Many-body localization via numerical methods

KITP MBL meeting

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Outline

1. For non-interacting systems, we understand essentially completely the effects of disorder, at least away from transitions.

Review of one-particle localization, where numerics are relatively easy.

For the simplest symmetries (orthogonal and unitary ensembles), disorder is localizing for essentially all states in 1D and 2D.

2. The combination of interactions and disorder in closed systems ("many-body localization", Basko et al.) is not nearly as well understood, even in 1D.

Different properties of the MBL phase lead to different possible numerical experiments. (Until very recently, "numerical experiments" were the only experiments! no longer.)

Examples: level statistics; entanglement of eigenstates; dynamics after a quench;…

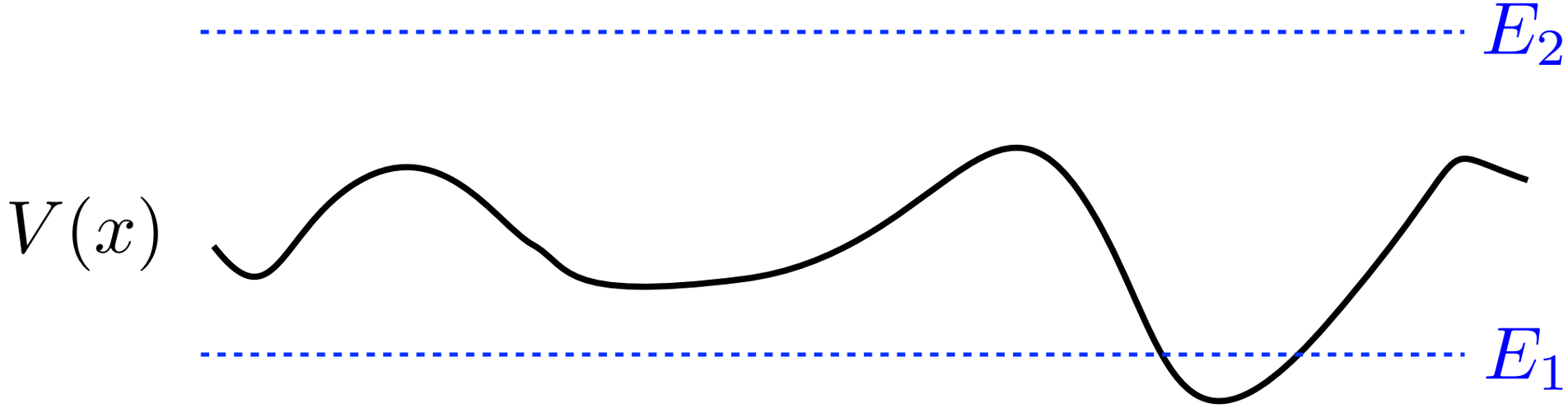
3. Closing comments

Subdiffusive scaling. Numerical implementations of strong-disorder RSRG.

Intro to disordered electronic systems

Consider a quantum particle, described by the Schrödinger equation, moving in a random potential.

Intuitively, we might expect:
at low energy, eigenstates are trapped ("localized") in potential minima
at high energy, eigenstates are scattering states

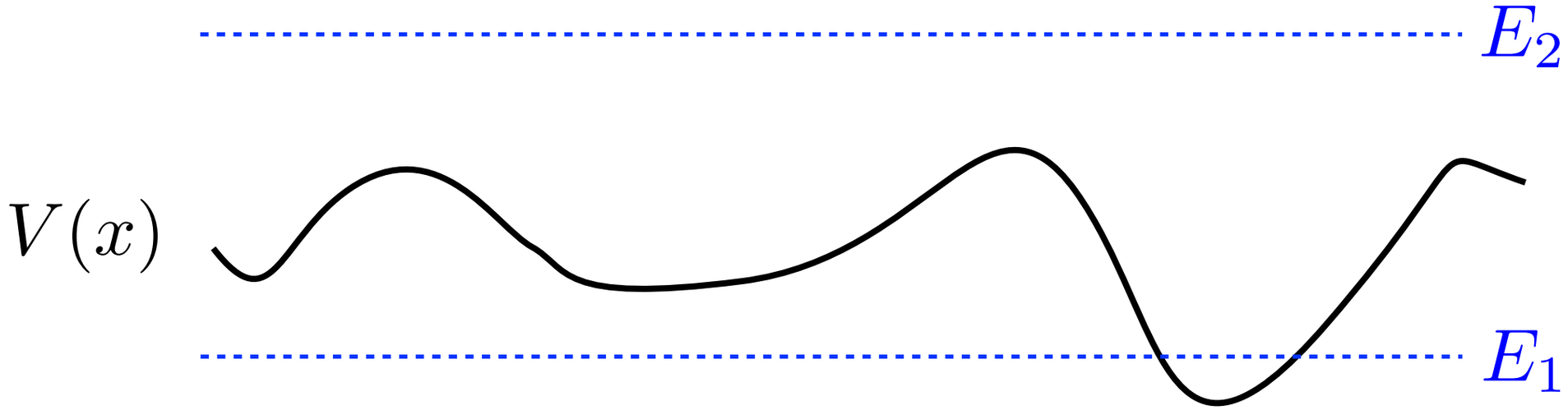


In 3D, this intuition is basically correct, and there is a specific energy (the "mobility edge") that separates localized from disordered states.

Argument for mobility edge: (Mott) coexistence of localized and extended states at same energy is unstable, as a small perturbation will mix and give only extended states.

Intro to disordered electronic systems

This intuition breaks down in one or two dimensions: all electronic states are localized up to arbitrarily high energies, although the localization length increases with *E*.



Why is 2D special? Consider the stability of scattering states. We can model the scattering state as a random walk.

A random walk above 2D revisits any point only a finite number of times on average, so a weak potential fluctuation cannot be amplified infinitely. In 2D or below, a point (say the starting point) is visited an infinite number of times, and a "weak" potential can become strong.

Intro to disordered electronic systems

For non-interacting systems, we understand essentially completely the effects of disorder, at least away from transitions.

For the simplest symmetries (orthogonal and unitary ensembles), disorder is localizing for essentially all states in 1D and 2D.

The combination of interactions and disorder in closed systems ("many-body localization") is not well understood even in 1D.

are the only two possibilities diffusive and localized? can there be subdiffusive scaling? (e.g., "glassy": r ~ log t)

CM experimental systems typically have "dephasing" from interactions with phonons, which ultimately leads to a finite diffusion constant.

Systems of atoms in an ultra cold lattice do not have phonons, so may be better.

Intro to disordered electronic systems

How do we see localization experimentally?

Localization in the sense described here requires *interference* (constructive interference of self-intersecting trajectories).

Hence it is a quantum property and disappears if the electrons lose their phase coherence by interacting with a their environment (e.g., a "bath" of phonons).

If that happens on a phase-breaking time scale τ_ϕ

then this acts as a cutoff on the effects of localization, e.g., on the reduction of conductivity.

Treating localization perturbatively ("weak localization theory") has been very powerful. Interaction effects can be incorporated (Altshuler-Aronov, Finkelstein, others) in this framework.

But in isolated systems (e.g., ultracold atomic systems), or possibly in femtosecond experiments on electrons, the system can be phase-coherent.

Including the bath also sidesteps some basic questions.

Intro to disordered electronic systems

So one-particle localization is very sensitive to dimensionality.

It is also sensitive to *symmetries*. For example, if we broke time-reversal symmetry with a magnetic field, then in 2D extended states survive at isolated energies.

If we assume that disorder breaks all symmetries except for two discrete symmetries *T* (time reversal) and *C* (chiral/charge conjugation), and that each of these can square to +1 or -1 if present, then there are 10 symmetry classes.

Why 10?

Just considering *T* gives 3 "Wigner-Dyson" classes: orthogonal ($T^2 = +1$), symplectic ($T^2 = -1$), and unitary (*T* broken).

Adding *C* gives 9 classes (3 times 3). There is also the possibility of having *CT* symmetry without either *C* or *T* separately, hence 10 "Altland-Zirnbauer" classes.

How do we see localization experimentally?
Why is it important for some basic physics questions?
Is there more to the story than symmetry and dimensionality?

Periodic table of insulators

Schnyder et al., Kitaev: 10-fold way classification, periodic in dimension
3 Wigner-Dyson cases + particle-hole symmetry in superconductors = 10
Better to think of as 2+8: see Freed and G. Moore, "Twisted Equivariant Matter"

There can be insulator-metal transitions, like the Anderson transition at the mobility edge, and also insulator-insulator transitions, like the quantum Hall plateau transition.

Numerics is not easy at these transitions but arguably easier than analytical theory.

TABLE II. Topological insulators (superconductors) with an integer ($\mathbb{Z}$) classification, (a) in the complex symmetry classes, predicted from the chiral U(1) anomaly, and (b) in the real symmetry classes, predicted from the gravitational anomaly (red), the chiral anomaly in the presence of background gravity (blue), and the chiral anomaly in the presence of both background gravity and U(1) gauge field (green).

Cartan\d	0	1	2	3	4	5	6	7	8	9	10	11	⋯
A	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	⋯
AIII	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	⋯
AI	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	⋯
BDI	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	⋯
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	⋯
DIII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	⋯
AII	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	⋯
CII	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	⋯
C	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	⋯
CI	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	⋯

MBL can be motivated by the basic question

Does an isolated quantum system with interactions and disorder show localization?

which is related to the equally basic question

When do isolated quantum systems thermalize?

The connection is that localization is the most plausible physical way to avoid thermalization: localized particles cannot move around and equilibrate. In a delocalized system, we expect that a test particle sees other particles as a thermal "bath".

Will focus on 1D. Besides symmetry and dimensionality, what else controls localization in the interacting case? What are the new properties of the localized phase? Which are interaction-specific?

Mostly will talk about 2 numerical methods:

Most numerics on MBL so far were done with "exact diagonalization" (ED): find all eigenvalues, or a subset, of the Hamiltonian matrix.

ED is great for small systems as it gives essentially complete information and its implementation and convergence are well understood.

It doesn't scale very well: cost for all eigenvalues goes as the cube of the matrix dimension, so beyond 20 spin-half sites becomes expensive.

Good news: there has been enormous progress 1992-present in DMRG/"matrix product state" methods to solve many-particle quantum problems in low spatial dimensions (especially 1D).

Understanding when these methods work well requires us to understand *entanglement*, which also leads to another useful definition of the MBL state.

Studying quantum correlations with classical algorithms: applied entanglement entropy

Basic (hazy) concept: "Entanglement entropy determines how much classical information is required to describe a quantum state."

Example:
how many classical real numbers are required to describe a *product* (not entangled) state of N spins?

simple product $$|\psi\rangle = A_{s_1} A_{s_2} A_{s_3} A_{s_4} |s_1 s_2 s_3 s_4\rangle$$

Answer: ~ N (versus exponentially many for a general state)

How do we efficiently manipulate/represent moderately entangled states?

Applied entanglement entropy

The remarkable success of the density-matrix renormalization group algorithm in one dimension (White, 1992; Ostlund and Rommer, 1995) can be understood as follows:

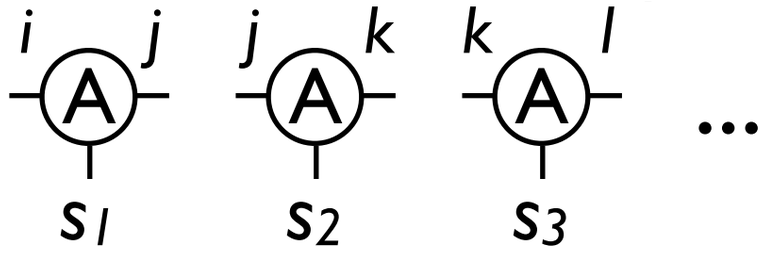
DMRG constructs "matrix product states" that retain local entanglement but throw away long-ranged entanglement.

