The Abdus Salam International Centre for Theoretical Physics

KITP, Santa Barbara,
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MULTIFRACTALITY OF RANDOM WAVE FUNCTIONS

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Two transitions in JJA?

many-body insulator (good or bad?)

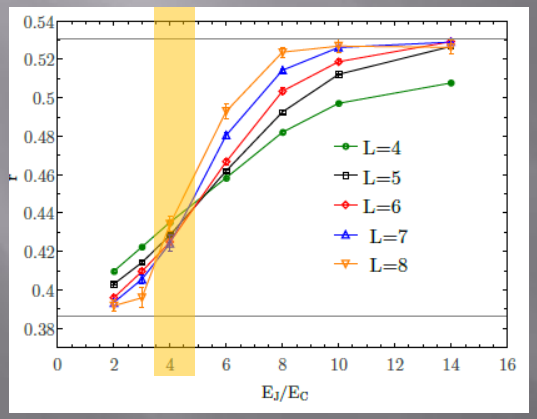
bad metal

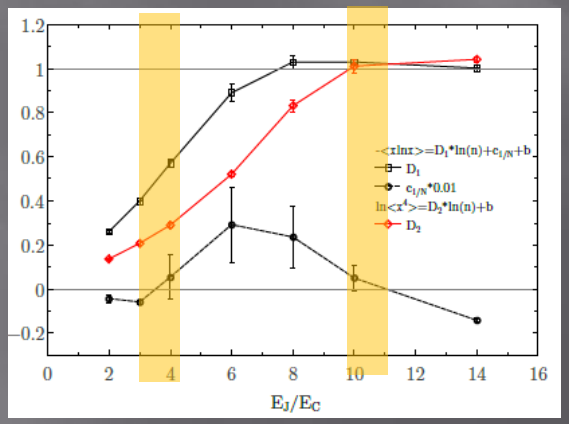
good metal

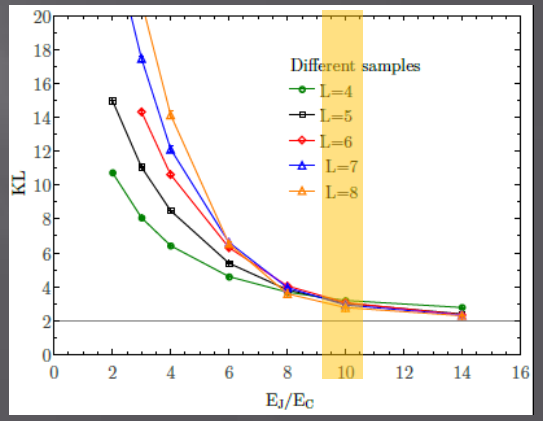
MBLT

ET

E_J/E_C







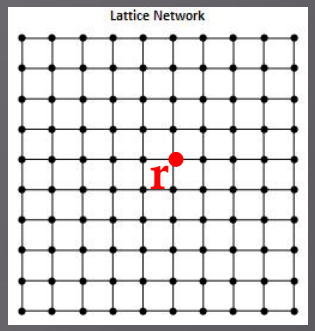
Manuel Pino, Altshuler, Ioffe, VEK

Single- and many-body wave functions

$$\Psi_n(r) = \langle n | r \rangle$$

basis vector in **real space**

$$r = (n_1, n_2, \ldots n_d), \quad Dim = L^d$$
$$n_i = 1, 2, \ldots L$$

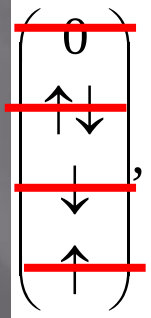


basis vector in **Fock Space**

L interacting S-spins

$$r = (s_1^z, s_2^z, \ldots s_L^z), \quad Dim = q^L$$
$$q = 2S + 1$$

L single-particle orbitals filled by spin-1/2 fermions

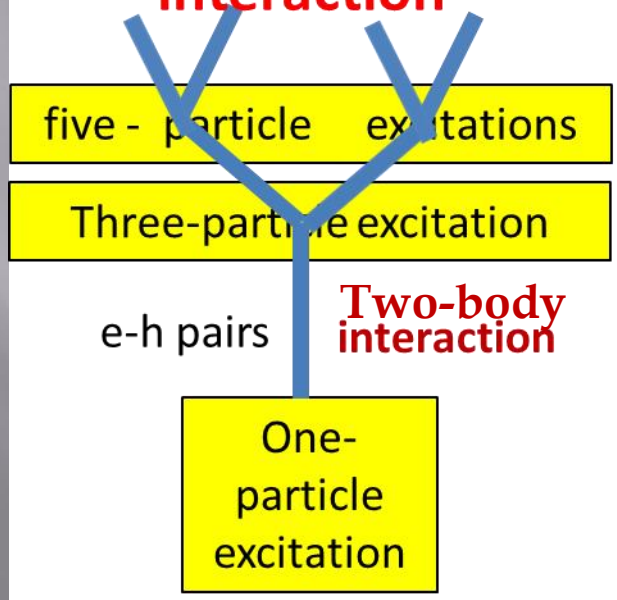


$$Dim = 4^L \quad S_{\text{eff}} = 3/2.$$

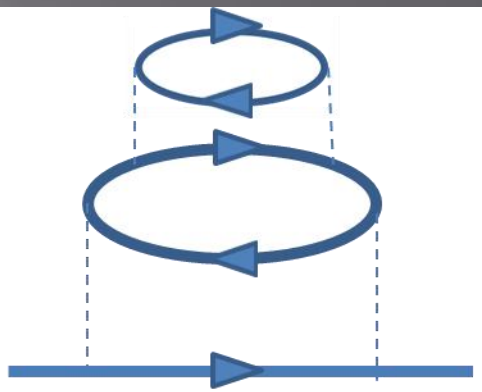
Exponentially large dimension of space

Hierarchical structure of many-body Fock space

Tree-like structure of many-body interaction



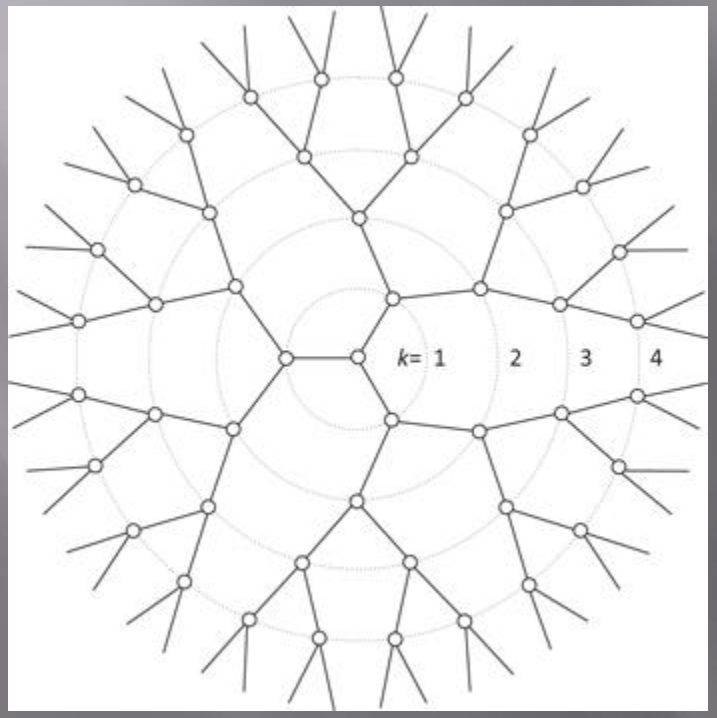
Altshuler, Gefen, Kamenev, Levitov , 1997



Basko, Aleiner, Altshuler, 2005

Real space analogues?

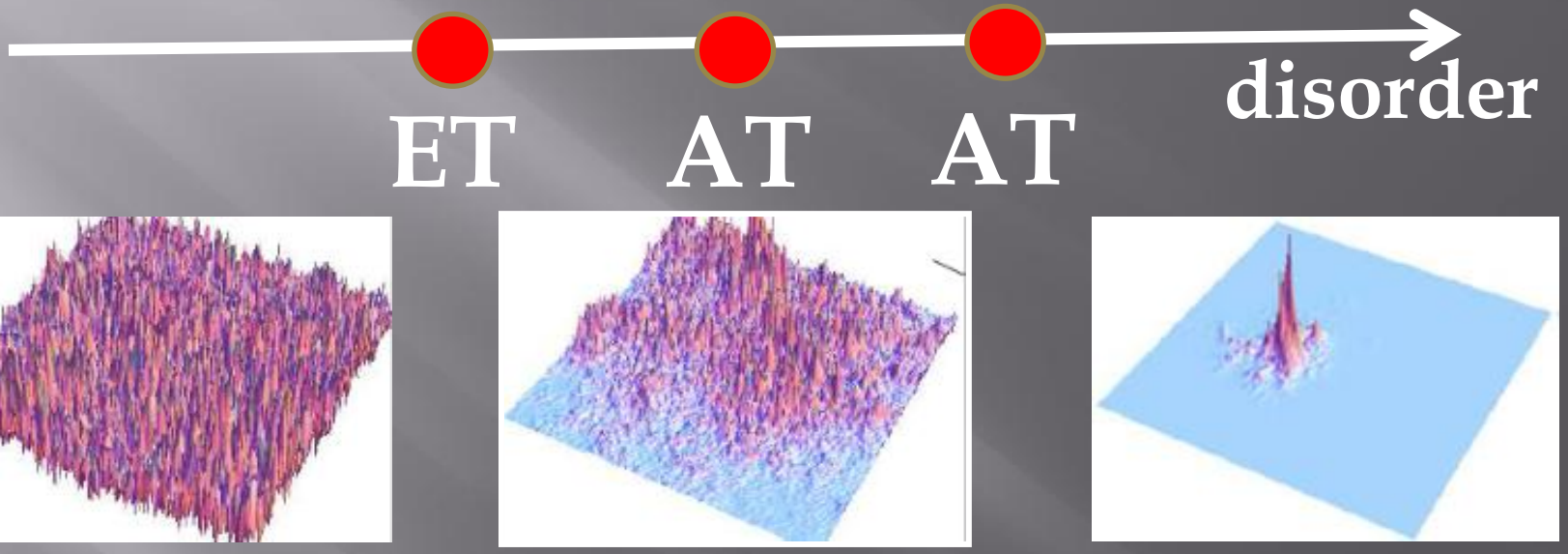
Bethe lattice?



