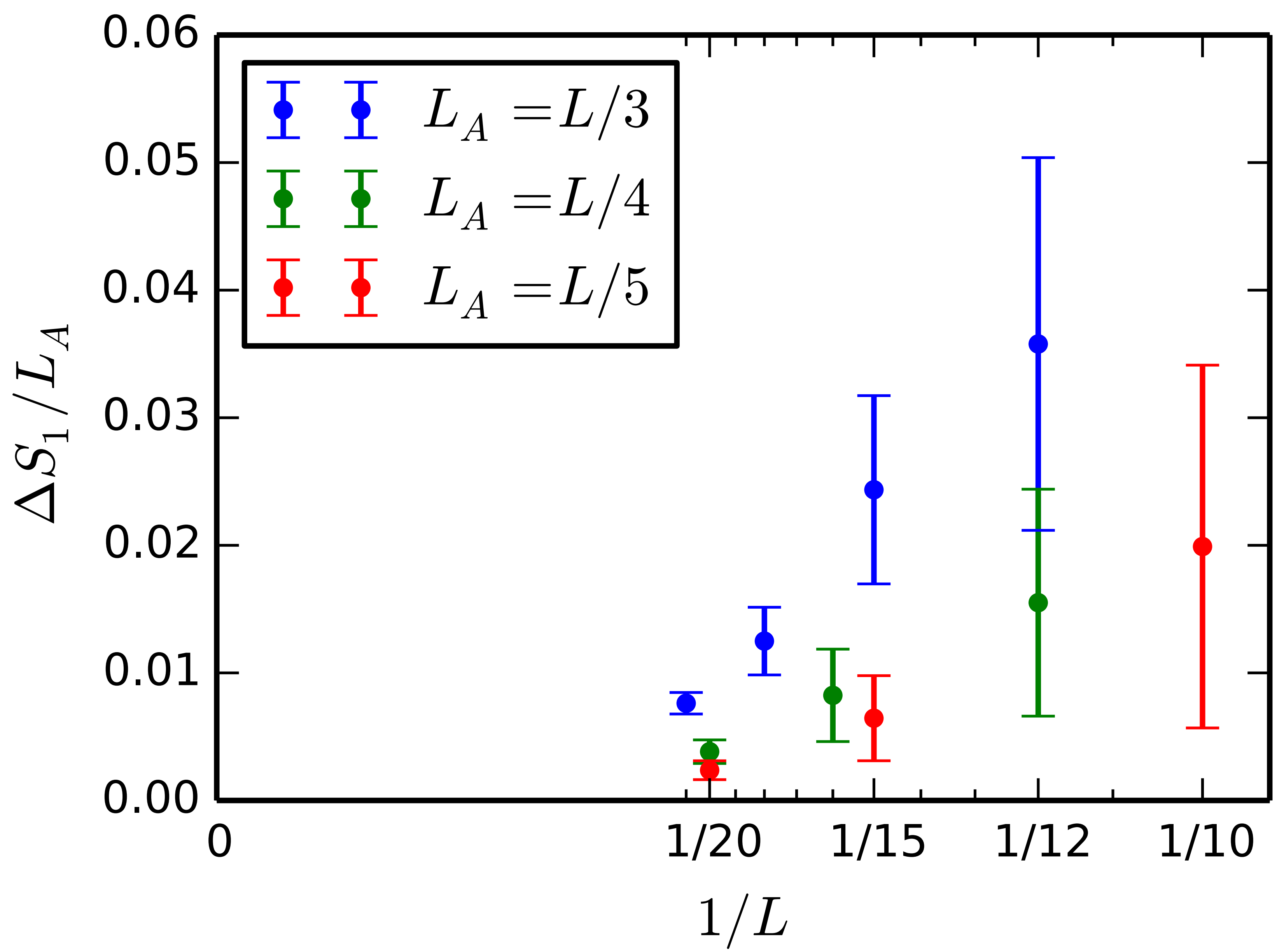
For random pure states, entanglement entropy density equals thermal entropy density as long as $V_A/V < 1/2$.



Lubkin1978; Lloyd, Pagels 1988; Page 1993

Does the above plot holds true for <u>finite energy density eigenstates</u> of an ergodic system?

