Symmetries and Shapes

1. Tiling patterns on plants
2. Mechanisms and Models
3. Symmetries, Universality
4. Rational Approximations of Irrational Numbers

Alan Newell

THE UNIVERSITY OF ARIZONA®

Patrick Shipman

Colorado State University

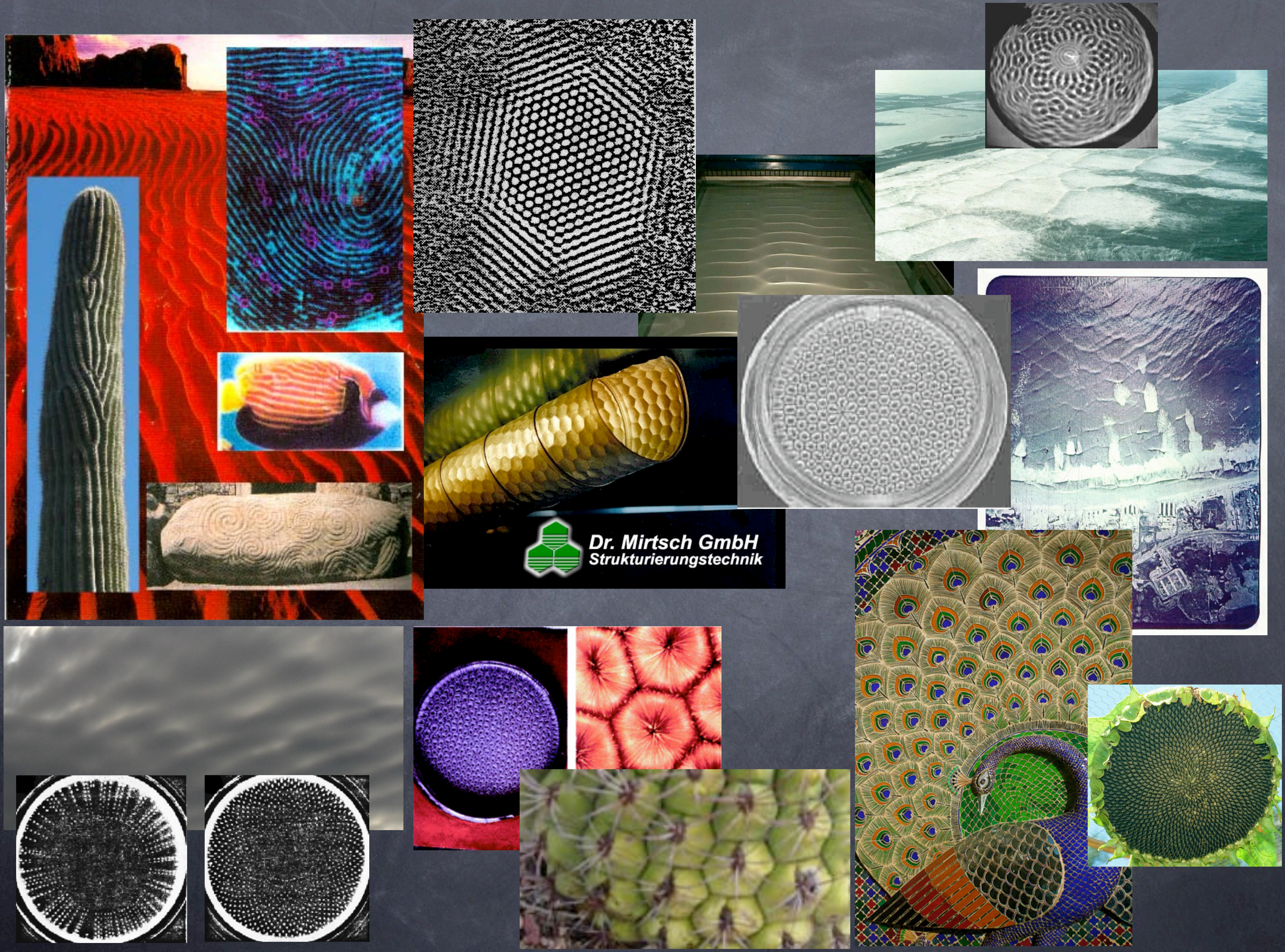
Todd Cooke

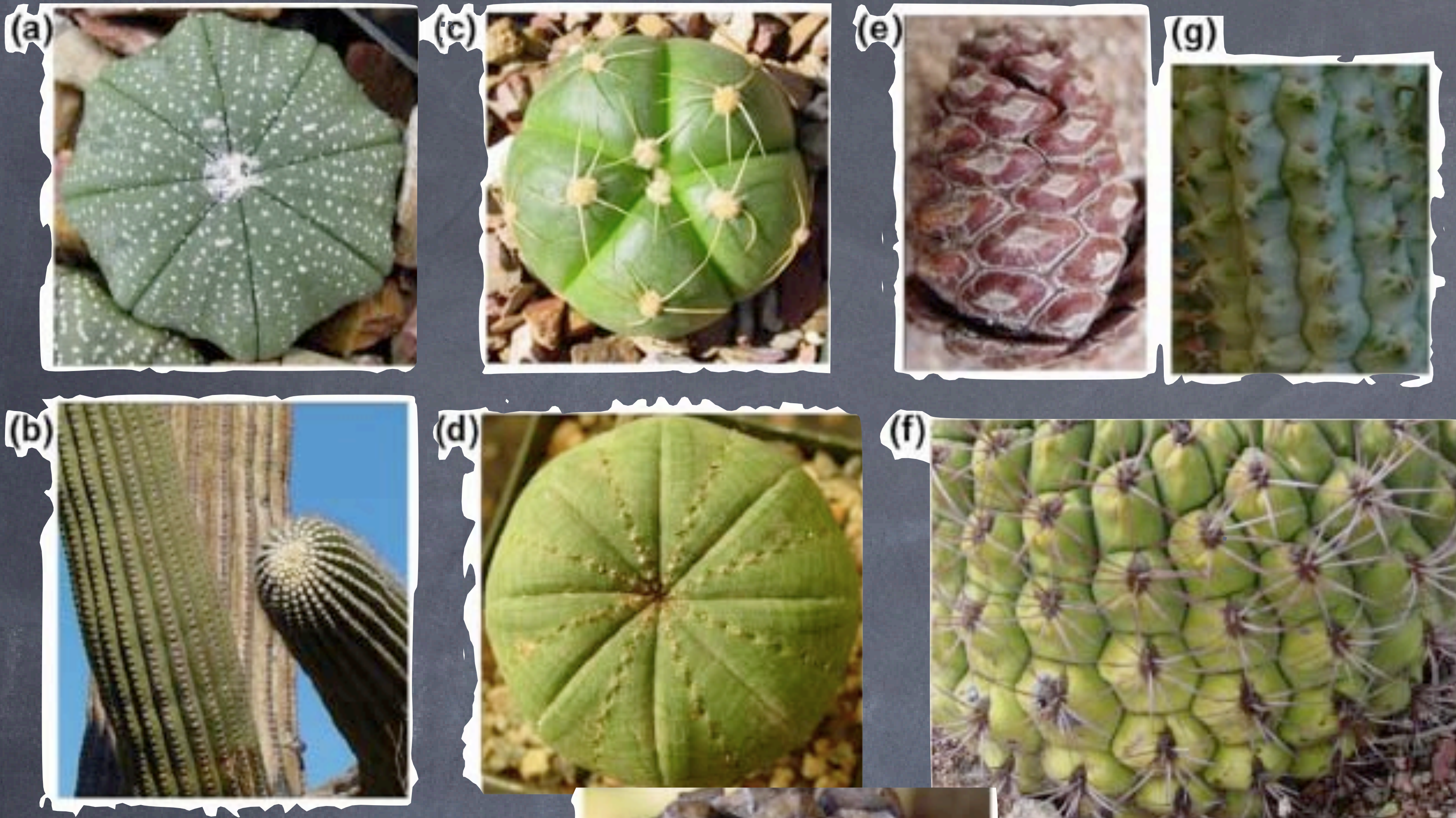
University of Maryland

UNIVERSITY OF MARYLAND 1856

Zhiying Sun

THE UNIVERSITY OF ARIZONA®


Dr. Mirtsch GmbH
Strukturierungstechnik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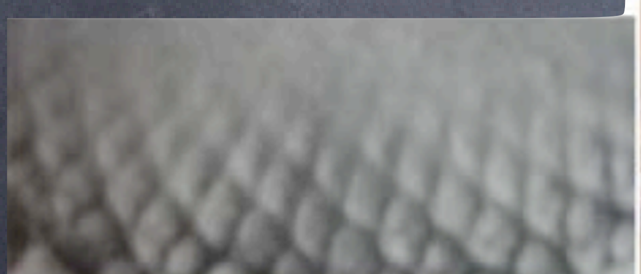


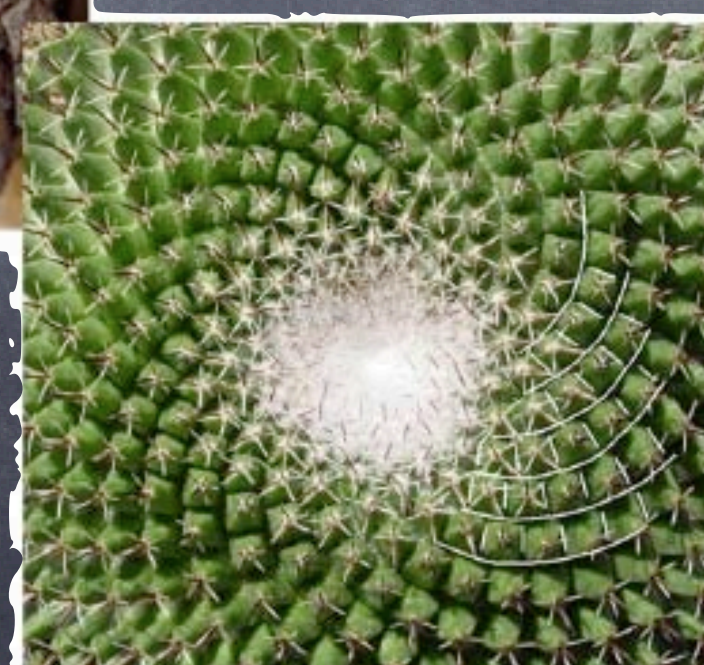

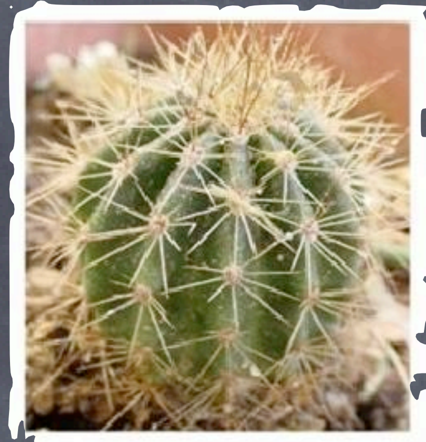


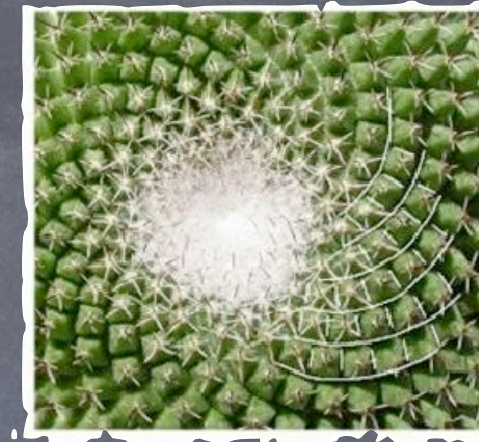
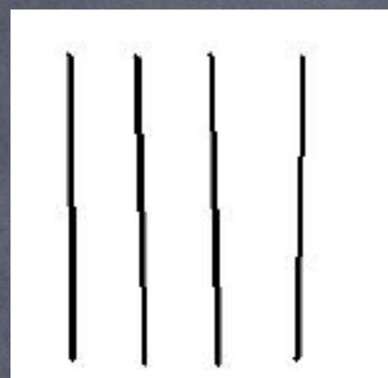
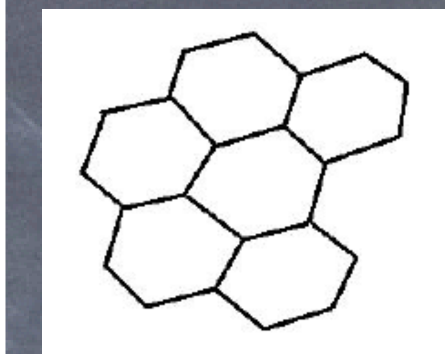
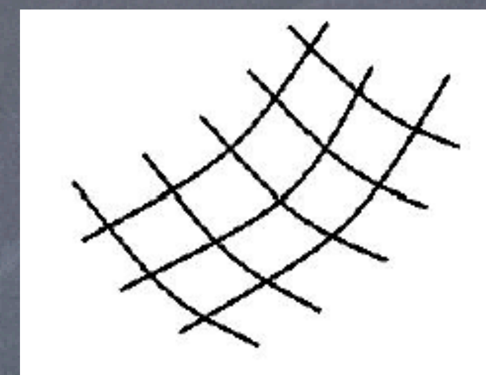
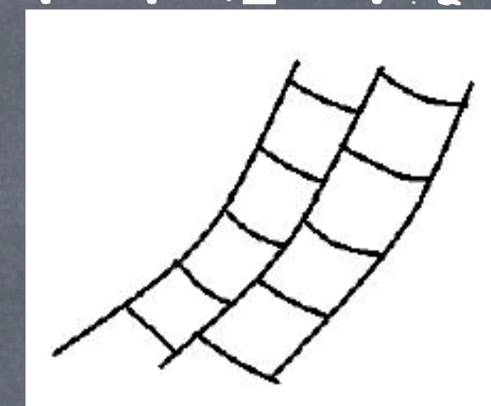
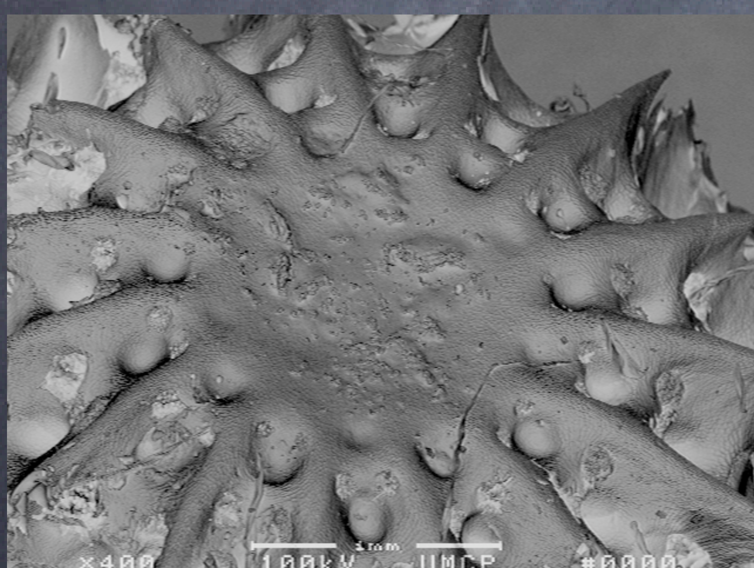
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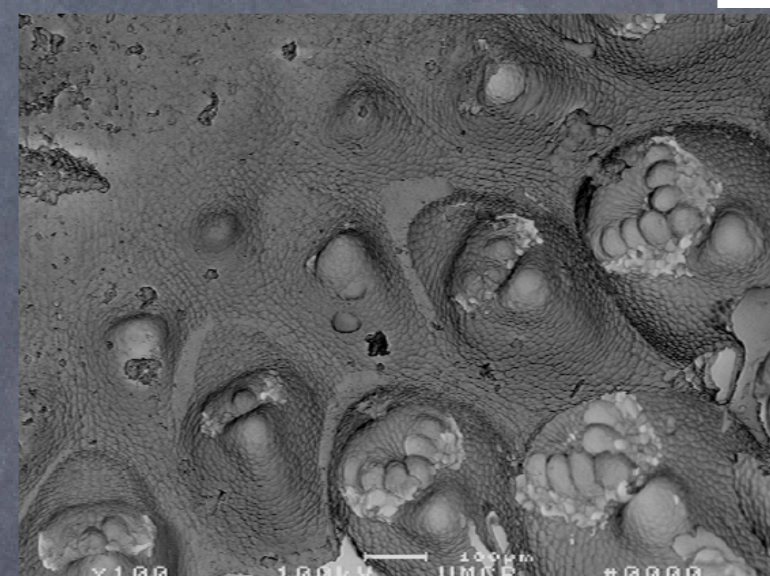
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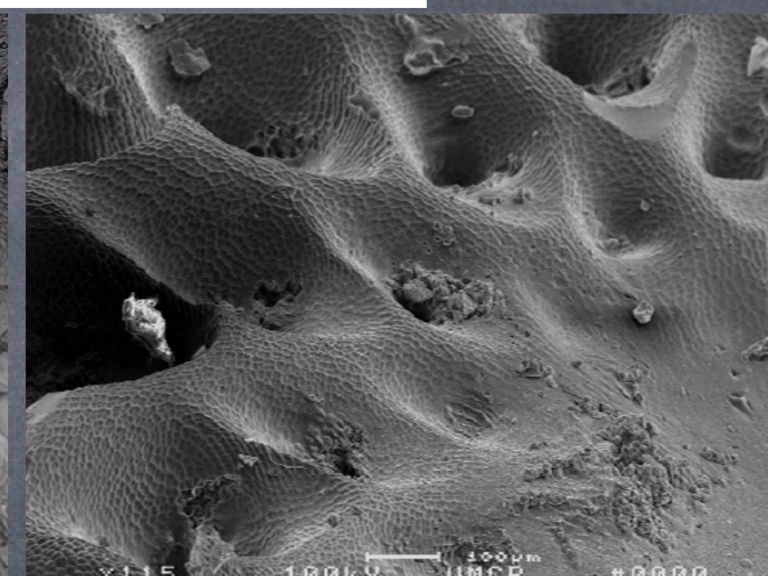
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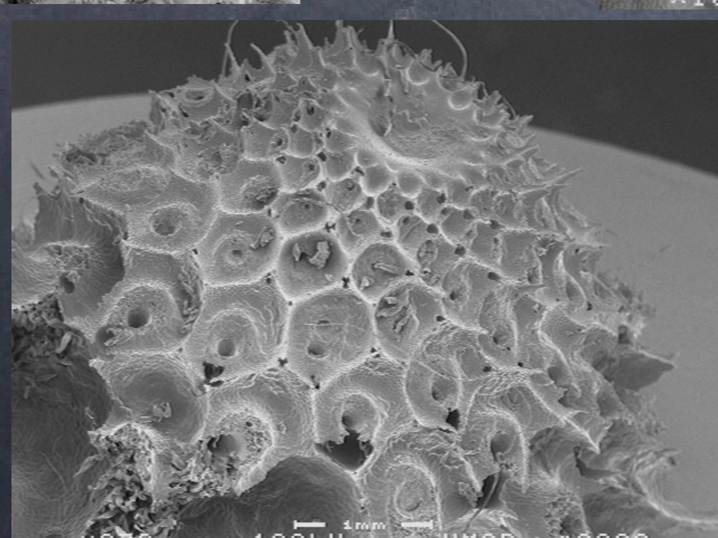
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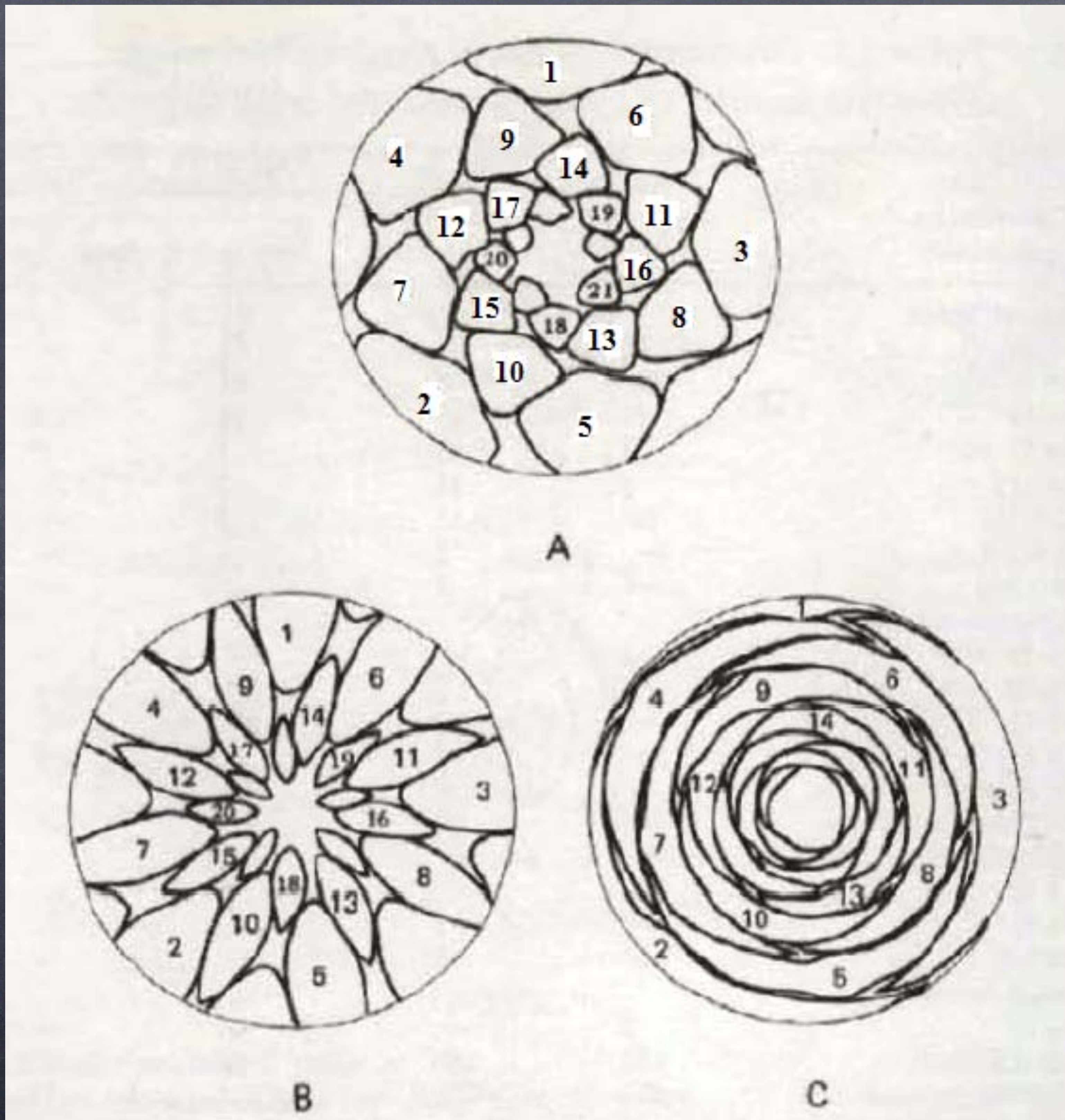

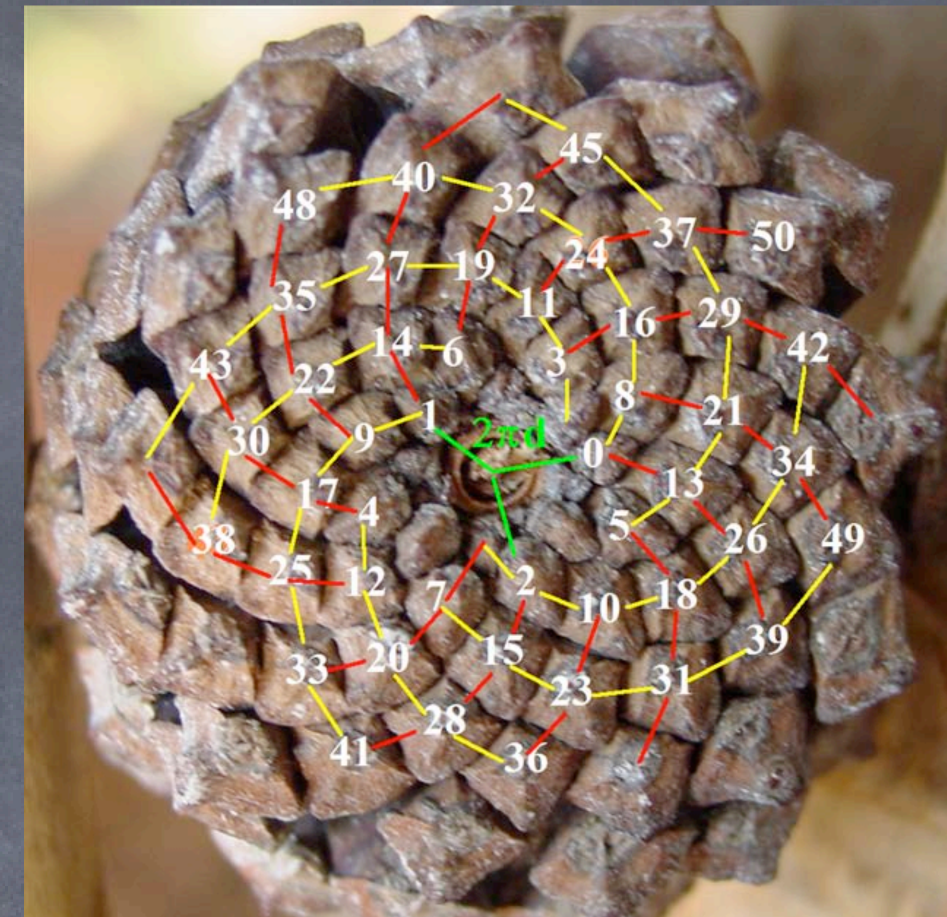

Williams, 1900‘s


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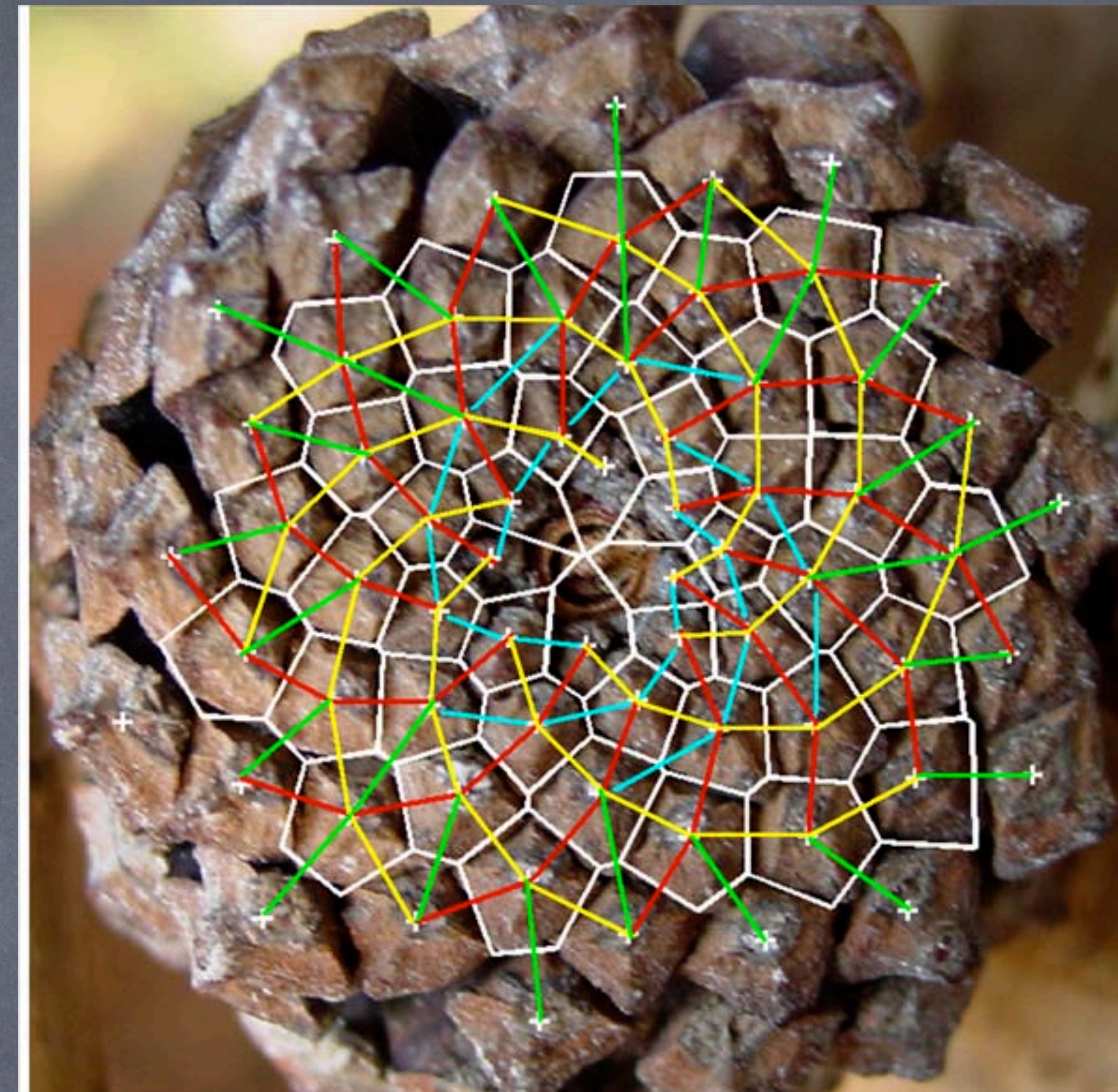
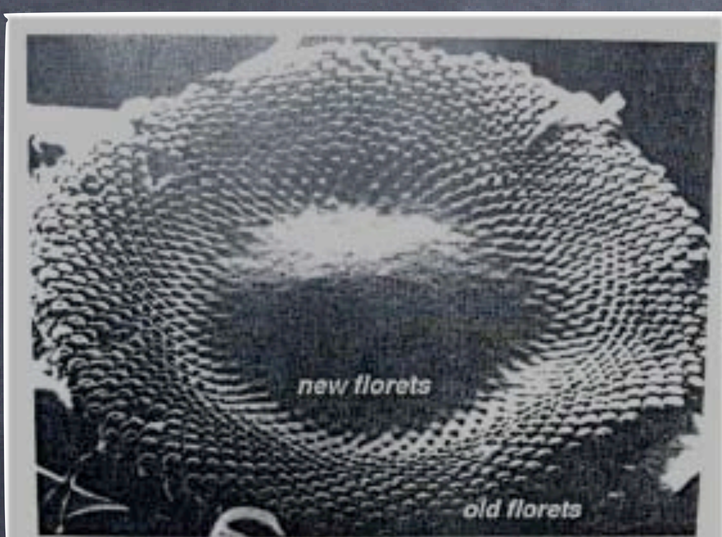