Hierarchical structure

Exponential number of sites

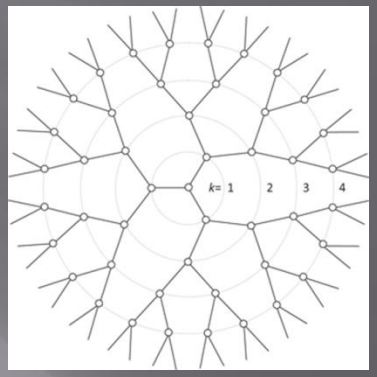
Real space analogue of “bad metal”?



Extended multifractal states

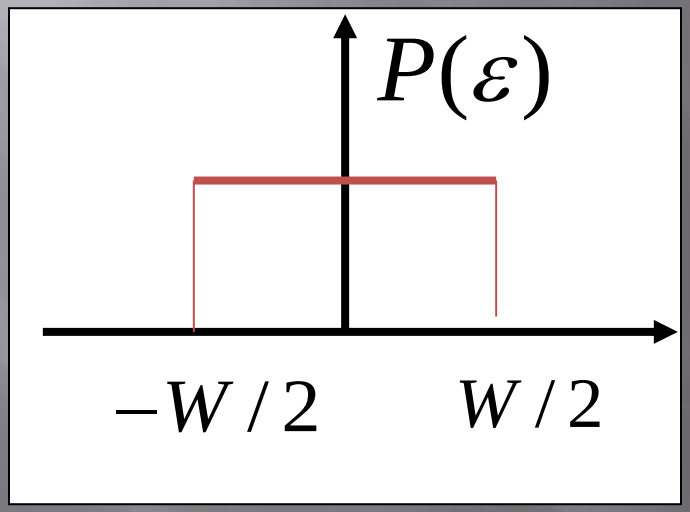
Anderson model on random regular graphs (RRG)

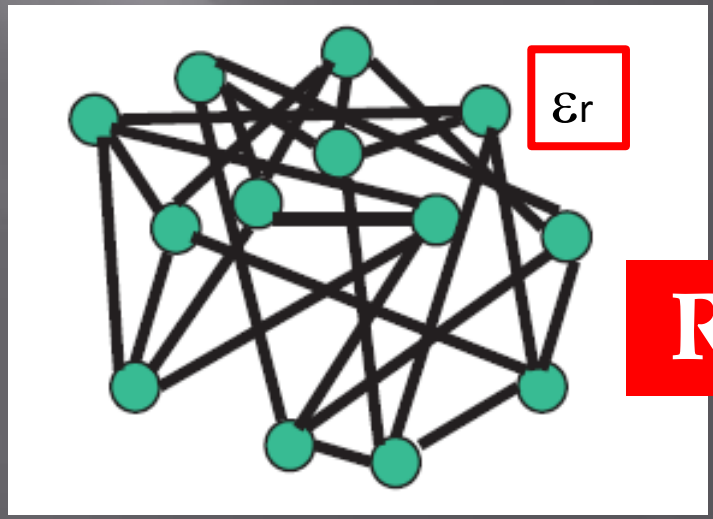
$$H = -I \sum_{<r,r'>} c_r^{+} c_{r'} + \sum_r \varepsilon_r n_r$$



Finite fraction of states on the boundary

Disorder strength W

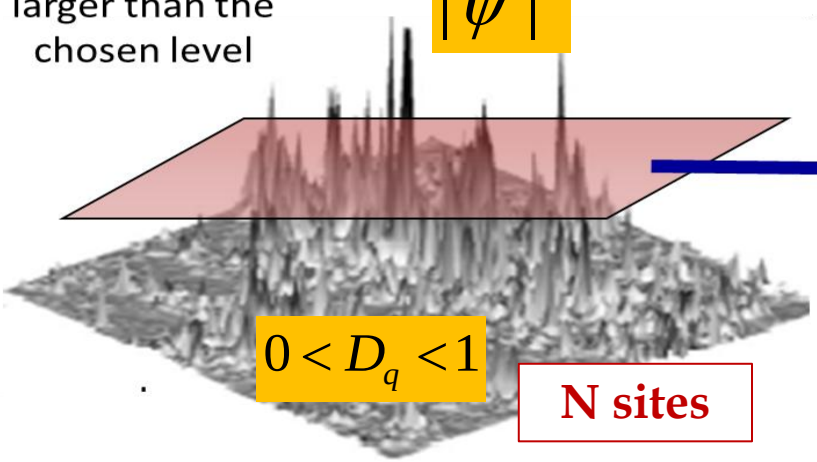




Ergodic and non-ergodic extended states

Map of the regions with amplitude larger than the chosen level

$|\psi|^2$



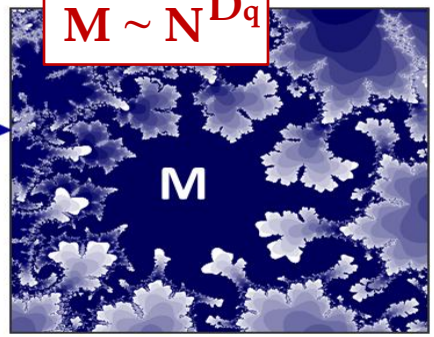
$0 < D_q < 1$

N sites

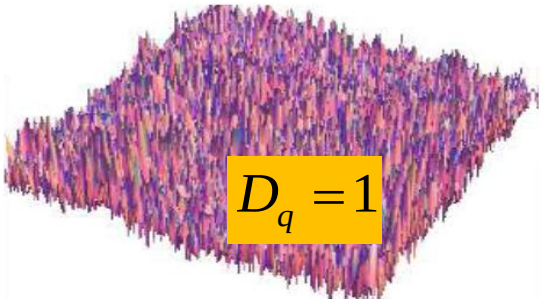
$$I_q = \left\langle \sum_r |\Psi(r)|^{2q} \right\rangle \propto \frac{1}{N^{\tau(q)}}$$

$$\tau(q) = (q-1)D_q$$

$M \sim N^{D_q}$



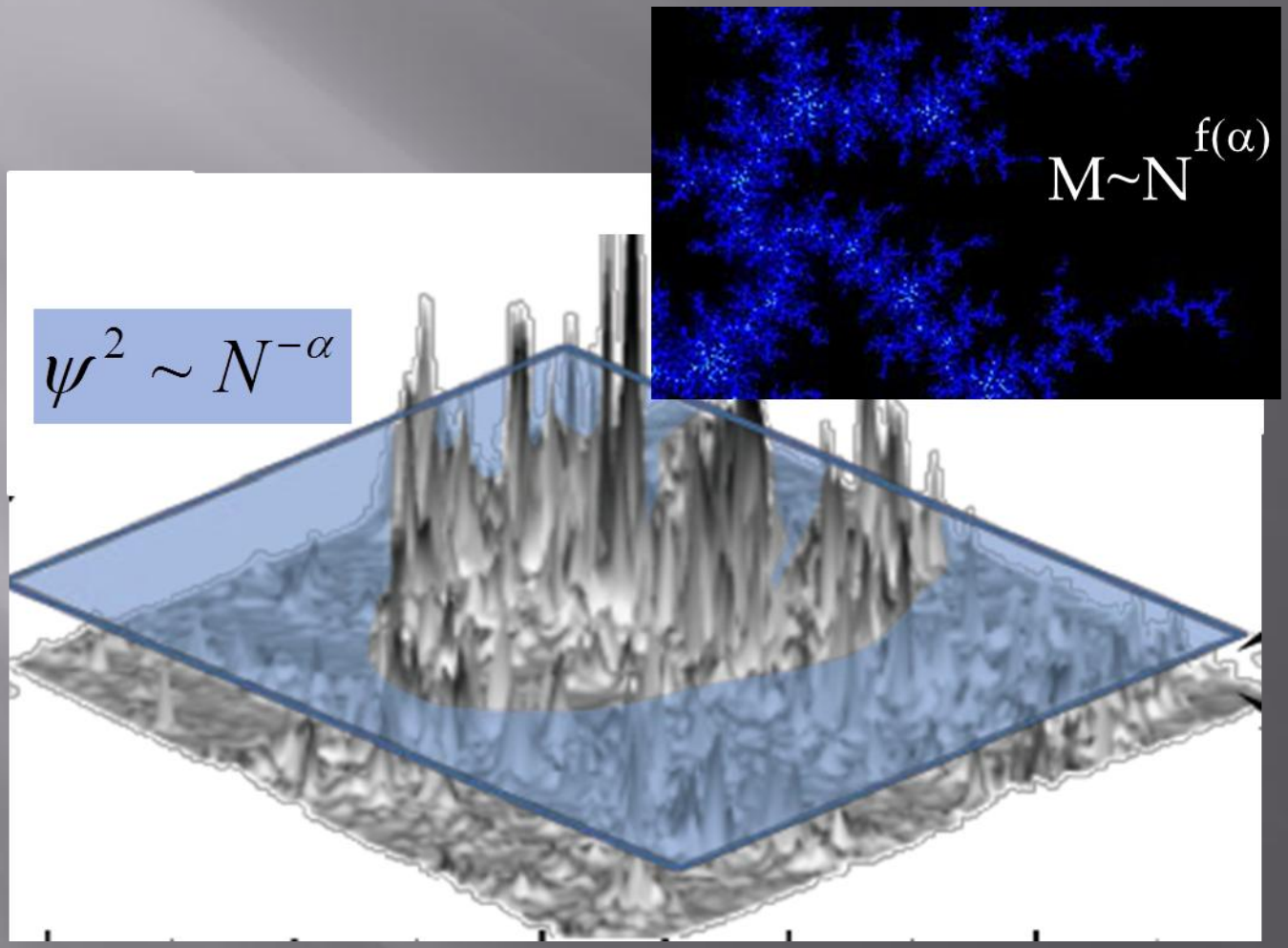
Non-ergodic, fractal states:
M/N->0
M->infty



$D_q = 1$

Ergodic states:
M/N->cst
M->infty

Function f(α)