Example states for four spins:

simple product $$|\psi\rangle = A_{s_1} A_{s_2} A_{s_3} A_{s_4} |s_1 s_2 s_3 s_4\rangle$$

matrix product $$|\psi\rangle = A^{ij}_{s_1} A^{jk}_{s_2} A^{kl}_{s_3} A^{li}_{s_4} |s_1 s_2 s_3 s_4\rangle$$

Graphical tensor network representation:



"Infinite system" methods

Note that we can impose translation invariance simply by requiring constant matrices *A*.

In other words, for quantities in a translation-invariant system, *we just calculate A, rather than a large finite system.*
(Idea 1 of renaissance; see Vidal '07, for example)

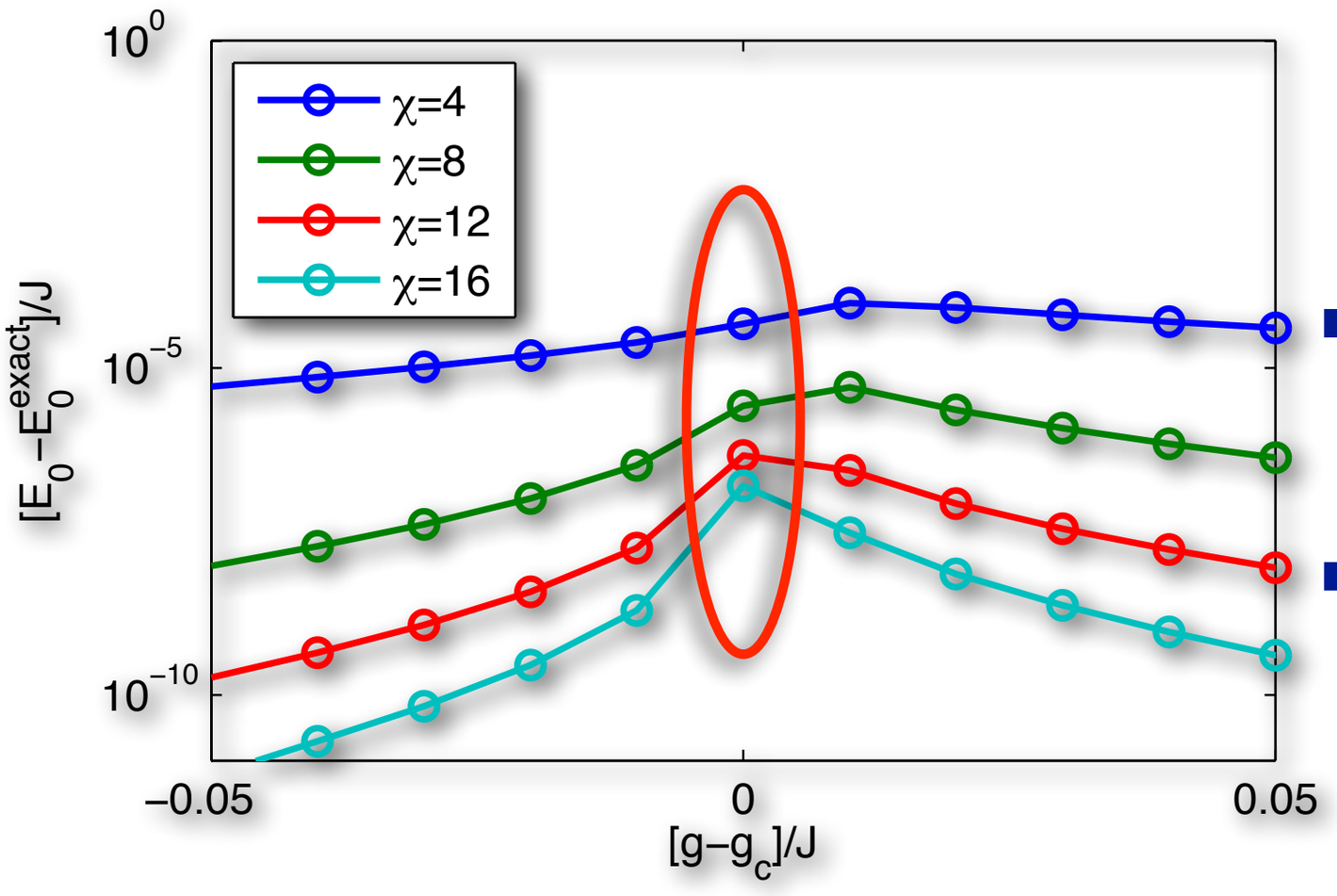
matrix product $$|\psi\rangle = A^{ij}_{s_1} A^{jk}_{s_2} A^{kl}_{s_3} A^{li}_{s_4} |s_1 s_2 s_3 s_4\rangle$$

So where is the approximation?
A finite matrix *A* can only capture a finite amount of entanglement.

In the early DMRG days, it was often thought:
1. To study an infinite system, we should study a large finite one.
2. Gapless/critical systems are hard. (Gapped uniform systems converge…)
3. Dynamical properties are hard
4. Finite temperature is hard

But none of these is strictly correct.

- find the ground state of a system by using **imaginary time evolution** (almost unitary for small time steps)
- parallel updates for **infinite/translational invariant** systems: **iTEBD** [Vidal '07]
- **example**, transverse Ising model: $H = \sum_i \left(J\sigma_i^z \sigma_{i+1}^z + g\sigma_i^x \right)$



➡convergence of wave function is worst at the **critical point**

➡**conformal invariance**

Criticality: finite-entanglement scaling

All numerical methods have difficulty with quantum critical points. In DMRG-type approaches, this can be understood from the divergence of entanglement entropy at such points: the entanglement in a matrix product state is limited by dim *A*.

matrix product $$|\psi\rangle = A^{ij}_{s_1} A^{jk}_{s_2} A^{kl}_{s_3} A^{li}_{s_4} |s_1 s_2 s_3 s_4\rangle$$

Quantitatively, it is found that dim *A* plays a role similar to imposing a finite system size:

(Tagliacozzo et al., PRB 2008). $$L_{\text{eff}} \propto \chi^{\kappa}, \quad \chi = \dim A$$

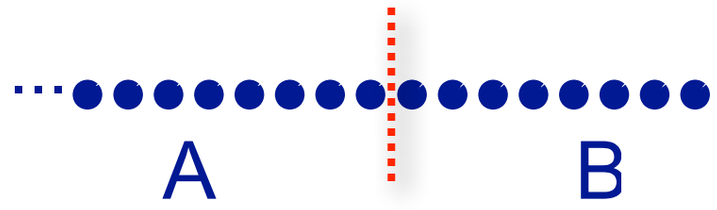
Finite matrix dimension effectively moves the system away from the critical point.

What determines this "finite-entanglement scaling"?
Is it like "finite-size scaling" of CFT's (cf. Blöte, Cardy, & Nightingale)

A way to picture the entanglement of a state

- **Schmidt decomposition** of the state (**SVD**):
-

$$
\begin{aligned}
|\psi\rangle &= \sum_{i=1}^{N_A}\sum_{j=1}^{N_B} C_{ij}|i\rangle_A|j\rangle_B \\
&= \sum_{\alpha=1}^{\min(N_A,N_B)} \lambda_\alpha|\phi_\alpha\rangle_A|\phi_\alpha\rangle_B
\end{aligned}
$$



with $\lambda_\alpha \geq 0$ and $\sum_\alpha \lambda_\alpha^2 = 1$

- a natural measure of the entanglement is the **entropy:**

$$S_A = S_B = S = -\sum_\alpha \lambda_\alpha^2 \log(\lambda_\alpha^2)$$

- **A heuristic argument** for the asymptotic case (using a continuum of Schmidt values and $\chi \to \infty$)

➡**universal finite-entanglement scaling relations**

$$\kappa = \frac{6}{c\left(\sqrt{\frac{12}{c}}+1\right)} \Rightarrow S = \frac{1}{\sqrt{\frac{12}{c}}+1}\log\chi$$

F. Pollmann, S. Mukerjee, A. Turner, and J.E. Moore, PRL 2009
Some checks for various critical theories are in that paper, and the recent work
B. Pirvu, G. Vidal, F. Verstraete, L. Tagliacozzo, arXiv:1204.3934

So critical points are worse than gapped points, but in a controlled way.
What does this mean in practice?

Remark: Entanglement spectra are qualitatively different for random critical spin chains than for pure ones, though entanglement entropies similar (M. Fagotti, P. Calabrese, JEM).

What about MBL versus ergodic states?

A thermalizing state should have volume-law entanglement of eigenstates according to the eigenstate thermalization hypothesis (ETH).

ETH=local measurements on an eigenstate of a thermalizing system are consistent with a thermal ensemble.

A picture of the MBL state is that it is similar to the ground state of a localized system and has an *area law* for entanglement.
(Bauer and Nayak, …)

So far we have three things we can look for to diagnose an MBL transition: vanishing of the conductivity, or absence of thermalization, or the change in the entanglement properties of eigenstates.

Note that the first two are slightly different: will hear later talks on how we might have a subdiffusive but thermalizing phase, for example.

See Bar Lev et al ’14, Hulin et al ‘90, Agarwal et al ’14, Potter et al ’14, Vosk et al ‘14

We can get two more signatures by an analogy to the clean case…

Many-body localization at infinite temperature

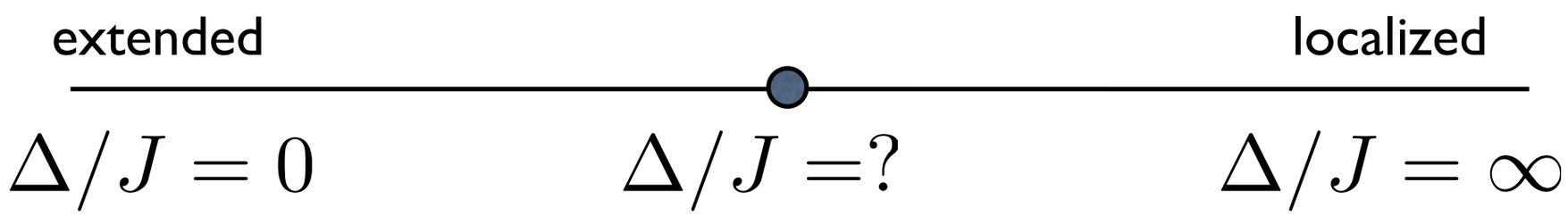
$$H = J_{xx} \sum_i \left(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y \right) + J_z \sum_i S_i^z S_{i+1}^z + \sum_i h_i S_i^z$$

Clean XXZ chain + random z-directed Zeeman field

$$h_i \in (-\Delta, \Delta)$$

Claim: look at "infinite-temperature" dynamics but with no dephasing; evolve an arbitrary initial state by the Schrödinger equation

Heisenberg phase diagram:
(Oganesyan-Huse spin chain version of BAA)

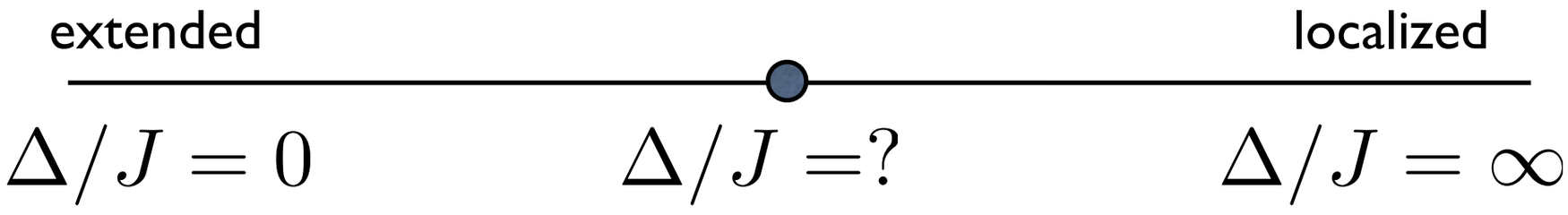


or is there an intermediate "ergodic non-metal"?