Mechanisms and Models

1. Biochemical agents

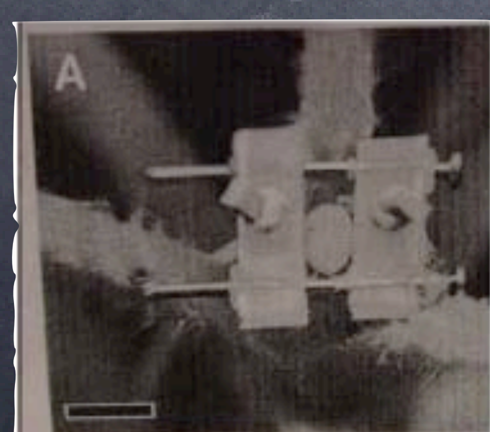
Rheinhardt, et. al. (2000
Jönsson, et. al. (2006)
Smith, et. al. (2006)



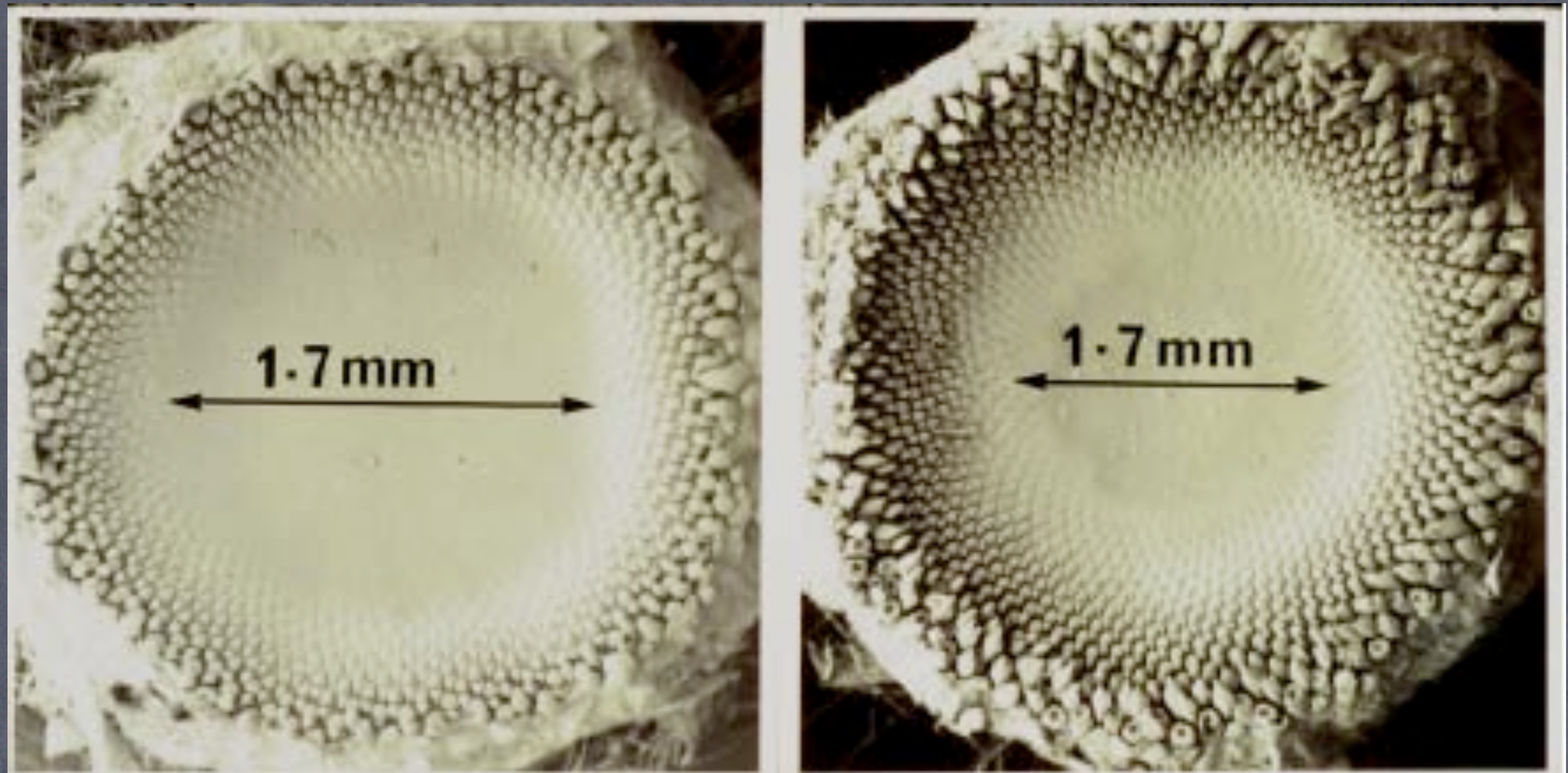
Dumais, Steele (2000)

2. Biomechanics

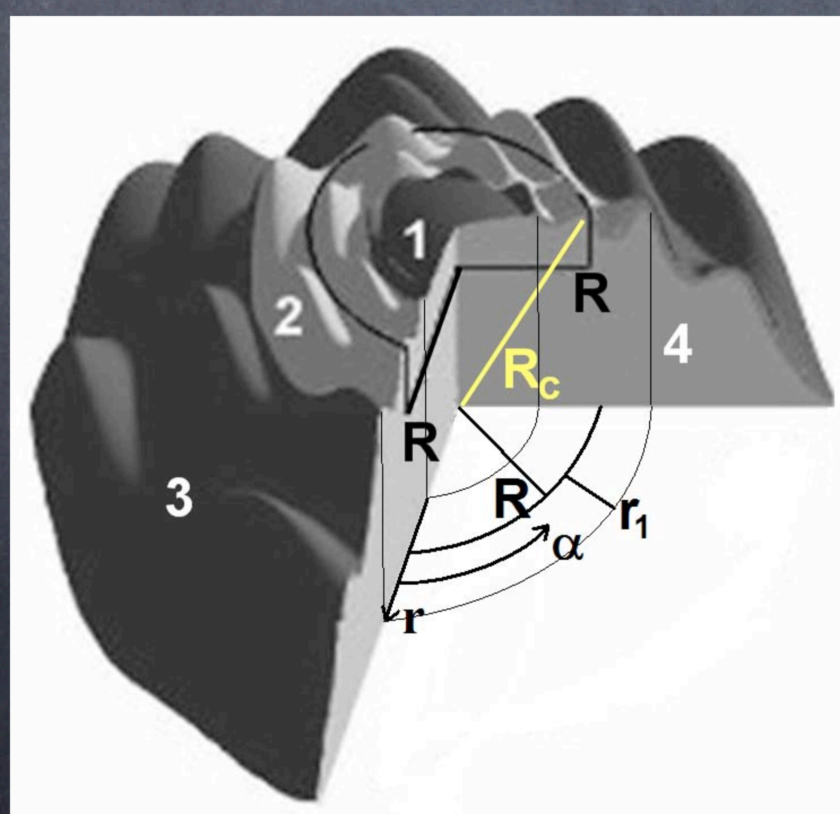
J. Dumais and C. Steele, 2000
L. Hernandez & P. Green, 1993
Green, Steele, Dumais (1990's)

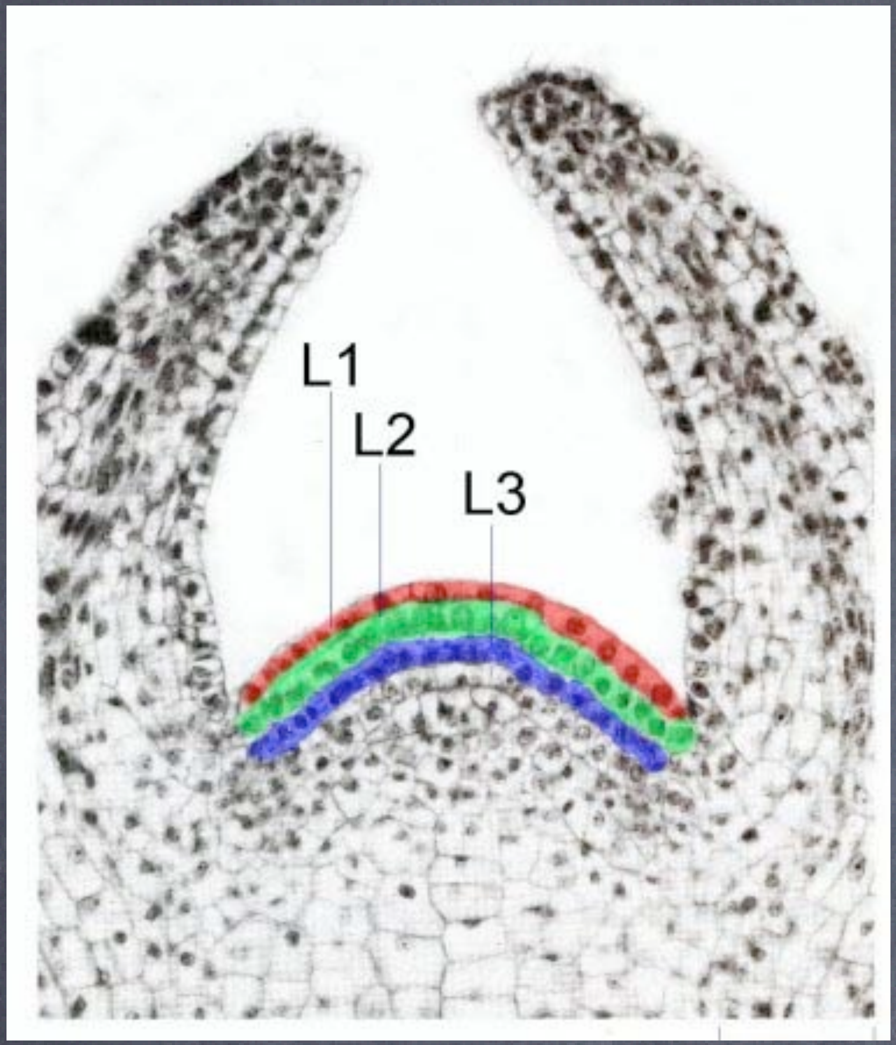


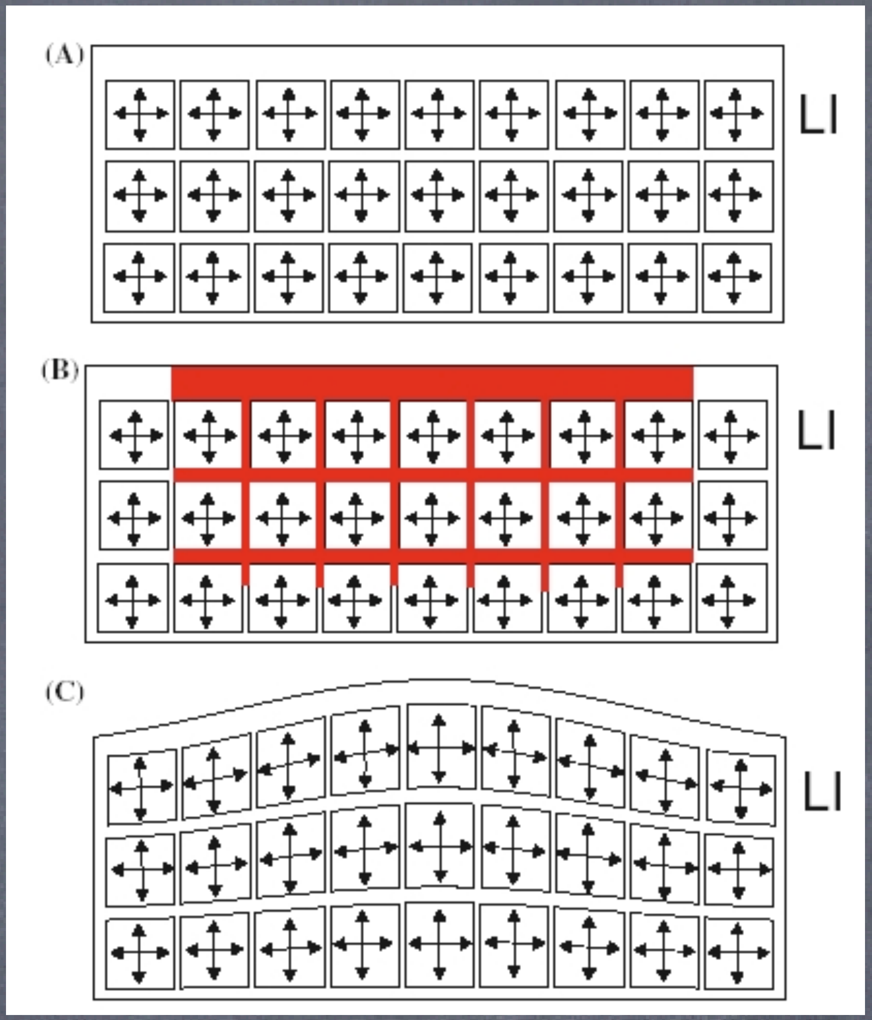
Hernandez, Green (1990's)



photos courtesy of John Palmer

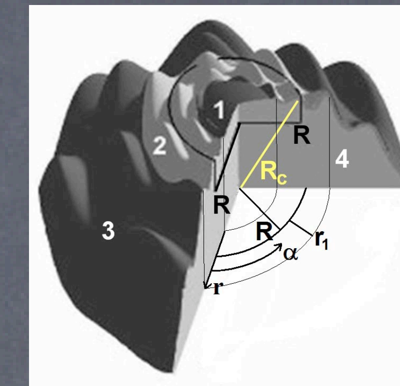






Fleming, 2005

Foppl-von Karman Equations



$w(r,\theta)$: vertical displacement $\qquad \chi(r,\theta)$: stress potential

Stress Equilibrium:

$$[\chi, w] = \chi_{xx} w_{yy} + w_{yy}\chi_{xx} - 2\chi_{xy} w_{xy}$$

$$\underbrace{D\Delta(\underbrace{\Delta w - H_g}_{\text{mean curvature}})}_{\text{bending}} - \underbrace{[\chi, w]}_{\text{stretching}} + \underbrace{\kappa w - \gamma w^3}_{\text{elastic foundation}} = 0$$

Compatability:

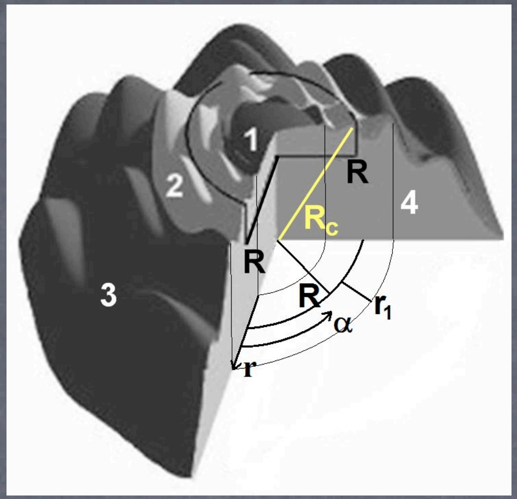
$$\frac{1}{Eh}\Delta(\Delta\chi) + \underbrace{\frac{1}{2}[w,w] + K_g}_{\text{Gaussian curvature}} = 0$$

$\mathbf{g}(\mathbf{r},\theta)$: Auxin fluctuation about stationary uniform concentration/ growth strain

$$K_g \doteq \Delta g$$

$$\frac{\partial}{\partial \mathbf{t}}\mathbf{g} = -\boldsymbol{\Delta}^2\mathbf{g} - \mathbf{H}\boldsymbol{\Delta}\mathbf{g} - \mathbf{L}\mathbf{g} - \kappa_1\nabla(\mathbf{g}\nabla\mathbf{g}) - \kappa_2\nabla(\nabla\mathbf{g}\boldsymbol{\Delta}\mathbf{g}) - \delta\mathbf{g}^3$$

Newell, et. al, 2008; Dervaux & ben Amar, 2008



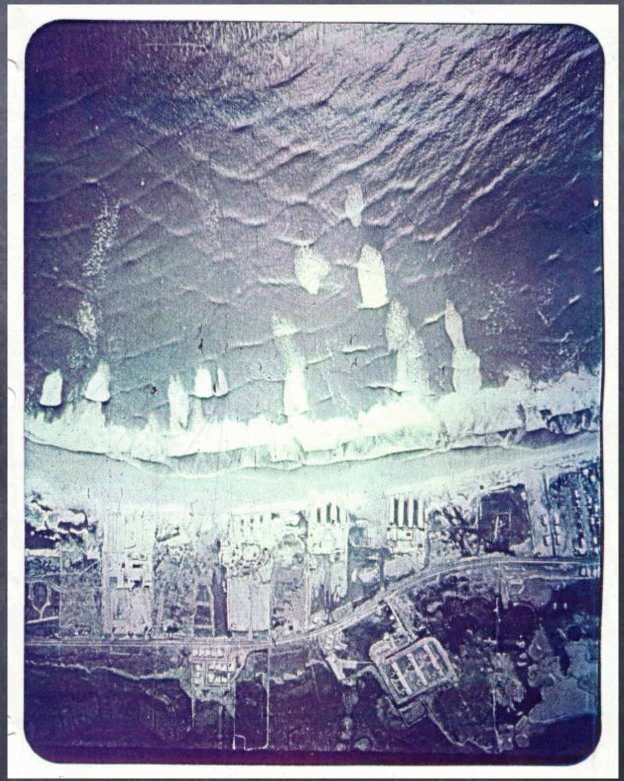
$$-\zeta_m w_t = \Delta^2 w + P\Delta w + \kappa w - C\Delta\chi + [\chi, w] + \gamma w^3$$

$$\Delta^2\chi + \Delta g - C\Delta w + \frac{1}{2}[w, w] = 0$$

$$-\zeta_g g_t = \Delta^2 g + H\Delta g + Lg + \kappa_1\nabla(g\nabla g) + \kappa_2\nabla(\nabla g\Delta g) + \delta w^3$$

$$L = \frac{\Lambda_m}{\Lambda_g}$$

Λ_m : natural mechanical wavelength Λ_g : natural biochemical wavelength

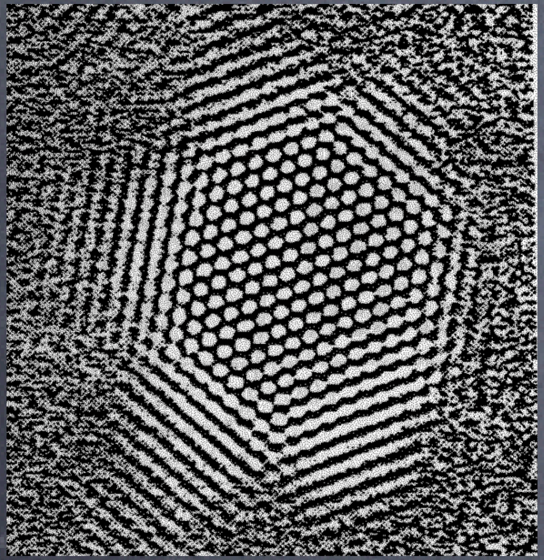


Analysis

$$-\zeta_m w_t = \Delta^2 w + P\Delta w + \kappa w - C\Delta\chi + [\chi, w] + \gamma w^3$$

$$\Delta^2 \chi + \Delta g - C\Delta w + \frac{1}{2}[w, w] = 0$$

$$-\zeta_g g_t = \Delta^2 g + H\Delta g + Lg + \kappa_1 \nabla(g\nabla g) + \kappa_2 \nabla(\nabla g \Delta g) + \delta w^3$$



Linear Analysis

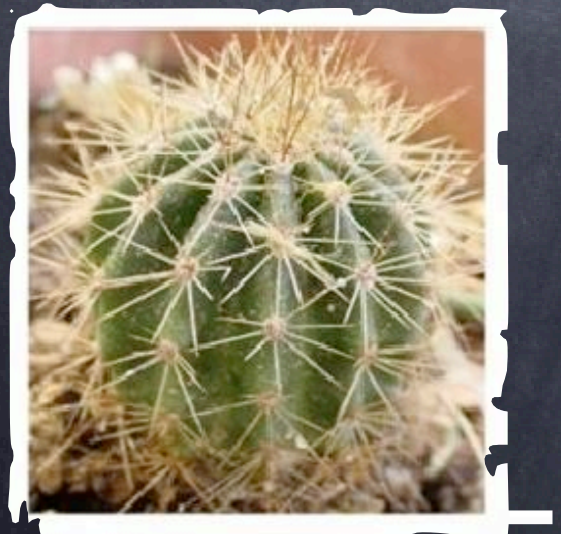
in one dimension

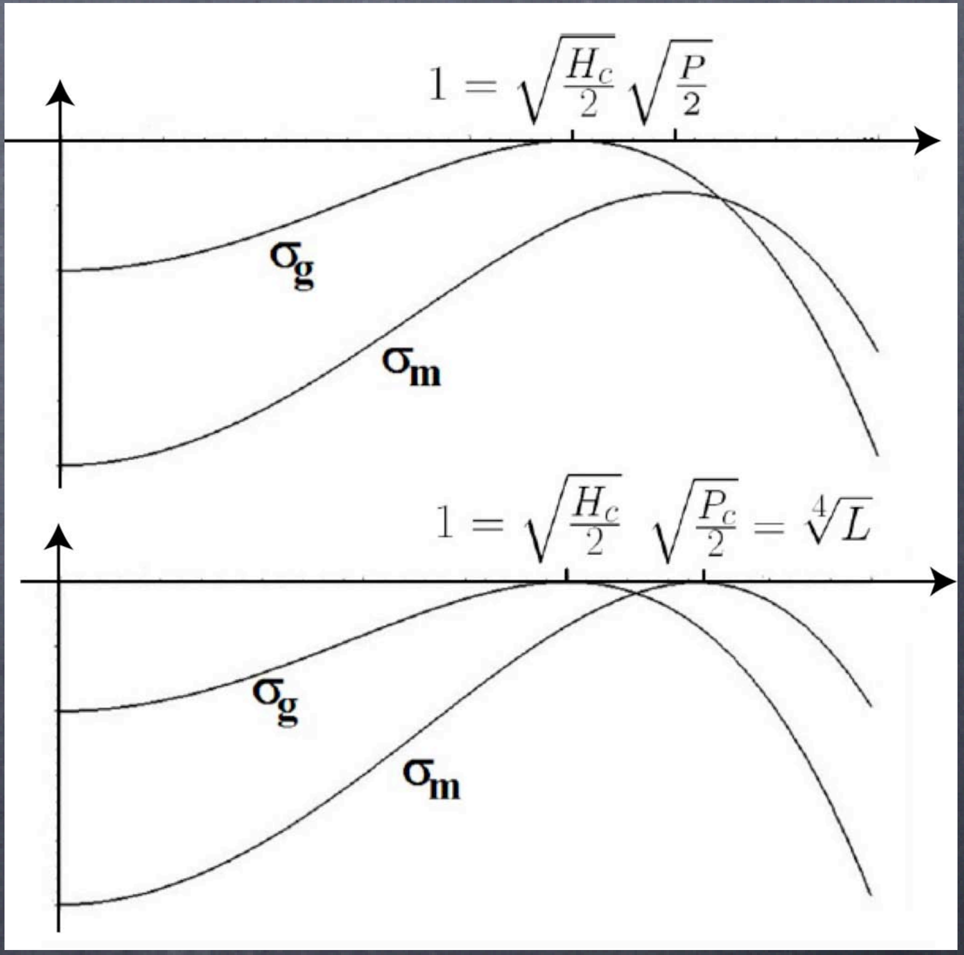
$$w(x) = e^{\sigma_m(k)t}\cos(kx)$$

$$\sigma_m(k) = -k^4 + Pk^2 - 1$$

$$g(x) = e^{\sigma_g(k)t}\cos(kx)$$

$$\sigma_g(k) = -k^4 + Hk^2 - L$$





Analysis

$$-\zeta_m w_t = \Delta^2 w + P\Delta w + \kappa w - C\Delta\chi + [\chi, w] + \gamma w^3$$

$$\Delta^2\chi + \Delta g - C\Delta w + \frac{1}{2}[w, w] = 0$$

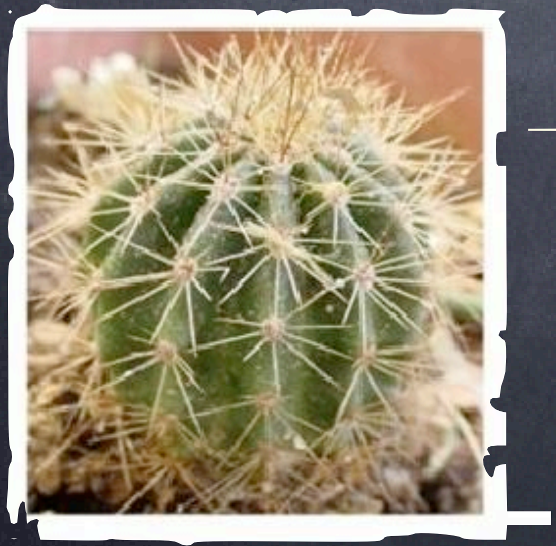
$$-\zeta_g g_t = \Delta^2 g + H\Delta g + Lg + \kappa_1\nabla(g\nabla g) + \kappa_2\nabla(\nabla g\Delta g) + \delta w^3$$