Multifractal PDF and the moments

$$P(\ln|\psi|^2) = A\,N^{f\left(-\frac{\ln|\psi|^2}{\ln N}\right)-1}$$

$$\alpha$$

$$I_q = \sum_r \left\langle \psi(r)^{2q} \right\rangle = N^{-\tau(q)}$$

$$\tau(q) = q - 1 - \Delta(q)$$

ergodic

anomalous dimension

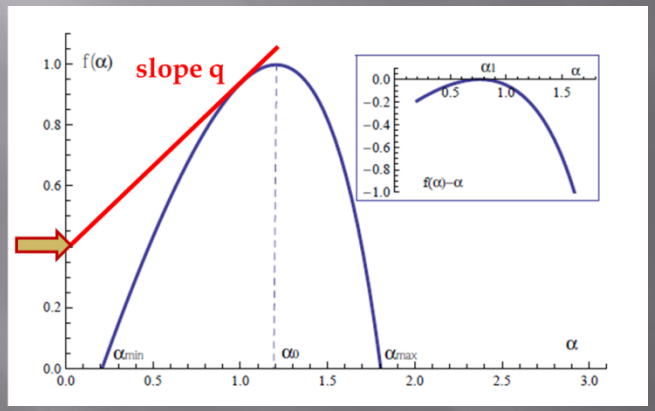
Legendre transform

$$I_q = \frac{\int_0^\infty d\alpha\, e^{\ln N\,[f(\alpha)-q\alpha]}}{\int_0^\infty d\alpha\, e^{\ln N\,[f(\alpha)-\alpha]}}$$

Saddle point approximation:

$$I_q = \sum_r \left\langle \psi(r)^{2q} \right\rangle = N^{-\tau(q)}$$

Set of critical exponents

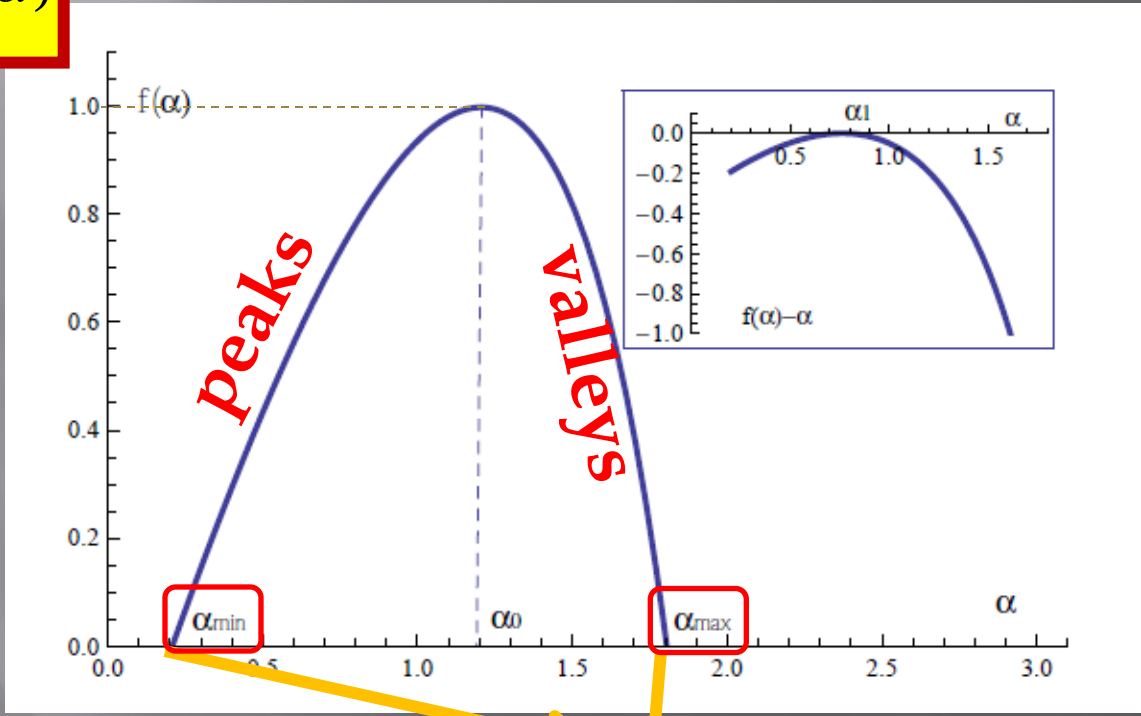


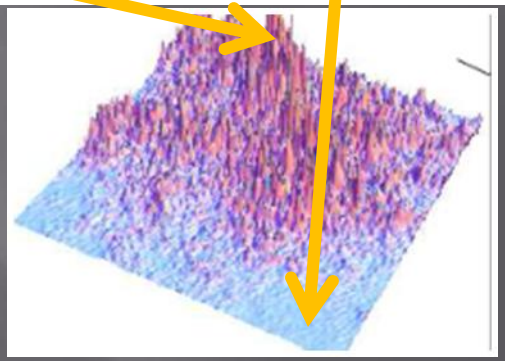
Legendre transform:

$$\tau_q = q\alpha_q - f(\alpha_q), \qquad f'(\alpha_q) = q$$

Peaks and valleys

$$M \sim N^{f(\alpha)}$$





highest peaks

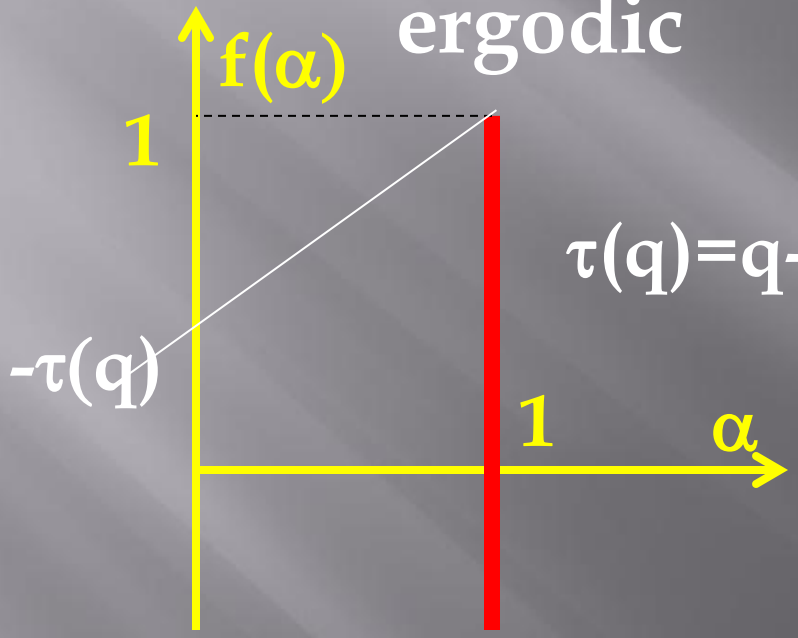
$$|\psi|^2_{\max} \sim N^{-\alpha_{\min}}$$

lowest valleys

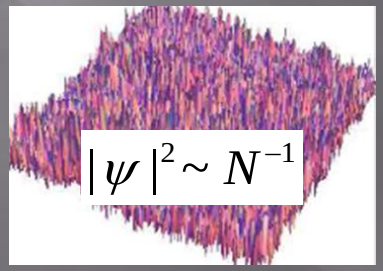
$$|\psi|^2_{\min} \sim N^{-\alpha_{\max}}$$

Degenerate cases: ergodic and exponentially localized w.f.

