GW invariants of $\mathbb{C}P^1$

$\overline{\mathcal{M}}_{g,n,d}$ stable maps $f:(C_g, z_1, \ldots, z_n) \to \mathbb{C}P^1$
of degree d

$$\psi_i = c_1(\mathcal{L}_i) \qquad (\mathcal{L}_i)_{f:(C,z_1,\ldots,z_n)\to \mathbb{C}P^1} = T^*_{z_i}\mathbb{C}P^1$$

$$ev_i(f:(C,z_1,\ldots,z_n)\to \mathbb{C}P^1) = f(z_i)$$

$$\langle \tau_{k_1,a_1} \ldots \tau_{k_n,a_n}\rangle_{g,n,d} = \int_{\overline{\mathcal{M}}_{g,n,d}} \psi_1^{k_1}\ldots\psi_n^{k_n}\, ev_1^*(\gamma_{a_1})\ldots ev_n^*(\gamma_{a_n})$$

$$\gamma_P = 1 \in H^0(\mathbb{C}P^1) \qquad t_{n,P} = s_n \qquad \partial = \partial/\partial s_0$$
$$\gamma_Q = \omega \in H^2(\mathbb{C}P^1) \qquad t_{n,Q} = t_n \qquad \partial_Q = \partial/\partial t_0$$

$$F_g = \sum_{d=0}^{\infty} q^d \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\substack{k_1\ldots k_n\\ a_1\ldots a_n}} t_{k_1,a_1}\ldots t_{k_n,a_n}\langle\tau_{k_1,a_1}\ldots\tau_{k_n,a_n}\rangle_{g,n,d}$$

$$F = \sum_{g=0}^{\infty} \varepsilon^{2g} F_g$$

Large phase space – coordinates $\{s_n, t_n\}$
Problem: calculate F
Genus 0: Dijkgraaf - Witten
uses topological recursion relations and string equation

$$v = \partial \partial_Q F_0 \qquad u = \partial^2 F_0$$
$$= s_0 + \cdots \qquad = t_0 + \cdots$$

$$\partial^n v = \begin{cases} 1 + s_1 + \cdots, & n = 1 \\ s_n + \cdots, & n > 1 \end{cases} \qquad \partial^n u = t_n + \cdots, \quad n > 0$$

Thus, $\{\partial^n v, \partial^n u\}$ give coordinates on large phase space.

<u>Toda equation</u> $$\boxed{\partial_Q^2 F_0 = q e^u}$$

Let $A = \mathbb{Q}[e^u, \partial^n u, \partial^n v]$

Poisson bracket on $A[p, p^{-1}]$:
$$\{f, g\} = p \frac{\partial f}{\partial p} \cdot \partial g - \partial f \cdot p \frac{\partial g}{\partial p}$$

Projection to Lie subalgebra : $+ : A[p, p^{-1}] \to A[p]$

$$L = p + v + e^u p^{-1}$$
$$\delta_n L = \{ L_+^n, L \} \qquad [\partial, \delta_n] = 0$$
$$\parallel$$
$$\delta_n v + e^u \delta_n u \cdot p^{-1}$$

<u>Theorem</u> $[\delta_n, \delta_m] = 0$

Toda hierarchy : $\delta_n = {}^{?} \, n! \, \partial_{n-1, Q}$

$$L_+^R = p + q v$$

$$\{L_+, L\} = \{p+v,\ p+v+e^u p^{-1}\}$$
$$= \{p+v,\ e^u p^{-1}\}$$
$$= \partial(e^u) + e^u\, \partial v\, p^{-1}$$

$$\delta_1 v = \partial(e^u) \qquad \partial^2(\delta_1^2 F) = \delta_1^2 u = \partial^2 e^u$$
$$\delta_1 u = \partial v$$