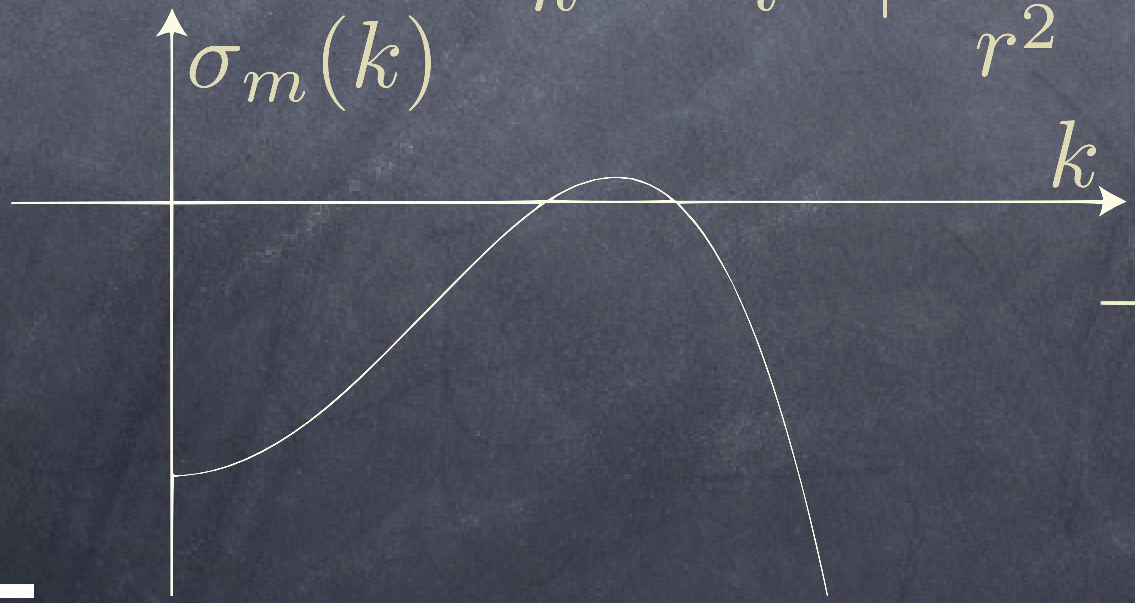
Linear Analysis in 2 dimensions

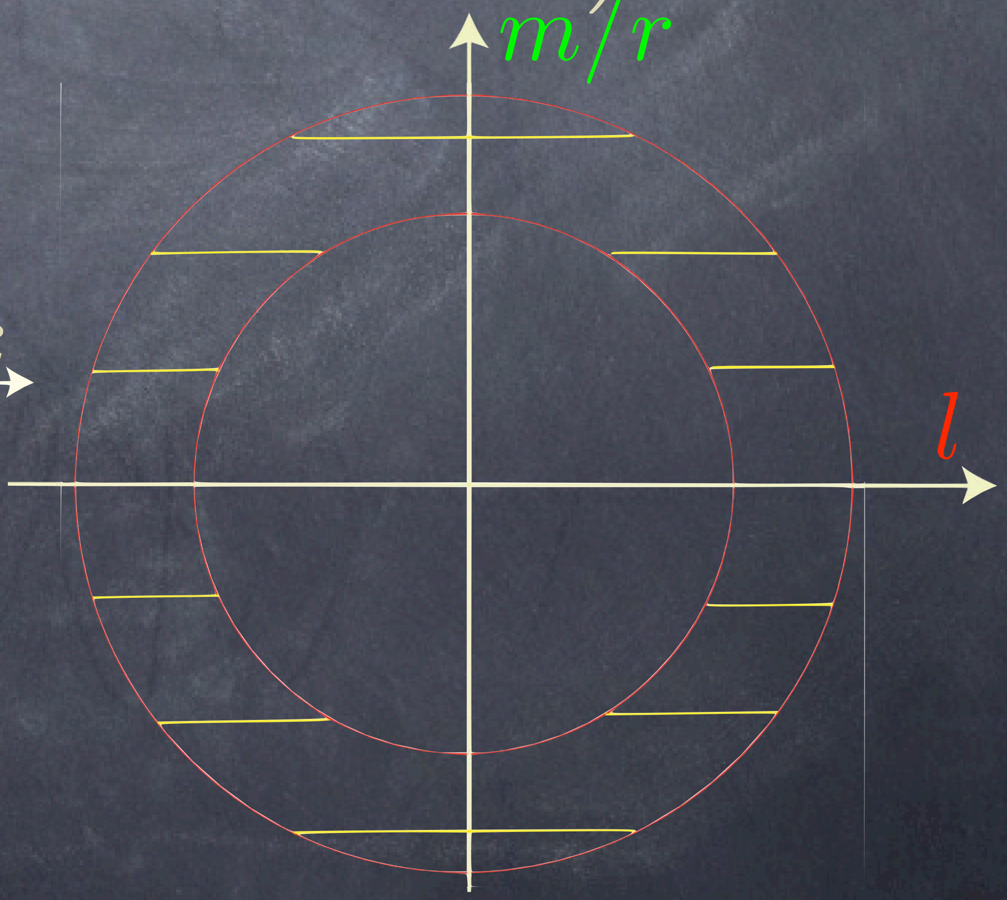
$$w(r,\theta) = \mathrm{e}^{\sigma_m(k)t}\cos\left(lr + m\theta = (l,m)\cdot(r,\theta) = \vec{k}\cdot\vec{x}\right)$$

$$\sigma_m(k) = -k^4 + Pk^2 - 1$$

$$k^2 = l^2 + \frac{m^2}{r^2}$$





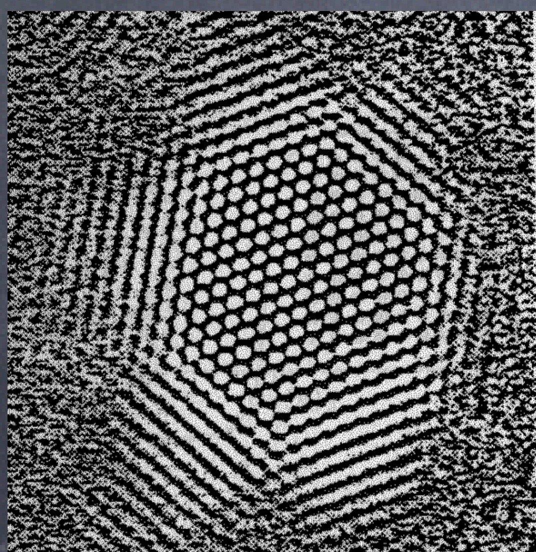


Analysis

$$-\zeta_m w_t = \Delta^2 w + P\Delta w + \kappa w - C\Delta\chi + [\chi, w] + \gamma w^3$$

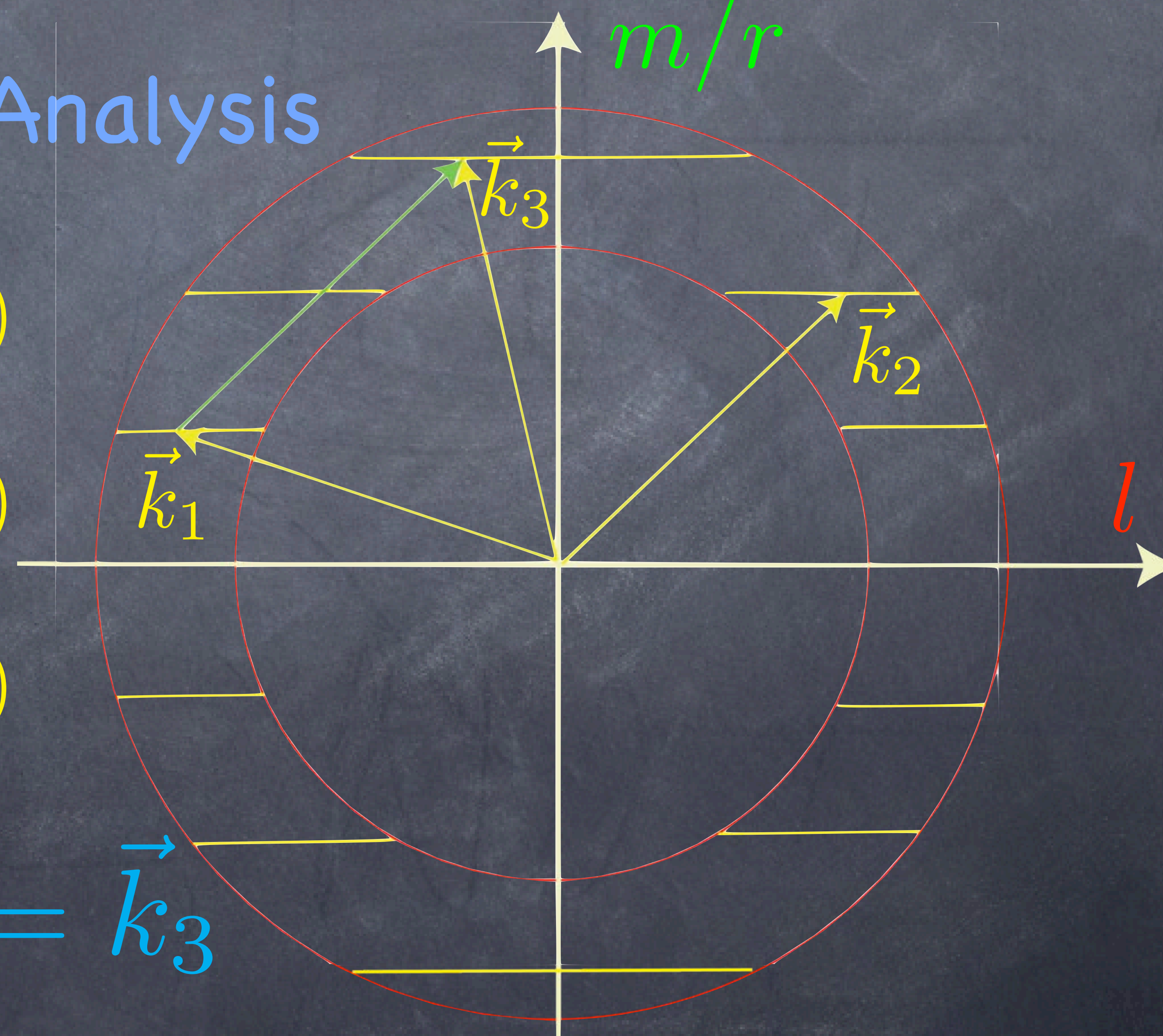
$$\Delta^2\chi + \Delta g - C\Delta w + \frac{1}{2}[w, w] = 0$$

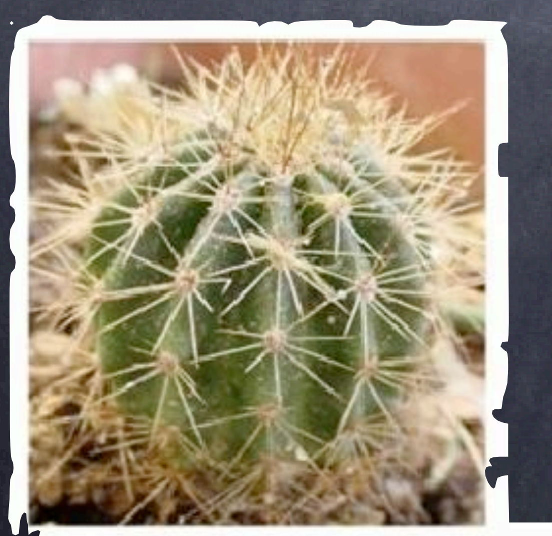
$$-\zeta_g g_t = \Delta^2 g + H\Delta g + Lg + \kappa_1\nabla(g\nabla g) + \kappa_2\nabla(\nabla g\Delta g) + \delta w^3$$



Nonlinear Analysis

$$\vec{k}_1 = (l_1, 1/r)$$
$$+$$
$$\vec{k}_2 = (l_2, 2/r)$$
$$=$$
$$\vec{k}_3 = (l_3, 3/r)$$





$$\vec{k}_1 + \vec{k}_2 = \vec{k}_3$$

Amplitude Equations

$$P \doteq P_c(1 + \epsilon P')$$

$$w = \sum_{\vec{k} \in \mathfrak{A}} A_{\vec{k}} \cos(\vec{k} \cdot \vec{x}) + \dots$$

Linear

$$\frac{\partial}{\partial t} A_{\vec{k}} + \left(2\mathrm{i} \frac{\partial}{\partial r} + \mathrm{i} \frac{\partial l}{\partial r} + \mathrm{i} \frac{l}{r} \right)^2 A_{\vec{k}} = \sigma(\vec{k}) A_{\vec{k}}$$

Quadratic Nonlinearity

$$+ \sum_{\vec{k}_r + \vec{k}_s = \vec{k}} \tau(\vec{k}_r, \vec{k}_s, \vec{k}) A_r^* A_s^*$$

Cubic Saturation

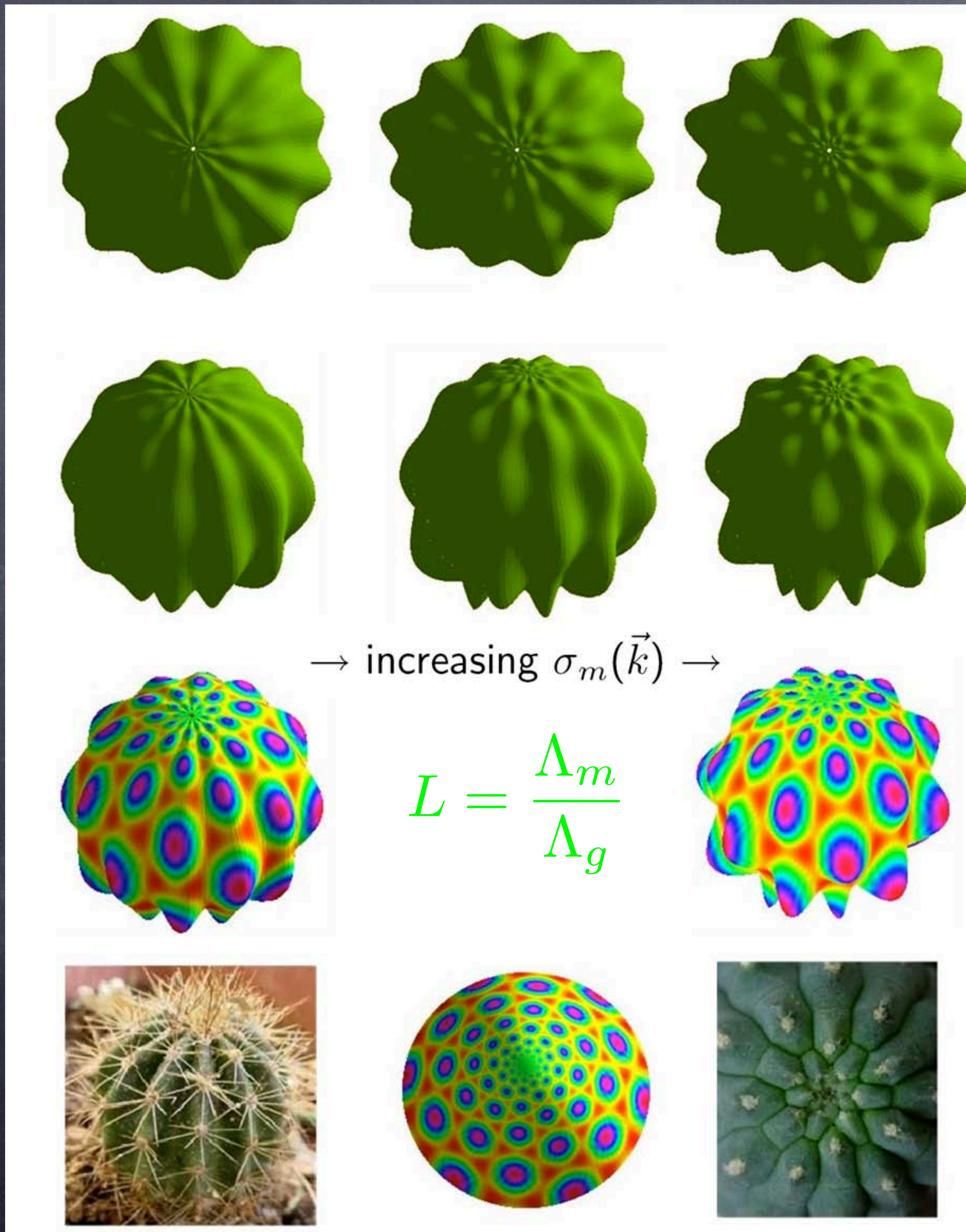
$$-3\gamma A_{\vec{k}} \left(|A_{\vec{k}}|^2 + 2 \sum_{\vec{k}_l \neq \vec{k}} |A_l|^2 \right)$$

Amplitude Equations

$$\frac{\partial}{\partial t}A_{\vec{k}} + \left(2\mathrm{i}\frac{\partial}{\partial r} + \mathrm{i}\frac{\partial l}{\partial r} + \mathrm{i}\frac{l}{r}\right)^2 A_{\vec{k}} = \sigma(\vec{k})A_{\vec{k}}$$
$$+ \sum_{\vec{k}_r+\vec{k}_s=\vec{k}} \tau(\vec{k}_r, \vec{k}_s, \vec{k})A_r^* A_s^*$$
$$-3\gamma A_{\vec{k}}\left(|A_{\vec{k}}|^2 + 2\sum_{\vec{k}_l \neq \vec{k}} |A_l|^2\right)$$

Task: Find wavevectors $\vec{k}$ and amplitudes $A_{\vec{k}}$ that minimize the associated energy $\mathfrak{E}(\vec{k} \in \mathfrak{A}; A_{\vec{k}})$

$$w(\vec{x}) = A_6 \cos(\vec{k}_6 \cdot \vec{x}) + A_6' \cos(\vec{k}_6' \cdot \vec{x}) + A_{12} \cos(\vec{k}_{12} \cdot \vec{x})$$



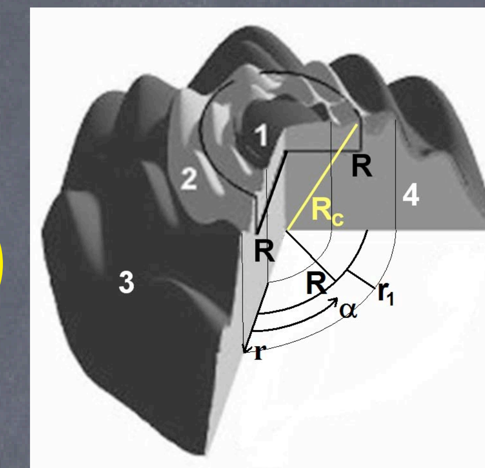
$$L = \frac{\Lambda_m}{\Lambda_g}$$

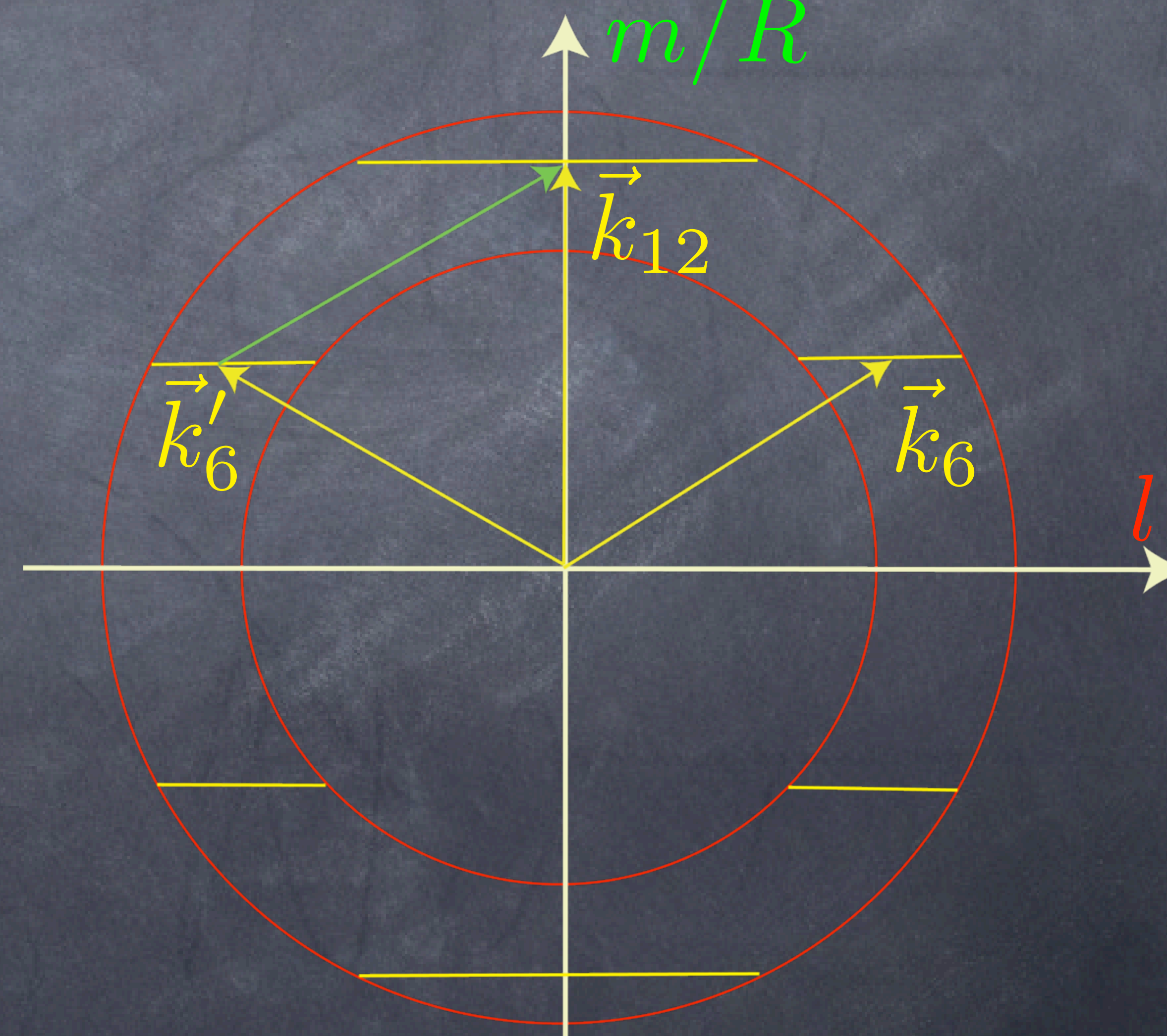
$$\vec{x} = (r, \theta)$$

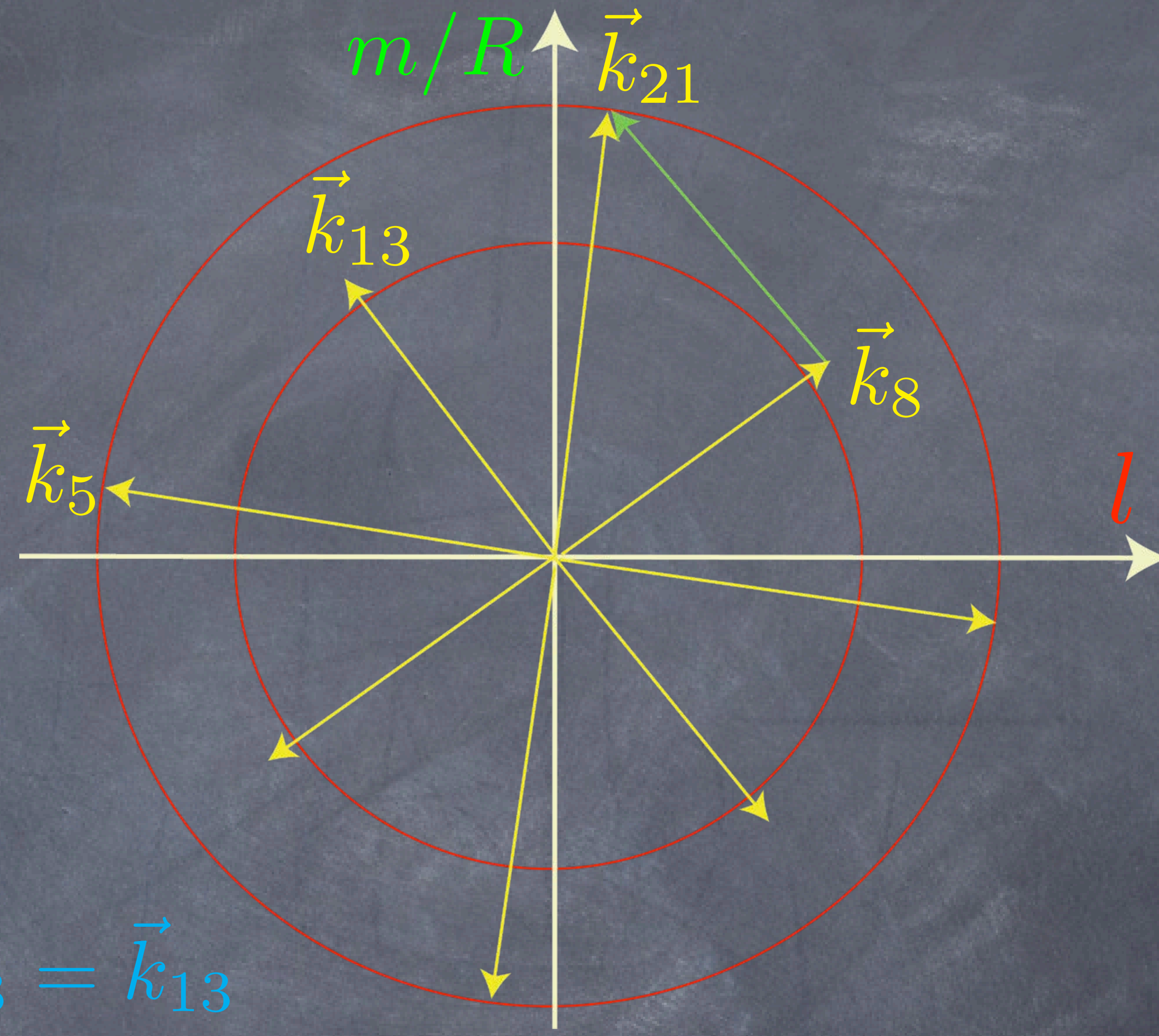
$$\vec{k}_6 = (l_6, 6/R)$$

$$\vec{k}_6' = (-l_6, 6/R)$$

$$\vec{k}_{12} = (0, 12/R)$$





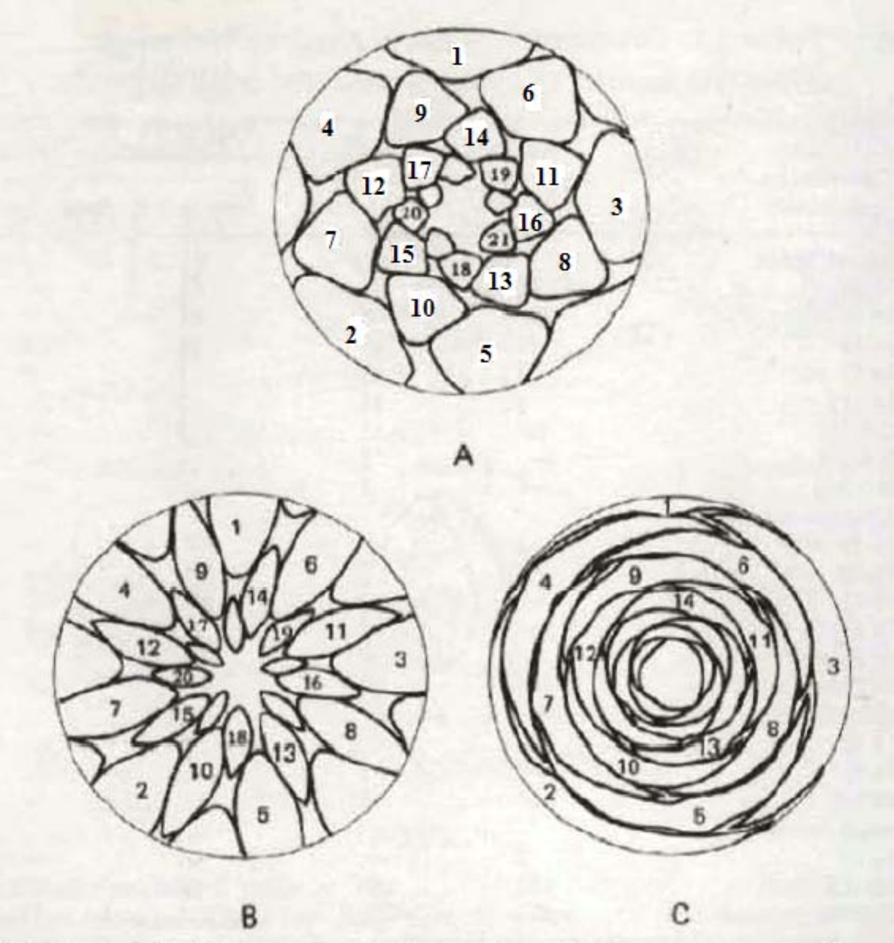


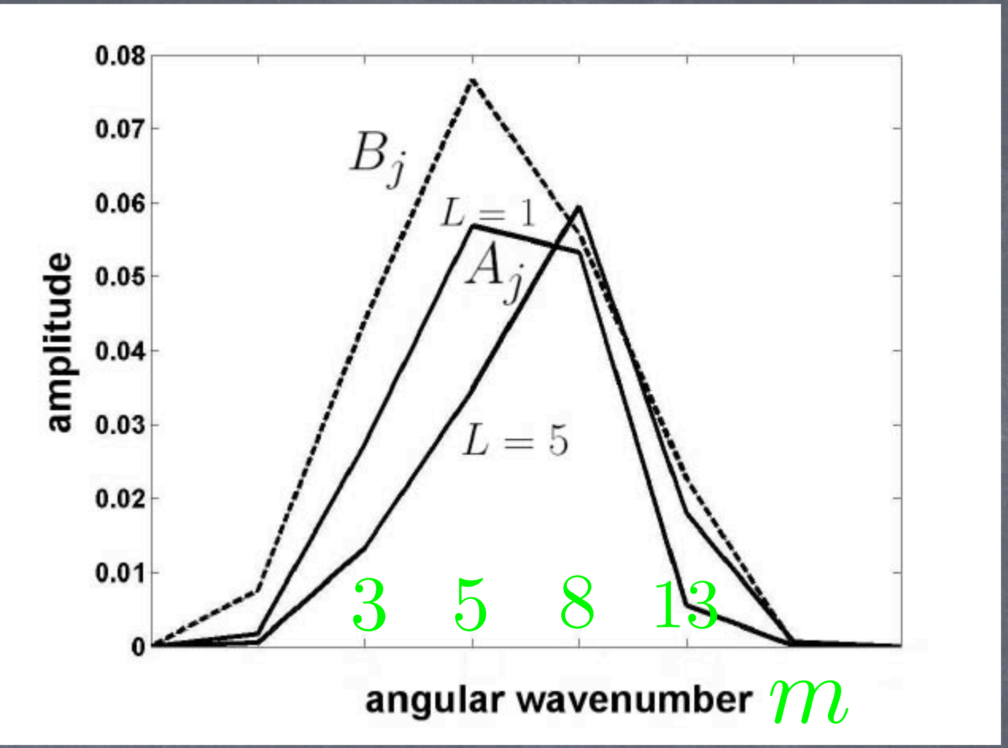
$$\vec{k}_5 + \vec{k}_8 = \vec{k}_{13}$$

$$\vec{k}_8 + \vec{k}_{13} = \vec{k}_{21}$$

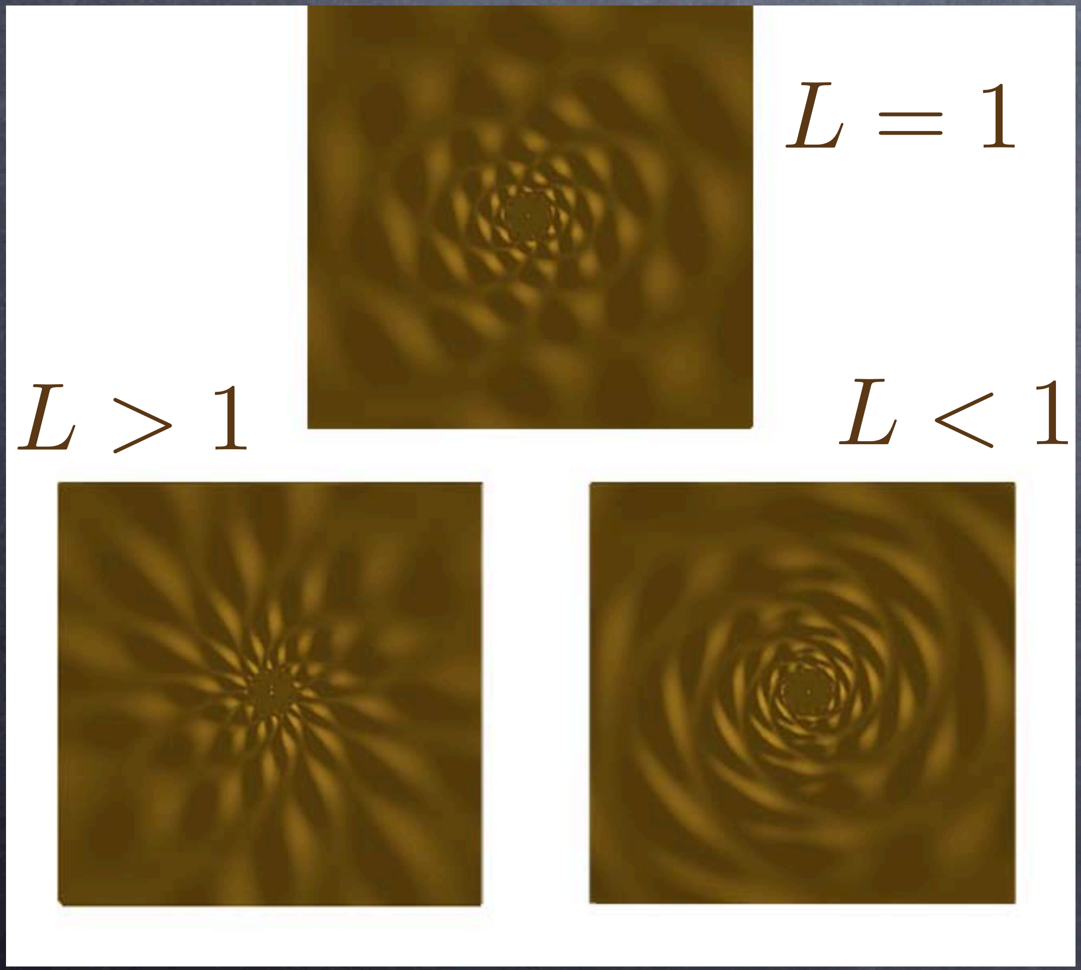
$$w(\vec{x}) = A_5 \cos(\vec{k}_5 \cdot \vec{x}) + A_8 \cos(\vec{k}_8 \cdot \vec{x}) + A_{13} \cos(\vec{k}_{13} \cdot \vec{x}) + A_{21} \cos(\vec{k}_{21} \cdot \vec{x})$$