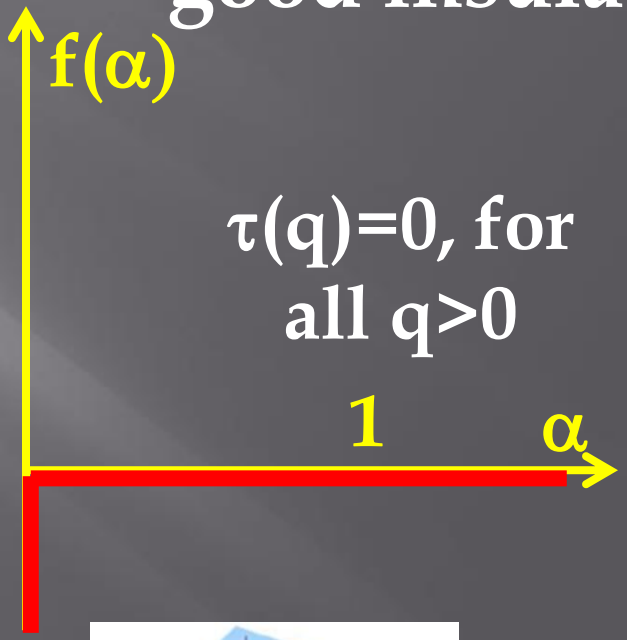
ergodic



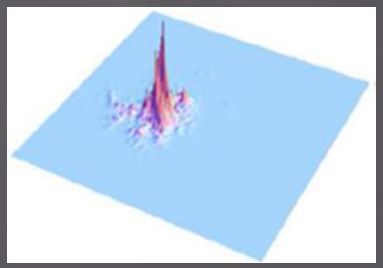
τ(q)=q-1



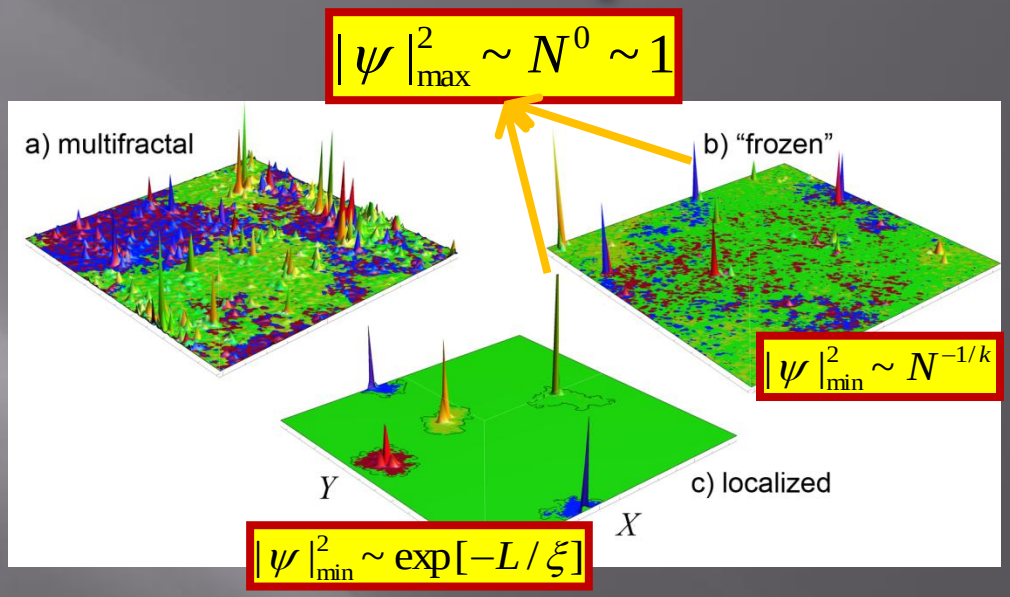
good insulator



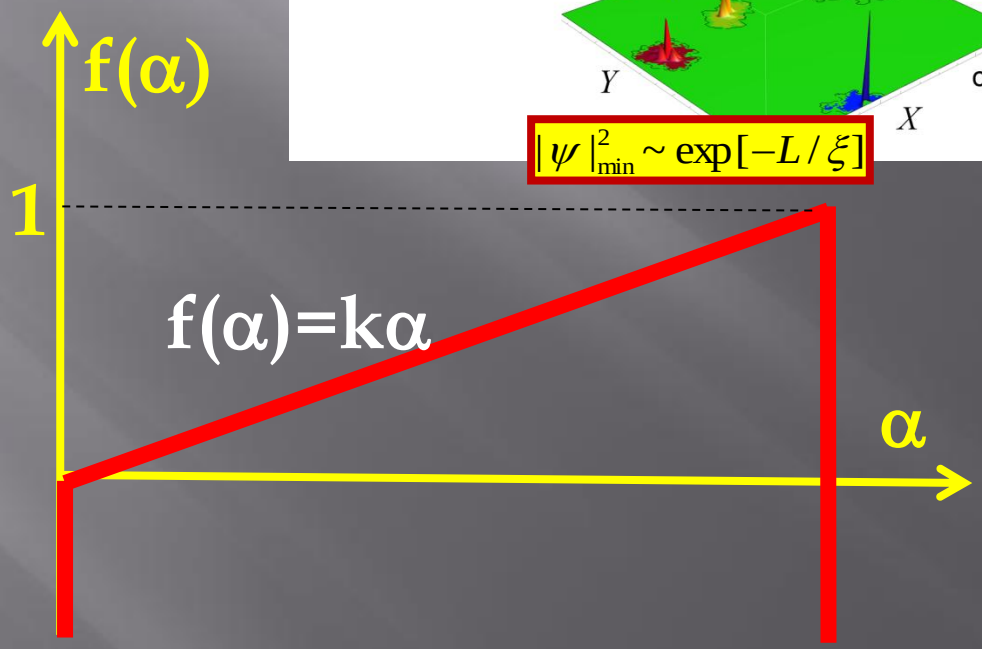
τ(q)=0, for all q>0



Bad insulator: “frozen phase”.



M.Foster et al.



$\tau(q)=0$, for all $q>k$ $\tau(q)=q/k-1$, for $0<q<k$

MF duality between peaks and valleys

Mirlin &Fyodorov 2006,

$$\Delta(q) = \Delta(1-q)$$

$$f(1+x) = f(1-x) + x$$

C. Monthus, B. Berche, and C. Chatelani,
J. Stat. Mech. P12002 (2009).

I. M. Khaymovich, J. V. Koski, O.-P. Saira,
V. E. K., J. P. Pekola
Nature Comm. 6, 7010 (2015)

MF duality is equivalent to the Crooks-like relation for PDF of

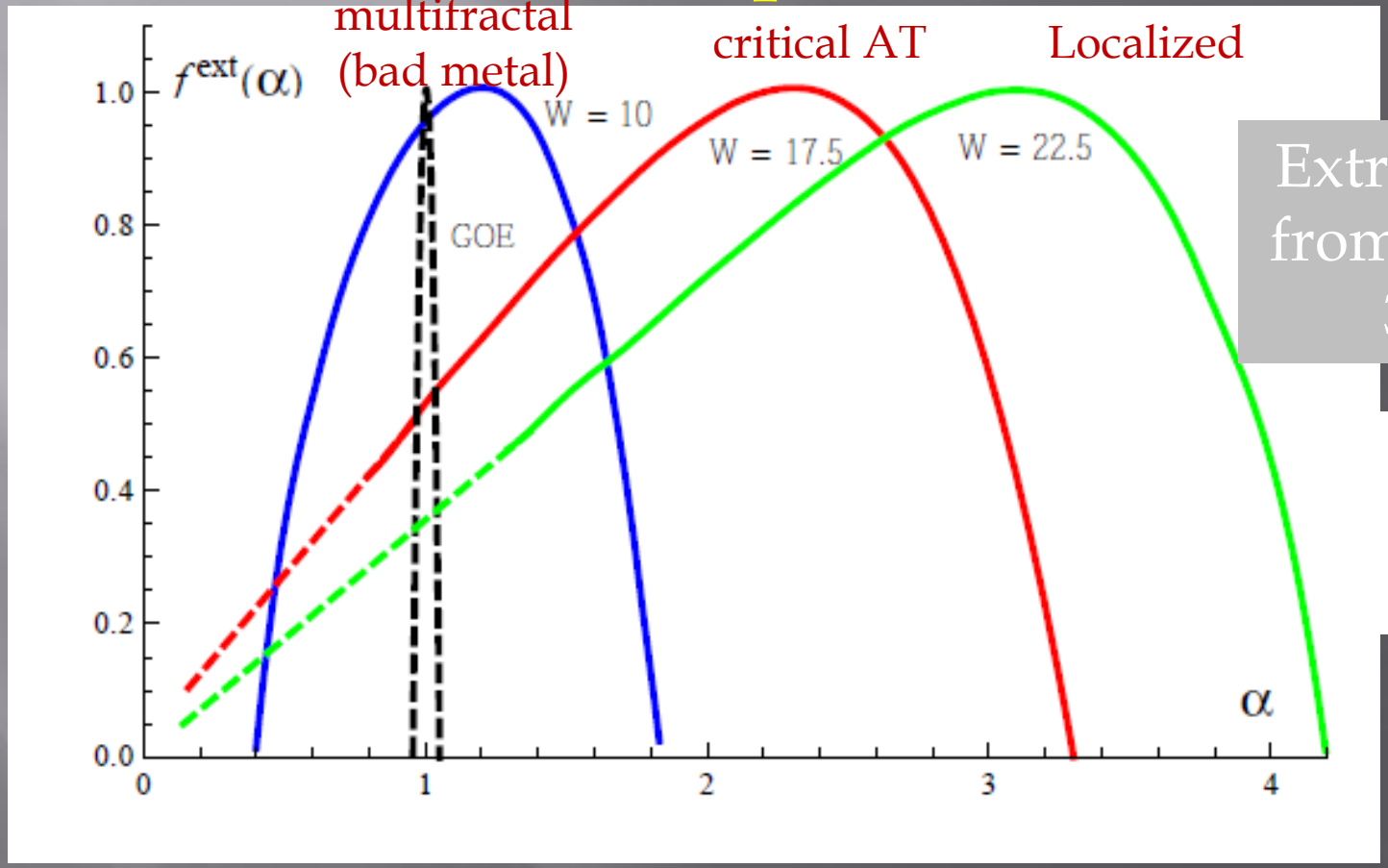
w= - ln(Nψ^2)/lnN,

$T^{-1} = \ln N$

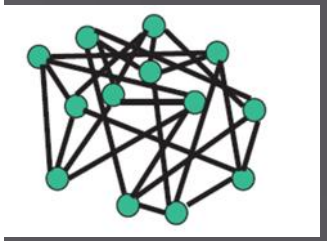
$$\frac{P(w)}{P(-w)} = \exp[w/T]$$

Search for ergodic transition on RRG

No ergodic transition was found: entire extended phase is multifractal?