$\Rightarrow$ Toda equation

Higher genus generalization — Eguchi, Yang
(recently proved by Okounkov, Pandharipande — they reduce it to a question about Hurwitz numbers, i.e. "2D QCD")

$$\nabla = \varepsilon^{-1}\left(e^{\varepsilon\partial/2} - e^{-\varepsilon\partial/2}\right), \qquad E$$

$$v = \nabla \partial_Q F \qquad u = \nabla^2 F$$

Algebra $A[\Lambda, \Lambda^{-1}]$

$$a\Lambda^k \cdot b\Lambda^l = \left(e^{-\varepsilon l \partial/2} a\right)\left(e^{\varepsilon k \partial/2} b\right)\Lambda^{k+l}$$

$+ : A[\Lambda, \Lambda^{-1}] \to A[\Lambda]$ as before

$$L = \Lambda + v + e^u \Lambda^{-1}$$

$$\delta_n L = [\varepsilon^{-1} L_+^n , L] , \quad [\partial, \delta_n] = 0 \quad \Big| \quad \begin{array}{l} \delta_1 v = \nabla e^u \\ \delta_1 u = \nabla v \end{array}$$

$$\| $$

$$\delta_n v + e^u \delta_n u . \Lambda^{-1}$$

$$\nabla^2 (\delta_1^2 F) = \delta_1^2 u = \nabla^2 e^u$$

<u>Toda equation</u> $\boxed{\partial_Q^2 F = e^u}$

<u>Toda conjecture</u> $\delta_n = n! \, \partial_{n-1, Q}$

<u>Another form of the conjecture</u> (G. - soon to appear, KIAS proceedings, Seoul, August 2000)

Let $\mathbb{D} = v \nabla + h\left(e^{\varepsilon\partial/2} + e^{-\varepsilon\partial/2}\right) \partial_Q$.

$$\boxed{\mathbb{D} \, \partial_{n-1, Q} F = (n+1) \nabla \partial_{n, Q} F}$$

This recursion is very powerful.

Example: $\lim\limits_{q \to 0} \partial_{k,Q} F =$

$$\sum_{g=0}^{\infty} (-1)^g \varepsilon^{2g} \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k_1 \dots k_n} s_{k_1} \dots s_{k_n} \int_{\overline{\mathcal{M}}_{g,n+1}} \psi_1^{k_1} \dots \psi_n^{k_n} \psi_{n+1}^{k} \lambda_g$$

Toda equation is trivial: $\partial_Q^2 F = 0$

$$D\,\partial_{n-1,Q} F = \nabla \partial_Q F \cdot \nabla \partial_{n-1,Q} F$$
$$= (n+1)\, \nabla \partial_{n,Q} F$$

$$\boxed{\nabla \partial_{n,Q} F = \frac{(\nabla \partial_Q F)^{n+1}}{(n+1)!}}$$

Evaluate ~~at origin~~ on line $\{ s_n = 0 \,,\ n > 0 \}$

$$\sum_{g=0}^{\infty} (-1)^g \varepsilon^{2g} \int_{\overline{M}_{g,1}} \psi^{2g-2} \lambda_g = \frac{\varepsilon/2}{\sinh(\varepsilon/2)}$$

(result of Faber, Pandharipande)

What about the flows $\partial_{n,P}$?

Two approaches : 1) Virasoro (Eguchi, Hori, Yang ; G)

2) Equivariant Toda (G, Pandharipande)

$$x_n = \mathbb{D}\, \partial_{n,P} F - (n+1) \nabla \partial_{n+1,P} F - 2 \nabla \partial_{n,Q} F$$

$$y_n = \mathbb{D}\, \partial_{n,Q} F - (n+2) \nabla \partial_{n+1,Q} F$$

Let Z_n = n^{th} Virasoro constraint

$$= e^{-\varepsilon^{-2} F} L_n e^{\varepsilon^2 F}$$

<u>Formula</u> $\hat{s}_m = \begin{cases} s_m & , m \neq 1 \\ s_1 - 1 & , m = 1 \end{cases}$

$$\mathbb{D}\, Z_{n+1} - \nabla Z_{n+1}$$

$$= \sum_{m=0}^{\infty} \Big((m+1)\dots(m+k+1) t_m y_{m+k} + m(m+1)\dots(m+k)\, \hat{s}_m x_{m+k}$$
$$+ 2m(n+1)\dots(m+k) \sum_{i=0}^{k} \frac{1}{m+i}\, \tilde{s}_m y_{m+k-1} \Big)$$
$$+ \sum_{i+j=k} i!\, j! \left(\varepsilon^2 \partial_{i-1,Q} y_{j-1} + \left(\left(e^{\varepsilon\partial/2} + e^{-\varepsilon\partial/2} \right) \partial_{i-1,Q} F \right) y_{j-1} \right)$$