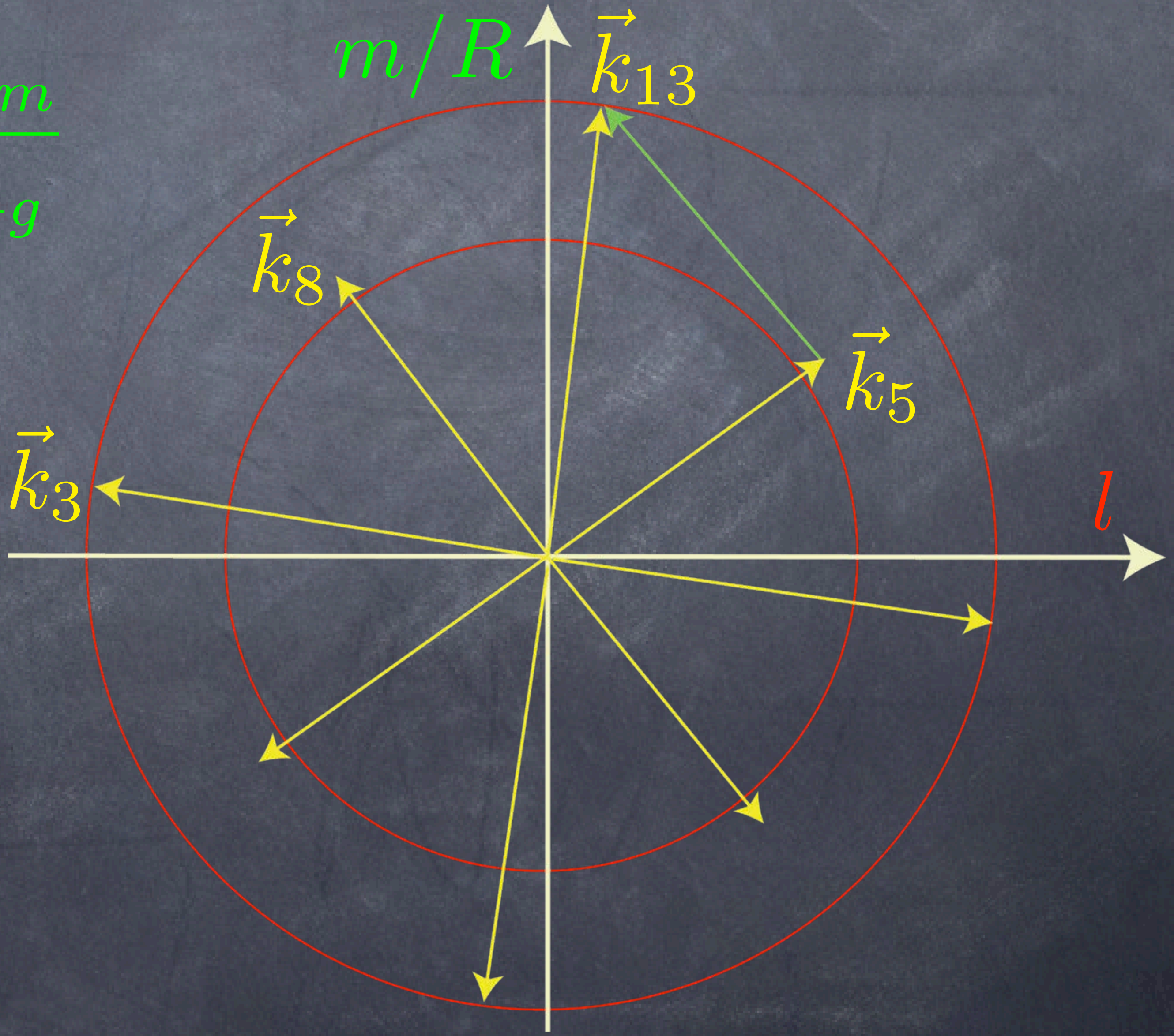
$$A_5 \simeq A_{21} < A_8 \simeq A_{21}$$

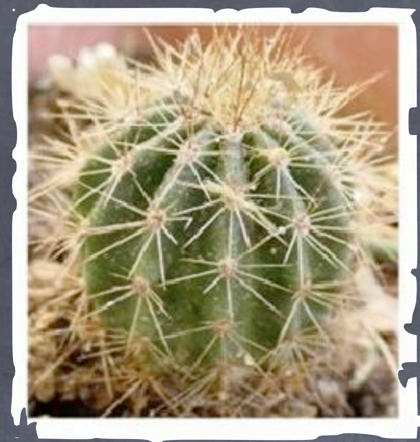


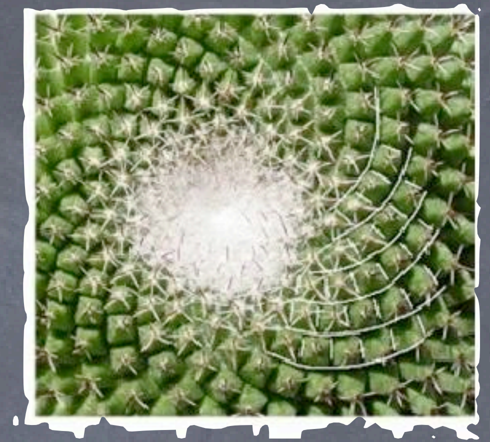


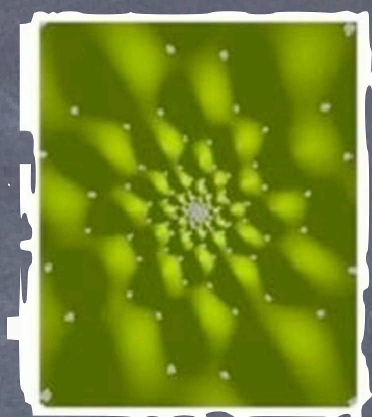
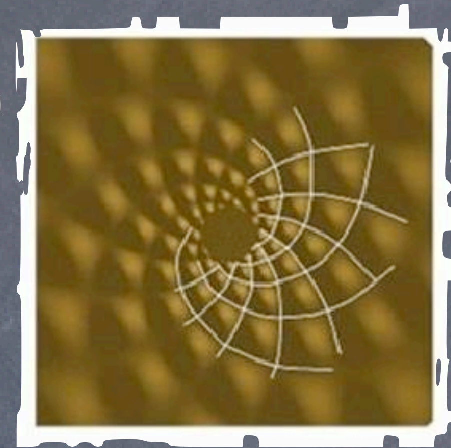
$$L = \frac{\Lambda_m}{\Lambda_g}$$

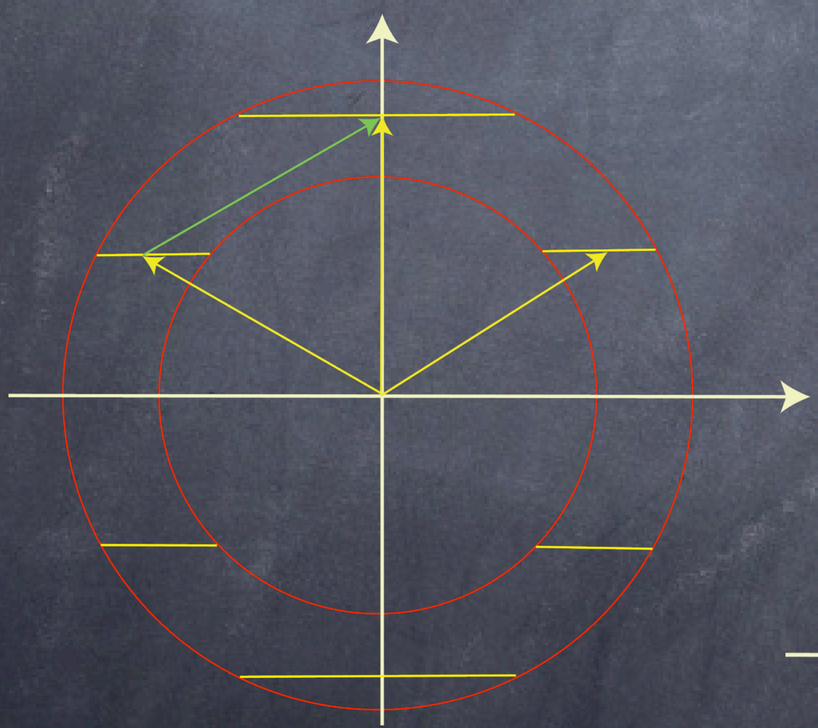
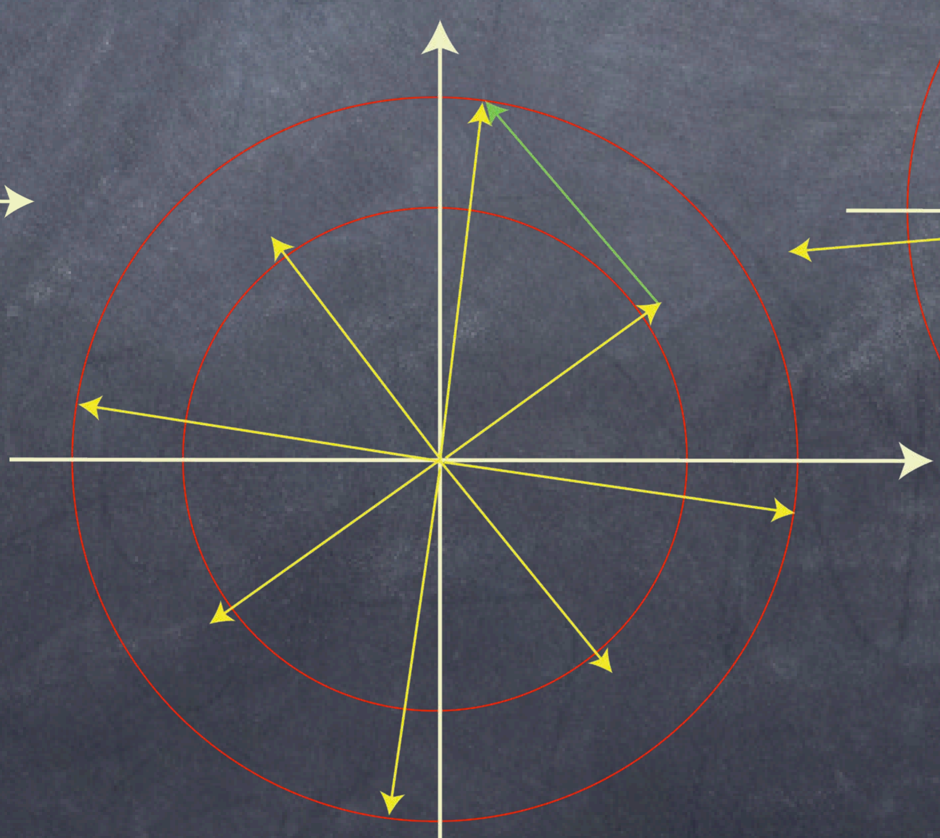
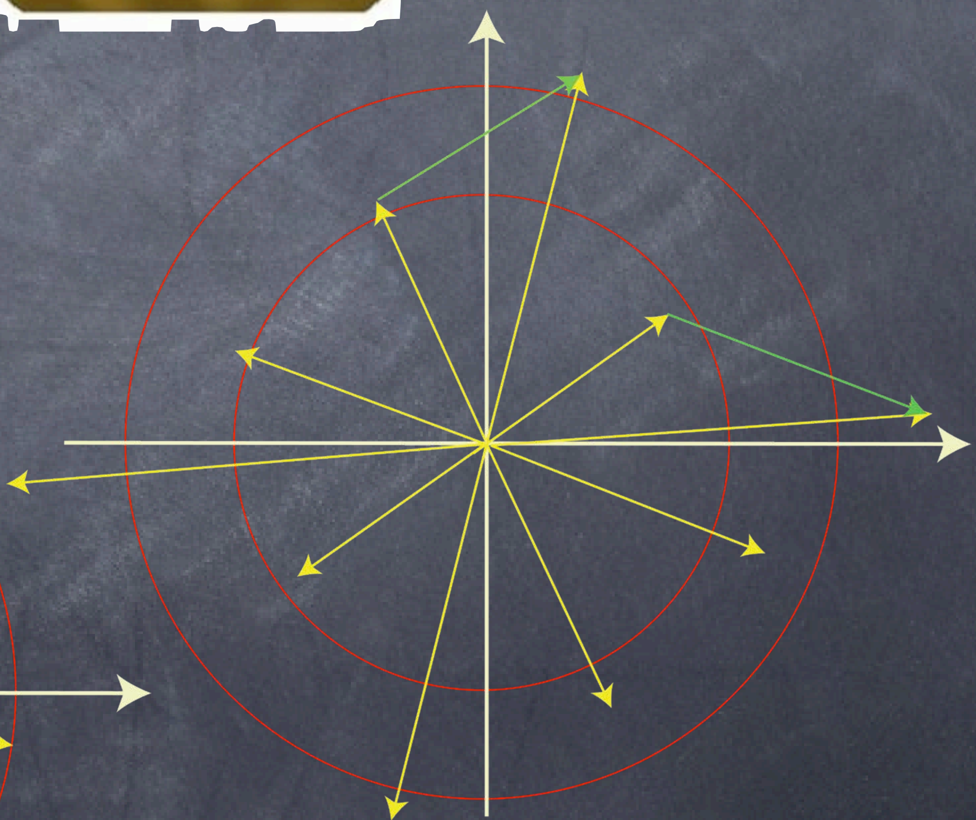


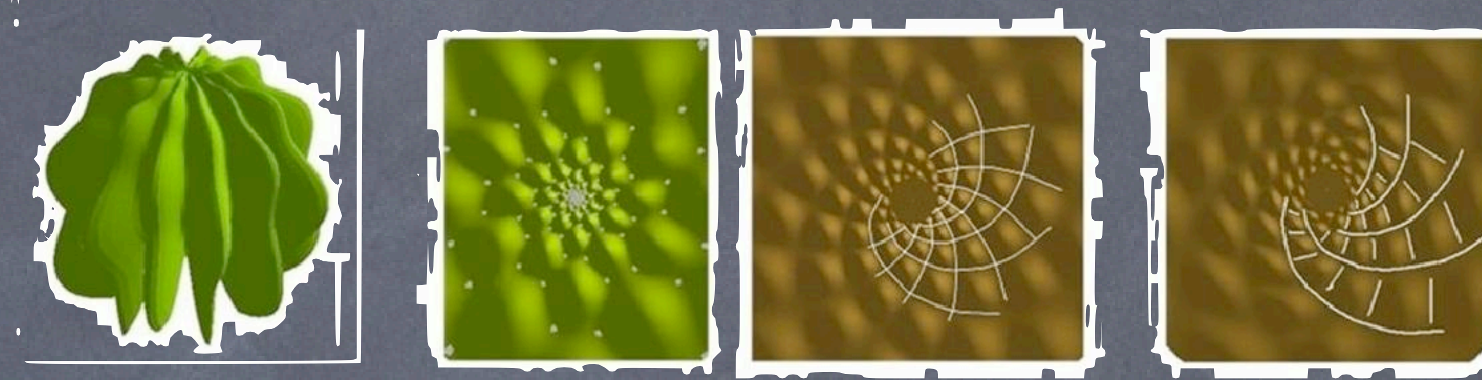


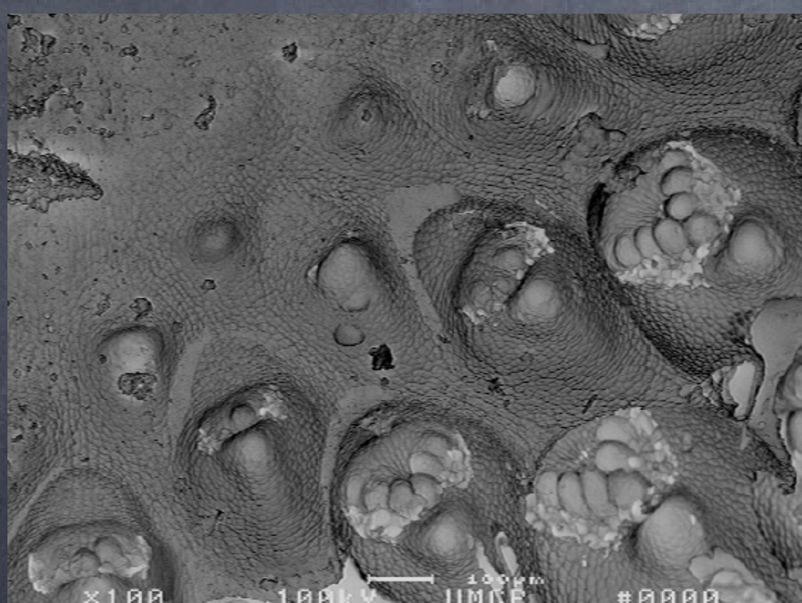


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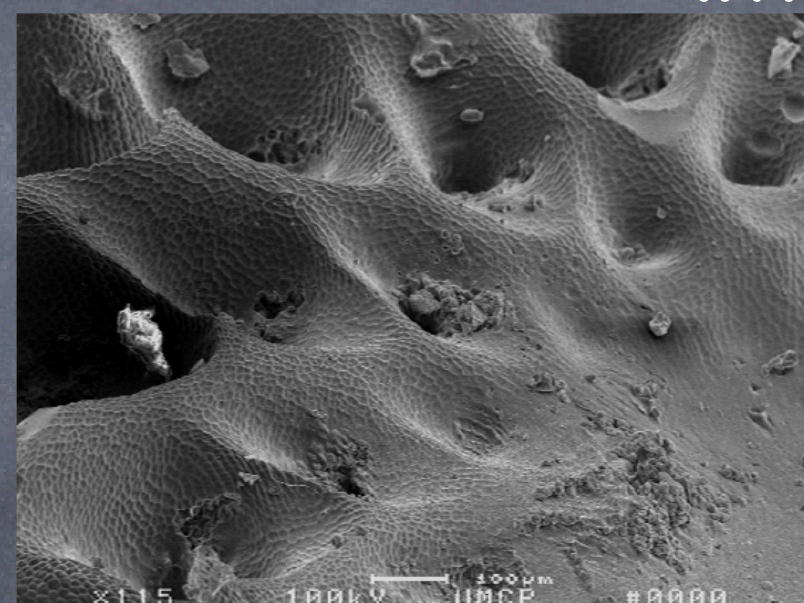
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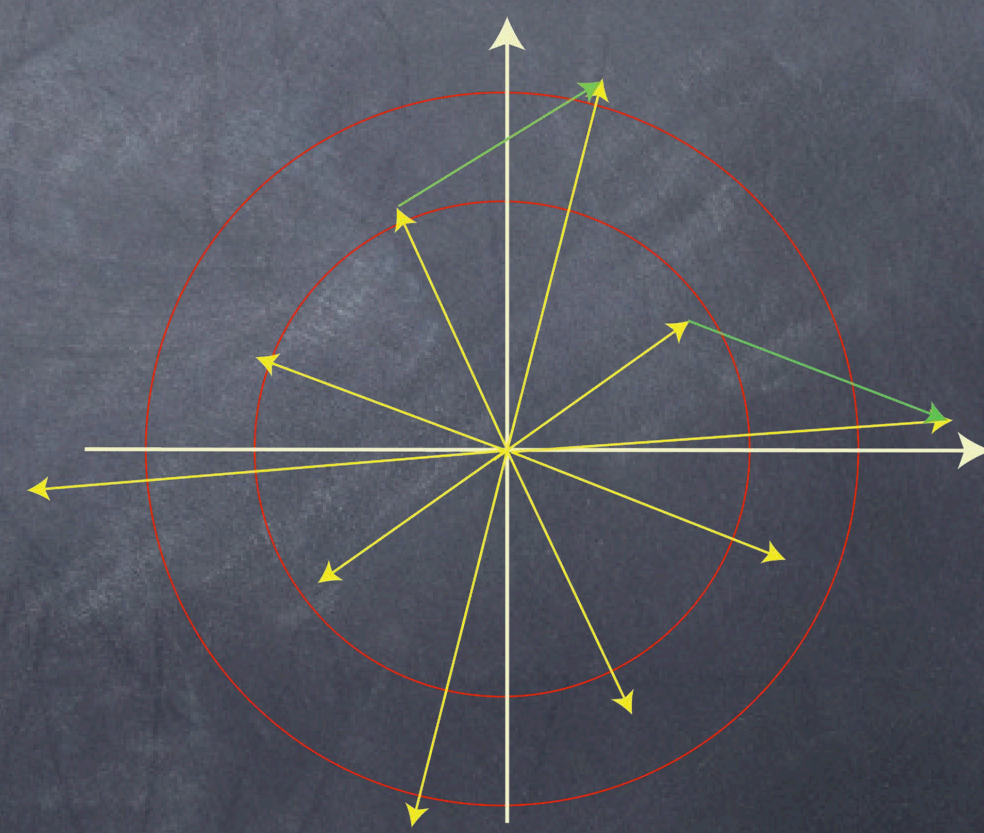
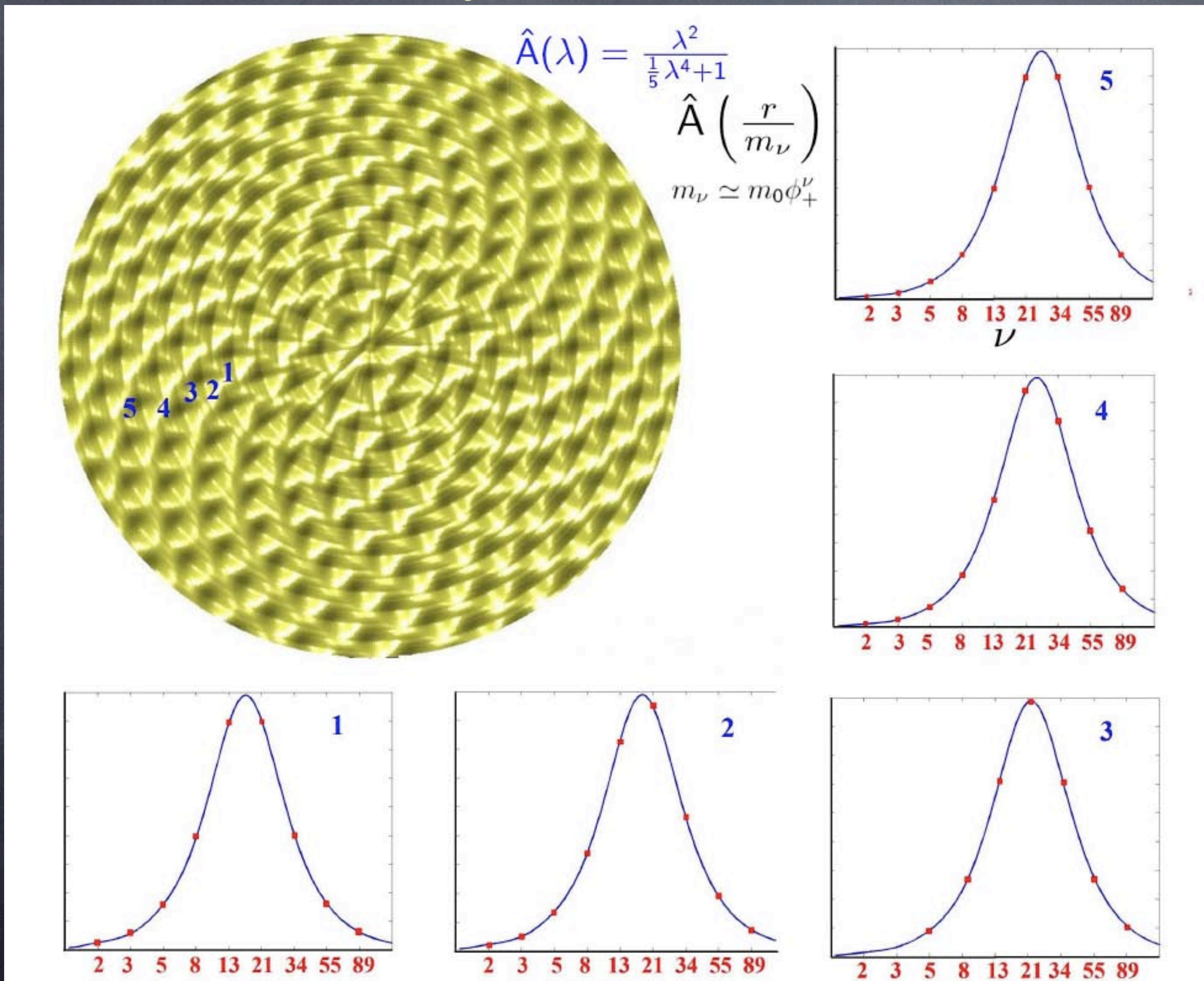
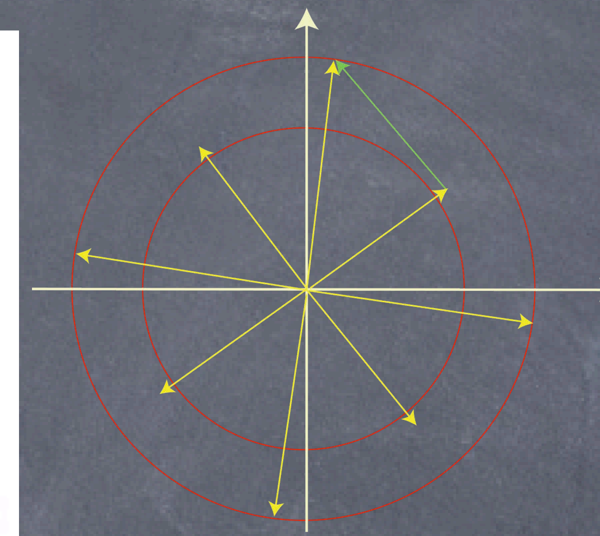
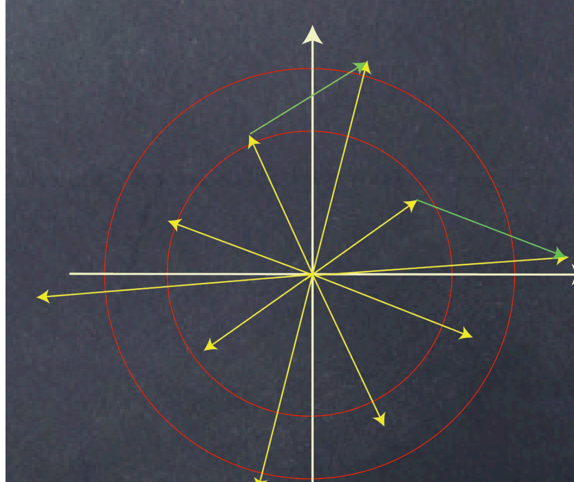