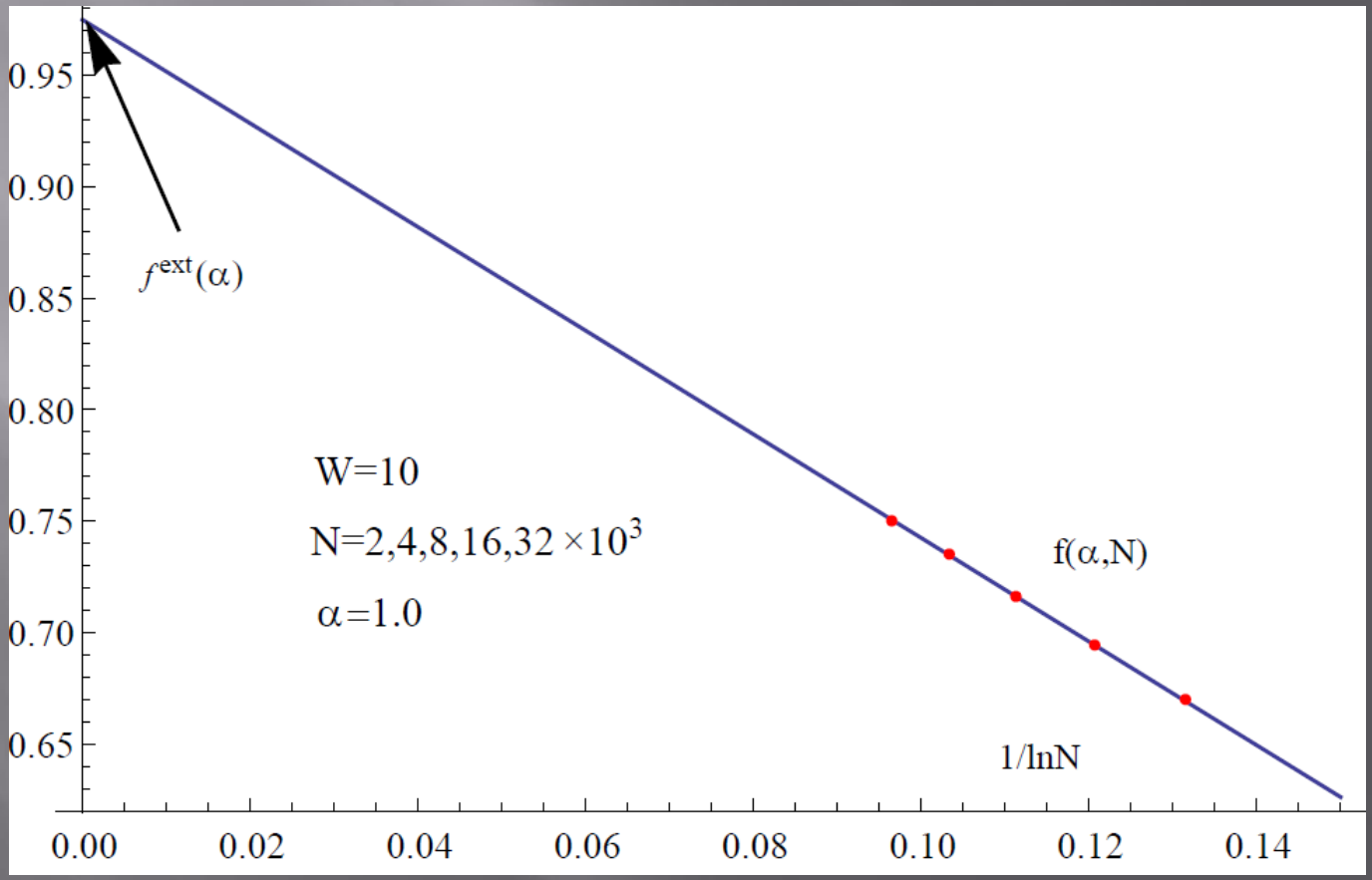
Extrapolation from N=2000-32000



De Luca et al. PRL v.113, 046806 (2014)

Extrapolation

$$\frac{\ln P_{rec}(\ln|\psi_{env}|^2)}{\ln N} = (f(\alpha, N) - 1) + \frac{A}{\ln N}$$



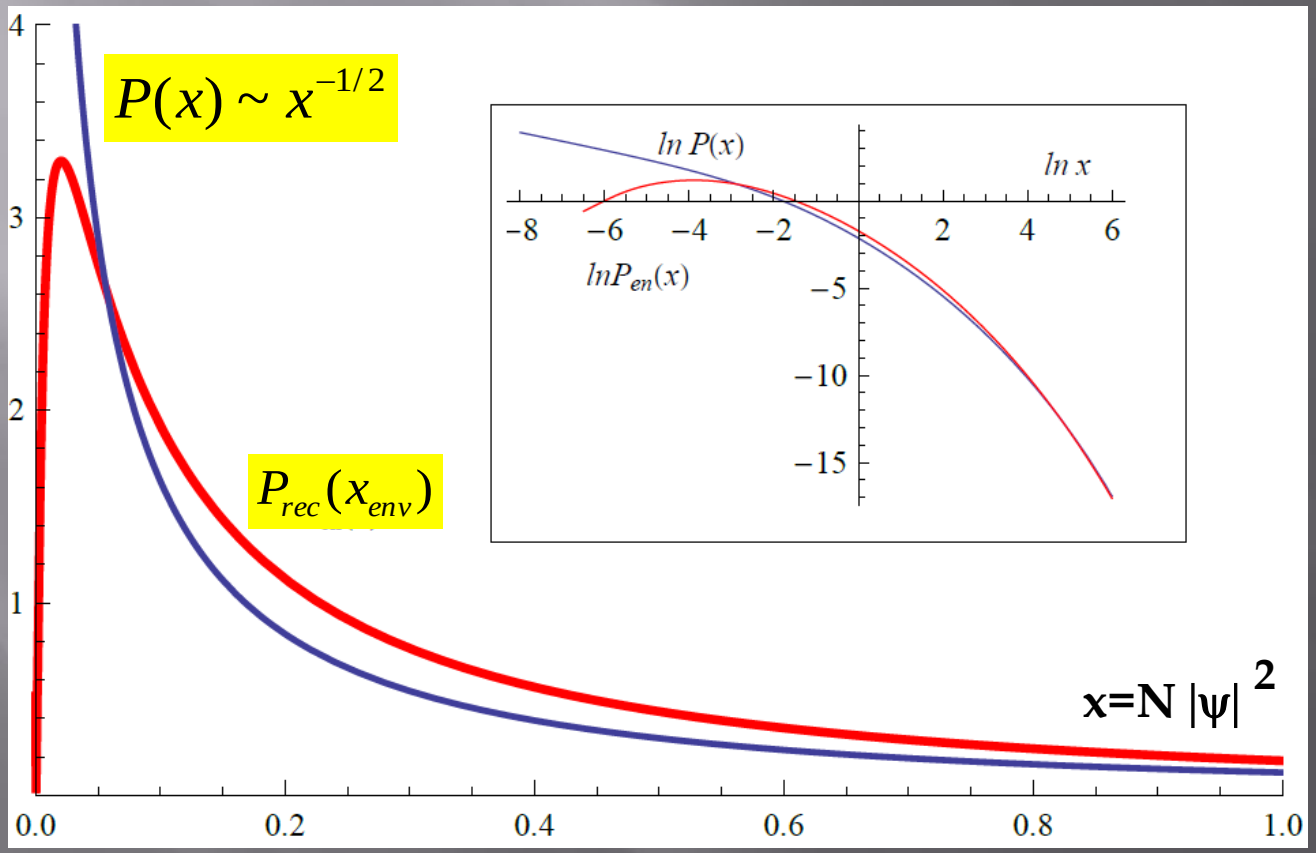
Rectification (de-convolution)

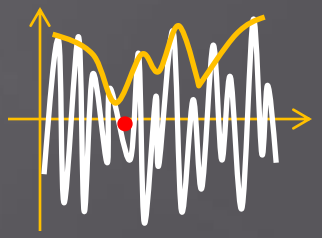
envelope oscillating

$$\psi = \psi_{env} \circ \psi_{GOE}$$

$$P(\ln|\psi|^2) = \text{convol}\left\{P_{rec}(\ln|\psi_{env}|^2) * P_{PT}(\ln|\psi_{GOE}|^2)\right\}$$

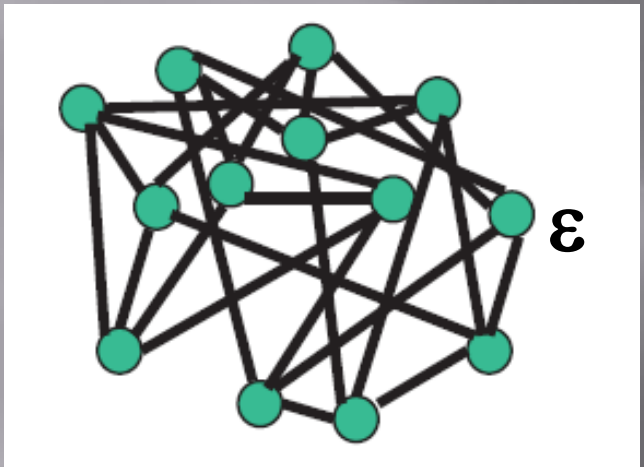
observed wanted known

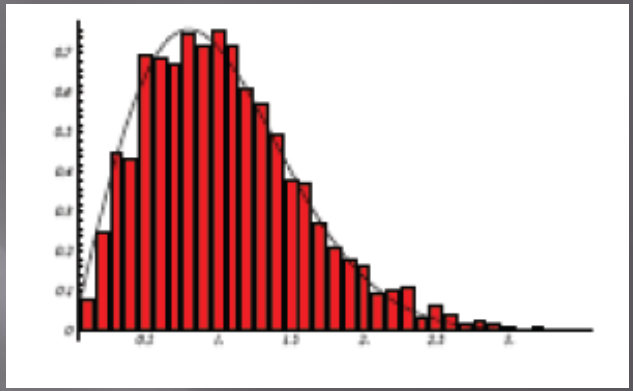




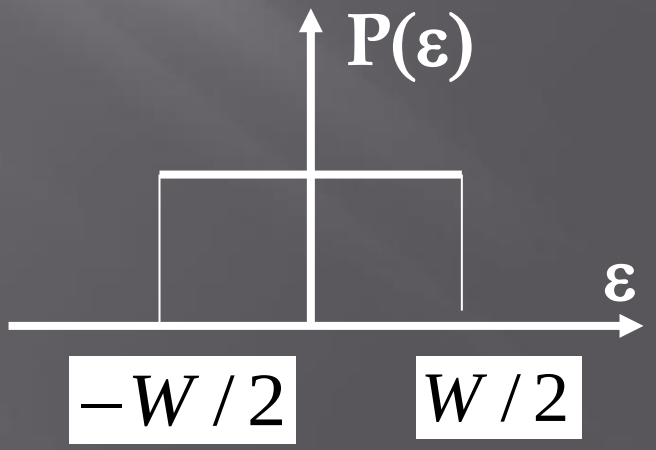
RRG: two disorder ensembles

Structural disorder: WD (Uzy Smiliansky)





On-site energy disorder:



A simpler model–relative of RRG?