Many-body localization at infinite temperature

$$H = J_{xx} \sum_i \left(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y \right) + J_z \sum_i S_i^z S_{i+1}^z + \sum_i h_i S_i^z$$

Heisenberg phase diagram:



Transition(s) should be detectable in:

level statistics: (Wigner-Dyson vs. Poisson) Oganesyan & Huse, 2008
dynamical correlation functions
correlation distributions Pal & Huse, 2010; Reichman et al. 2010
entanglement growth/thermalization (JHB,FP,JEM 2012)
entanglement variance (recent work of Alet et al., Bardarson et al., …)

This spin chain problem is a numerically easier reformulation of many-body localization in continuum Fermi systems at nonzero T (Basko, Aleiner, Altshuler 2007)
Hoped to be generic for 1D local interactions, disorder, U(1) symmetry.

Many-body localization at infinite temperature

$$H = J_{xx} \sum_i \left(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y \right) + J_z \sum_i S_i^z S_{i+1}^z + \sum_i h_i S_i^z$$

level statistics: (Wigner-Dyson vs. Poisson) Oganesyan & Huse, 2008

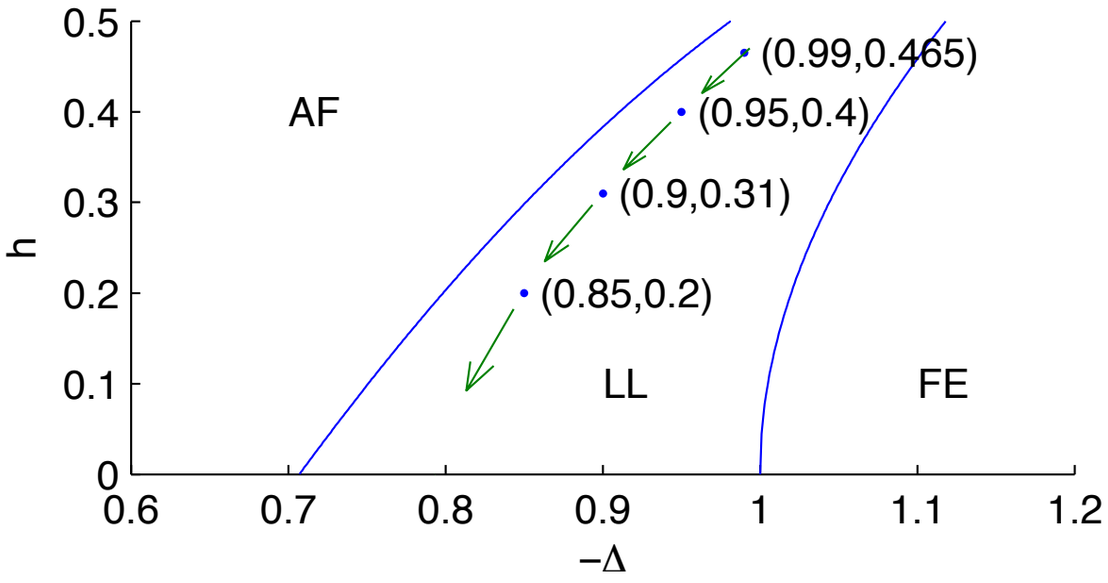
The idea is that diffusive and integrable systems have different level statistics, which is a simple property of the eigenvalues alone.

An MBL system is like an *integrable* system, which normally means a translation-invariant system with a complete set of conservation laws (return to this point in a moment).

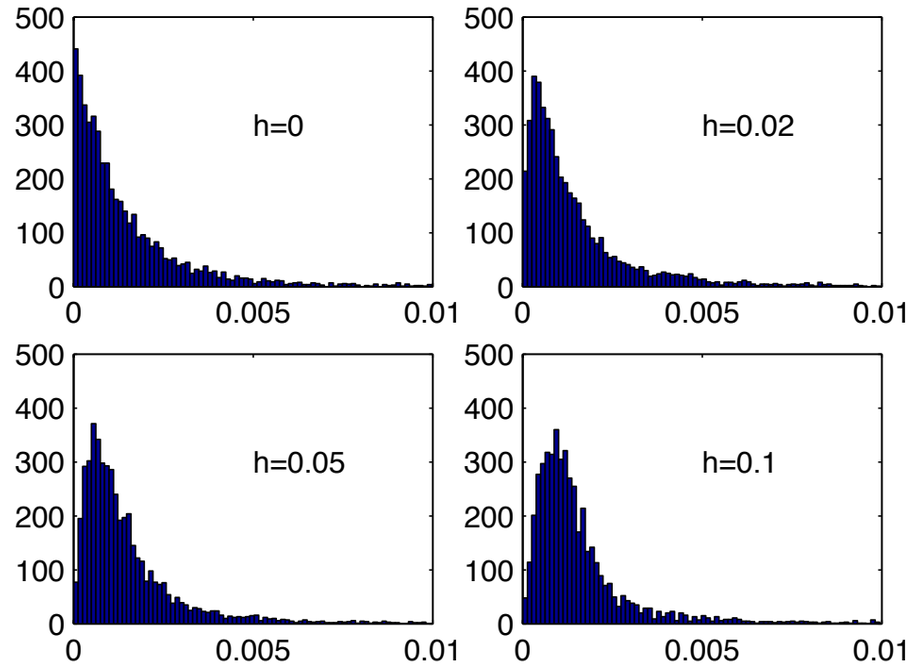
The key difference (and let's look for it numerically) is that the integrability of an MBL system is stable to disorder, while conventional integrability is not.

Staggered field and non-integrability

$$H = \sum_{i=1}^{L} \left[S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta S_i^z S_{i+1}^z + (-1)^i h S_i^z \right]$$



In one region, of the phase diagram, h is irrelevant (system remains Luttinger liquid), and we can track RG flow

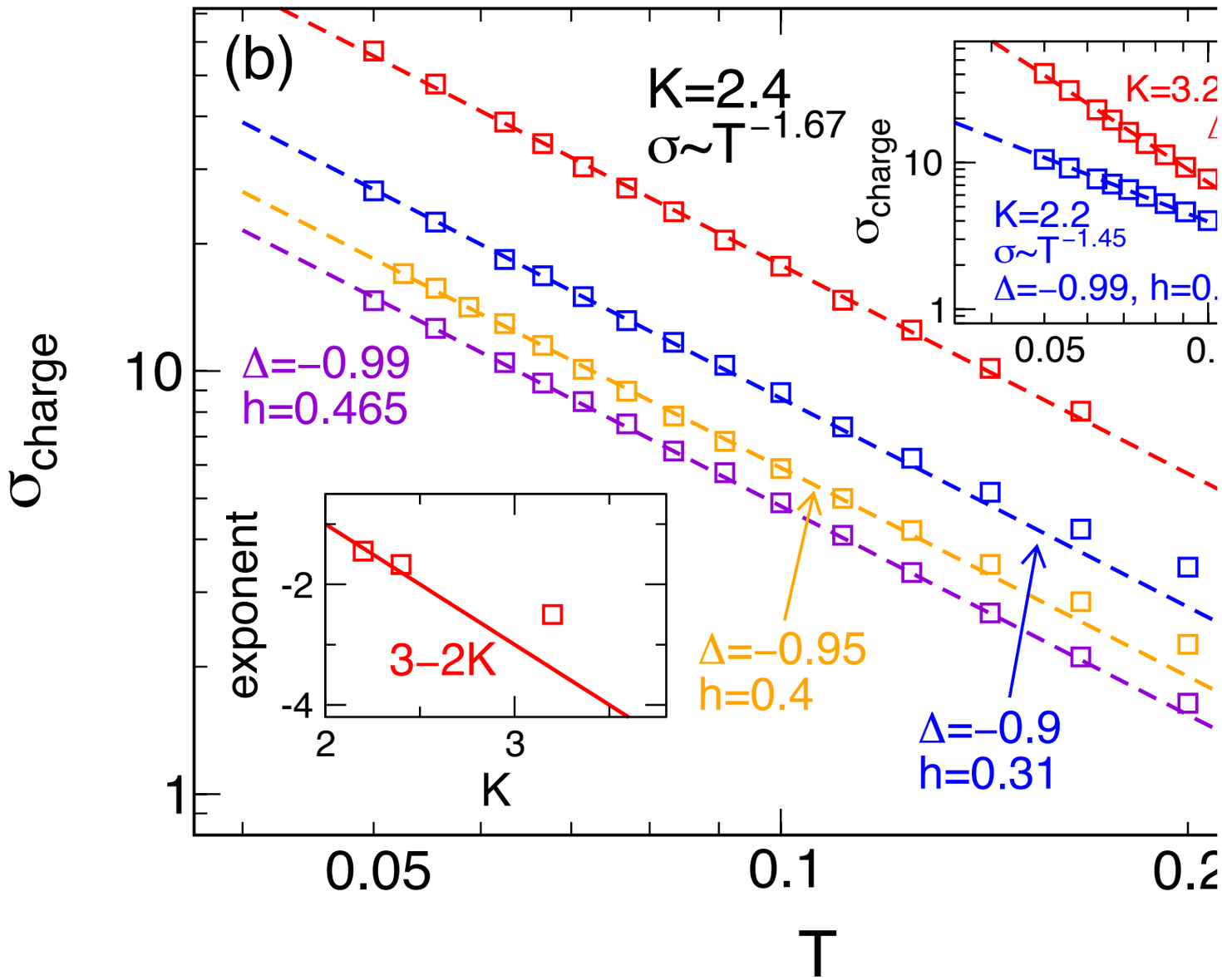


Level statistics become Wigner-Dyson (level repulsion) rather than Poisson

Argument for Poisson statistics: two nearby states are likely to be in different symmetry sectors, and hence do not repel each other as they are not mixed by a perturbation.

Conductivity scaling

$$\sigma = \lim_{t_M \to \infty} \lim_{L \to \infty} \frac{1}{LT} \mathrm{Re} \int_0^{t_M} \langle J(t) J(0) \rangle \, dt.$$



For K not too large, linear prediction is self-consistent and power-laws are observed that are consistent with bosonization predictions.

Conductivity diverges at low temperature as the integrability-breaking perturbation is irrelevant.

(Huang, Karrasch, Moore PRB 2013)

Integrability in MBL

$$H = J_{xx} \sum_i \left(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y \right) + J_z \sum_i S_i^z S_{i+1}^z + \sum_i h_i S_i^z$$

Another way to define the MBL phase and explain its lack of thermalization is in terms of a complete set of conserved quantities that are genuinely local (Serbyn, Papic, Abanin; Imbrie)

(i.e., local as in the non-interacting case, not translation-invariant sums of local objects)

We expect to see Poisson statistics in the MBL phase simply because nearby states are likely to be localized in different parts of the sample, and hence not repel each other. Will come back to this.

Many-body localization at infinite temperature

$$H = J_{xx} \sum_i \left(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y \right) + J_z \sum_i S_i^z S_{i+1}^z + \sum_i h_i S_i^z$$

Numerical experiment: start with an arbitrary product state (local Sz eigenstate) and evolve under *H*. Can view as a "global quench".

"Extended phase": expect S grows linearly with t (Calabrese and Cardy)

"One-particle localized phase": (Jz = 0) eigenstates are Slater determinants of localized one-particle states; S saturates to a finite value.

What happens if we add interactions to the localized phase?

Note: this is efficiently simulable because for early times the system has small entanglement (Prelovsek et al., 2007)

Jens Bardarson, Frank Pollmann, and JEM, PRL **109**, 017202 (2012).