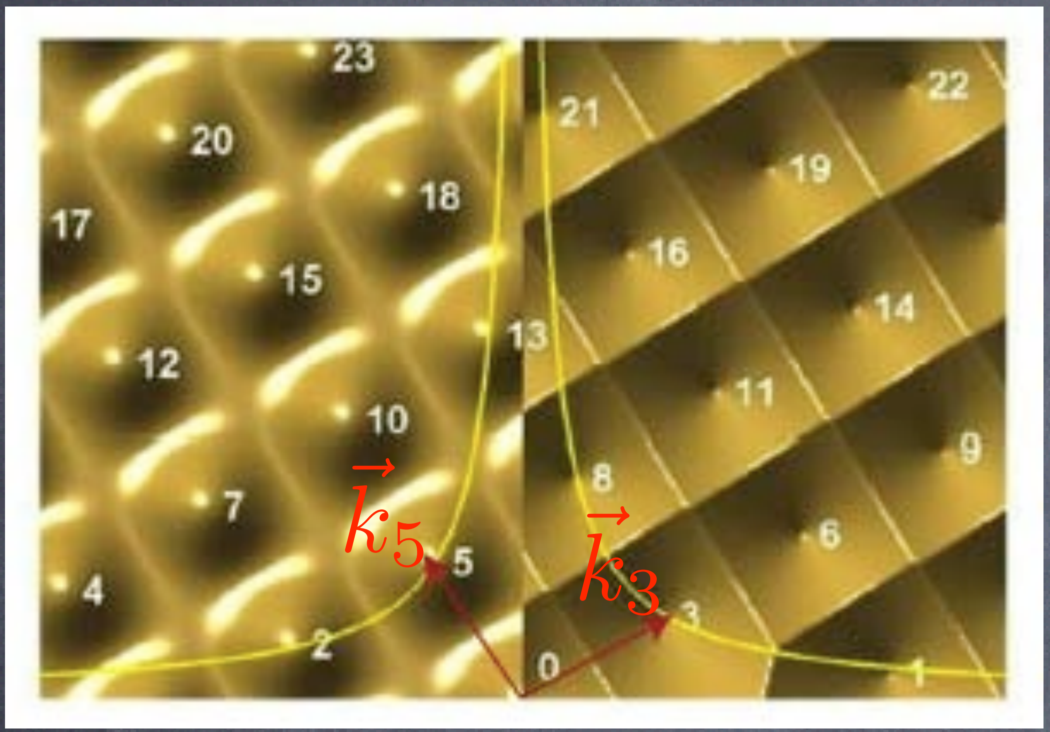
$$w(r,\theta) = \sum_{j=1}^{10} \hat{A}\left(\frac{r}{m_j}\right)\cos(\vec{k}_j(r)\cdot\vec{x})$$



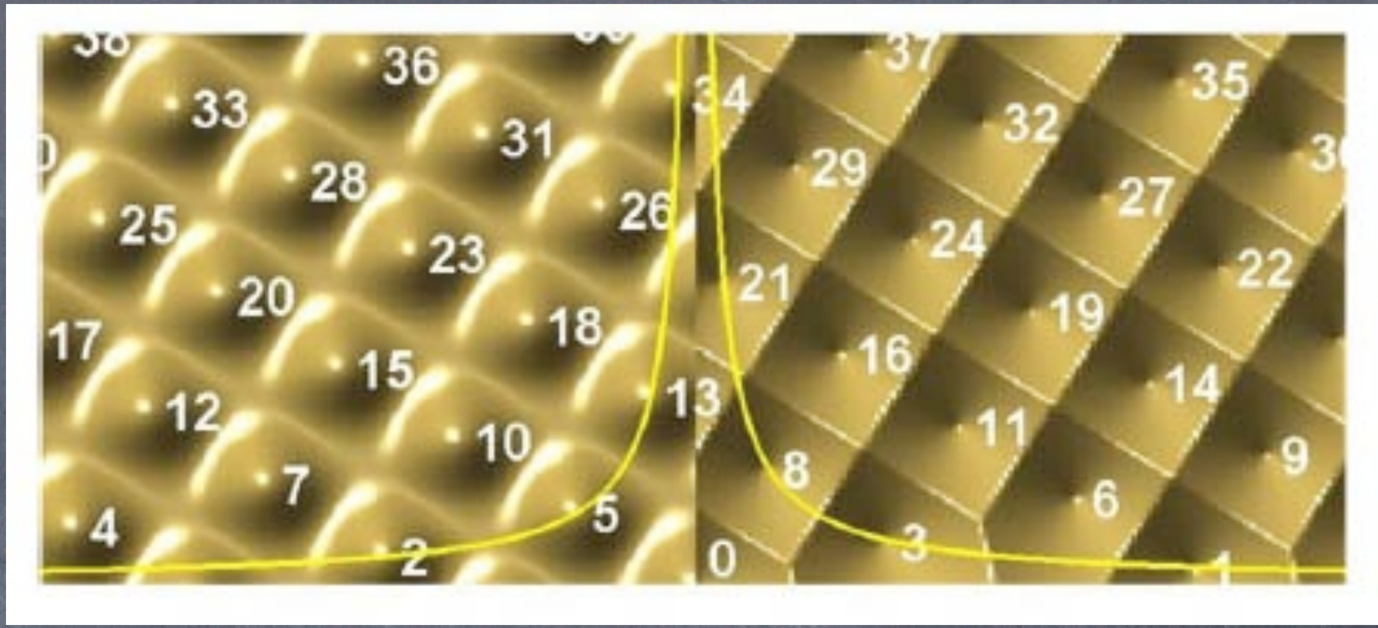




$$E\left(\ldots A_2, A_3, A_5, A_8, \ldots, \vec{k}_m = \left(l_m, \frac{m}{R}\right), \vec{k}_n = \left(l_n, \frac{n}{R}\right)\right)$$

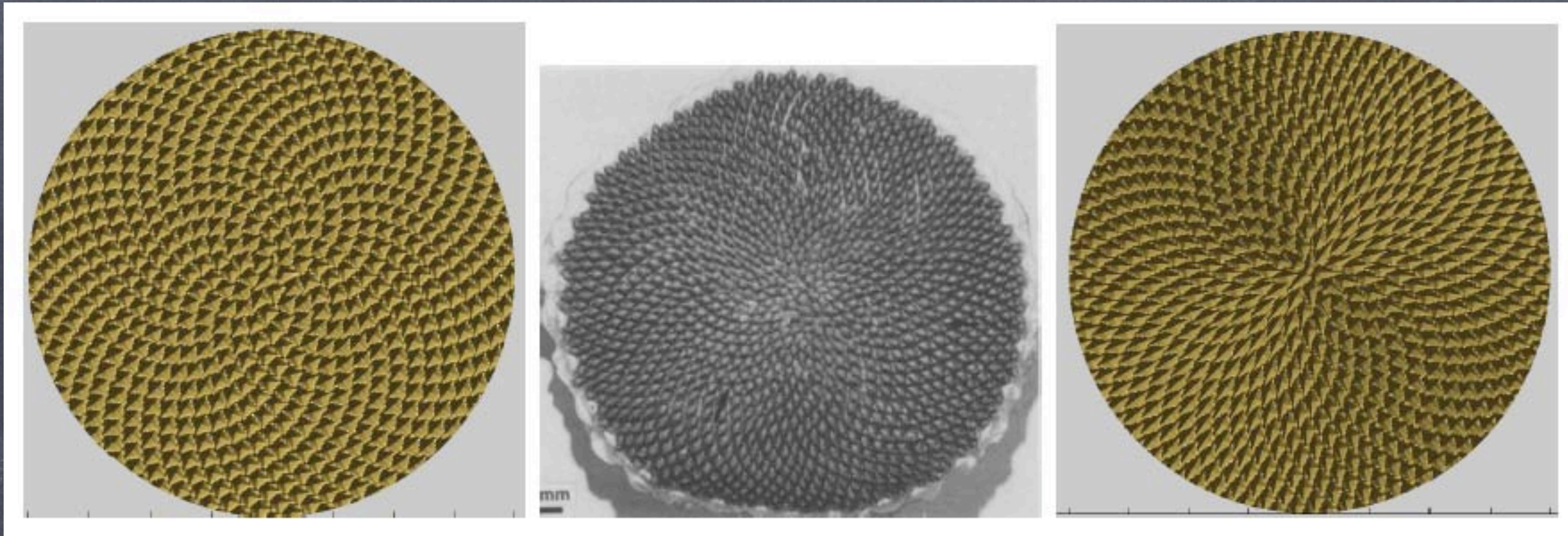


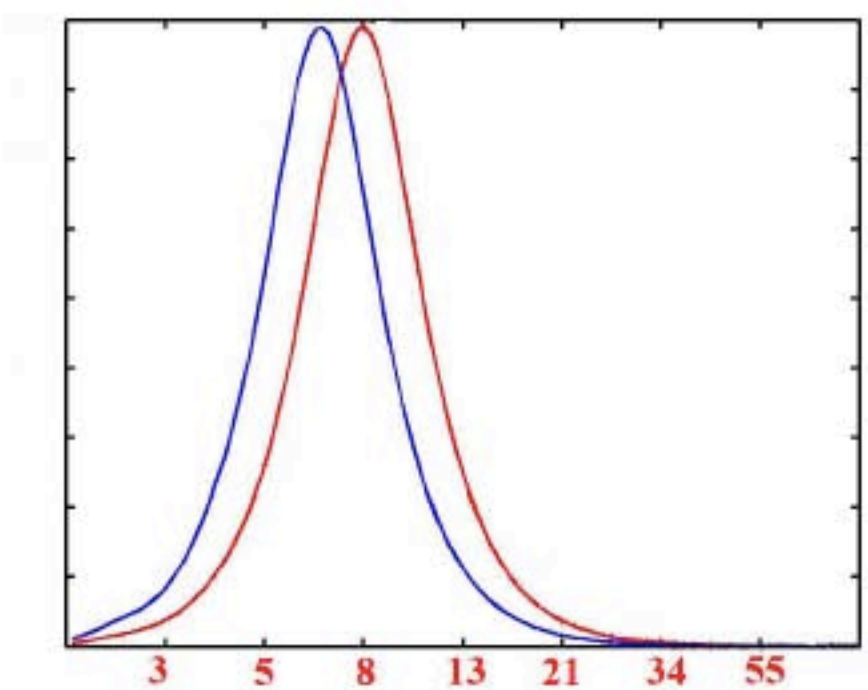
R



$R\phi_+$

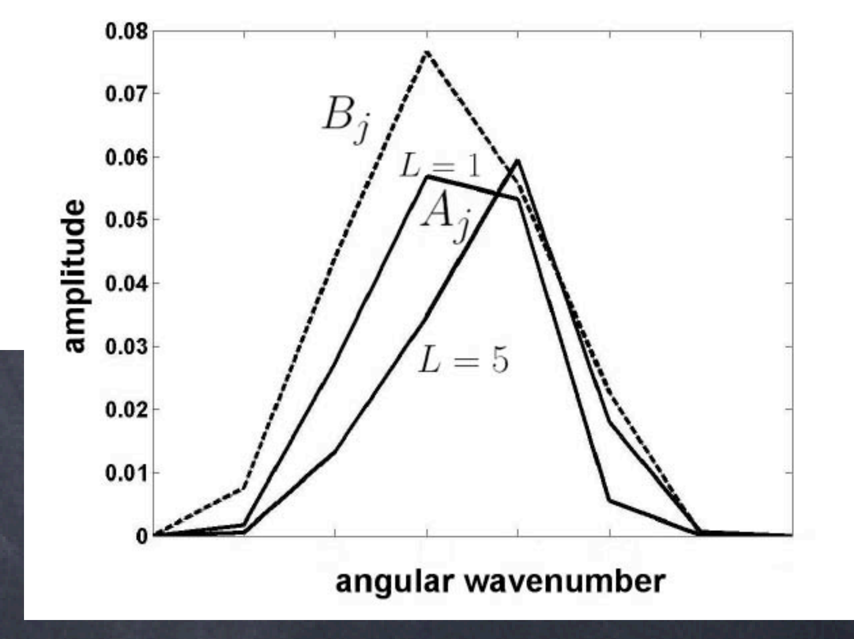
$$\phi_+ = \frac{1+\sqrt{5}}{2}$$

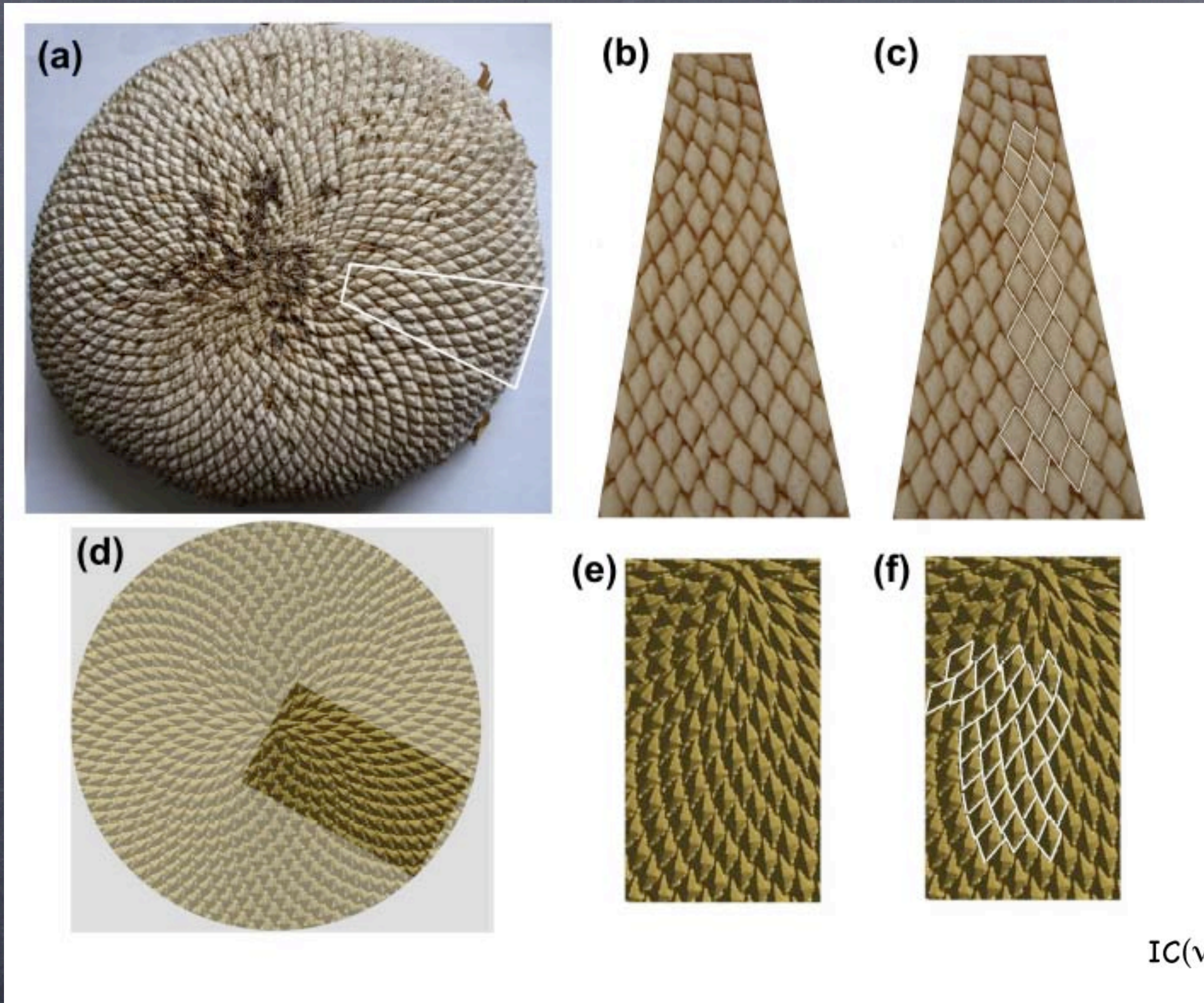


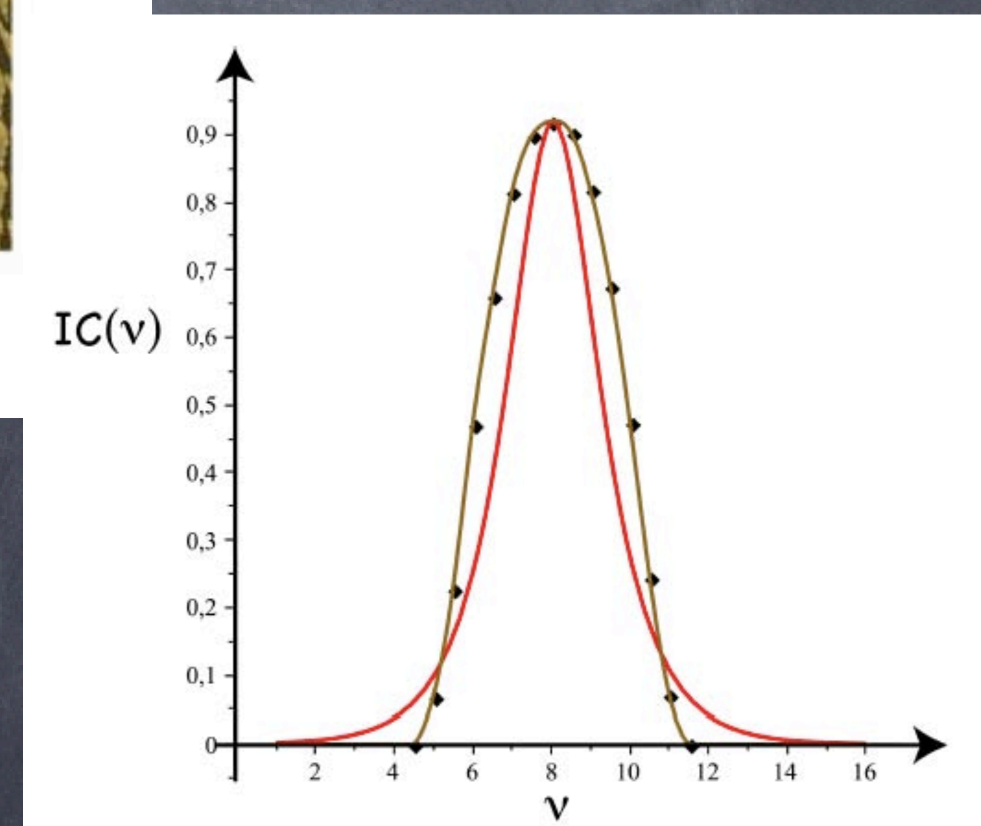


$$w(r,\alpha) = \sum_{\nu=1}^{10} \hat{\mathsf{A}}\left(\frac{r}{m_\nu}\right) \cos(\vec{k}_\nu \cdot \vec{x})$$

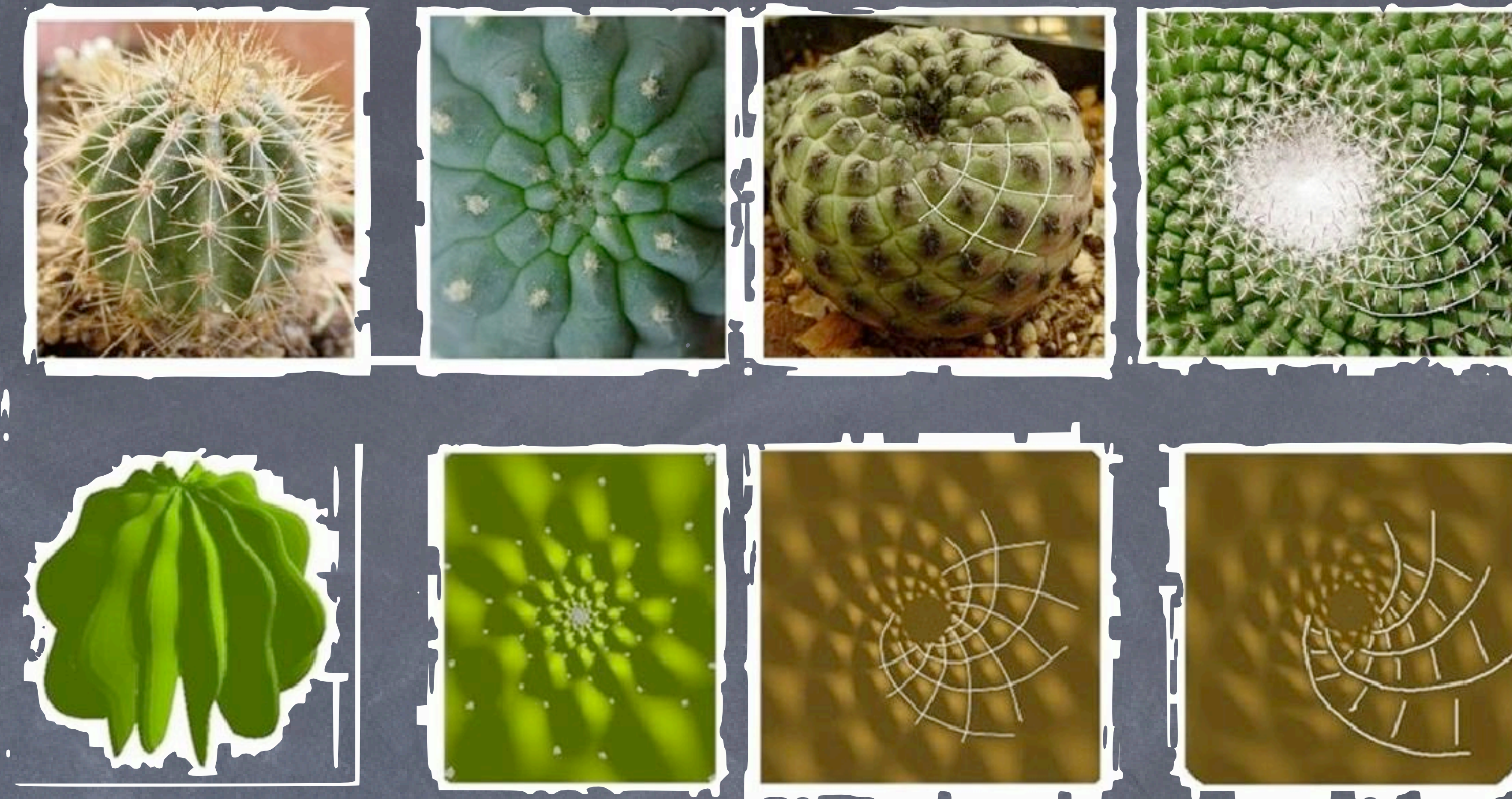
$$\hat{\mathsf{A}}(\rho) = \frac{\rho^2}{\frac{1}{5}\rho^4 + a > 1}$$

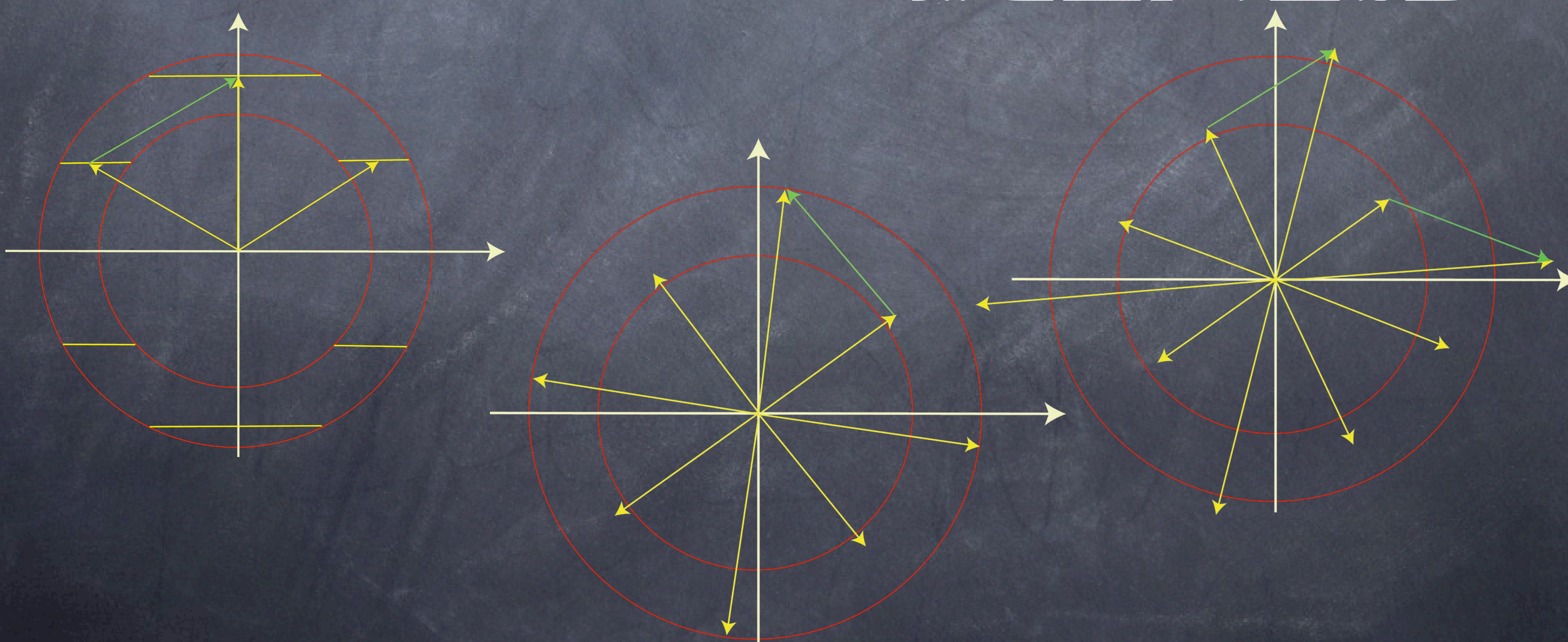


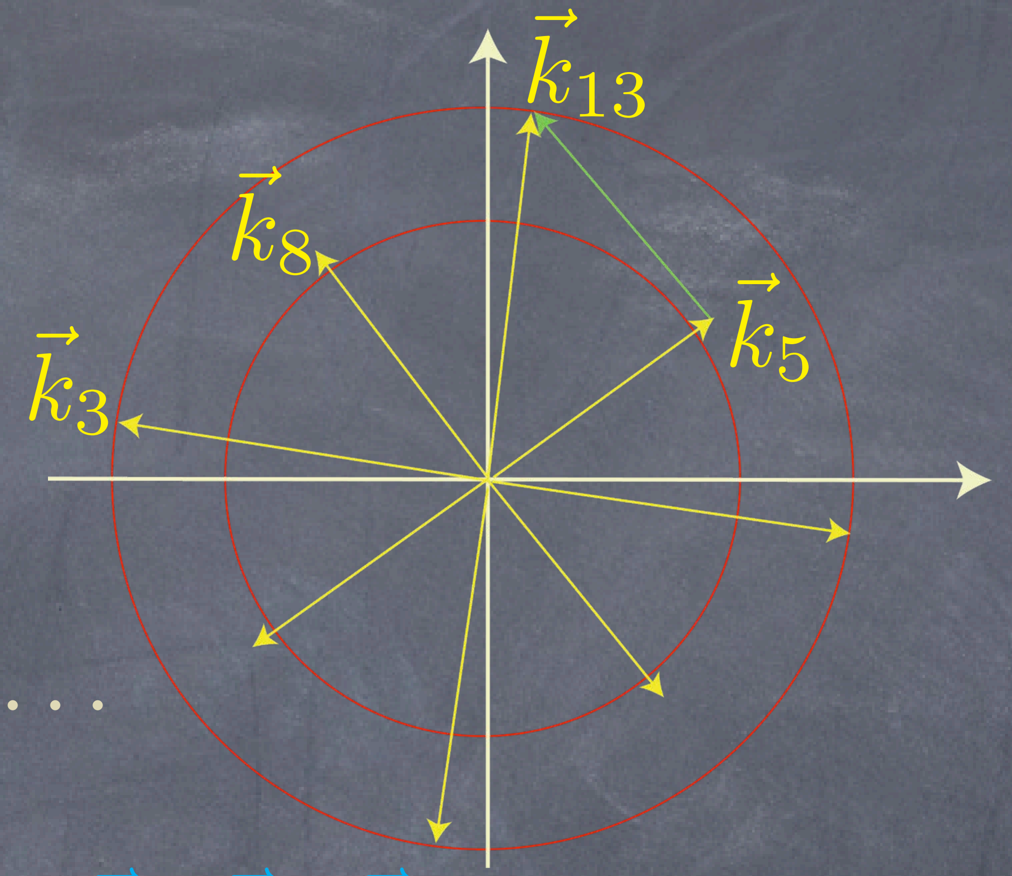

(a)
(b)
(c)
(d)
(e)
(f)