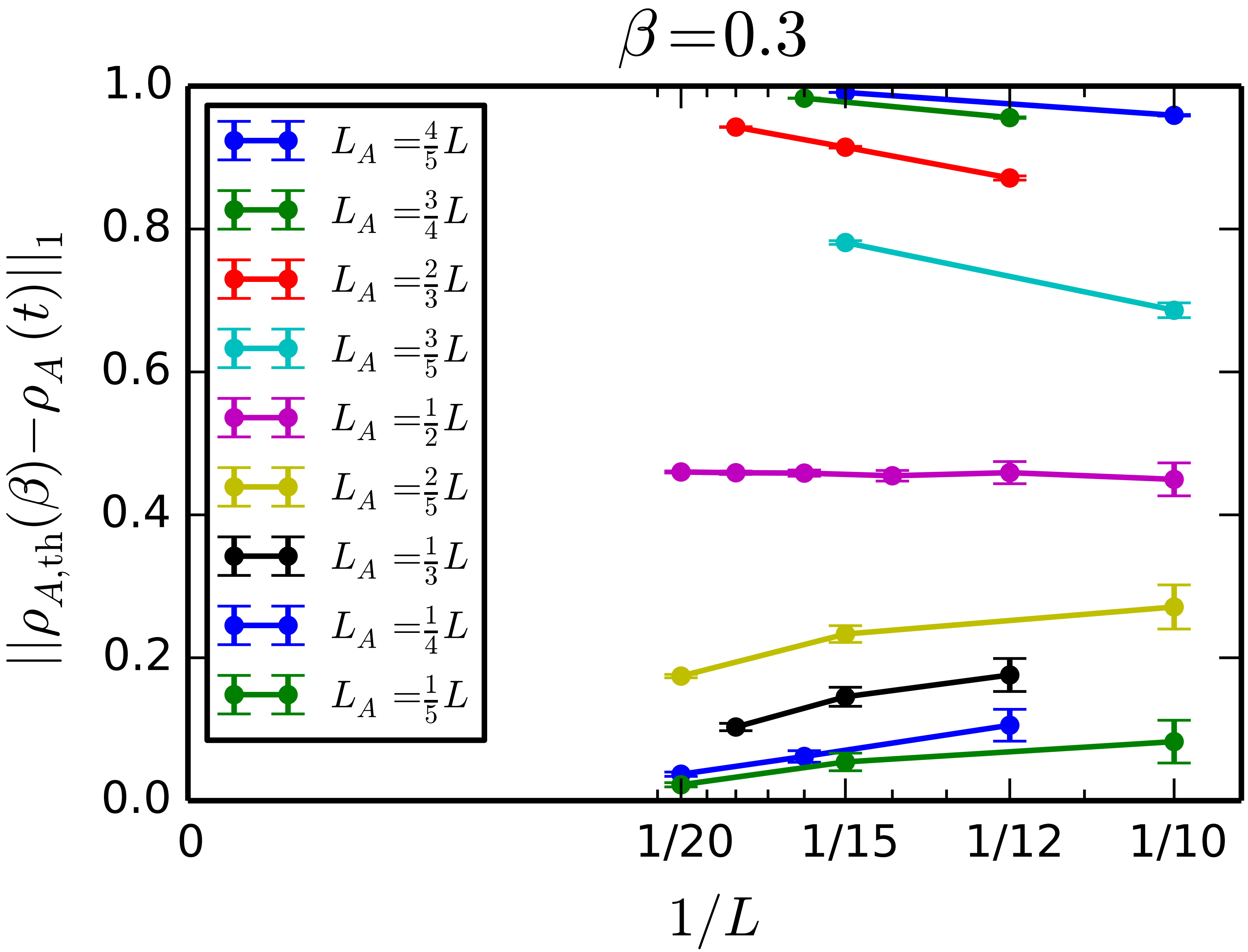
Scaling of Entanglement Entropy at fixed L_A/L



Entanglement entropy seemingly equals thermal entropy even at fixed $L_A < L/2$ at non-infinite temperatures as well.

Analytical evidence of conjecture from recent work on large central charge CFTs (Hartman et al (2014), Kaplan et al (2014)).

Long-time scaling of trace distance for fixed V_A/V after quantum quench



Jim Garrison, TG (2015).

Similar result obtained by Hosur.