Many-body localization at infinite temperature

$$H = J_{xx} \sum_i \left(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y \right) + J_z \sum_i S_i^z S_{i+1}^z + \sum_i h_i S_i^z$$

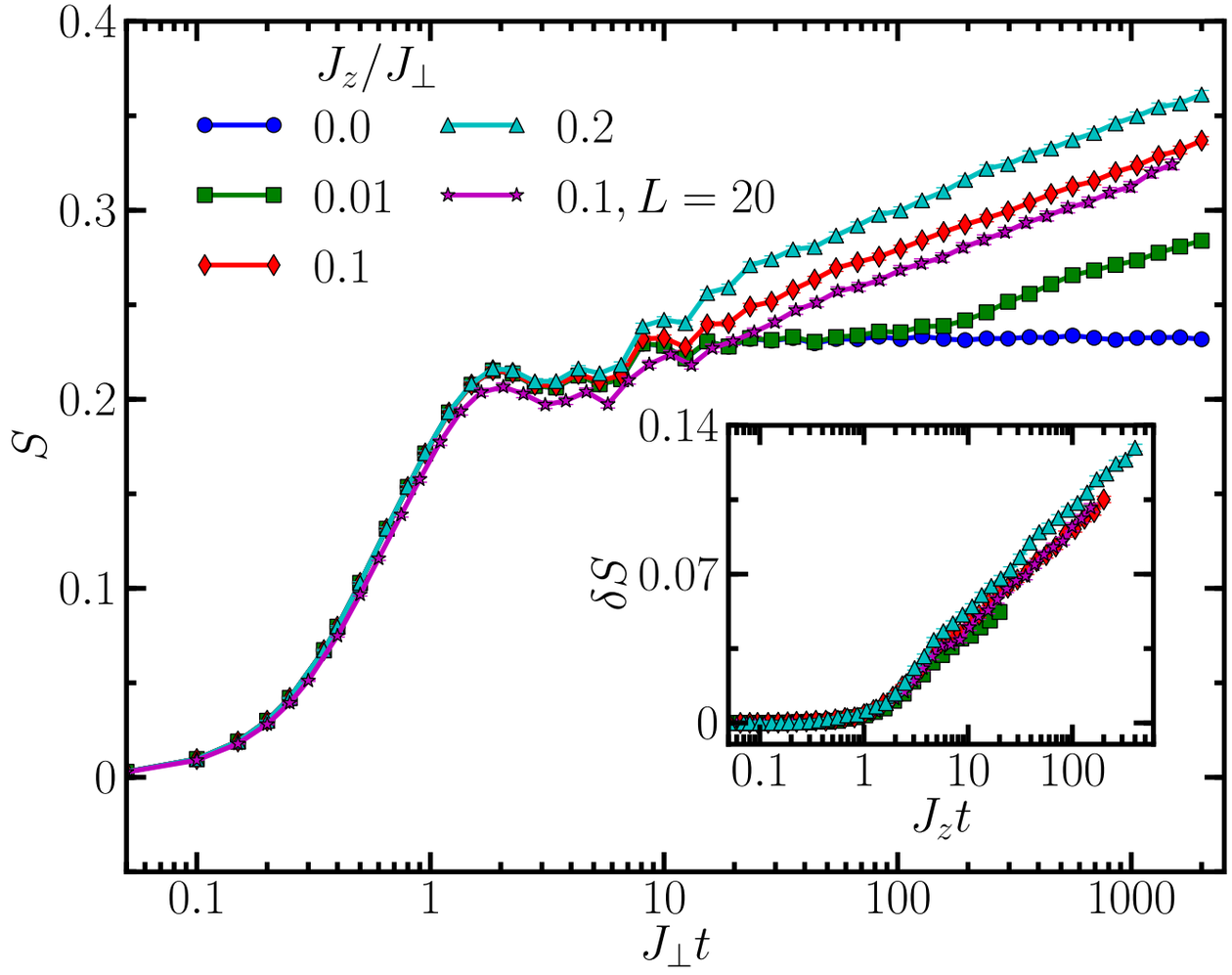
Numerical experiment: start with an arbitrary z-product state (local Sz eigenstate) and evolve under *H*. Can view as a "global quench".

Half-chain entanglement saturates with no interactions.

Interactions increase entanglement growth (consistent with previous work: De Chiara et al., Prelovsek et al.).

Surprise:
Interactions are a singular perturbation.

Even a very weak interaction leads eventually to a *slow* but *unbounded* increase of entanglement.



Many-body localization at infinite temperature

$$H = J_{xx} \sum_i \left(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y \right) + J_z \sum_i S_i^z S_{i+1}^z + \sum_i h_i S_i^z$$

Numerical experiment: start with an arbitrary z-product state (local Sz eigenstate) and evolve under *H*. Can view as a "global quench".

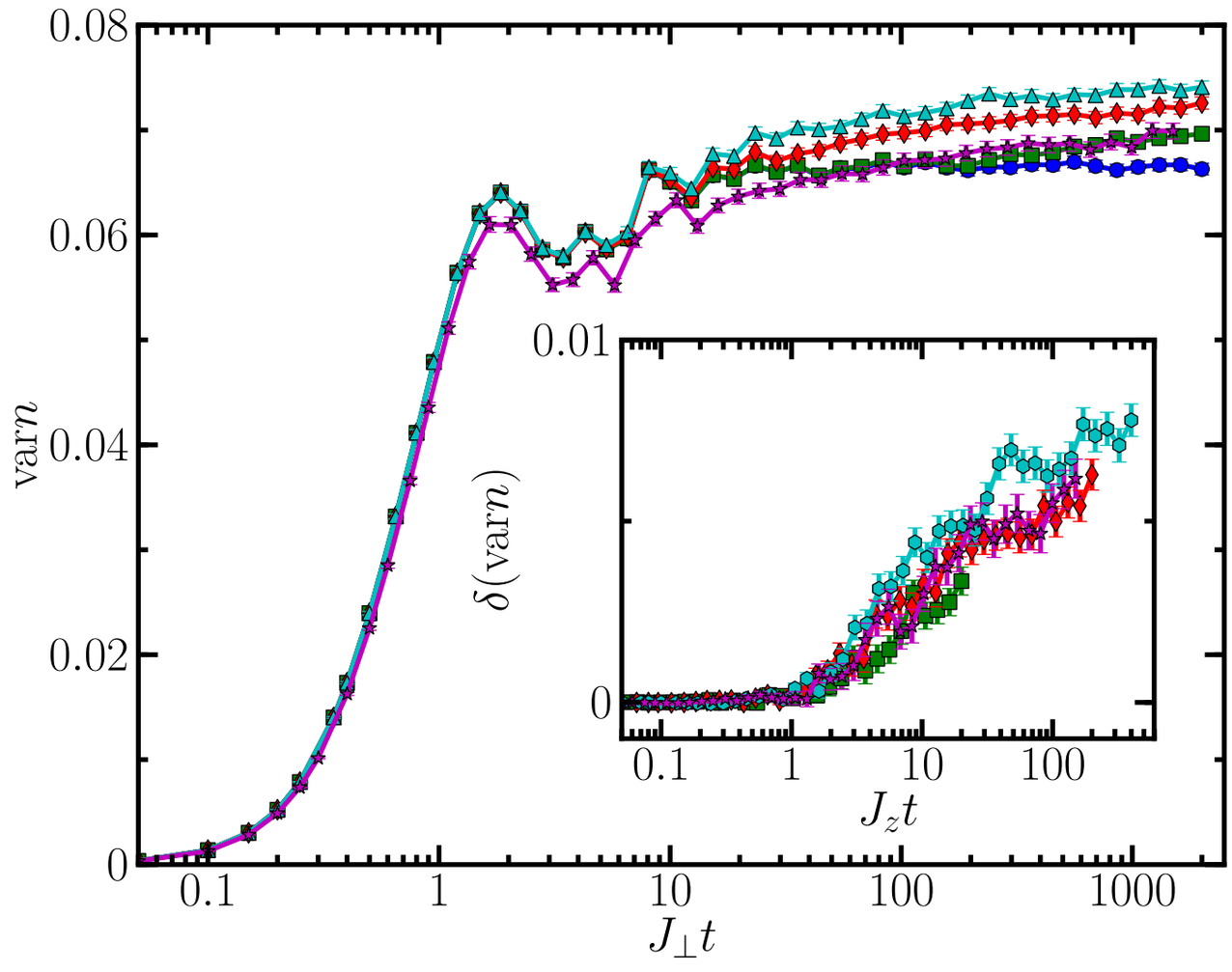
What about transport of the U(1) quantity?

Effect of interactions is less obviously singular--it could be that conductivity is zero.

We cannot rule out that the only physics with interactions is extended and that there is eventually thermalization.

But there is a long, possibly infinite, time range over which dynamics is very slow.

(Slower log log dynamics at low energy in random singlet phase--Igloi et al. PRB 2012)



Question: Is entanglement "physical"?

Yes, but hard to measure (although see Greiner et al. 2015); are other properties sensitive to this logarithmically slow dynamics?

Eigenstates versus dynamics of observables

One way to view the MBL phase: all eigenstates are basically similar, because a slight change in the potential will change which eigenstate is the ground state.

For example, all (or almost all) eigenstates are area-law (cf. Bauer-Nayak).

Different from the diffusive case, where the ground state is special (area law versus volume law, for example).

The arbitrariness in the MBL phase suggests that it may be difficult to prepare a single excited eigenstate; more generally, it is nontrivial to connect dynamics of observables (e.g., after a quench) to the properties of eigenstates.

Testing "dephasing without delocalization"

Favored scenario: (Huse-Oganesyan, Papic-Serbyn,-Abanin, Vosk-Altman, …)

The entanglement increase can be understood in terms of independent pairs with interaction energy scale

$$J_{\text{eff}} = J_0 \exp(-L/\xi_0)$$

which under the (short-time) assumption that pairs contribute independently to entanglement gives

$$S \sim \xi_0 \log(J_0 t)$$

An experimentally practical way to test this log: Romain Vasseur, Siddharth Parameswaran, and JEM, PRB 2015

"Revivals": how often, in a single realization of disorder, does a single spin's expectation return to its original value?

This is basically a probe of how many frequencies are involved in the spin's dynamics. That increases dramatically between Anderson localization and MBL.

The dephasing picture has to break down as we approach the transition to a delocalized phase.

Model: XXZ chain plus "probe spin" at edge

$$
\begin{aligned}
H &= H_{\mathrm{XXZ}}[\{\sigma_i\}] + \frac{\lambda}{2}\left(S^+\sigma_1^- + S^-\sigma_1^+\right) \\
H_{\mathrm{XXZ}} &= \frac{1}{4}\sum_{i=1}^{L-1} J_\perp\left(\sigma_i^+\sigma_{i+1}^- + \sigma_i^-\sigma_{i+1}^+\right) + J_z\sigma_i^z\sigma_{i+1}^z + \frac{h_i}{2}\sigma_i^z
\end{aligned}
$$

Question: "revivals"

If probe spin is initially polarized, how frequently does its polarization return to nearly the initial value?

Qualitative motivation:
already in a classical system, Poincare recurrence time is a measurement of phase space volume.
Larger phase space to explore = lower rate of revivals.