IC(ν)
0,9
0,8
0,7
0,6
0,5
0,4
0,3
0,2
0,1
0
2
4
6
8
10
12
14
16
ν

Rhombicity







$$\frac{\partial}{\partial t}A_3 = \sigma(k_3^2)A_3 + \tau(\vec{k}_3,\vec{k}_5,\vec{k}_8)A_5^*A_8 - \ldots$$

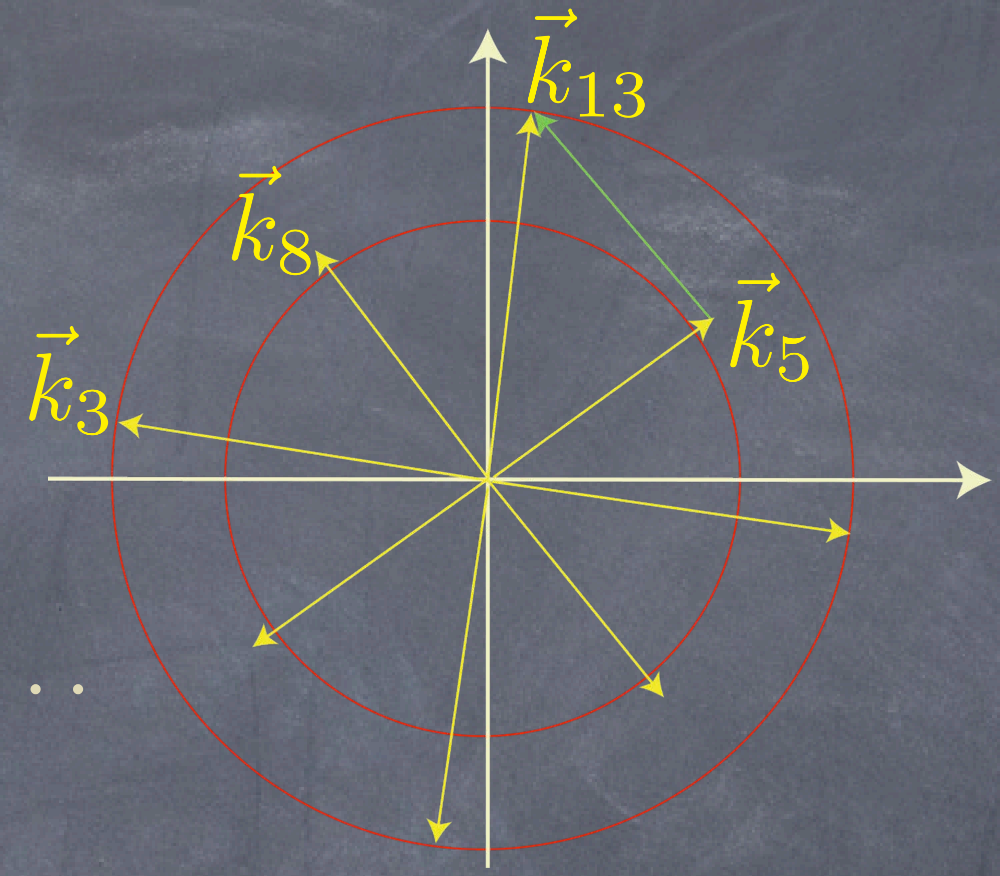
$$\frac{\partial}{\partial t}A_5 = \sigma(k_5^2)A_5 + \tau(\vec{k}_3,\vec{k}_5,\vec{k}_8)A_3^*A_8 + \tau(\vec{k}_5,\vec{k}_8,\vec{k}_{13})A_8^*A_{13} - \ldots$$

$$\frac{\partial}{\partial t}A_8 = \sigma(k_8^2)A_8 + \tau(\vec{k}_3,\vec{k}_5,\vec{k}_8)A_3A_5 + \tau(\vec{k}_5,\vec{k}_8,\vec{k}_{13})A_5^*A_{13} - \ldots$$

$$\frac{\partial}{\partial t}A_{13} = \sigma(k_{13}^2)A_{13} + \tau(\vec{k}_5,\vec{k}_8,\vec{k}_{13})A_5A_8 - \ldots$$

ansatz: $A_3 = A_{13}$

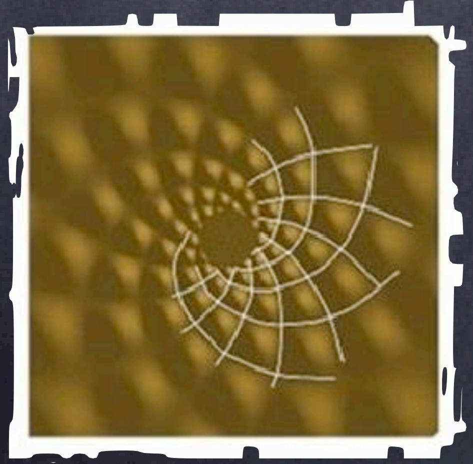
$A_5 = A_8$



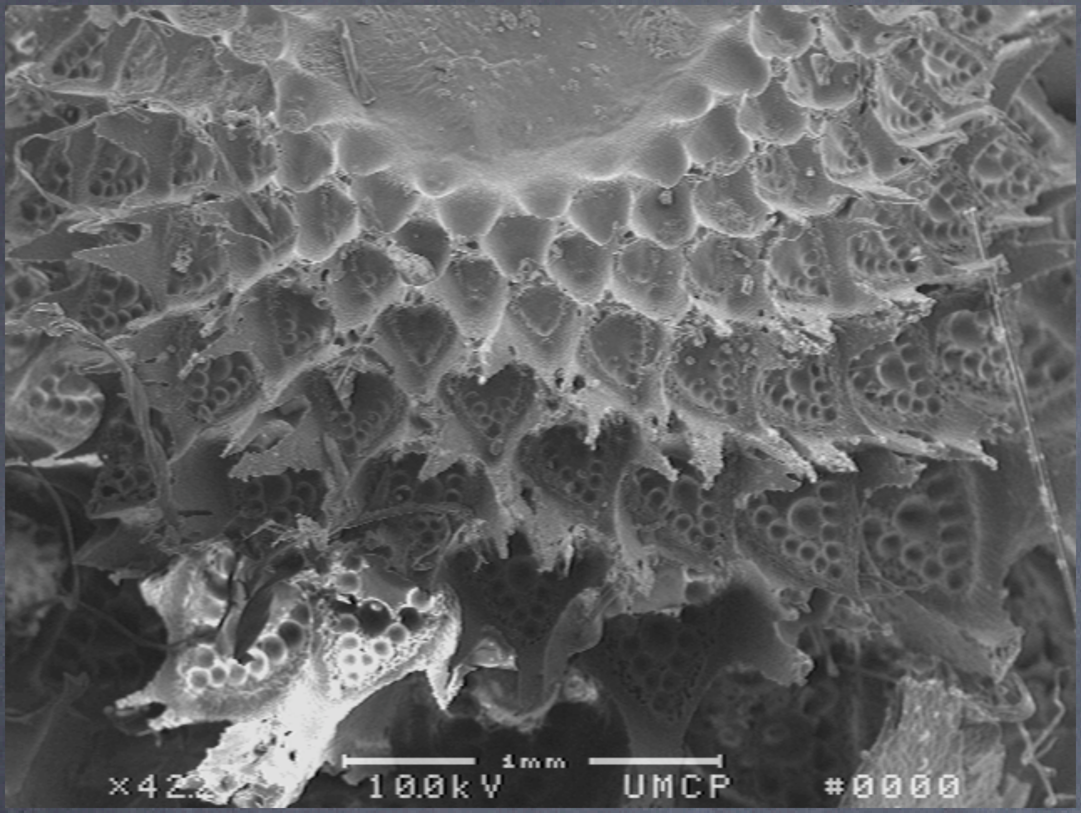
$$\frac{\partial}{\partial t}A_3 = \sigma(k_3^2)A_3 + \tau(\vec{k}_3,\vec{k}_5,\vec{k}_5)|A_5|^2 - \dots$$

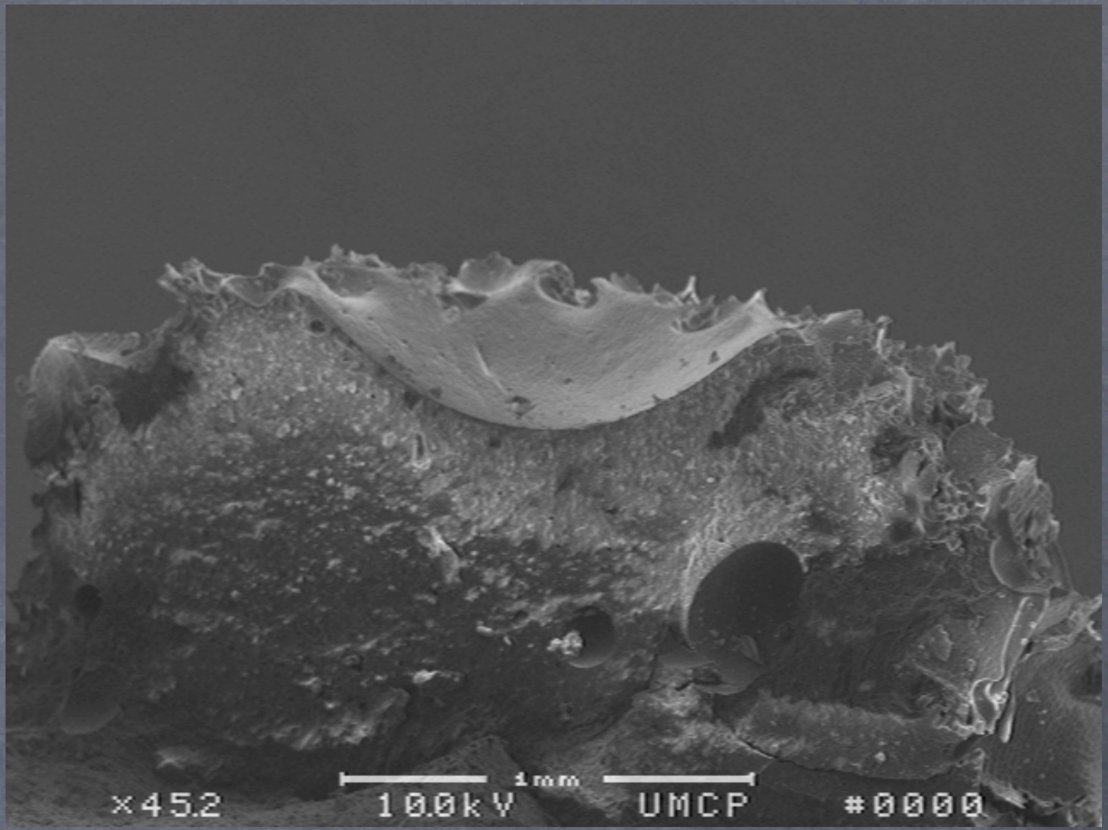
$$\frac{\partial}{\partial t}A_5 = \sigma(k_5^2)A_5 + \tau(\vec{k}_3,\vec{k}_5,\vec{k}_5)A_3^*A_5 + \tau(\vec{k}_5,\vec{k}_5,\vec{k}_3)A_5^*A_3 - \dots$$

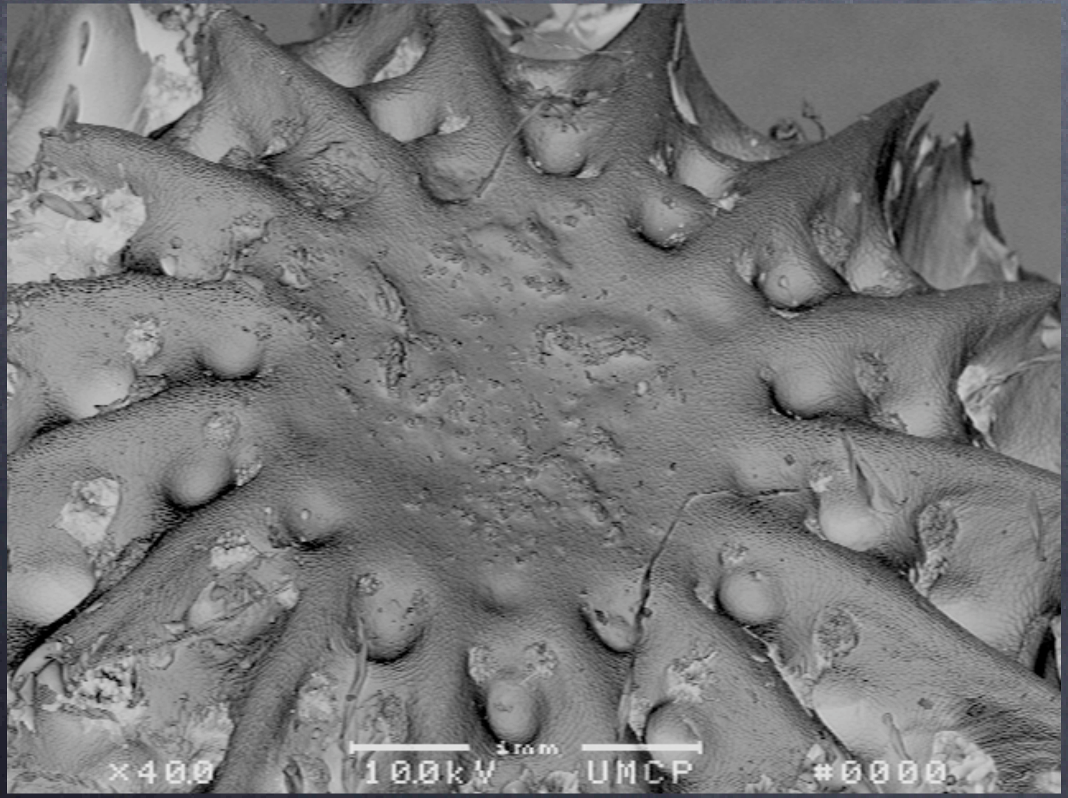
solution: $A_3(=A_{13}) < A_5(=A_8)$

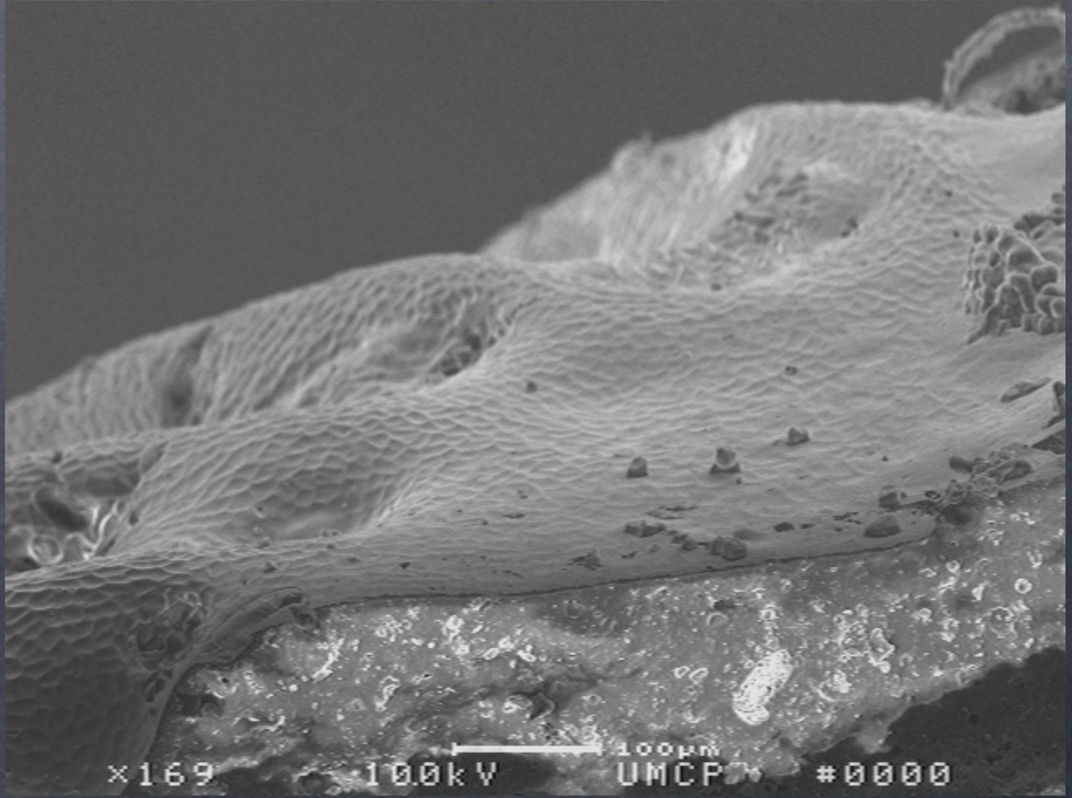


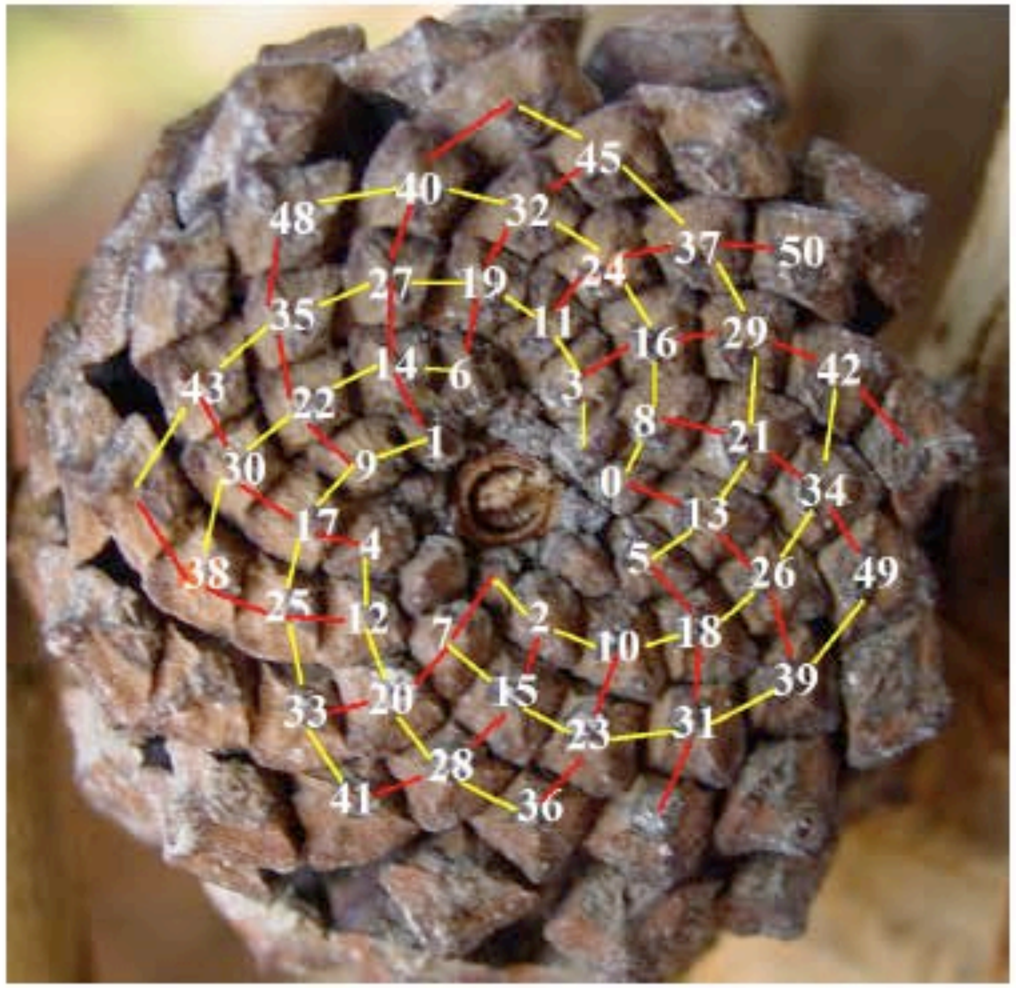
$$\tau(\vec{k}_1, \vec{k}_2, \vec{k}_3) = -C(\vec{k}_1 \times \vec{k}_2)^2 \sum_{j=1}^{3} \frac{1}{k_j^2}$$





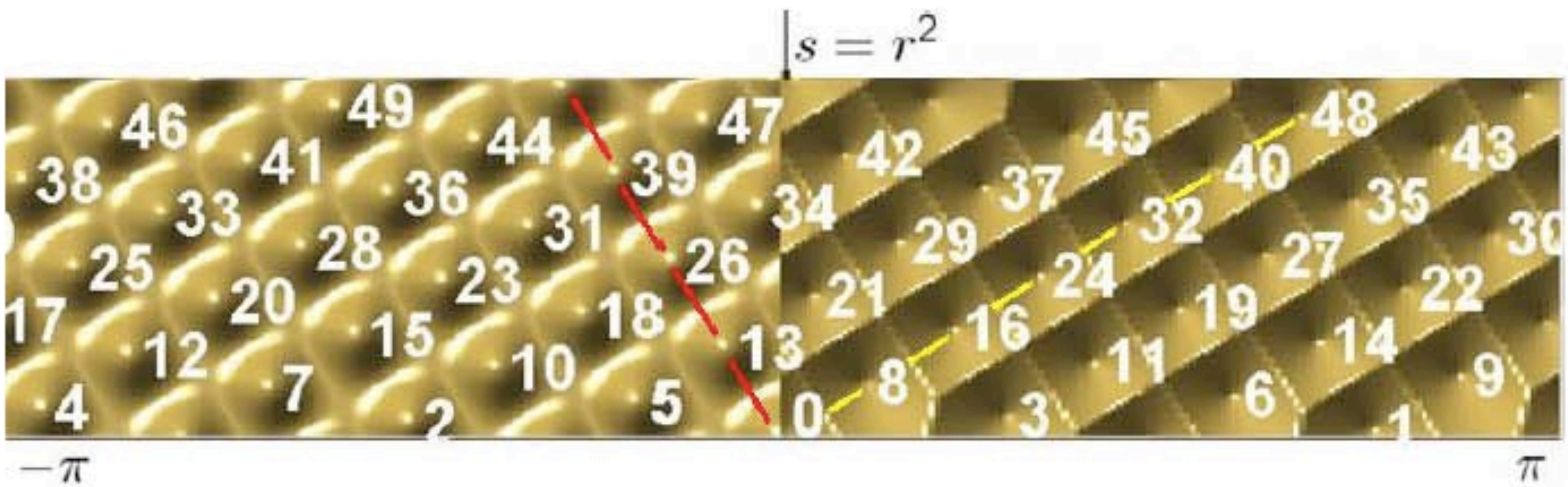


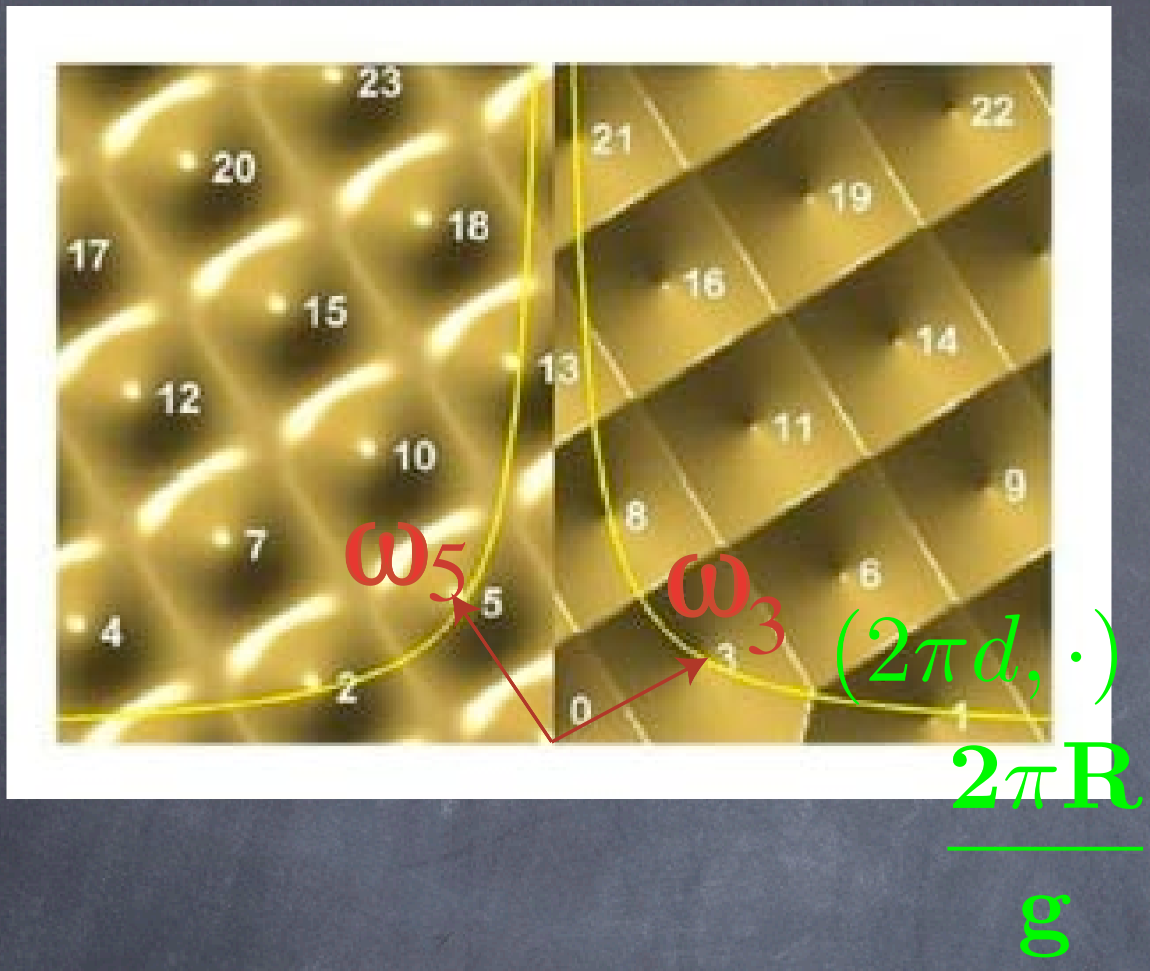


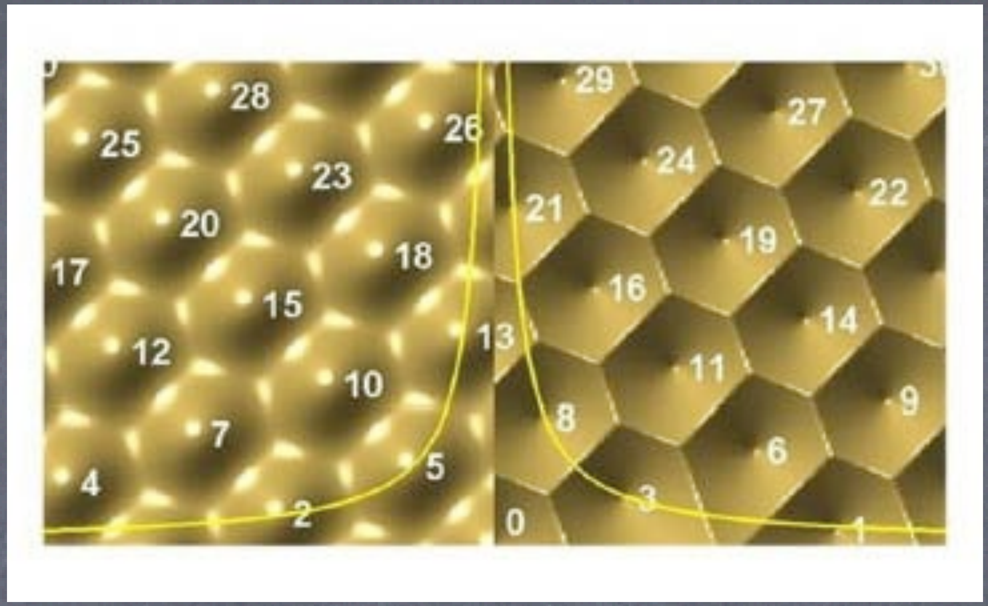


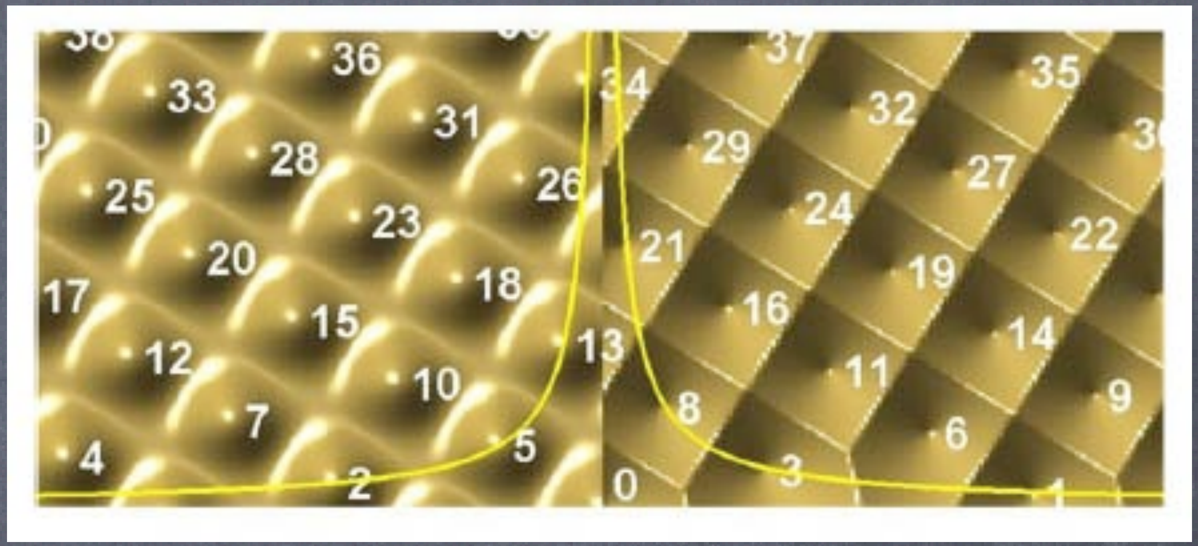
$$(r_z, \alpha_z) \simeq (\sqrt{\lambda z}, 2\pi z d)$$