<u>Corollary</u> Toda + Virasoro $\Rightarrow$ recursion

$$\mathbb{D}\, \partial_{n-1,P} F = n \nabla \partial_{n,P} F + 2 \nabla \partial_{n-1,Q} F$$

Taking $\lim_{q \to 0}$, we get a nice formula for

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{k_1} \dots \psi_n^{k_n} \lambda_{g-1}$$

In particular (Faber, Pandharipande) $\sum (-1)^g \varepsilon^{2g} \int_{\overline{\mathcal{M}}_{g,1}} \psi_1^{2g-1} \lambda_{g-1} = \log\left(\frac{\varepsilon/2}{\sinh(\varepsilon/2)} \right)$

Equivariant Toda

Genus 0 :

$$\partial_Q \left(\partial_Q - \lambda \, \partial_P \right) F = e^{u}$$

δ_1 $\bar{\delta}_1$

Toda Field Theory

$$L = p + \sum_{n=0}^{\infty} u_n \, p^{-n}$$

$$\bar{L} = \sum_{n=1}^{\infty} v_n \, p^{n}$$

Commuting flows ($\approx$ 2 component KP)

$$\delta_n L = \{ L_+^n , L \} \qquad \bar{\delta}_n L = - \{ \bar{L}_-^{-n} , L \}$$

$$\delta_n \bar{L} = \{ L_+^n , \bar{L} \} \qquad \bar{\delta}_n \bar{L} = - \{ \bar{L}_-^{-n} , L \}$$

$$[\partial, \delta_n] = 0 \qquad [\partial, \bar{\delta}_n] = 0$$

But

$$\{ L_+ - \bar{L}_-^{-1} , L \} = \lambda \, \partial L$$

$$\{ L_+ - \bar{L}_-^{-1} , \bar{L} \} = \lambda \, \partial \bar{L}$$

Hypothesis : $u_0 = v \qquad v_1 = e^{-u}$

$$\Leftrightarrow \quad L_+ - \bar{L}_-^{-1} = p + v + e^{u} p^{-1} .$$

Idea : Basis of $H^*_{S^1}(\mathbb{C}P^1)$ associated to two fixed points is better than homogeneous basis of $H^*(\mathbb{C}P^1)$

Promote to higher genus:

Let $L(0) = \Lambda + v + e^{u}\Lambda^{-1}$

$$L(\lambda) = L(0) + O(\lambda)$$
$$\bar{L}(\lambda) = L(0)^{-1} + O(\lambda)$$

$$[\varepsilon^{-1}L(0), L(\lambda)] = \lambda\,\partial L(\lambda)$$
$$[\varepsilon^{-1}L(0), \bar{L}(\lambda)] = \lambda\,\partial \bar{L}(\lambda)$$

Unique solution – determines $\partial_{n,Q}$, $\partial_{n,P}$ by

$$\delta_n = n!\,\partial_{n-1,Q}$$
$$\bar{\delta}_n = n!\,(\partial_{n-1,Q} - \lambda\,\partial_{n-1,P})$$

Puzzle: recover Virasoro in limit $\lambda \to 0$.

Very powerful – determines

$$\sum_{g=0}^{\infty} (-1)^g \varepsilon^{2g} \int_{\bar{\mathcal{M}}_{g,1}} \frac{1 + \lambda_1 + \lambda_2 + \cdots + \lambda_g}{1 - t\psi}$$

$$= t^{-2}\frac{\varepsilon/2}{\sinh(\varepsilon/2)} + t^{-1}\log\left(\frac{\varepsilon/2}{\sinh(\varepsilon/2)}\right) + \cdots$$

Matrix model?