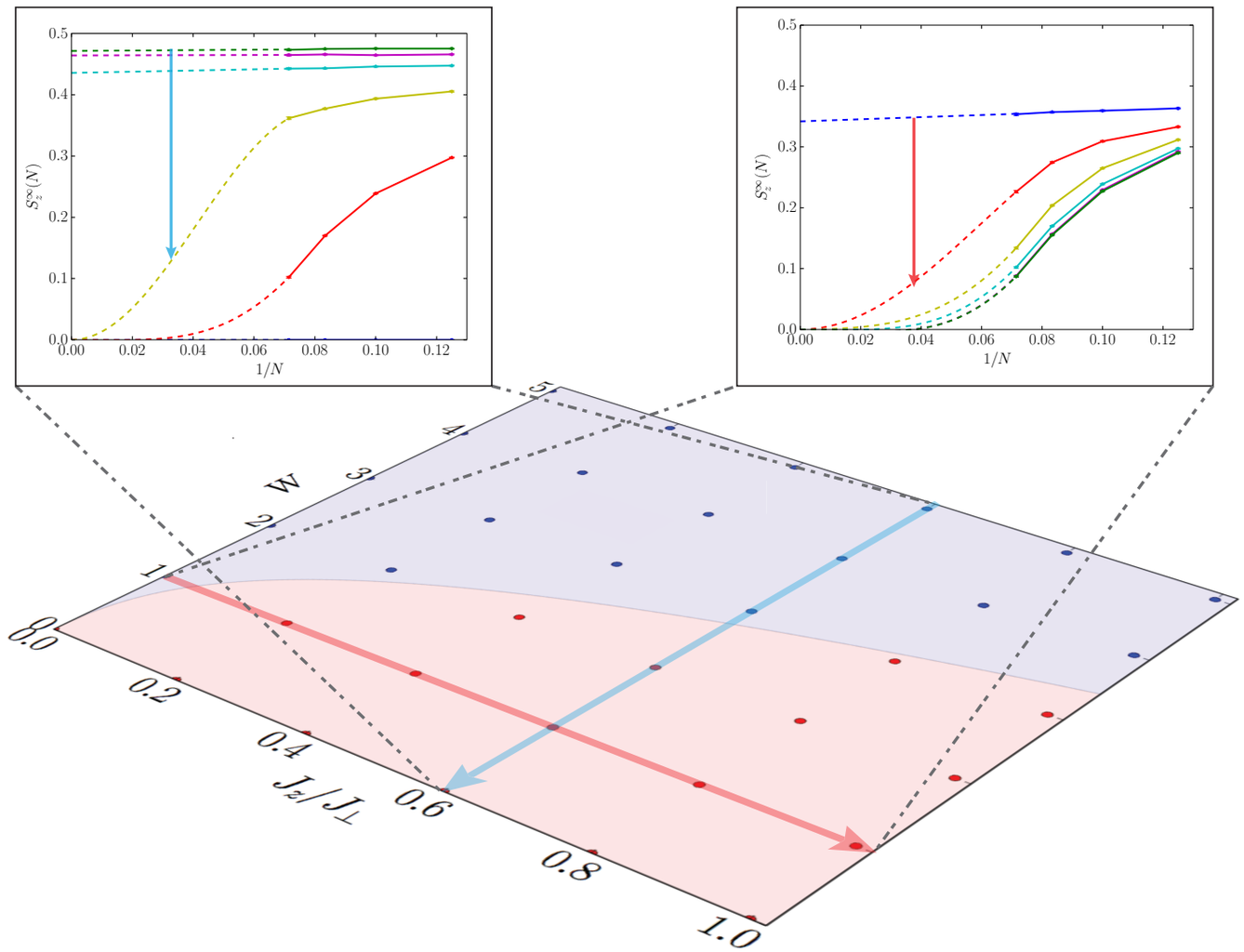
Numerical experiment

Initial state is probe spin up and random initial state of chain (with and without constraint of total Sz = 0).

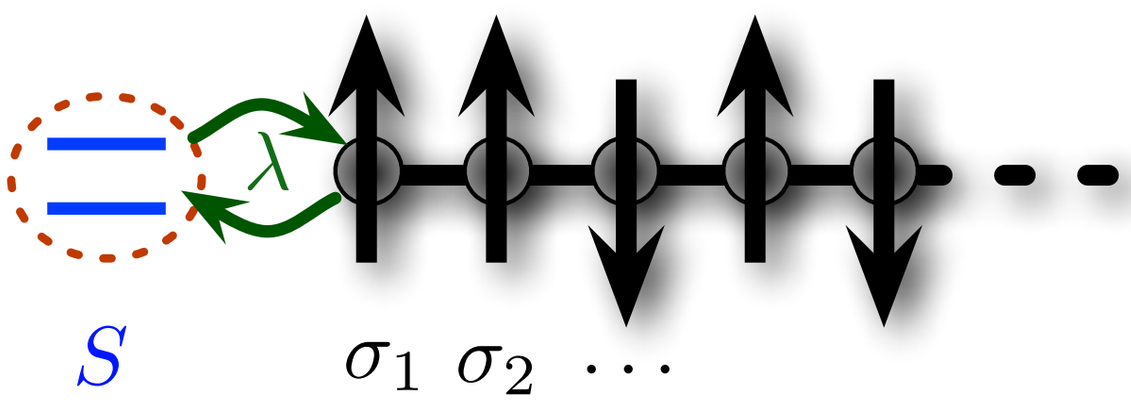
Evolve in time and record a "revival" whenever average probe spin is within (1-epsilon) of initial value.

1. Estimate phase diagram via residual magnetization

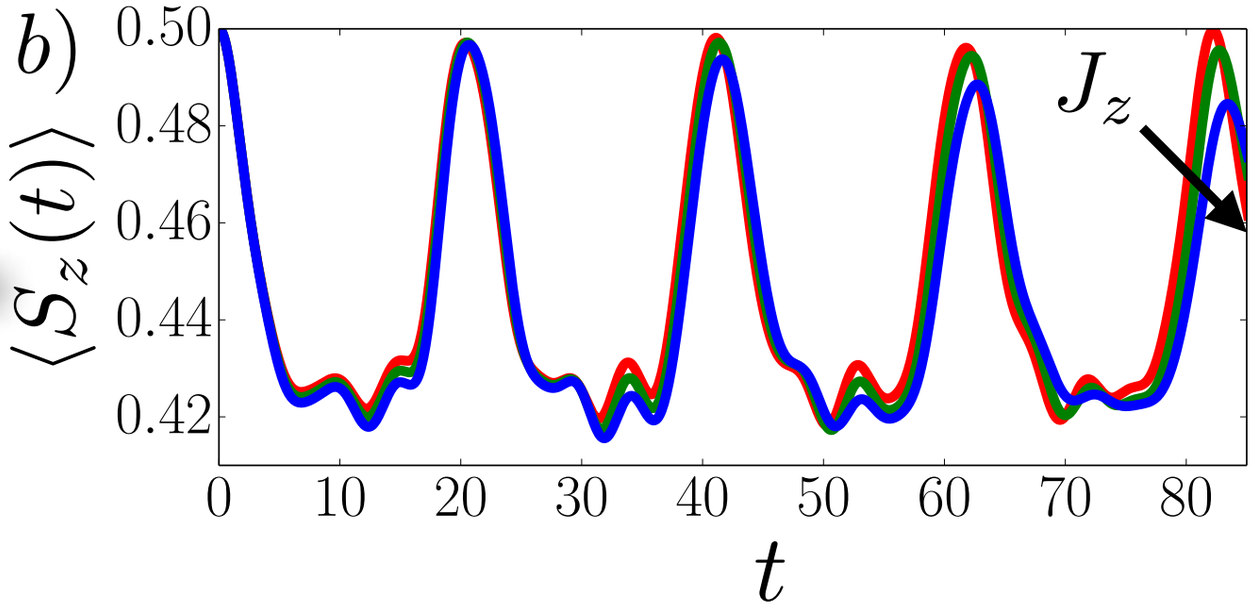
$$\overline{S^z_\infty} \equiv \overline{\sum_\alpha \langle\alpha|S^z|\alpha\rangle \, |\langle\Psi_0|\alpha\rangle|^2},$$



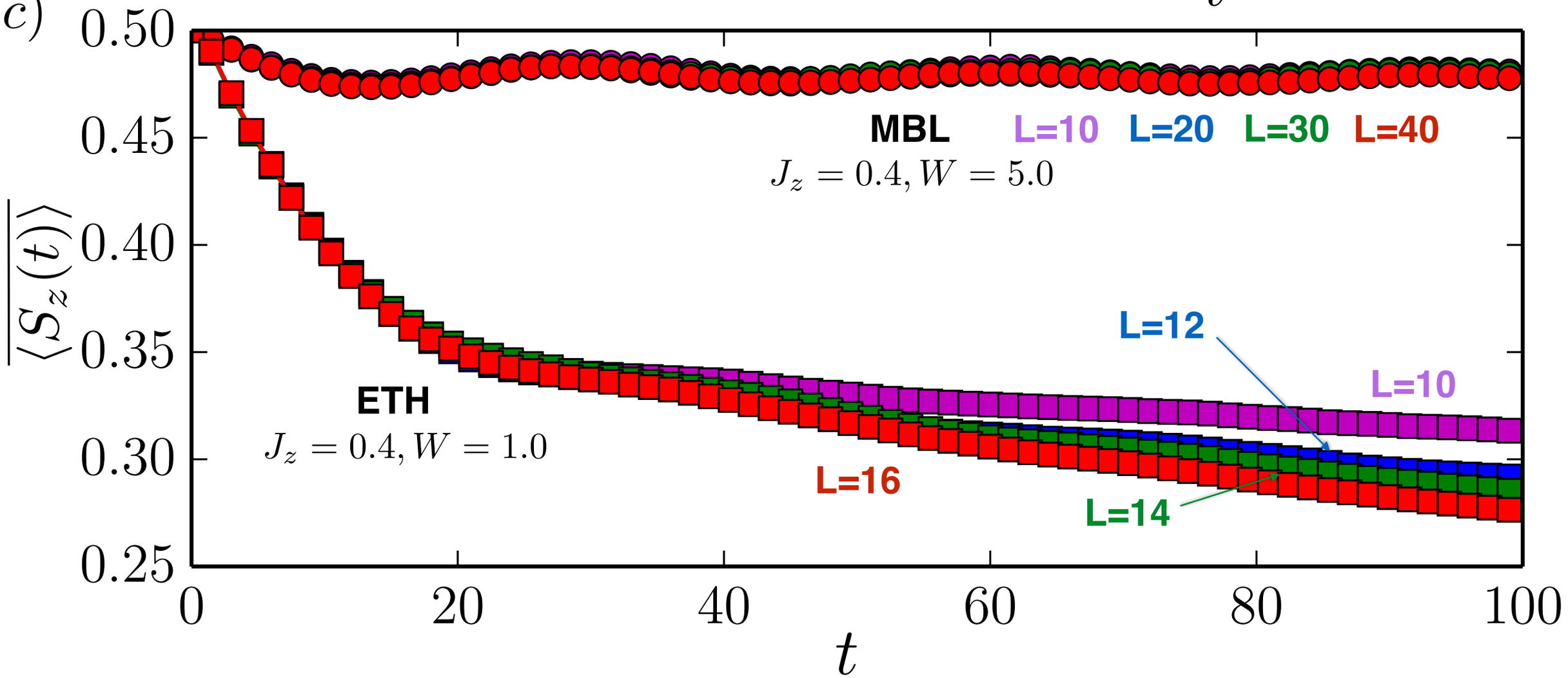
a)


λ
S
σ_1 σ_2 $\cdots$

b)