$$(s_z, \theta_z) = (\lambda z, 2\pi z d) = z(\lambda, 2\pi d)$$







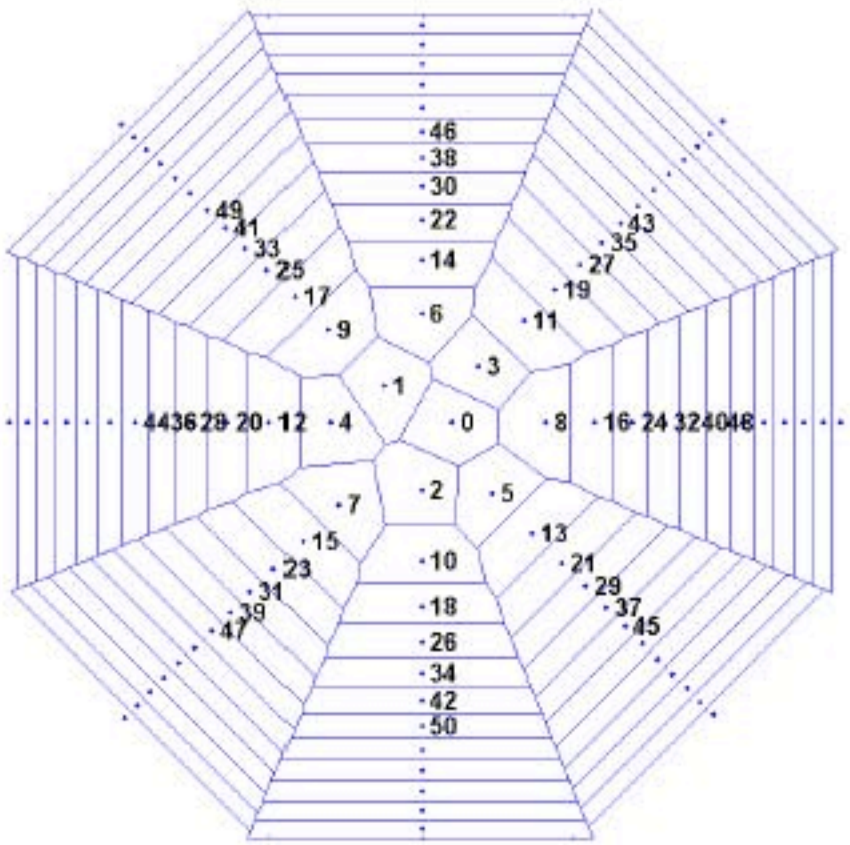


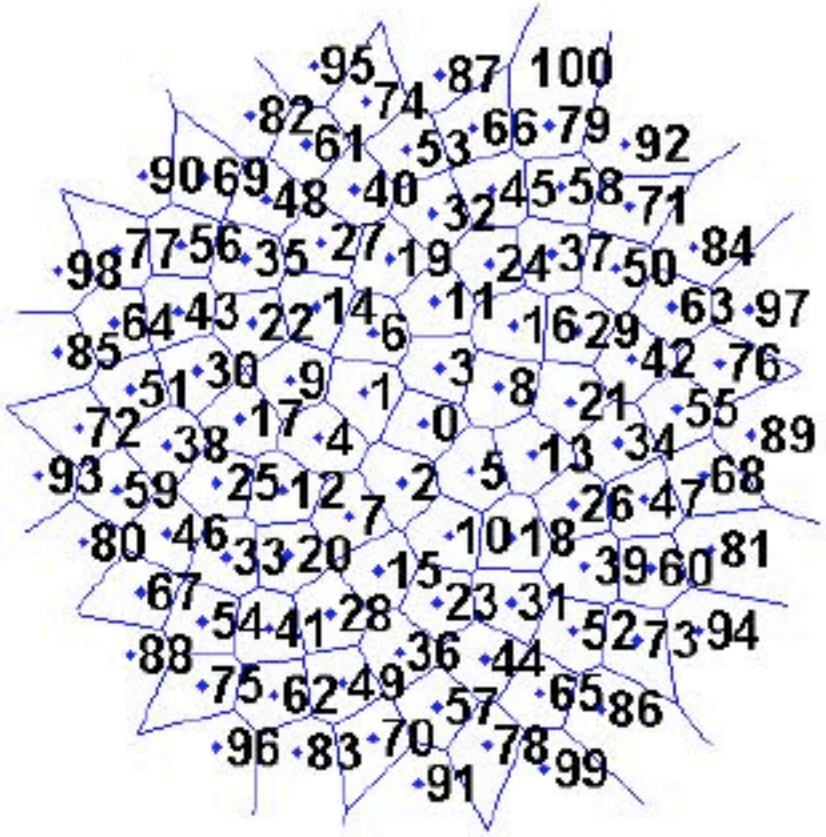
$$\Gamma = \frac{2\pi R}{\sqrt{A}}$$

$$\Omega_{(m,n)}(A,\Gamma,d,g) = \sqrt{A}\begin{pmatrix} \frac{n}{\Gamma} & \frac{\Gamma}{g}(nd-p) \\ \frac{m}{\Gamma} & \frac{\Gamma}{g}(md-q) \end{pmatrix}$$

basis matrix for a lattice $\mathbb{L}(A,\Gamma,d,g)$

$S = \{x_i\}$; Voronoi cell $V_{x_i} = \{\vec{x} \in \mathbb{R}^2 : |\vec{x} - \vec{x}_i| < |\vec{x} - \vec{x}_j| \forall j, j \neq i\}$





$\mathbb{L}(A, \Gamma = 1, d = \frac{3}{8}, g = 1)$

$\mathbb{L}(A, \Gamma = 1, d = \frac{1}{2}(3 - \sqrt{5}), g = 1)$

$$\frac{1}{2}(3 - \sqrt{5}) - \frac{3}{8} \simeq 0.006966$$

Theorem (see, e.g. Hardy and Wright, Theorem 185): Any irrational ξ has an infinite number of rational approximations $\frac{p}{z}$ which satisfy

$$\left|\frac{p}{z} - \xi\right| < \frac{1}{z^2\sqrt{5}}.$$

Rewritten, $z|z\xi - p| < \frac{1}{\sqrt{5}}$.

$\sqrt{5}$ is the best one can do.

For $\xi = \frac{q_0+q_1\phi_+}{m_0+m_1\phi_+}$, $q_0 m_1 - q_1 m_0 = 1$, the theorem would fail for $\sqrt{5}$ replaced by $\eta > \sqrt{5}$.

For $m_0 = 0$, $m_1 = 1$, $q_0 = -1$, $q_1 = 1$,

$$\{m_\nu\} = \{0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \dots\}$$

$$\hat{d} = \frac{-1+\phi_+}{0+\phi_+} = 1 + \phi_+ = \frac{1}{2}(3 - \sqrt{5}) = 0.381966\dots$$

Proposition: Given two Fibonacci-like sequences q_ν and m_ν, respectively generated by q_0, q_1 and m_0, m_1, define $g = q_1 m_0 - q_0 m_1$, $\hat{d} = \frac{q_0 + q_1\phi_+}{m_0 + m_1\phi_+}$. Then,

$$\lim_{k\to\infty} \frac{m_{2k}}{g}(m_{2k}\hat{d} - q_{2k}) = \frac{1}{\sqrt{5}},$$

and

$$\lim_{k\to\infty} \frac{m_{2k+1}}{g}(m_{2k+1}\hat{d} - q_{2k+1}) = -\frac{1}{\sqrt{5}}.$$

Proof: Using Binet's formula, we write

$$\frac{m_\nu}{g}\left(m_\nu \frac{q_0 + q_1\phi_+}{m_0 + m_1\phi_+} - q_\nu\right) = \frac{(-1)^\nu (m_0 + m_1\phi_+) + \phi_-^{2\nu-2}(m_0 + m_1\phi_-)}{\sqrt{5}(m_0 + m_1\phi_+)}.$$

Now use that $|\phi_-| < 1$. ■

Fibonacci-like sequences $x_0, x_1 \in \mathbb{R}^n$, $x_{\nu+1} = x_\nu + x_{\nu-1}$

$$\text{For } Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} x_{\nu+1} \\ x_\nu \end{pmatrix} = Q^\nu \begin{pmatrix} x_1 \\ x_0 \end{pmatrix}.$$

$$\begin{pmatrix} x_{\nu+1} \\ x_\nu \end{pmatrix} = Q^\nu \begin{pmatrix} x_1 \\ x_0 \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} \phi_+^{\nu+1} - \phi_-^{\nu+1} & \phi_+^\nu - \phi_-^\nu \\ \phi_+^\nu - \phi_-^\nu & \phi_+^{\nu-1} - \phi_-^{\nu-1} \end{pmatrix} \begin{pmatrix} x_1 \\ x_0 \end{pmatrix}$$

$\phi_\pm = \frac{1 \pm \sqrt{5}}{2}$ are the roots of $\eta^2 - \eta - 1 = 0$.

Binet's formula: $x_\nu = \frac{1}{\sqrt{5}} \left[(\phi_+^\nu - \phi_-^\nu) x_1 + (\phi_+^{\nu-1} - \phi_-^{\nu-1}) x_0 \right]$

For $m_0, m_1, q_0, q_1 \in \mathbb{Z}$, define $\Omega_1(A,\Gamma,d,g) \doteq \sqrt{A}\begin{pmatrix} \frac{m_1}{\Gamma} & \frac{\Gamma}{g}(m_1 d - q_1) \\ \frac{m_0}{\Gamma} & \frac{\Gamma}{g}(m_0 d - q_0) \end{pmatrix}$

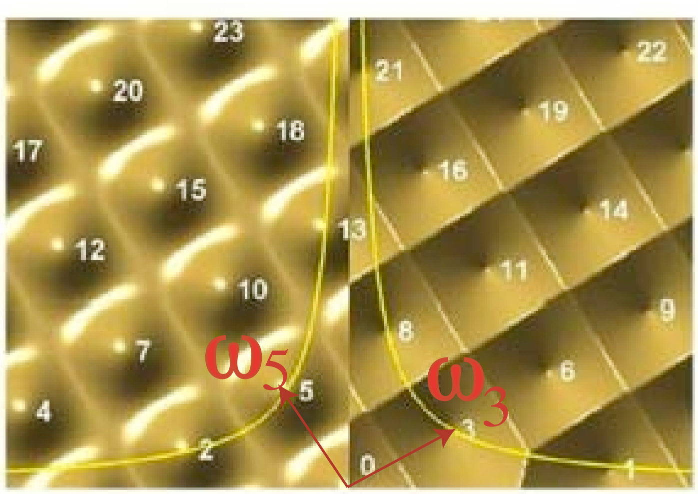
$$\Omega_{\nu+1} \doteq Q\Omega_\nu; \qquad \Omega_\nu(A,\Gamma,d,g) = \sqrt{A}\begin{pmatrix} \frac{m_\nu}{\Gamma} & \frac{\Gamma}{g}(m_\nu d - q_\nu) \\ \frac{m_{\nu-1}}{\Gamma} & \frac{\Gamma}{g}(m_{\nu-1} d - q_{\nu-1}) \end{pmatrix}$$

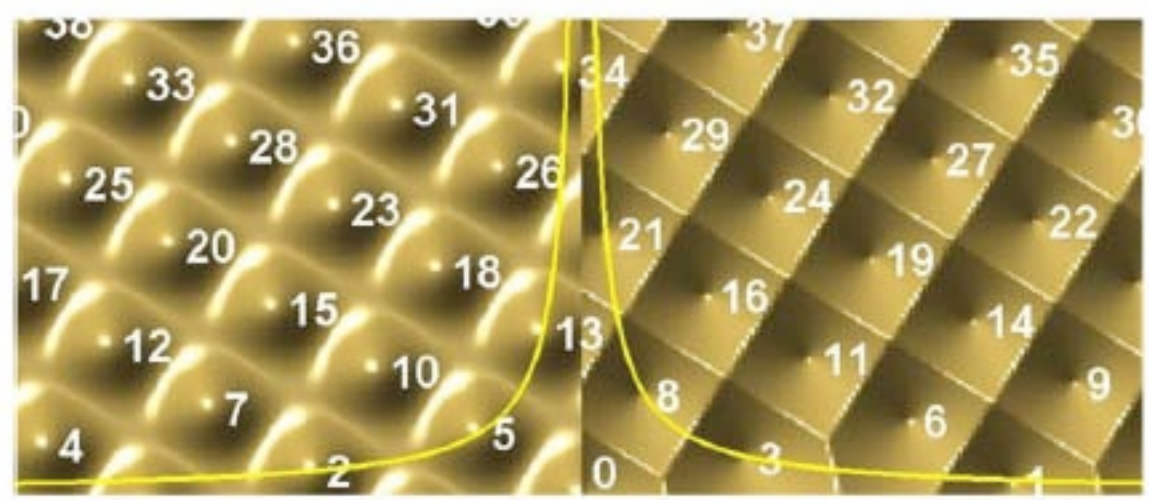
$$\hat{d} = \frac{q_0 + q_1\phi_+}{m_0 + m_1\phi_+}; \qquad \hat{\Omega}_\nu(A,\Gamma) = \sqrt{A}\begin{pmatrix} \frac{m_\nu}{\Gamma} & \frac{\Gamma}{g}(m_\nu \hat{d} - q_\nu) \\ \frac{m_{\nu-1}}{\Gamma} & \frac{\Gamma}{g}(m_{\nu-1} \hat{d} - q_{\nu-1}) \end{pmatrix}$$

$$\lambda_\nu \doteq \frac{\Gamma}{m_\nu}. \qquad \hat{\Omega}_\nu(A,\Gamma) = \sqrt{A}\begin{pmatrix} \frac{1}{\lambda_\nu} & \lambda_\nu \frac{1}{g} m_\nu(m_\nu \hat{d} - q_\nu) \\ \frac{1}{\lambda_{\nu-1}} & \lambda_{\nu-1}\frac{1}{g} m_{\nu-1}(m_{\nu-1}\hat{d} - q_{\nu-1}) \end{pmatrix}$$

$$\simeq \sqrt{A}\begin{pmatrix} \frac{1}{\lambda_\nu} & \lambda_\nu \frac{\pm 1}{\sqrt{5}} \\ \frac{1}{\lambda_{\nu-1}} & \lambda_{\nu-1}\frac{\mp 1}{\sqrt{5}} \end{pmatrix}$$

The sequence $\vec{\omega}_\nu$ of basis vectors lies near the curves $\sqrt{A}\left(\frac{1}{\lambda}, \pm\frac{\lambda}{\sqrt{5}}\right)$.





Inverse Length Functions

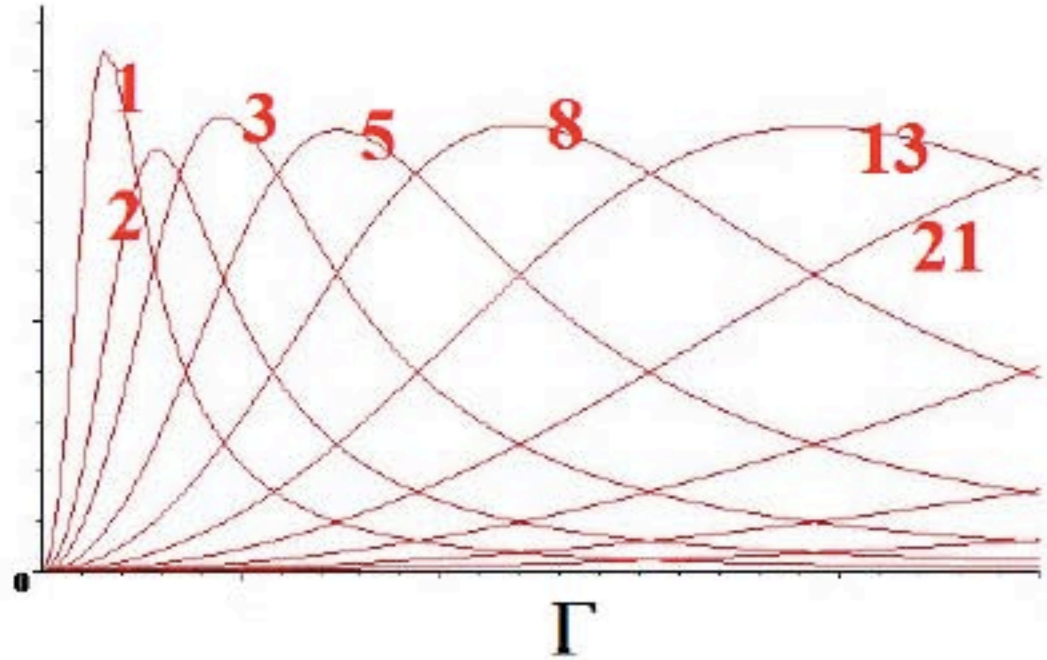
$$\mathsf{A}_\nu \doteq \frac{A}{||\vec{\omega}_\nu^2||^2} = \frac{1}{\frac{\Gamma^2}{g^2}(m_\nu d - q_\nu)^2 + \frac{m_\nu^2}{\Gamma^2}} = \frac{1}{\lambda_\nu^2 \frac{m_\nu^2}{g^2}(m_\nu d - q_\nu)^2 + \frac{1}{\lambda_\nu^2}}$$

$$\lambda_\nu = \frac{\Gamma}{m_\nu}$$

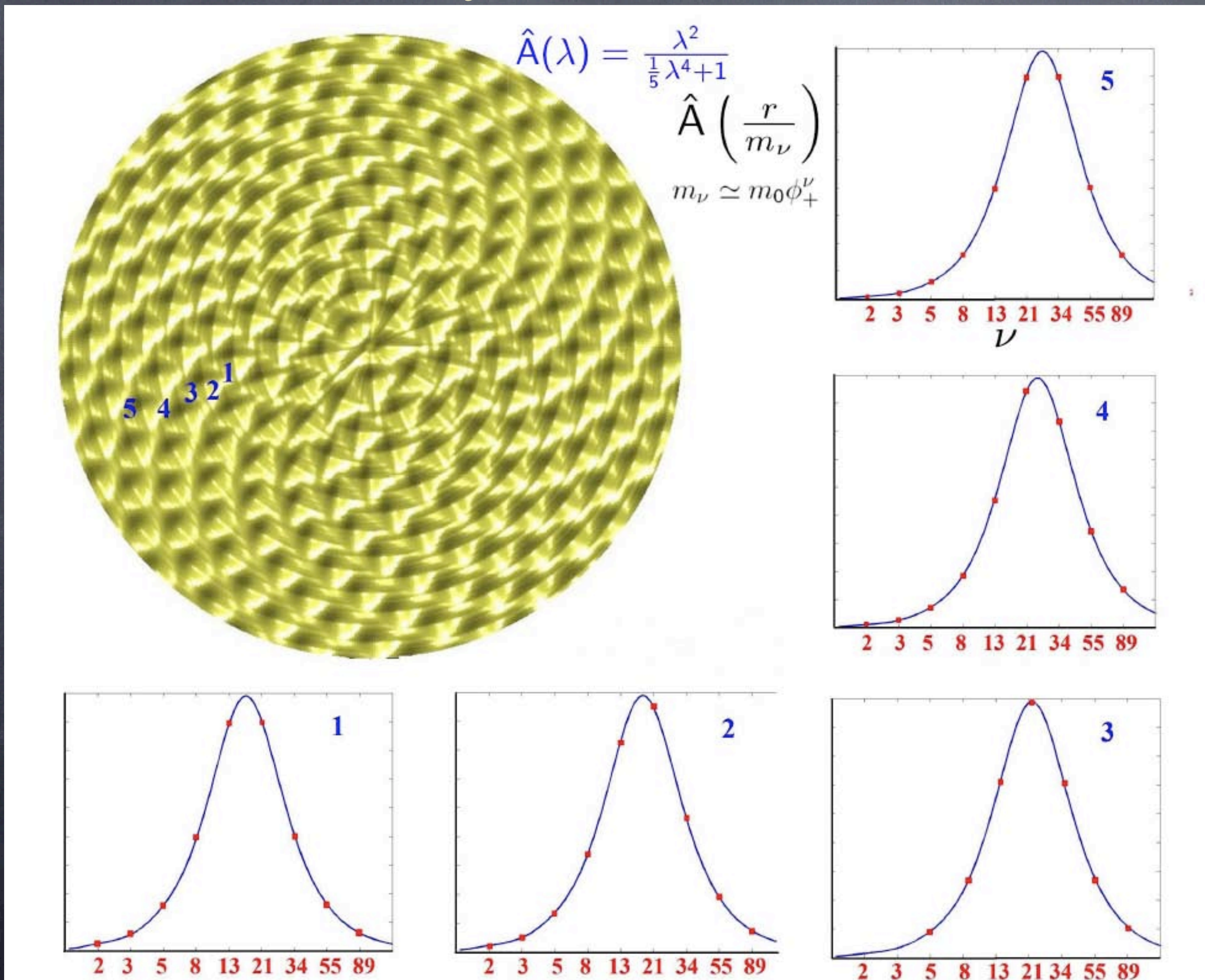
$$\hat{\mathsf{A}}_\nu \doteq \frac{1}{\lambda_\nu^2 \frac{m_\nu^2}{g^2}(m_\nu \hat{d} - q_\nu)^2 + \frac{1}{\lambda_\nu^2}} \simeq \frac{\lambda_\nu^2}{\lambda_\nu^4 \frac{1}{5} + 1}$$

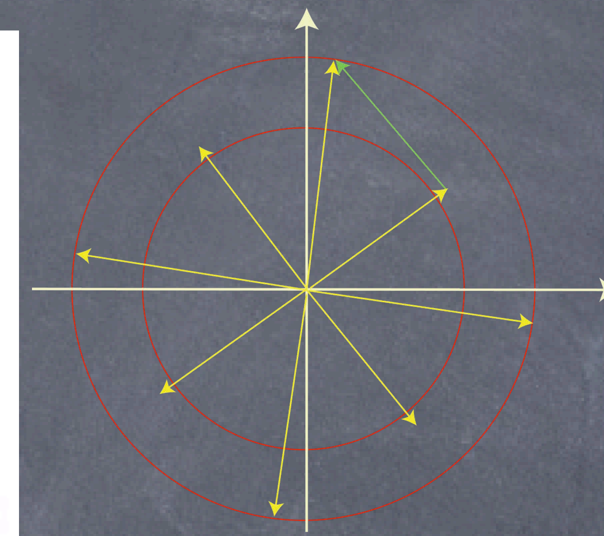
$\hat{\mathsf{A}}\left(\frac{\Gamma}{m}\right)$

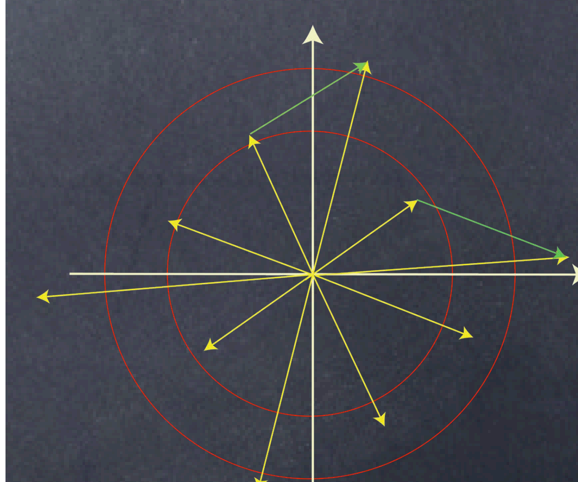
$\hat{\mathsf{A}}_\nu \simeq \hat{\mathsf{A}}(\lambda_\nu)$ for $\hat{\mathsf{A}}(\lambda) = \frac{\lambda^2}{\frac{1}{5}\lambda^4 + 1}$

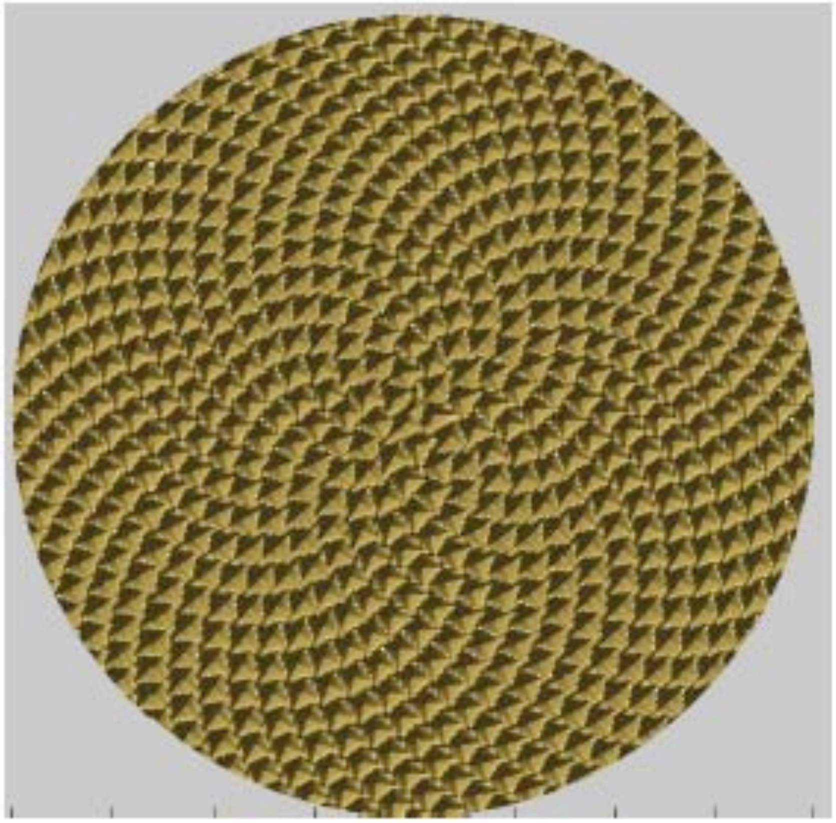


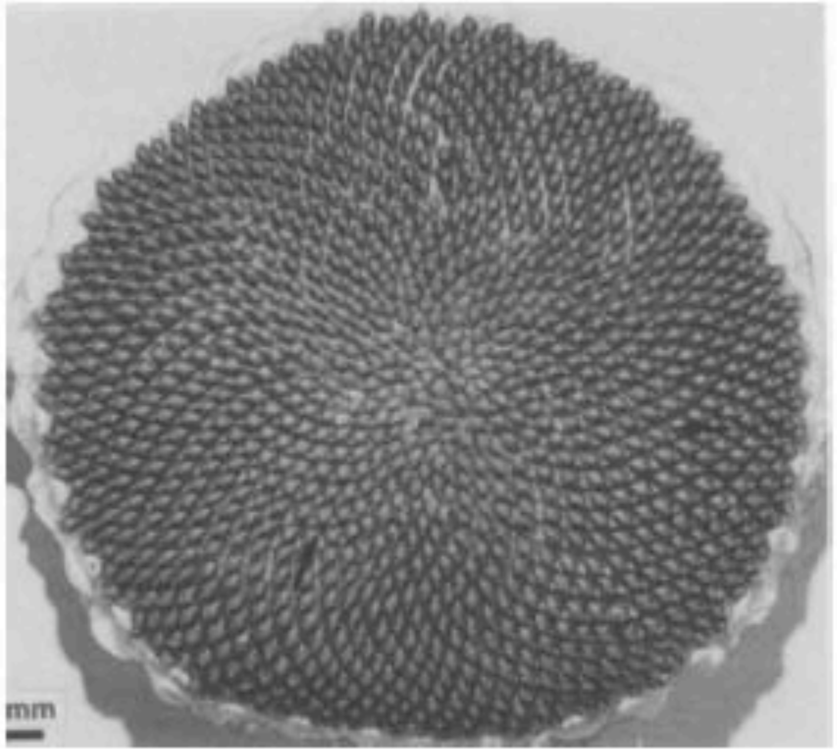
$$w(r,\theta) = \sum_{j=1}^{10} \hat{A}\left(\frac{r}{m_j}\right)\cos(\vec{k}_j(r)\cdot\vec{x})$$