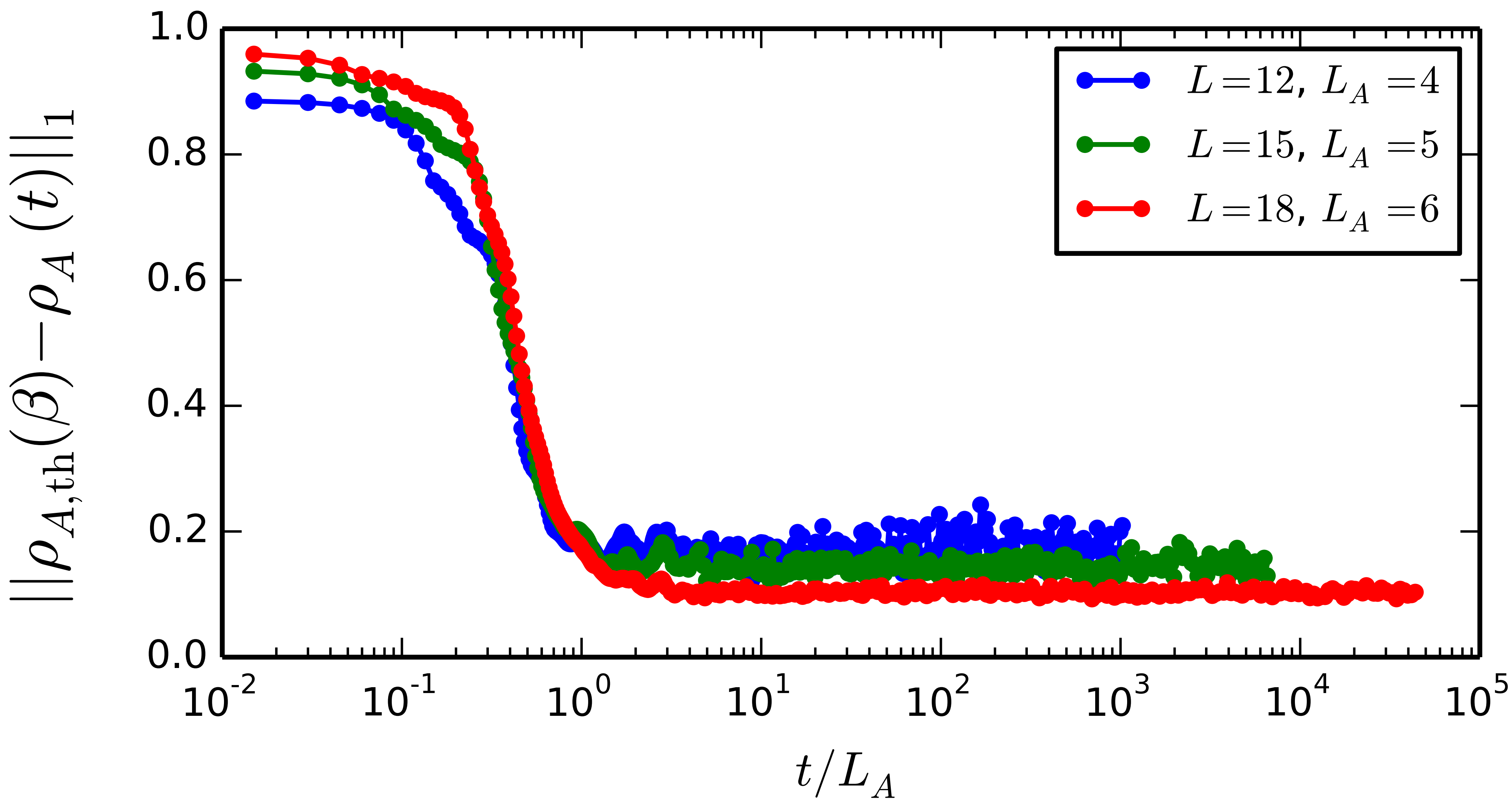
Questions

- Can one calculate properties of a system at all temperatures using a single eigenstate?

- Does thermalization occurs in a region A even when V_A/V is held fixed i.e. subsytem not much smaller than the total system?