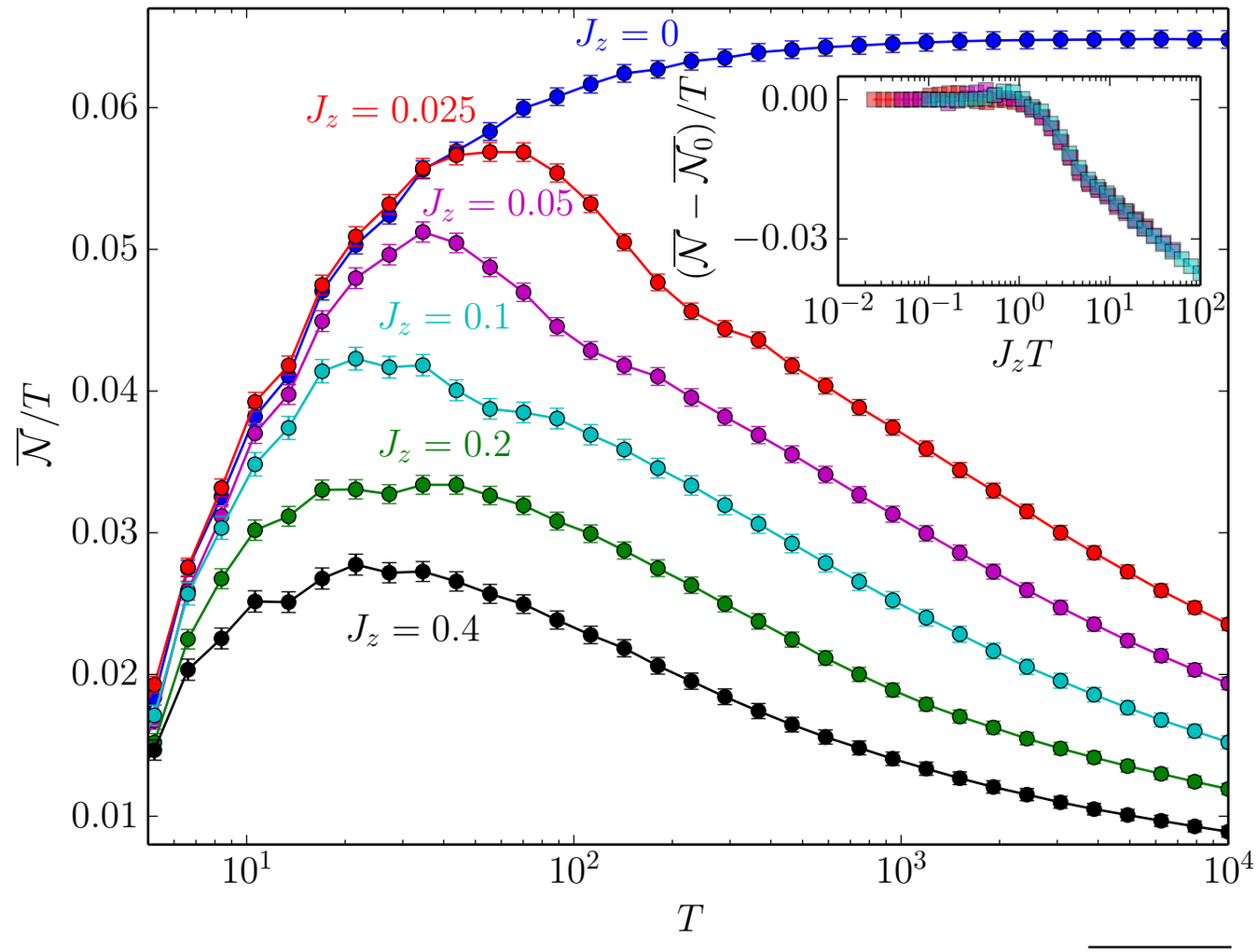

$\langle S_z(t) \rangle$
0.50
0.48
0.46
0.44
0.42
0
10
20
30
40
50
60
70
80
t
J_z

c)


$\overline{\langle S_z(t) \rangle}$
0.50
0.45
0.40
0.35
0.30
0.25
0
20
40
60
80
100
t
MBL
L=10
L=20
L=30
L=40
$J_z = 0.4, W = 5.0$
ETH
$J_z = 0.4, W = 1.0$
L=12
L=10
L=16
L=14

Numerical experiment

2. How does the revival rate show the effects of interactions?



Quantum Revivals. Disorder-averaged revival rate $\overline{\mathcal{N}(T)}/T$ as function of total time, T. Upon adding interactions of strength J_z, revivals are suppressed beyond $T^* \sim J_z^{-1}$. (Inset) The same data collapses onto a universal curve when plotted against $J_z T$.

Scaling collapse

We can do better than just saying that interactions = fewer revivals. Actually the same phenomenology that explains entanglement growth appears here as well: the difference in revival rates is

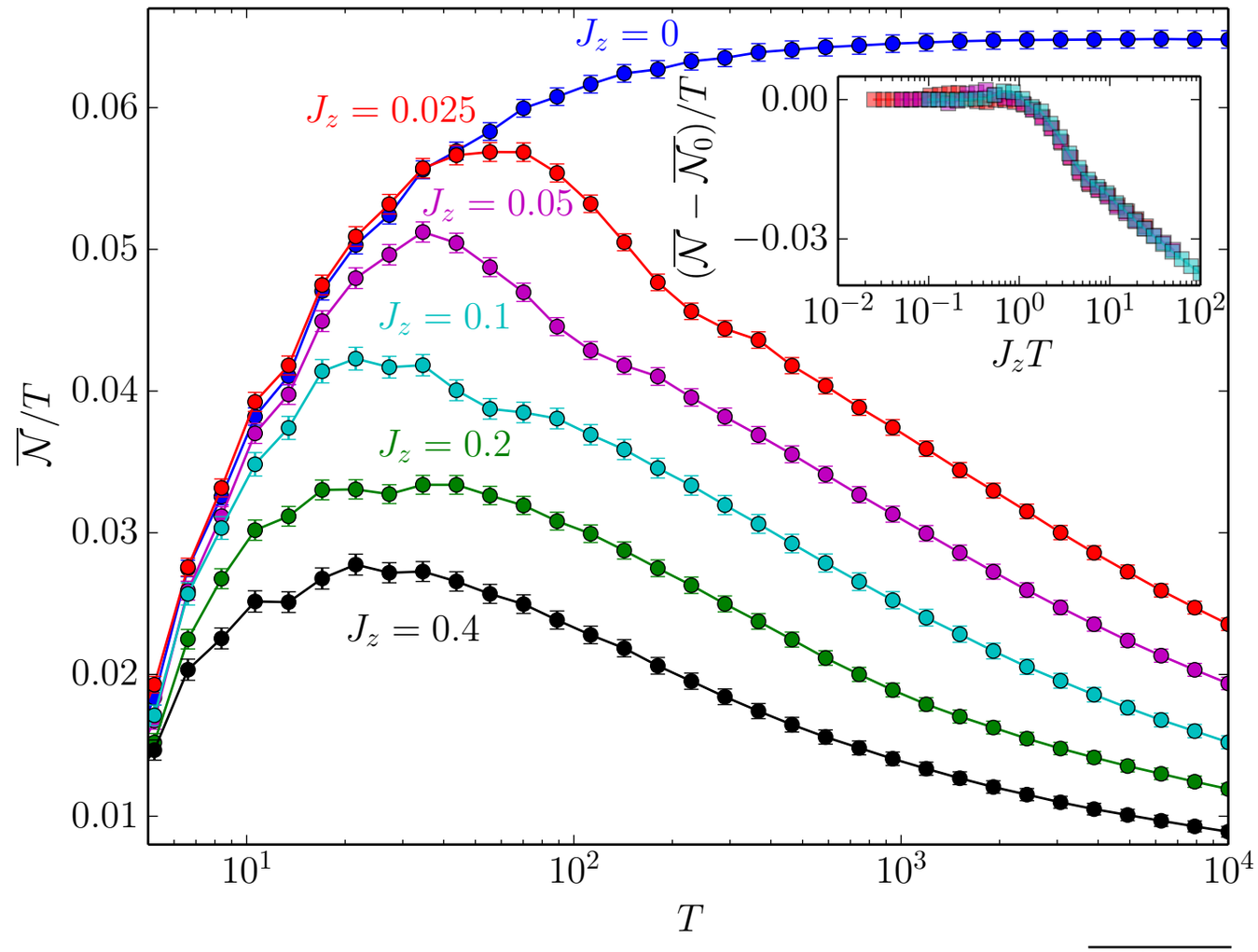
$$\frac{\overline{\mathcal{N} - \mathcal{N}_0}}{T} \approx \nu(N + \alpha \log J_z t) - \nu(N),$$

where \nu(N) is the revival rate when N different frequencies matter (expect an exponential dependence, but details turn out to be irrelevant).

The numerics show that the revival rate indeed shows a collapse with logarithmic time over most of the MBL phase (presumably not all of it)…

Numerical experiment

2. How does the revival rate show the effects of interactions?



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Result: a simple picture

The "real-space Fermi liquid" form

$$H = \sum_i \epsilon_i n_i + \sum_{i,j} U_{ij} n_i n_j + \dots$$

controls not just entanglement growth but more "physical" observables over a wide range of the MBL phase.

The resulting logarithmic time evolution (assuming U falls off exponentially) is likely to be a generic property of dynamics of observables in the MBL phase. This log scaling may be the most important observable difference between MBL and Anderson phases.

Point: two simple guesses (revival rate saturates as in Anderson case, or as 1/exp(xi)) are wrong.

It would be nice to understand (a) what is the long-time state of a block in the MBL phase starting from some physical preparation process (typically volume law but not ETH); (b) how *H* becomes more complicated (3-body, etc.) close to the transition.

Many-body localization at infinite temperature

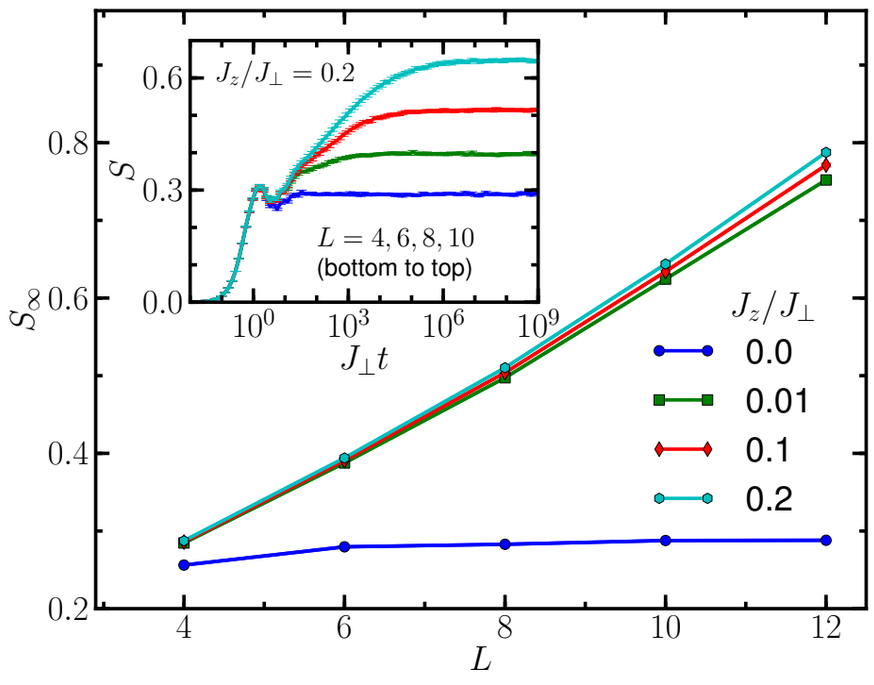
$$H = J_{xx} \sum_i \left(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y \right) + J_z \sum_i S_i^z S_{i+1}^z + \sum_i h_i S_i^z$$

What does entanglement entropy growth mean?

The entanglement entropy comes from the reduced density matrix, which governs any local experiment.

So any measurement of entropy in a subsystem will show that the interacting system is "more thermalized" than the Anderson one.

However, studies of the saturation of small blocks suggest that the full thermal entropy is not reached: O(L) but small.



What happens near the MBL transition?

One big piece of news is that there could be two transitions:

as disorder decreases, in 1D we first get a thermalizing but subdiffusive phase, before (eventually?) transitioning to diffusion.

The resulting logarithmic time evolution (assuming U falls off exponentially) is likely to be a generic property of dynamics of observables in the MBL phase. This log scaling may be the most important observable difference between MBL and Anderson phases.

Point: two simple guesses (revival rate saturates as in Anderson case, or as 1/exp(xi)) are wrong.

It would be nice to understand (a) what is the long-time state of a block in the MBL phase starting from some physical preparation process (typically volume law but not ETH); (b) how *H* becomes more complicated (3-body, etc.) close to the transition.

What happens near the MBL transition?

Even the vicinity of the thermalizing transition is complicated.