WD RMT:

$$\sigma = \frac{\lambda^2}{N^\gamma} << 1$$

Mimics pristine RRG

$$\left\langle H_{nm}^2 \right\rangle = \begin{pmatrix} \sigma & \sigma & \sigma & \sigma \\ \sigma & \sigma & \sigma & \sigma \\ \sigma & \sigma & \sigma & \sigma \\ \sigma & \sigma & \sigma & \sigma \end{pmatrix}$$

Adding diagonal disorder

Special diagonal: Rosenzweig-Porter (1960) ensemble

$$\left\langle H_{nm}^2 \right\rangle = \begin{pmatrix} 1 & \sigma & \sigma & \sigma \\ \sigma & 1 & \sigma & \sigma \\ \sigma & \sigma & 1 & \sigma \\ \sigma & \sigma & \sigma & 1 \end{pmatrix}$$

Two transition points

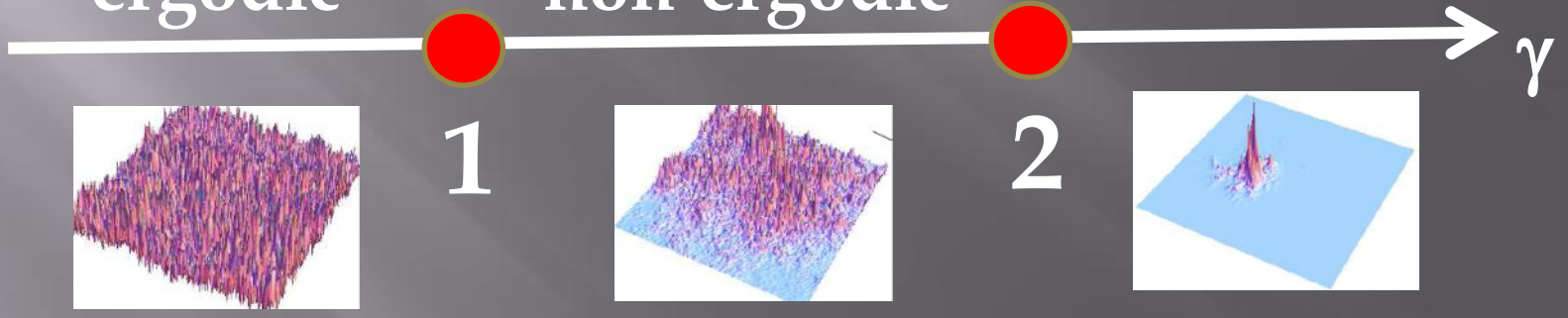
$$\sigma = \frac{\lambda^2}{N^\gamma} << 1$$

$$\left\langle H_{nm}^2 \right\rangle = \begin{pmatrix} 1 & \sigma & \sigma & \sigma \\ \sigma & 1 & \sigma & \sigma \\ \sigma & \sigma & 1 & \sigma \\ \sigma & \sigma & \sigma & 1 \end{pmatrix}$$

Extended ergodic

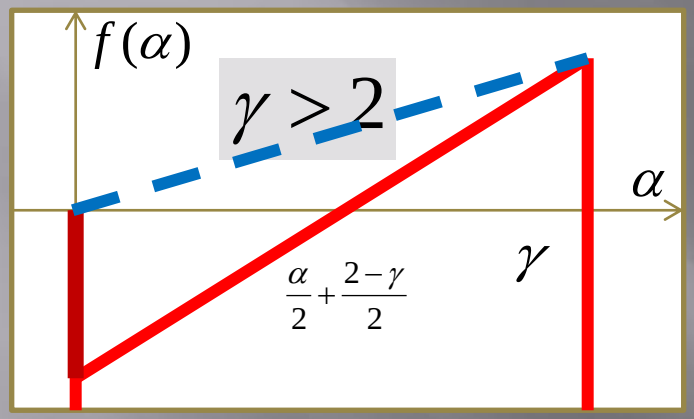
Extended, non-ergodic

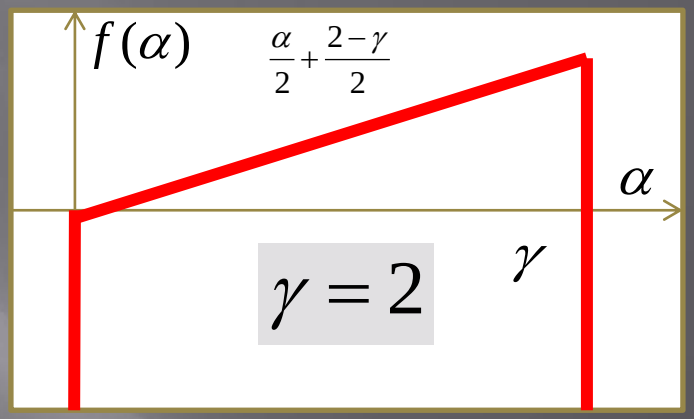
localized

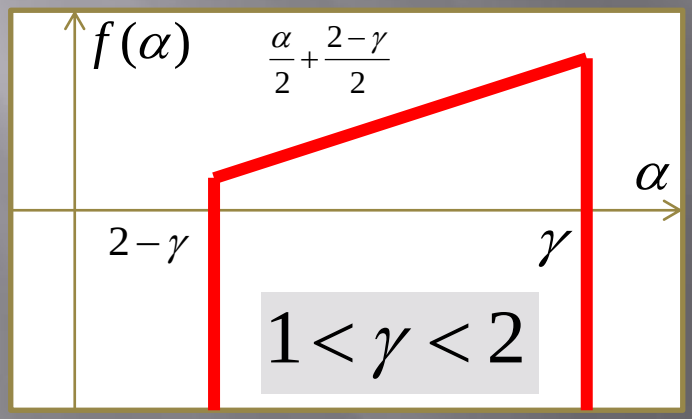


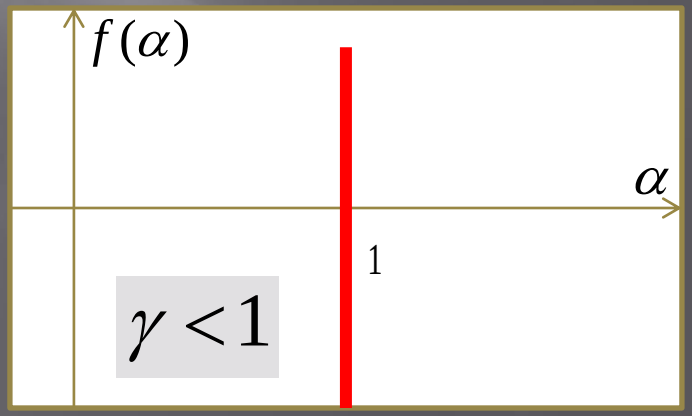
VEK, I.Khaymovich, E.Cuevas, M.Amini, arXiv: 1508.01714

Multifractality spectrum f(α)



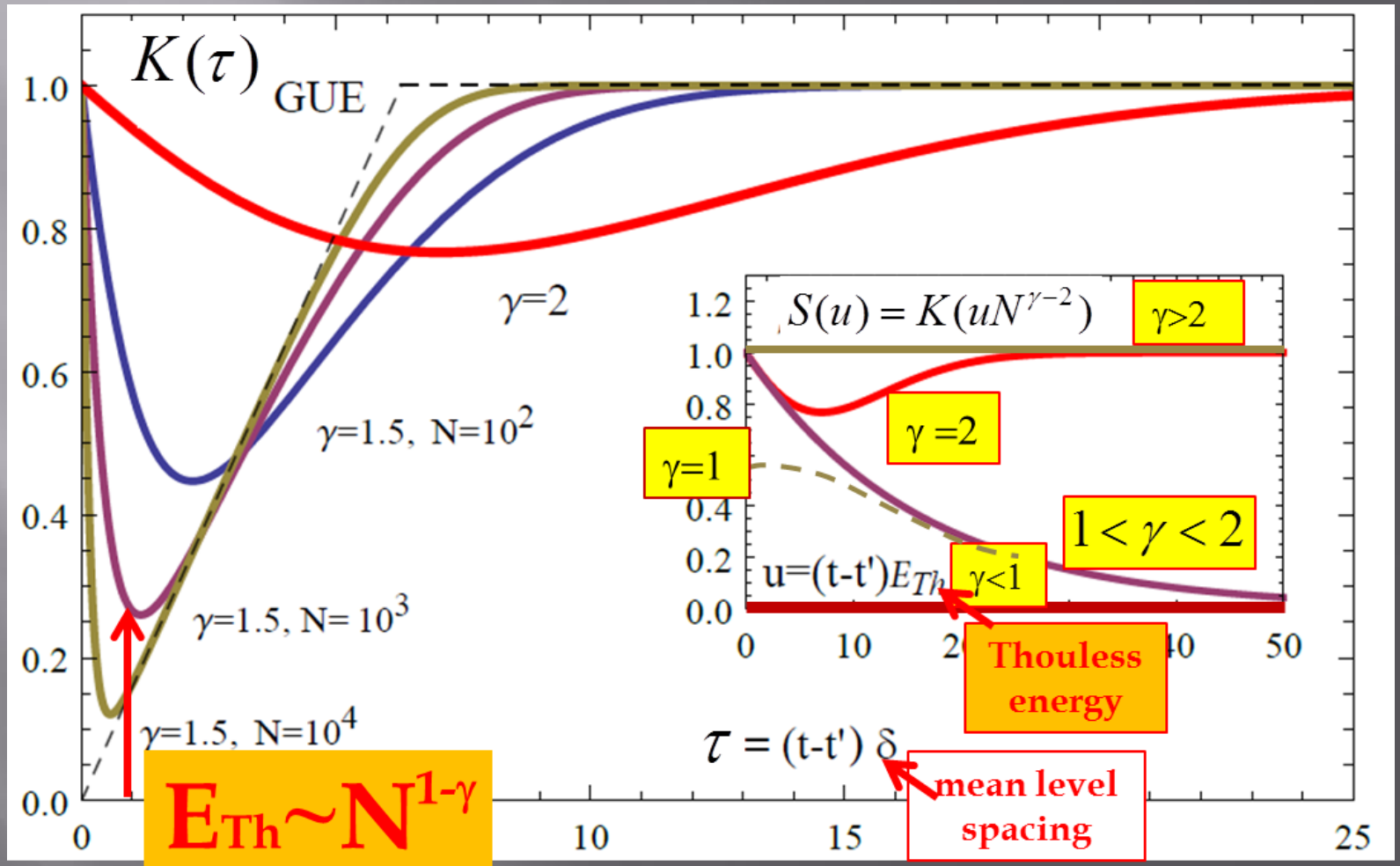






$$D_{q>1/2} = 2-\gamma$$

Spectral form-factor $\langle \exp[i\tau(E_n - E_m)] \rangle$ exact analytic result