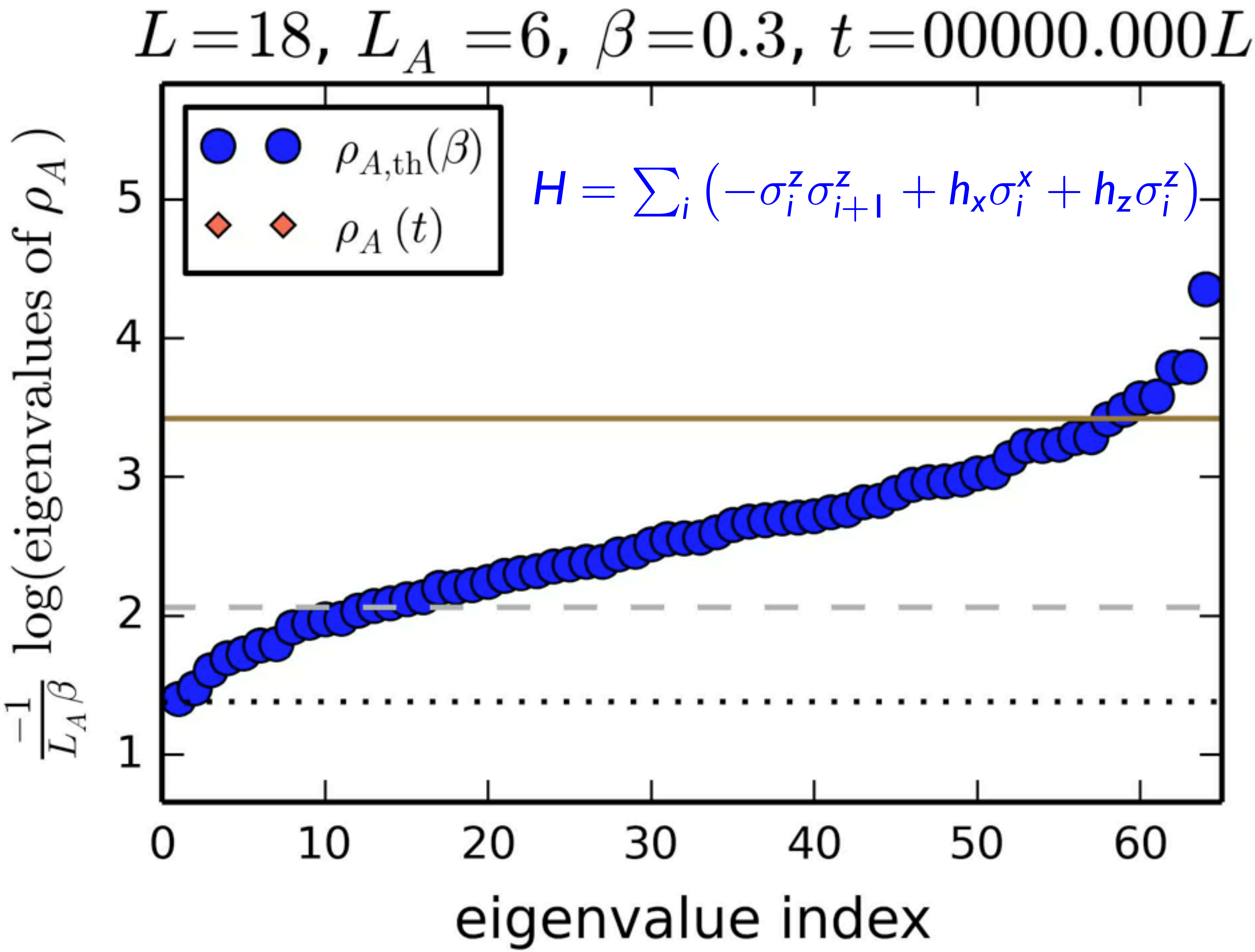
- Is the thermalization time for local and non-local operators vastly different?