M. Serbyn and JEM, 2015: analyze MBL transition with tools from Anderson transitions (generalized level statistics and multifractality)

For eigenvalues, use Dyson-type plasma model (an interaction between eigenvalues *V* and an overall confining potential *U*)

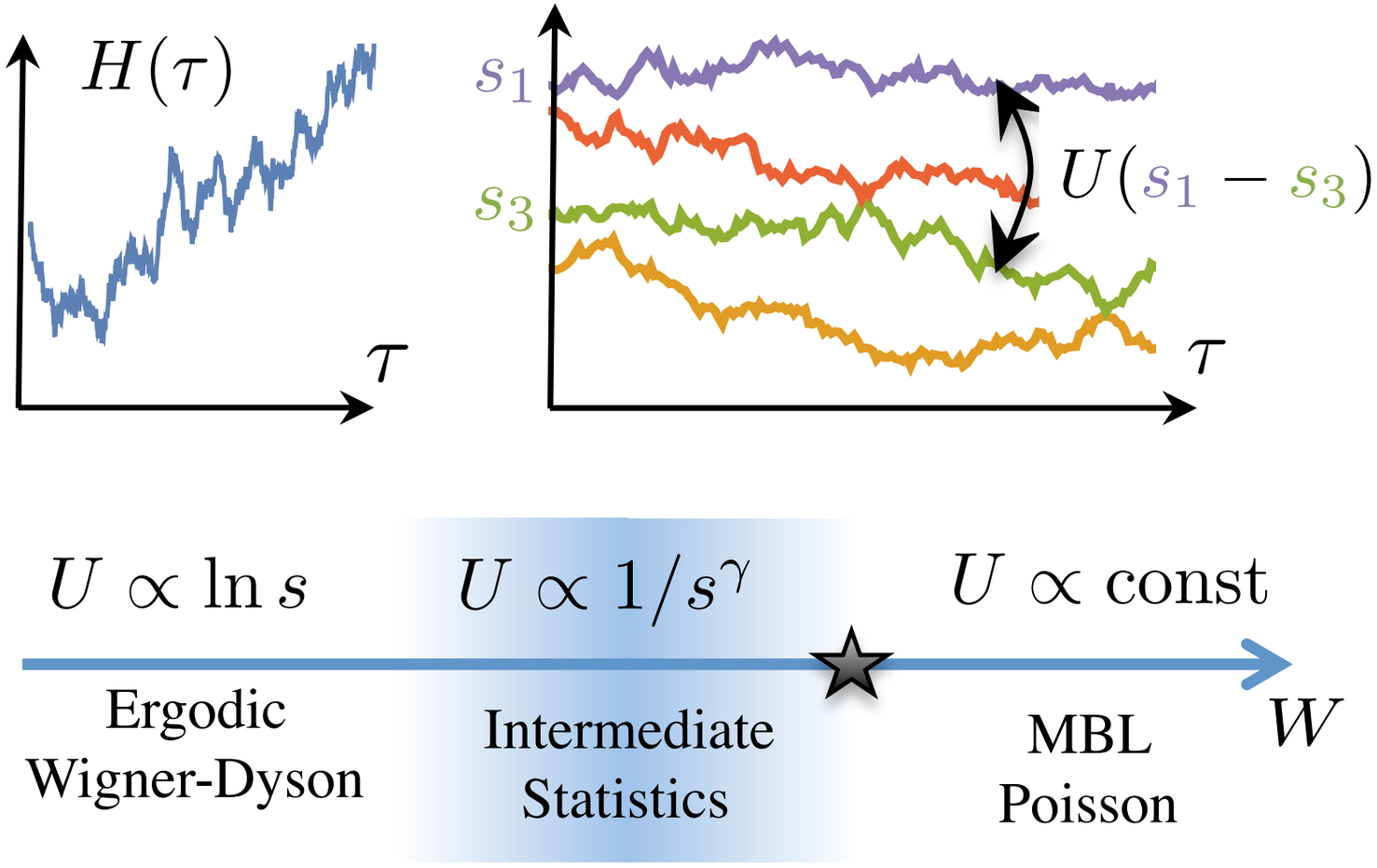
$$P(\{s_i\}) = \frac{e^{-\beta H}}{Z},\ H = \sum_i W(s_i) + \sum_{i<j} U(s_i - s_j),$$

$\beta = 1$ for orthogonal matrix ensemble which will be of primary interest. The confining potential $W(s) = s^2/2$ is parabolic, and interaction is $U(s_i - s_j) = \ln|s_i - s_j|$.

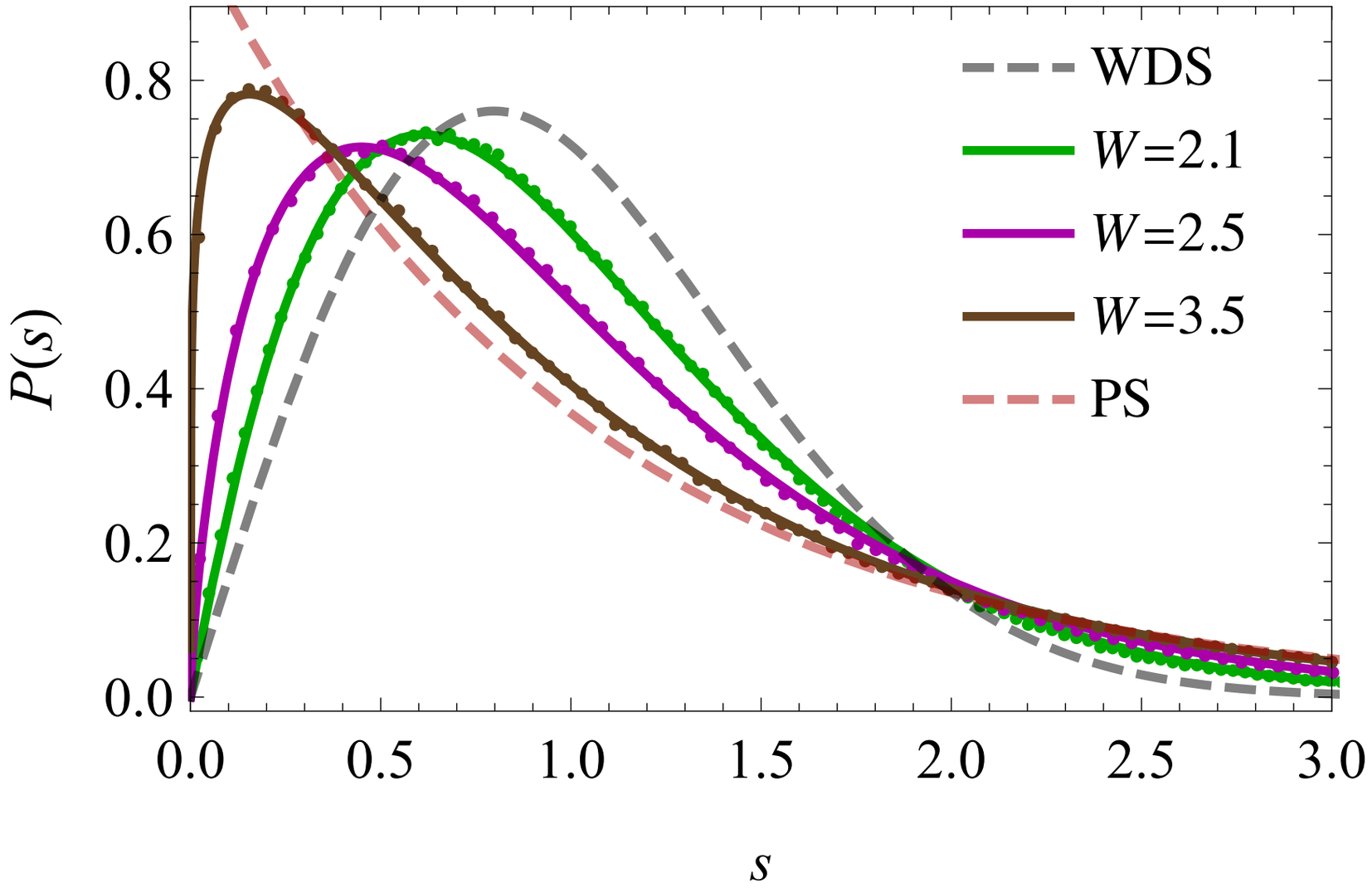
Challenges: statistics

We need to pick a way to fit intermediate level statistics:
There are ~100 proposals to generalize/interpolate between Poisson and Wigner-Dyson statistics, most of which have little microscopic justification.

We find a good fit by changing the interaction in the plasma model.
Dynamical "random walk" picture:



Challenges: statistics



We propose the following form of the level spacing distribution and spectral rigidity to interpolate between WDS and PS,

$$P(s;\beta,\gamma_P) = C_1 x^\beta \exp\left(-C_2 x^{2-\gamma_P}\right), \ \text{var}\, N = \chi N^{\gamma_{\text{var}}}, \tag{1}$$

where the parameter $1 \geq \gamma_P, \gamma_{\text{var}} \geq 0$ controls the tails of the statistics and level rigidity, and $1 \geq \beta \geq 0$ determines the level repulsion.

Challenges: multifractality of wavefunctions

One reason why even Anderson transitions are hard: "multifractality" means that there is a continuum of critical exponents.

$$\langle\langle|\psi(0)\psi(r)|^n\rangle\rangle \sim \frac{1}{r^{2\Delta(n)}}$$

Very different from a simple thermal phase transition (e.g., Ising has only *two* fundamental exponents).

Our strategy to test the model: compute several properties from both eigenvalues and eigenvectors, and see if they "collapse" (many results are explained by few parameters)

What happens near the MBL transition?

For eigenfunctions, follow Chalker RW approach, specialized to a spin chain:

$$\langle V_{nn} V_{mm} \rangle = v^2 d_{nm} = \frac{1}{\delta} \langle n|S_i^z|n\rangle \langle m|S_i^z|m\rangle, \quad (1)$$

$$\langle V_{nm} V_{mn} \rangle = v^2 c_{nm} = \frac{1}{\delta} |\langle m|S_i^z|n\rangle|^2, \quad (2)$$

where we normalized matrix elements by δ, the many-body level spacing, so that s_n represent unfolded energy spectrum. The correlator (**??**) sets the spectrum of a random noise, while c_{nm} determines the interaction between levels in the ensemble.

To make analytic progress we use a mean-field like approximation [**?**], assuming that d_{nm} and c_{nm} can be replaced by their ensemble averages,

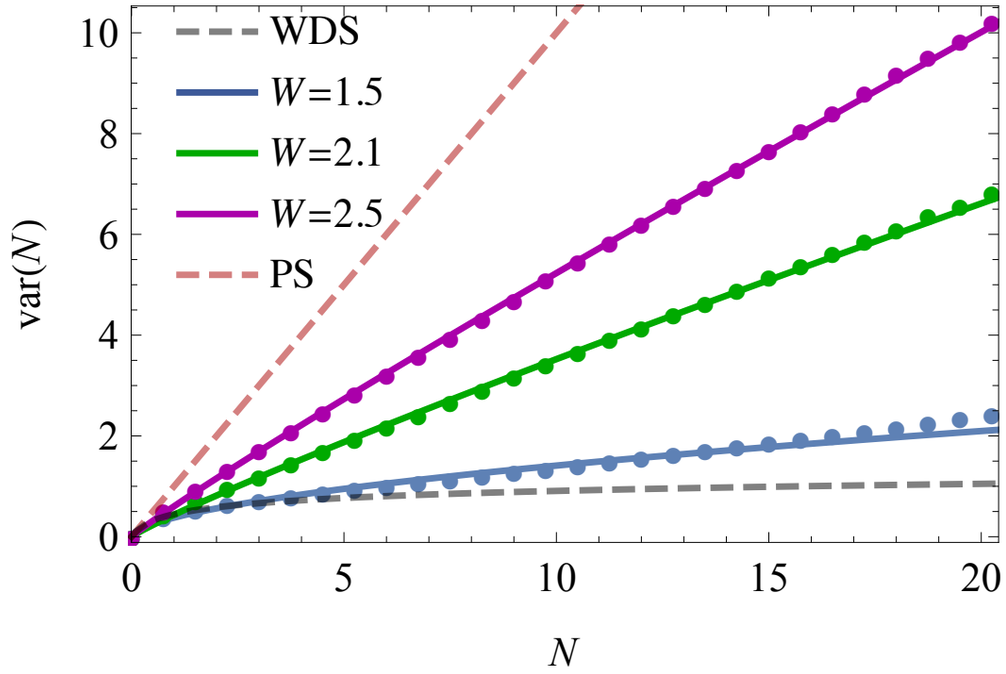
$$c(\omega) = \langle c_{nm} \delta(s_n - s_m - \omega) \rangle, \quad (3)$$