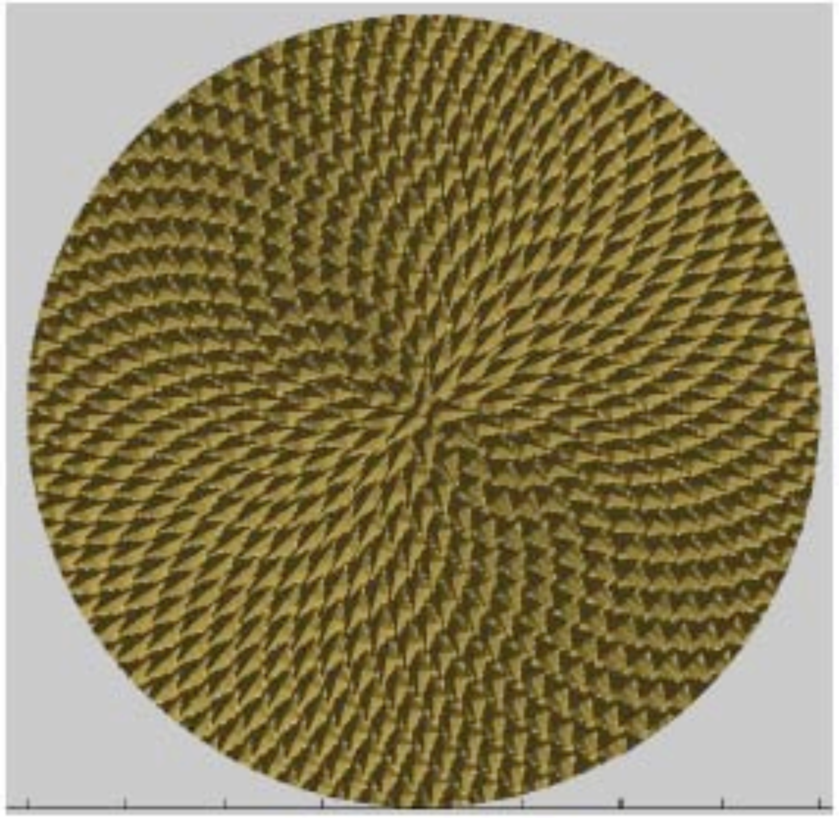


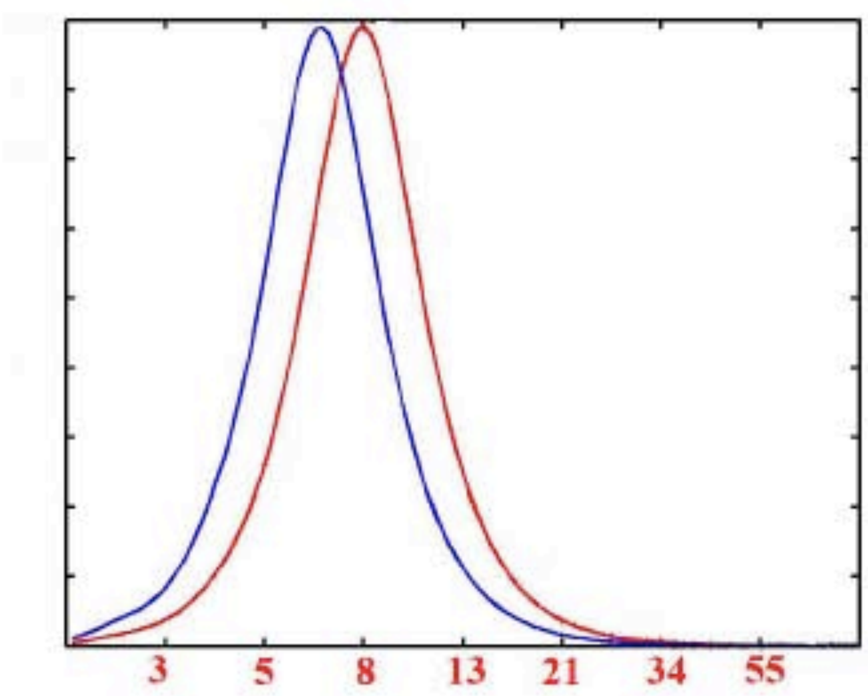






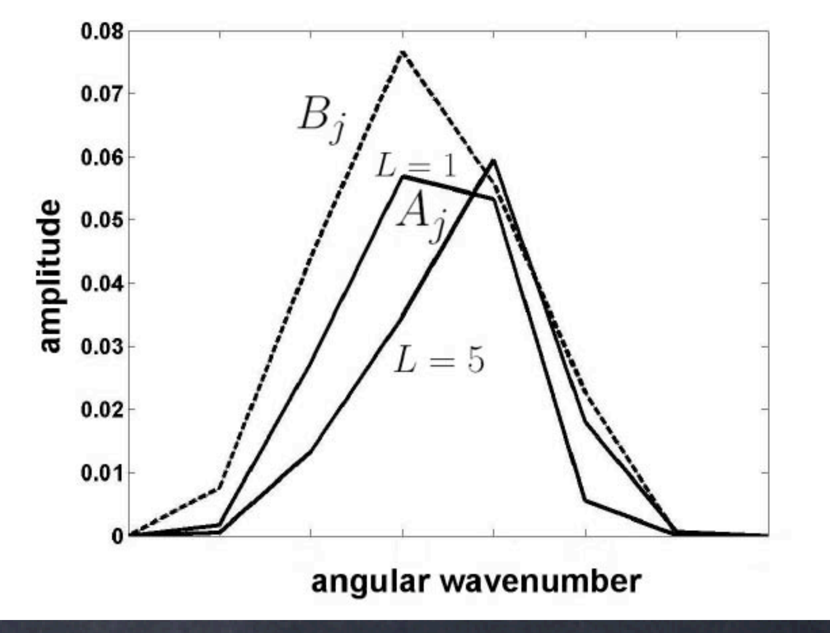


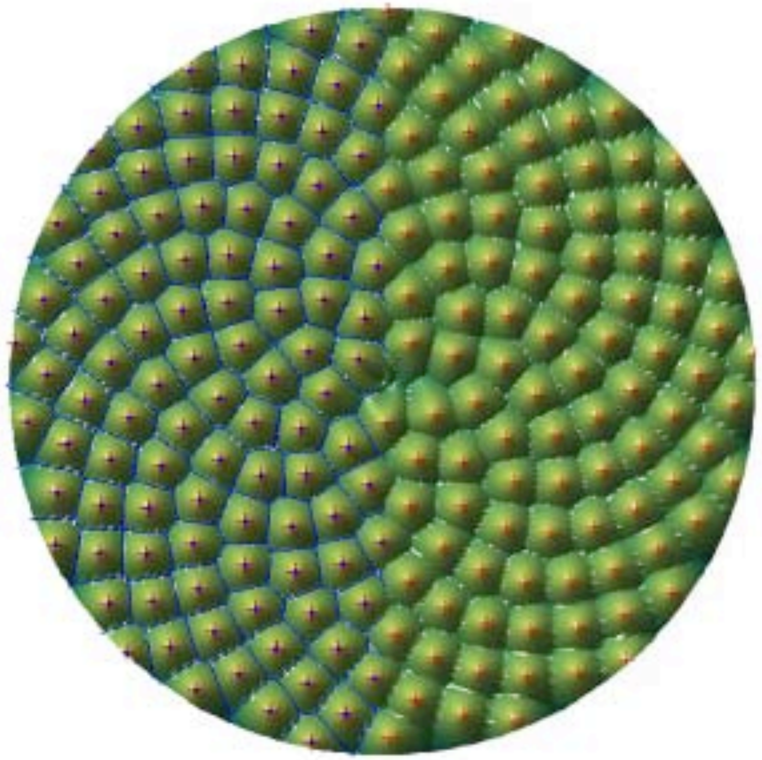




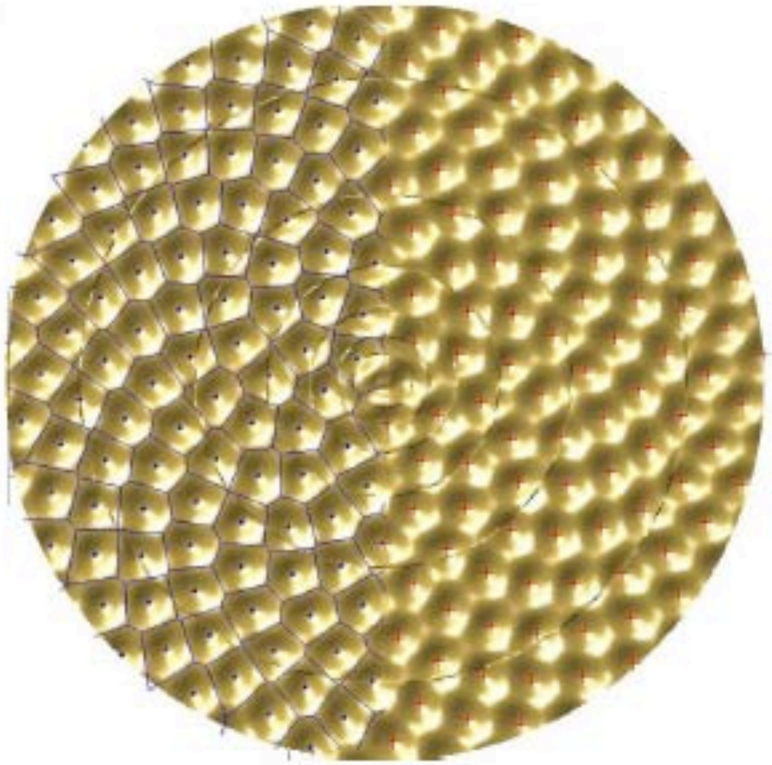
$$w(r,\alpha) = \sum_{\nu=1}^{10} \hat{\mathsf{A}}\left(\frac{r}{m_\nu}\right)\cos(\vec{k}_\nu \cdot \vec{x})$$

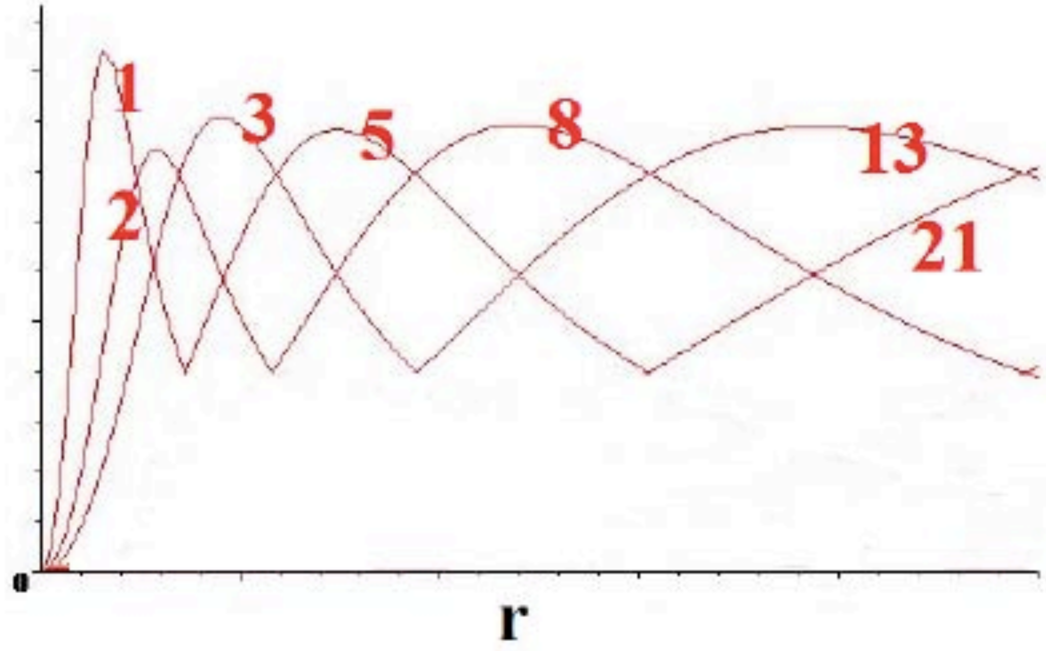
$$\hat{\mathsf{A}}(\rho) = \frac{\rho^2}{\frac{1}{5}\rho^4 + a > 1}$$





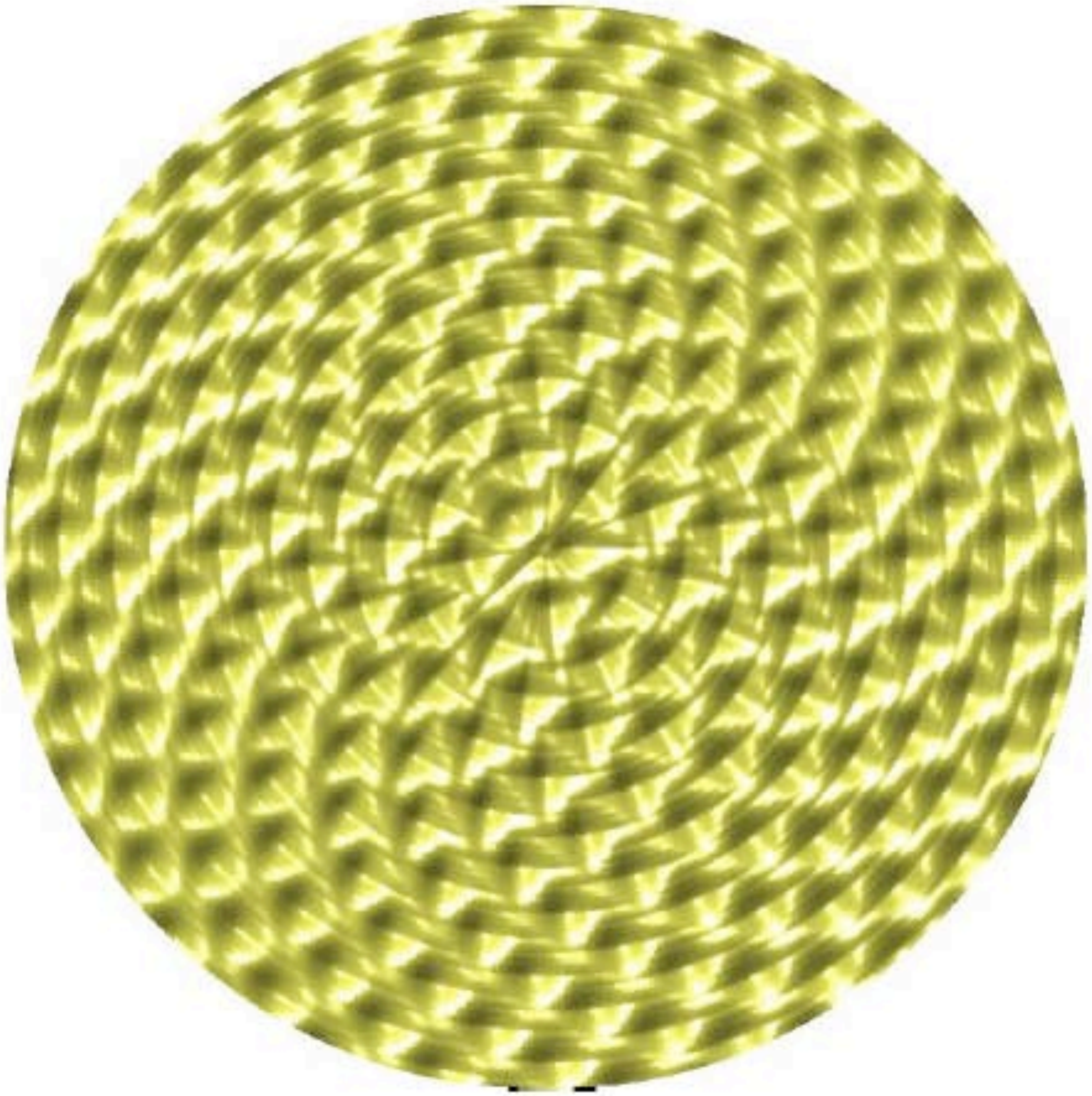
$$w(r,\alpha) = \hat{A}\left(\frac{r}{m}\right)\cos(\vec{k}_m \cdot \vec{x})$$
$$+\hat{A}\left(\frac{r}{n}\right)\cos(\vec{k}_n \cdot \vec{x}) + \hat{A}\left(\frac{r}{m+n}\right)\cos(\vec{k}_{m+n} \cdot \vec{x})$$
$$\vec{k}_m = \frac{2\pi}{\sqrt{A}}\left(\frac{r}{g}(q - md), \frac{m}{r}\right)$$
$$\vec{k}_n = \frac{2\pi}{\sqrt{A}}\left(\frac{r}{g}(p - nd), \frac{n}{r}\right)$$

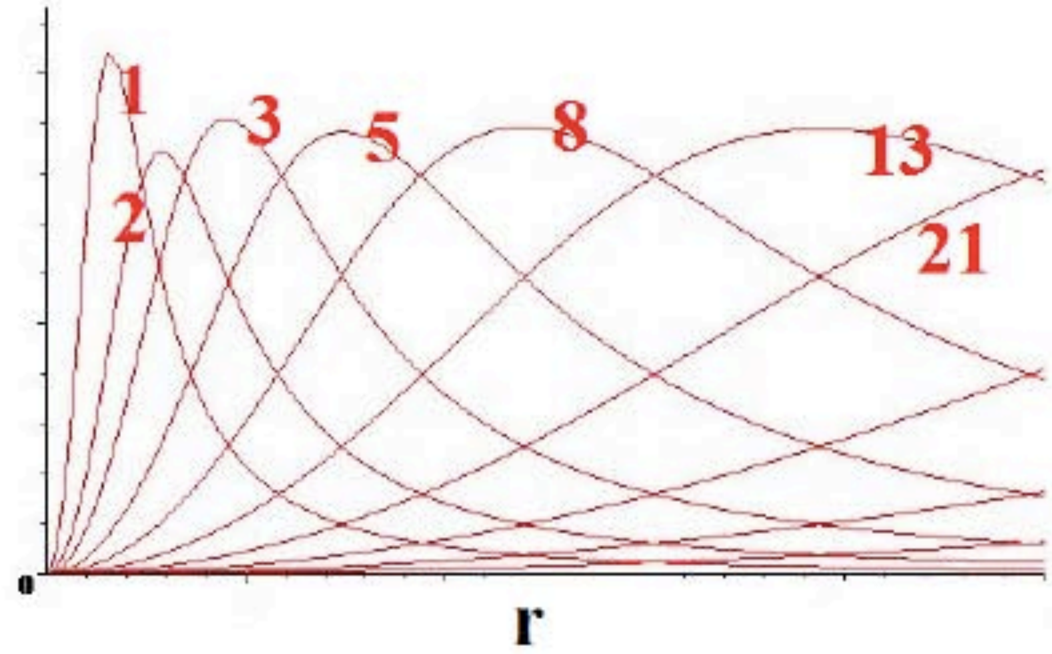


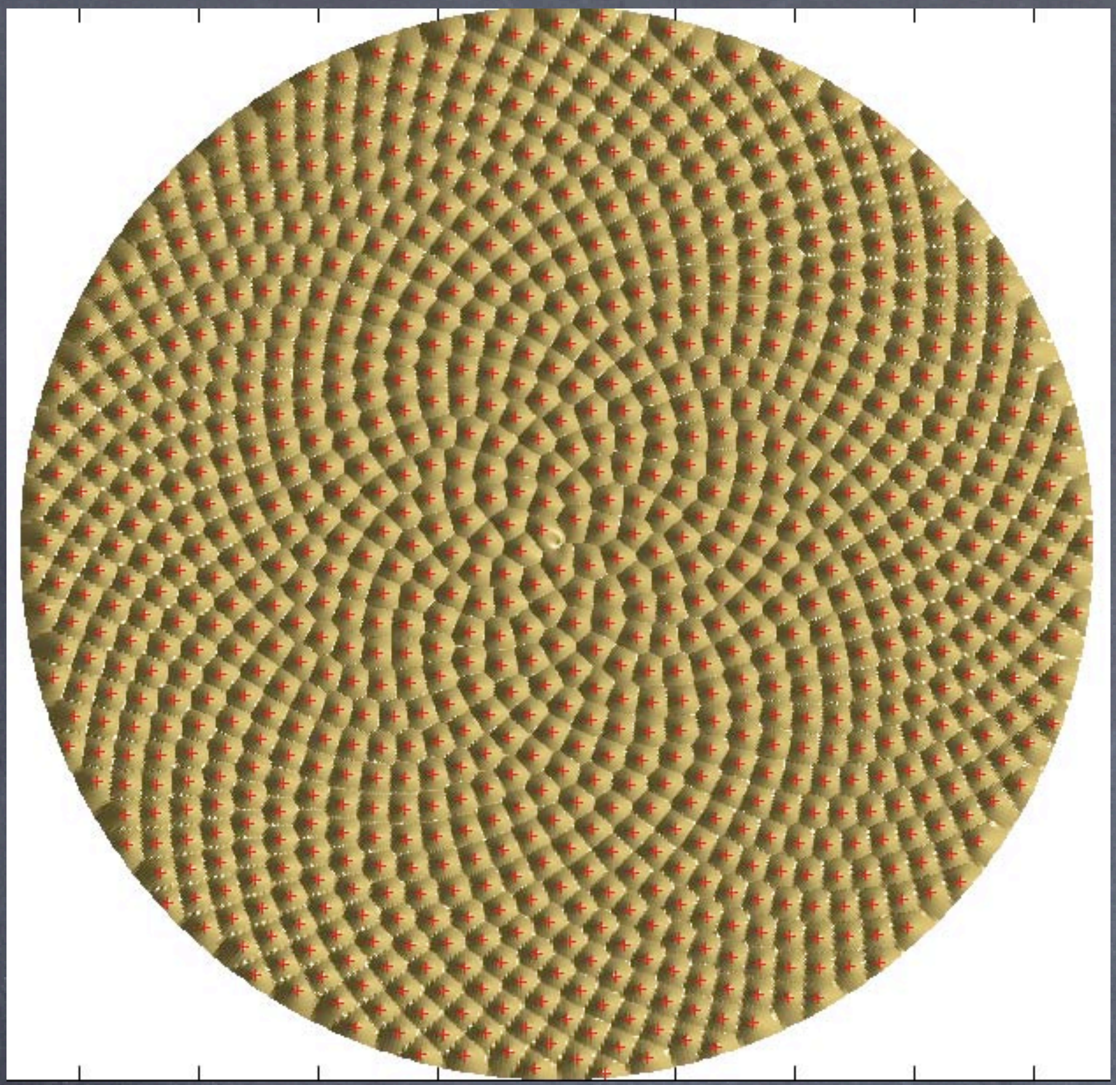


Multiple Triad Surface

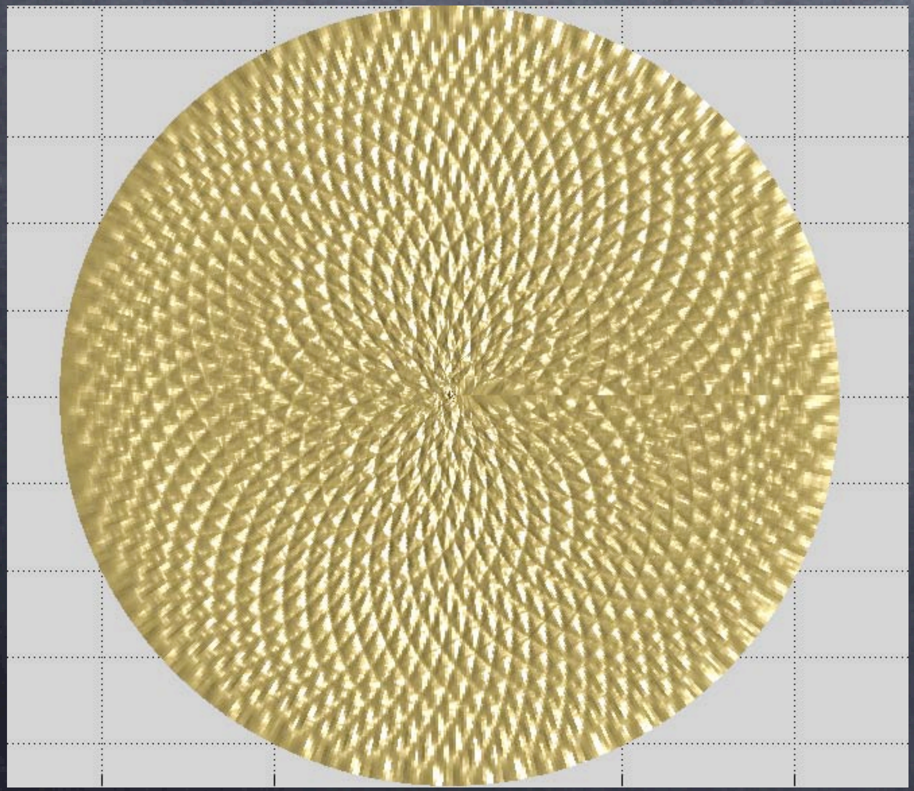
$$w(r,\alpha) = \sum_{\nu=1}^{10} \hat{A}\left(\frac{r}{m_\nu}\right)\cos(\vec{k}_\nu \cdot \vec{x}) \qquad \vec{k}_\nu(r) = \frac{2\pi}{\sqrt{A}}\left(\frac{r}{g}(q_\nu - m_\nu d), \frac{m_\nu}{r}\right)$$

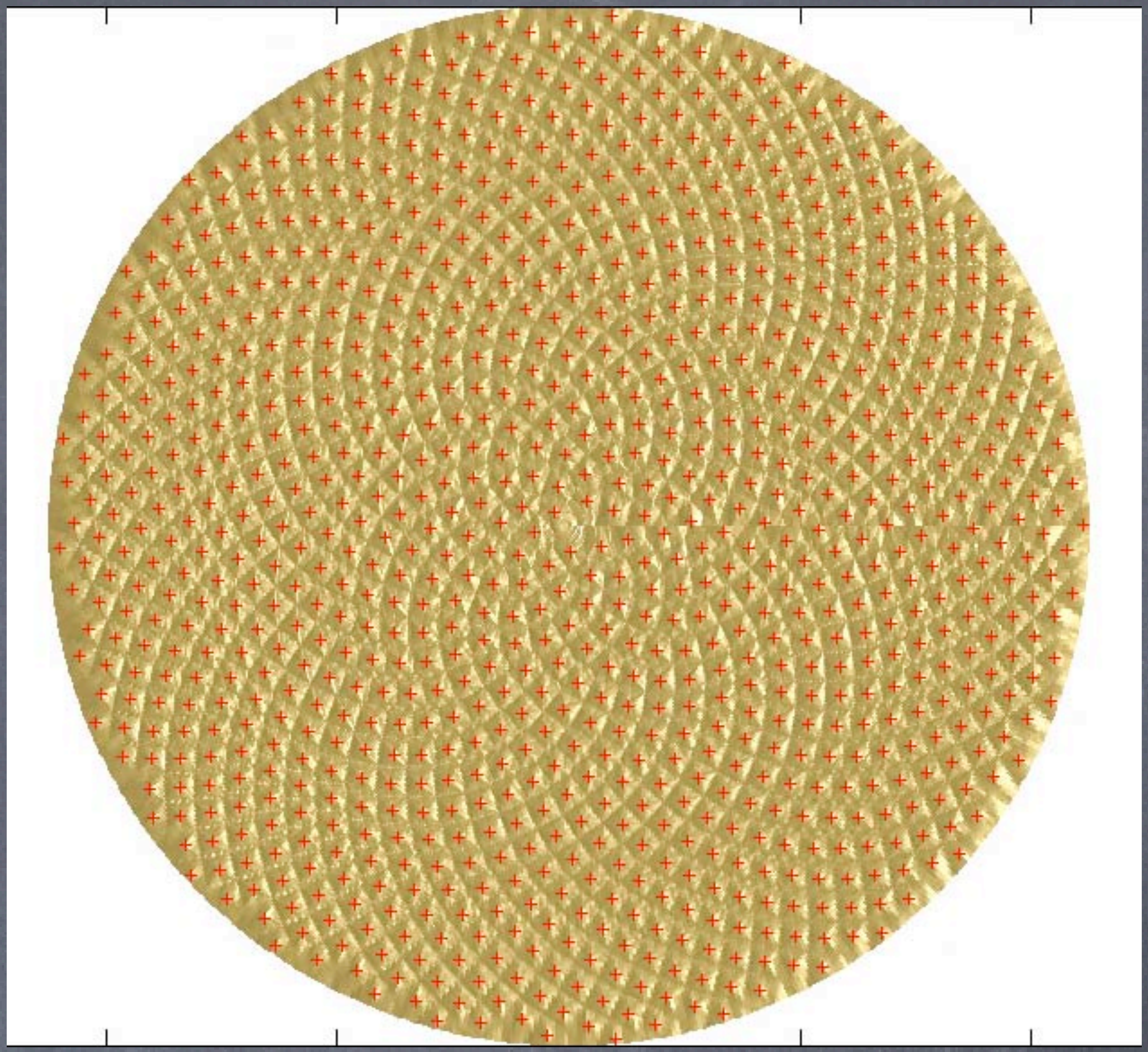






$$w(\vec{x}) = -\mathrm{dist}^2(\vec{x}, \Omega)$$





$$w(\vec{x}) = -\sum \alpha_n \mathrm{dist}^2(\vec{x}, \Omega_n)$$