Time evolution of trace norm distance

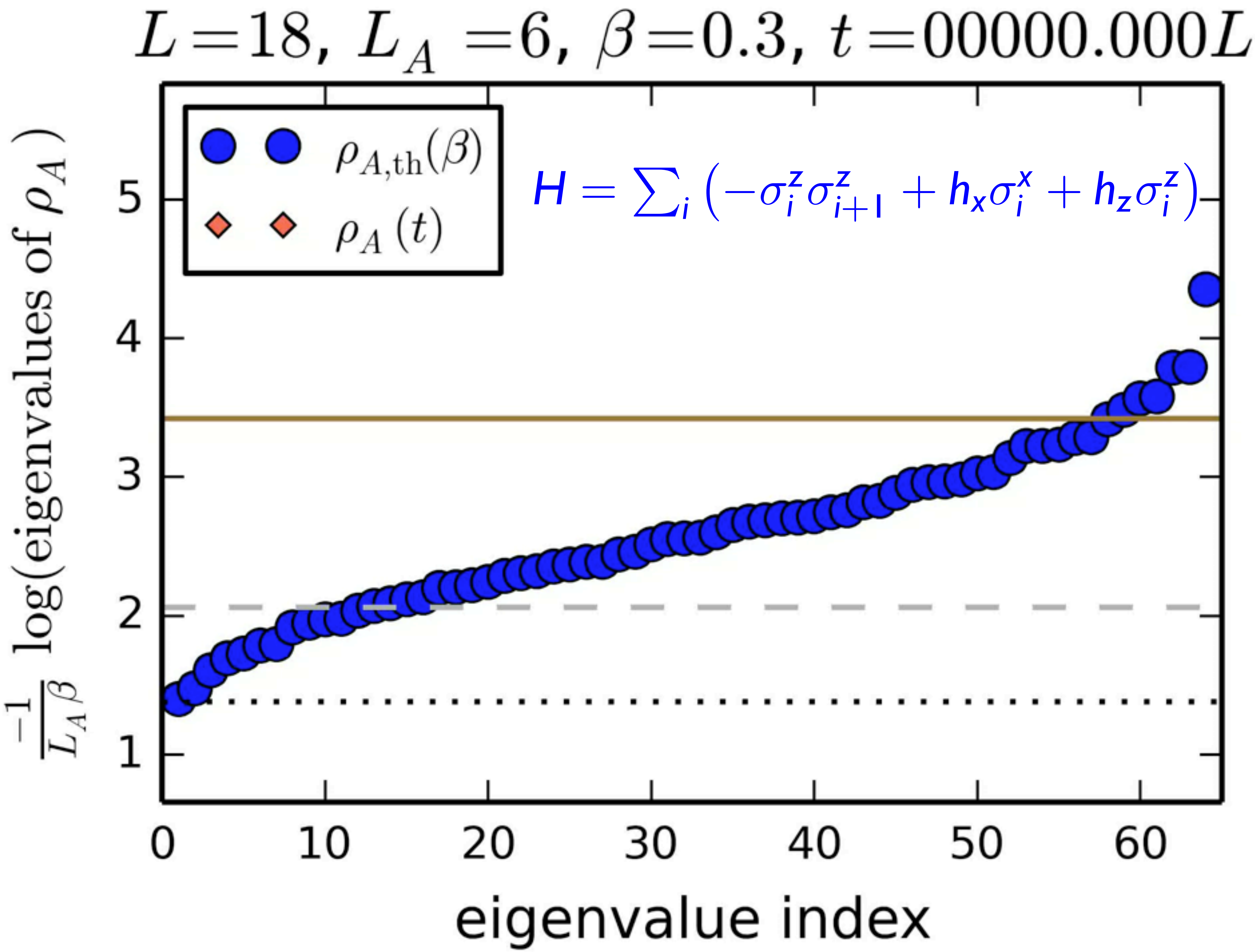


"Nothing happens" after time scale t > L^a where a ≈ 2.

But non-local operators do thermalize <u>slower</u> than local operators! (cf. quench video)



$L = 18,\ L_A = 6,\ \beta = 0.3,\ t = 00000.000L$
$\rho_{A,\mathrm{th}}(\beta)$
$\rho_A(t)$
$H = \sum_i \left(-\sigma_i^z \sigma_{i+1}^z + h_x \sigma_i^x + h_z \sigma_i^z\right)$
$\frac{-1}{L_A \beta}$ log(eigenvalues of ρ_A)
eigenvalue index
0
10
20
30
40
50
60
1
2
3
4
5



$L = 18,\ L_A = 6,\ \beta = 0.3,\ t = 00000.000L$
$\rho_{A,\mathrm{th}}(\beta)$
$\rho_A(t)$
$H = \sum_i \left(-\sigma_i^z\sigma_{i+1}^z + h_x\sigma_i^x + h_z\sigma_i^z\right)$
$\frac{-1}{L_A\beta}\log(\text{eigenvalues of }\rho_A)$
1
2
3
4
5
0
10
20
30
40
50
60
eigenvalue index

Summary and Open Questions

- Under reasonable assumptions, strong subadditivity of entanglement implies that eigenstates at a continuous MBL transition ergodic (volume law entanglement with thermal coefficient). Fully ergodic? Off-diagonal matrix elements of operators?

- Single eigenstate enough to extract Hamiltonian properties at arbitrary temperatures!

- Entanglement entropy corresponding to a finite energy density eigenstates equals thermal entropy as long as $V_A < V/2$. One doesn't need $V_A << V$.

- Can MBL transition be first-order? Perhaps it is always first-order?

- Analytical results for thermalization of non-local operators?