Ergodic transition shows up only at energy scales >> mean level spacing

Cannot be seen in r-statistics

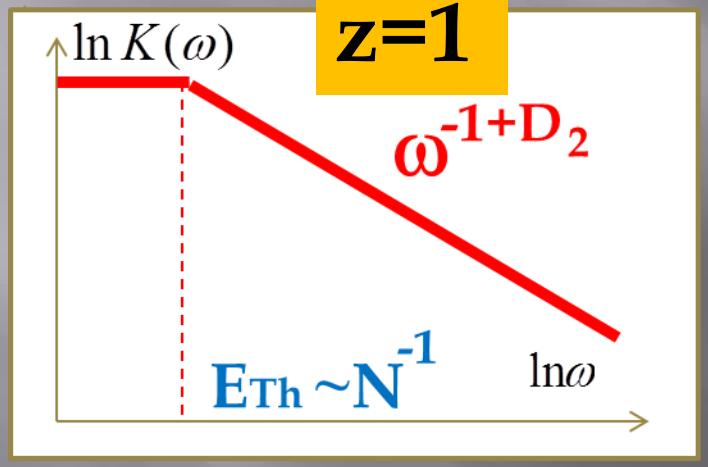
CHALKER'S SCALING FOR TWO MULTIFRACTAL STATES

3D Anderson, PLBRM

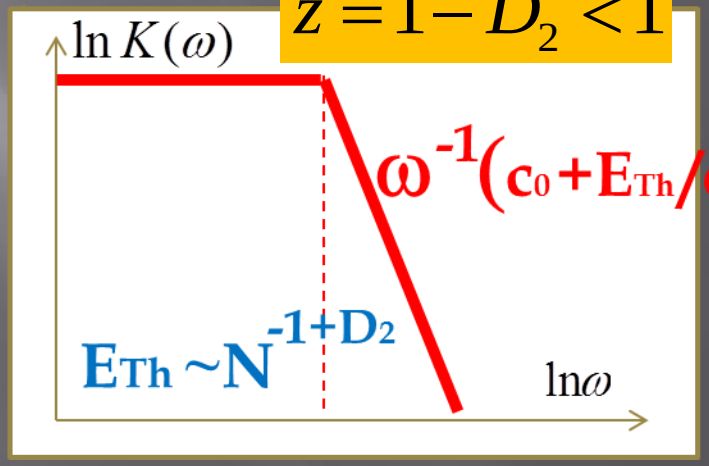
$$K(\omega) = N\left\langle \sum_r |\Psi_E(r)|^2 \, |\Psi_{E+\omega}(r)|^2 \right\rangle$$