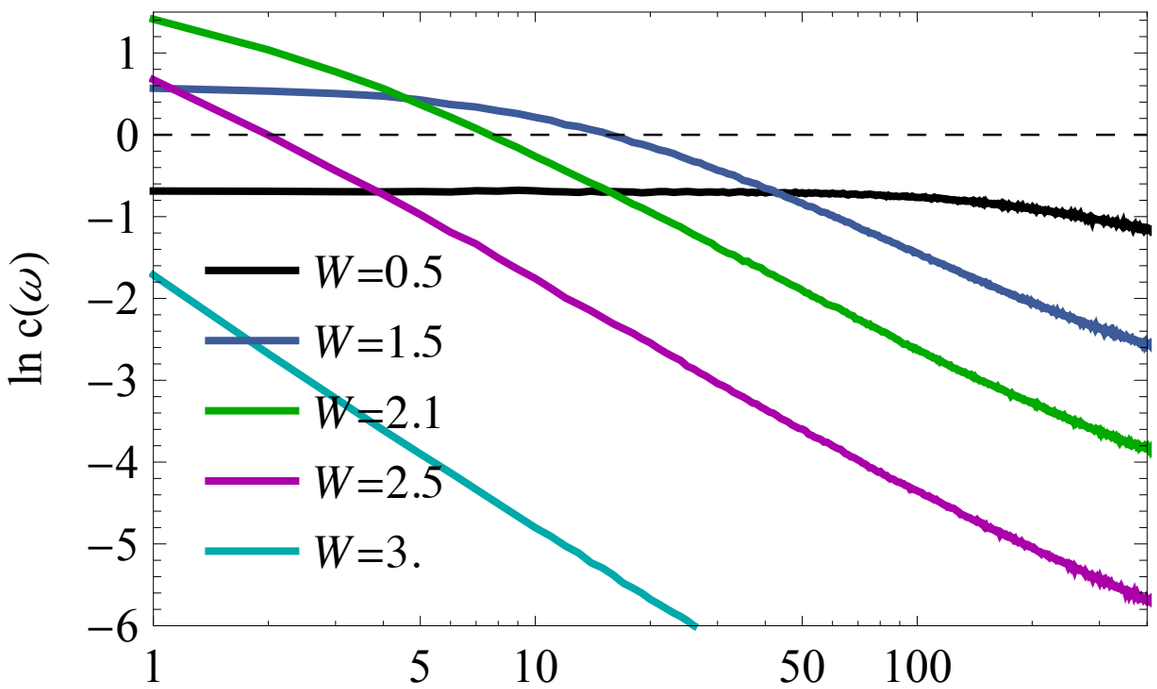
(and similar expression for d_{nm}) which now depend only on the energy difference between eigenstates.

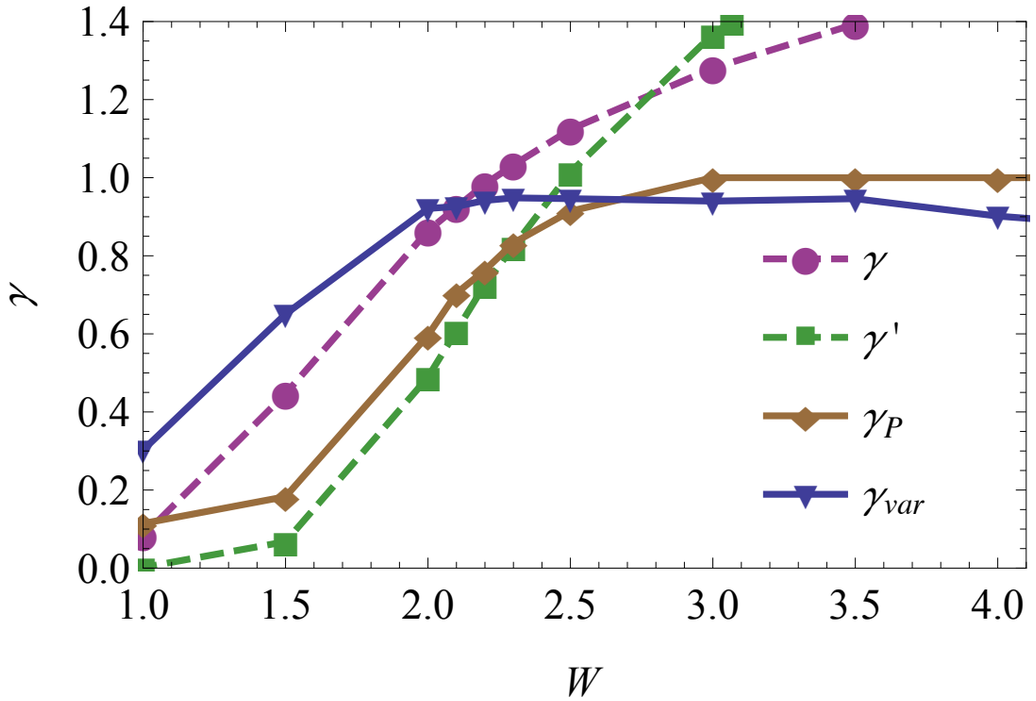
For the single-particle Anderson localization, the c_{nm} and d_{nm} necessarily coincide [**?**], as they both are given by wave functions overlaps, $c_{nm} = d_{nm} \propto \int dx |\psi_n(\tau, x)|^2 |\psi_m(\tau, x)|^2$. **Not true for MBL!**

Collapse of different properties

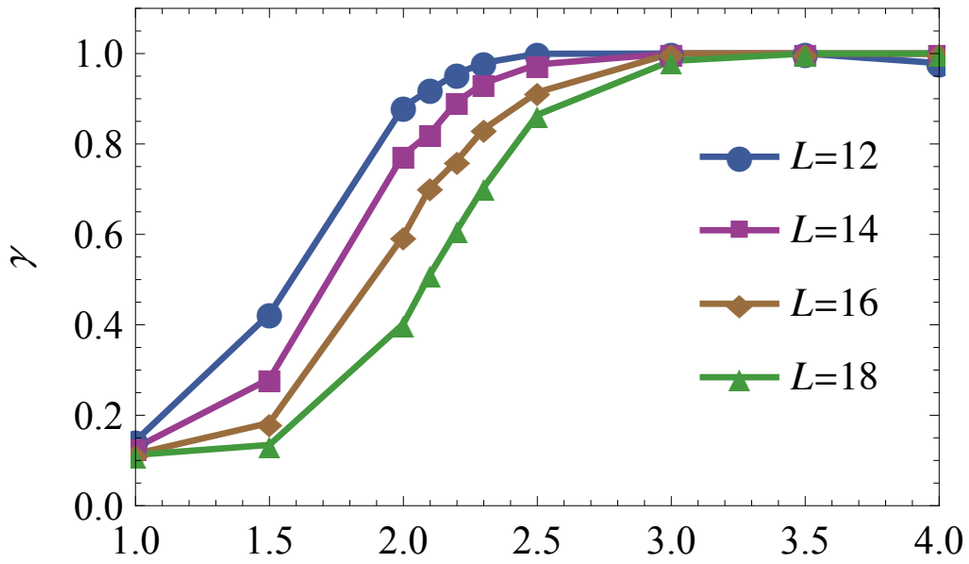
For additional information on multifractality and possible extrapolation to thermodynamic limit, see Kravtsov talk.

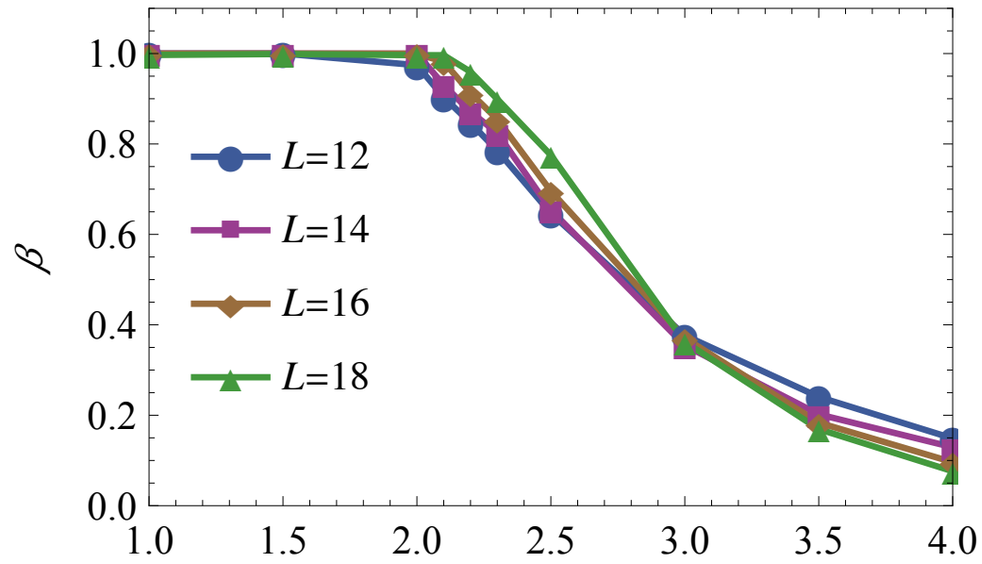






What happens near the MBL transition?





Within the picture of Brownian motion [**?**, **?**], the level statistics is controlled by the effective interaction between energy levels, see Fig. 1. In particular, deep in the metal phase, the WD statistics emerges from the partition function of a one-dimensional Coulomb gas, where particles interact with a logarithmic potential $U(s) = \log|s|$. At a first stage, upon approaching the MBL transition, the effective interaction starts to decay as a power-law: $U(s_i - s_j) = |s_i - s_j|^{-\gamma}$ when $|s_1 - s_2| \geq N_{\rm erg}$. The power-law interaction changes tails of the level statistics, so it can be approximately described by the plasma model, and is intermediate between PS and WDS case. At the second stage, when exponent γ becomes bigger than one, the interaction becomes effectively short-ranged, and level spacing distribution tends to the semi-Poisson distribution [**?**]. In this regime it is the range of the interaction which changes with disorder/system size. As soon as the range of interactions reaches zero, we arrive at Poisson statistics.

Two topics I didn't discuss:

How does MBL interact with conventional order (either symmetry-breaking or topological)?

What are MBL-like features in translation-invariant systems? (M. Mueller et al., Yao et al., Papic et al.)

Newish development: there are now actual "universality class" type predictions about the MBL transition from (Potter, Altman talks later this week).

These are obtained (so far) from numerical simulation of real-space renormalization group equations for excited states ("RSRG-X", Pekker et al.) but they make predictions for quantities like correlation lengths.

Can we test these against "microscopic" numerics? Can we compare the predictions for other quantities such as entanglement?

One specifically numerical bit of progress to be covered later this week:

since excited states in the MBL phase are structurally similar to ground states, can we isolate one with DMRG?

Yes… (DMRG-X, Khemani, Pollmann, Sondhi, arXiv: 1509.00483; Yu, Pekker, Clark , arXiv:1509.01244)

General conclusions:

1. Numerics have been very important in advancing our understanding of MBL, finding several surprises.
2. We are still pretty far away from quantitative numerics at the level of those in either non-interacting disordered problems (in any D) or interacting non-disordered problems (in 1D).
3. There is gradual progress in understanding how to bring matrix-product-state methods to bear on MBL.
4. We can view the MBL transition(s) as about conductivity, thermalization, entanglement, level statistics, multifractality, MPS representability, …