RRG, RP

z=1



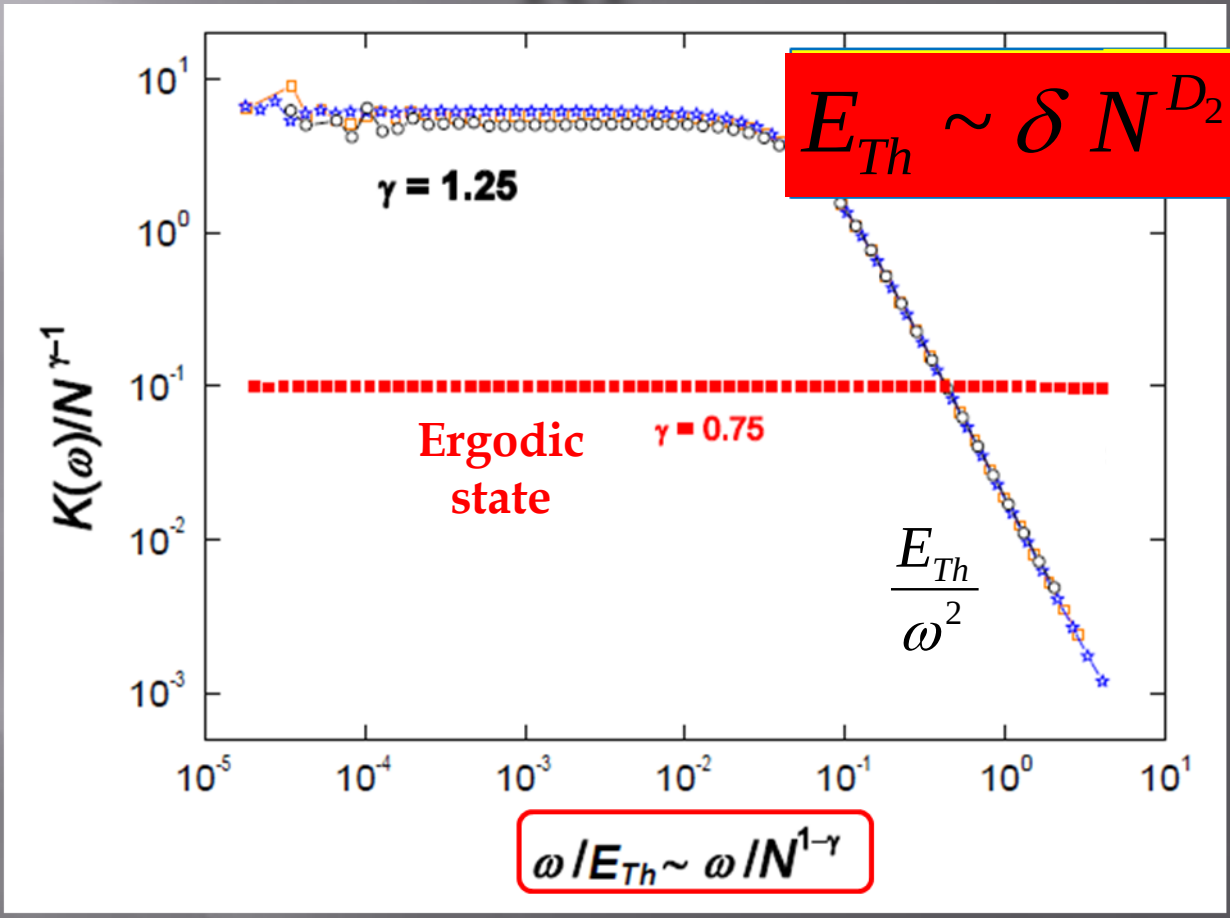
$z = 1 - D_2 < 1$



$$E_{Th} \sim N^{-z} = \delta\, N^{1-z}$$

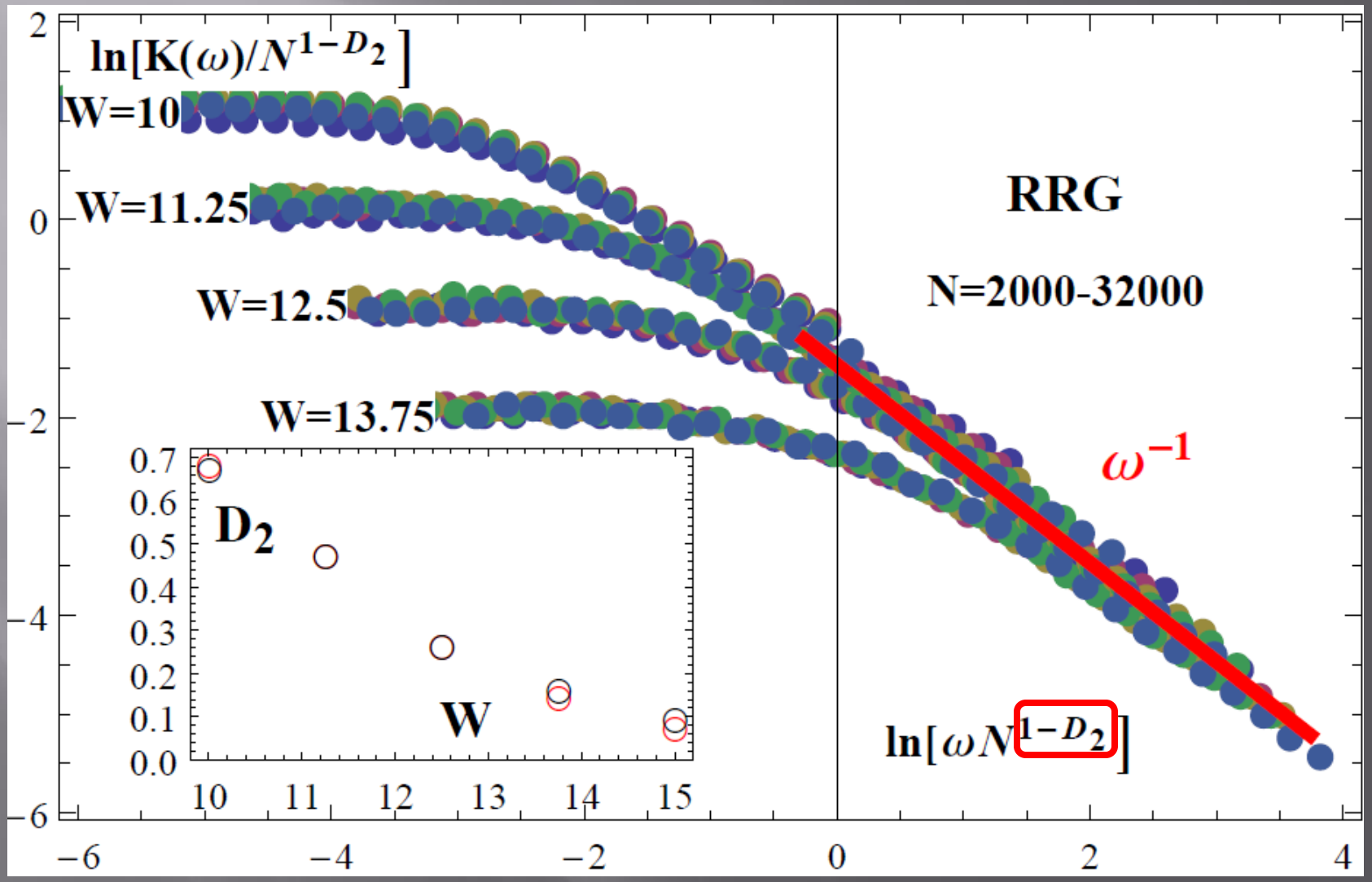
$$\mu = \frac{1 - D_2}{z}$$

Collapse of data for K(ω) for RP

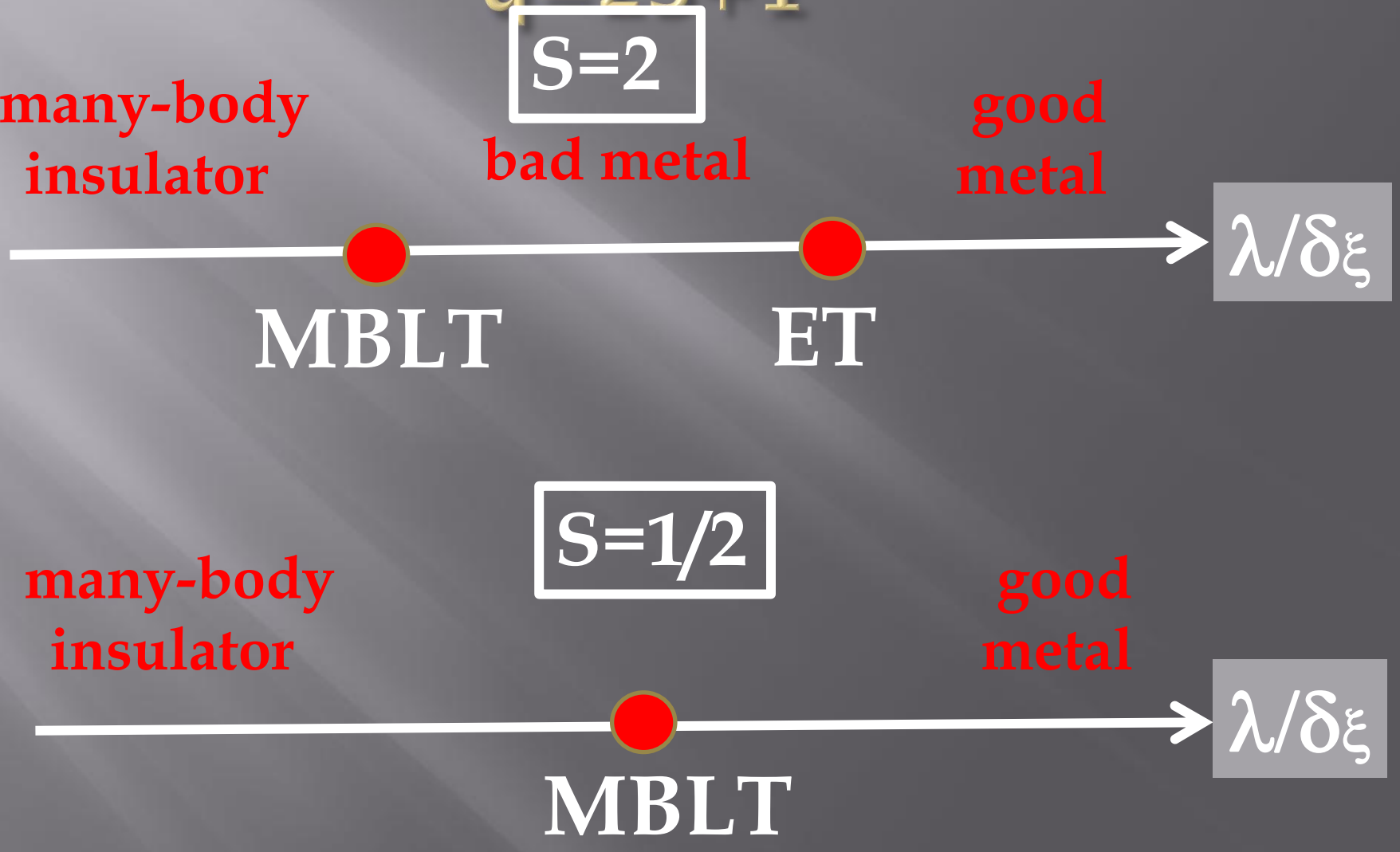


The same scaling of Thouless energy as for spectral form-factor

THOULESS ENERGY ON RRG

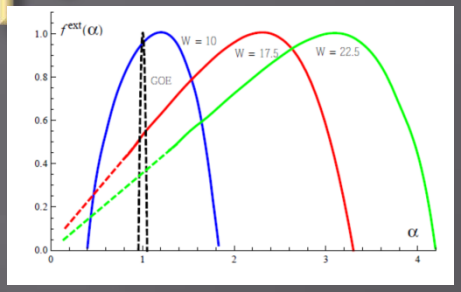


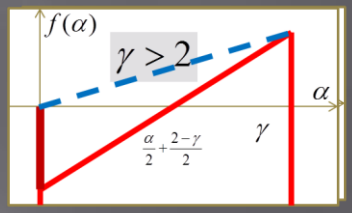
MBL: existence of intermediate phase and the multiplicity $q=2S+1$

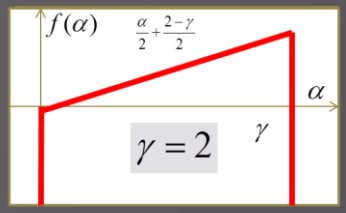


Localization on hierarchical lattices and MBL:

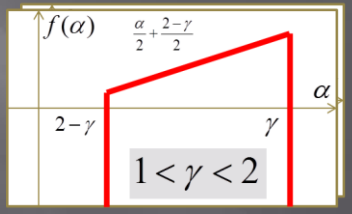
No qualitative difference between localized and AT critical states:
MBL with q>2

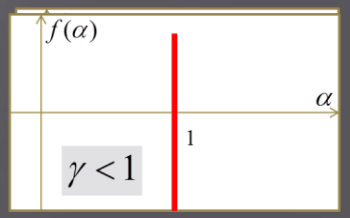




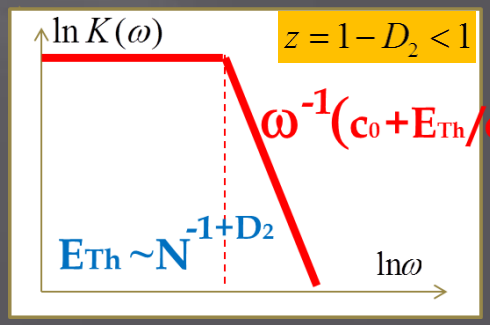


Existence of non-ergodic extended phase and ergodic transition:
MBL with q>2



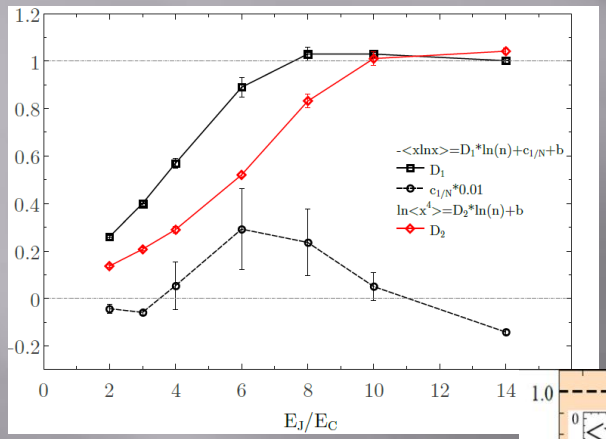


Anomalous Chalker's scaling:
yet to be studied in MB systems

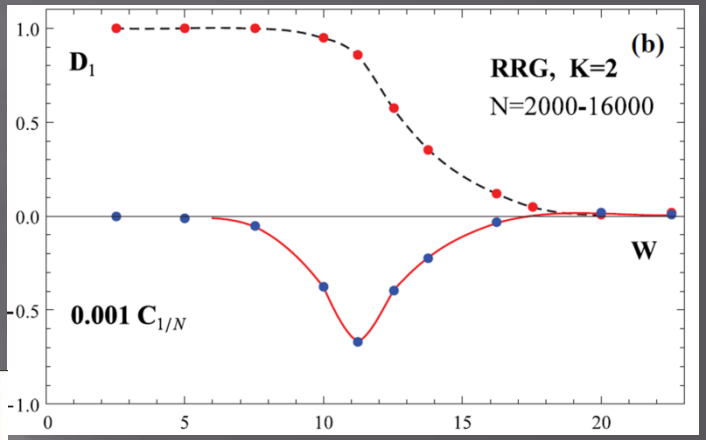


$$\langle x \ln x \rangle = (1 - D_1) \ln N + b + \frac{C_{1/N}}{N}$$

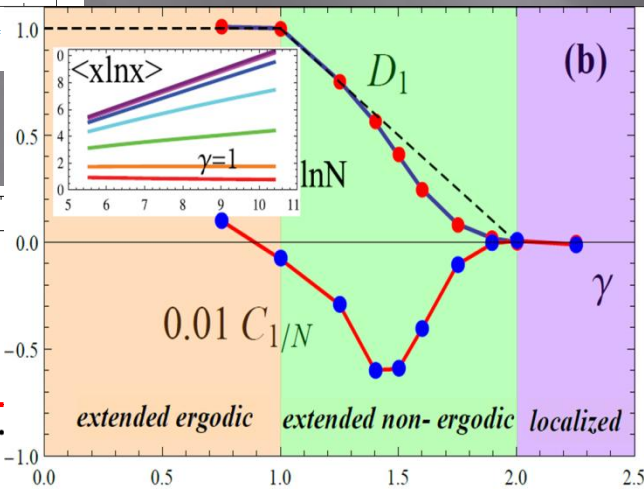
XXZ q=2S+1=5



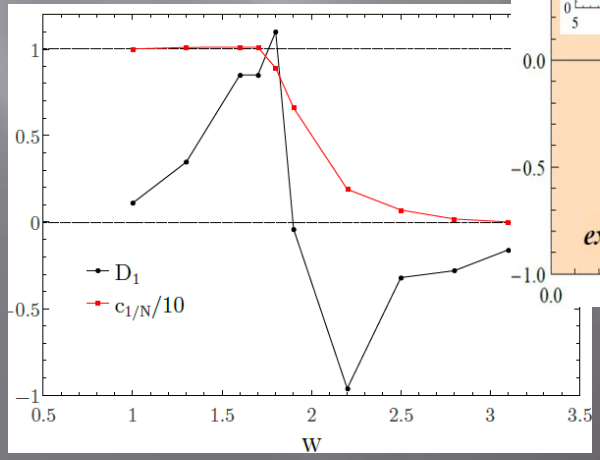
RRG K=2



Rosenzweig-Porter



XXZ q=2S+1=2



3D Anderson