Higgs bundles and D-branes - 2

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Higgs bundles (appropriately generalized) are also useful for describing D-branes in type IIA backgrounds. Given a smooth Calabi-Yau 3-fold X and a smooth slag submanifold $M \subset X$, one can argue that a SUSY configuration of n $D6$ branes wrapping M is described classically by a pair (V, A) consisting of a complex rank n vector bundle $V \to M$ and a flat connection A on V.

Comments:

• When $n = 1$ this translates into the usual statement that a single $D6$ brane wrapping M is described by a complex field

$$A = a + \sqrt{-1}\varphi,$$

where

a is a $U(1)$ flat connection on M;
φ is a harmonic 1-form on M.

• The derivation of the above statement is based on the assumption that when quantum corrections are suppressed the moduli space of A-branes can be described to first order near any point as the moduli of A-branes on the symplectic linearization of X near M, i.e. on $\mathrm{tot}(T^{\vee}M)$ taken with its standard symplectic structure. In this setup, the data describing n $D6$ branes wrapping M is interpretted as the result of a dimensional reduction of a hermitian Yang-Mills instanton on $\mathrm{tot}(T^{\vee}M)$ along the leaves of the foliation $T^{\vee}M \to M$. The resulting object is a triple (V, a, φ), where:

• V is a complex vector bundle of rank n;

• a is a connection on V preserving some hermitian metric h;

• $\varphi \in \Gamma(M, \mathrm{ad}(V) \otimes A^1_M)$, satisfying $\varphi^* = -\varphi$ where φ^* is the adjoint of φ w.r.t. h;

Furthermore (V, a, φ) must satisfy two systems of PDE:

- (F-flatness condition)

$$F_a = -\varphi \wedge \varphi, \qquad D_a \varphi = 0.$$

- (D-flatness condition)

$$D_a * \varphi = 0.$$

Given (a, φ) satisfying the F-flatness condition, it is clear that the combination $A = a + \sqrt{-1}\varphi$ is a complex flat connection on V.

Problem: How can one ensure the D-flatness condition starting simply with a complex flat connection?

Note: If we start with a hermitian flat connection A, then

$$A = a, \qquad \varphi = 0$$

and the D-flatness holds automatically. In this way we recover the standard description of n-tuples of A branes as pairs (M, A) consisting of a slag submanifold and a unitary flat connection A on a rank n complex vector bundle on M.

More generally, if A is any complex flat connection on a vector bundle V and if h is a hermitian metric on V, then A decomposes canonically as $A = a + \sqrt{-1}\varphi$, where a is a connection on V which preserves h. Now:

F-flatness of $(a, \varphi) \iff$ flatness of A.

D-flatness for $(a, \varphi) \iff$ a constraint on the metric h.

Metrics h satisfying the D-flatness constraint on a flat complex bundle (V, A) are called *harmonic metrics* have been studied before by Corlette and Donaldson.

Theorem [K.Corlette] *Suppose M is a compact Riemannian manifold and let V is a complex rank n vector bundle on M equipped with a flat connection A. Then:*

- *If A has reductive monodromy, then a harmonic metric on (V, A) exists;*
- *If A has irreducible monodromy, then the harmonic metric h on (V, A) is unique. Furthermore the h-preserving piece a of A is also flat.*

Thus, for an irreducible (V, A) we get a unique decomposition $A = a + \sqrt{-1}\varphi$ so that

$$\begin{aligned} a \text{ is hermitian }, &\qquad \varphi^* = -\varphi \\ F_a = 0, &\qquad \varphi \wedge \varphi = 0 \\ D_a\varphi = 0, &\qquad D_a * \varphi = 0. \end{aligned}$$

In other words - the data (V, a, φ) describing n type A branes wrapping the manifold M should be thought of as the analogue of a Higgs bundle in Riemannian geometry.

This analogy can be made even more geometric and leads to some interesting insights on the Strominger-Yau-Zaslow picture of mirror symmetry.

Questions

- *Is there a quantum deformation of the harmonicity equation on the metric h?*
- *Is there a deformation of the notion of a complex local system that captures the quantum corrected configurations of n branes wrapping M?*

Example: Let $T = \mathbb{R}^3/\mathbb{Z}^3$ be a three dimensional real torus taken with the flat metric. Let x_1, x_2, x_3 be the coordinates on $\mathbb{R}^3$ and let

$$\begin{aligned}\mathbb{D} &= \boldsymbol{\mu}_2 \ltimes \boldsymbol{\mu}_4 \\ &= \left\langle \alpha, \beta \middle| \alpha^2 = 1, \beta^4 = 1, \alpha\beta\alpha = \beta^3 \right\rangle,\end{aligned}$$

be the dihedral group of order 8. The group $\mathbb{D}$ acts freely on T by the affine transformations:

$$\alpha(\boldsymbol{x}) = \left(-x_1,\ -x_2 + \frac{3}{4},\ x_3 + \frac{1}{2}\right)$$

$$\beta(\boldsymbol{x}) = \left(x_1 + \frac{1}{4},\ -x_2 + \frac{1}{4},\ -x_3\right),$$

and so $M := T/\mathbb{D}$ is a flat compact three manifold.

Consider:

- $X := T^*M$ with its natural flat structure - a non-compact Calabi-Yau threefold;
- $M \subset X$ as the zero section - a compact special Lagrangian submanifold.

Let Y be the complex Calabi-Yau which is a mirror of X. Mirror symmetry predicts the existence of an isomorphism of moduli spaces:

$$\left\{\begin{matrix}\text{The moduli space} \\ \text{of flat } GL(n,\mathbb{C}) \\ \text{connections on } M\end{matrix}\right\} \cong \left\{\begin{matrix}\text{The Hilbert scheme} \\ \text{of length } n \text{ zero} \\ \text{dimensional sub-} \\ \text{schemes of } Y\end{matrix}\right\}$$

We can test this prediction by independently computing both sides.

<u>Moduli of flat connections:</u>

The moduli space is

$$\mathrm{Loc}_n(M) = \mathrm{Hom}(\pi_1(M), GL(n,\mathbb{C}))^{ss}/GL(n,\mathbb{C}).$$

We also have the $SL(n,\mathbb{C})$ version $\mathrm{Loc}_n^o(M)$ of the moduli space.

Note that

$$\begin{aligned} Z(\mathbb{D}) \quad &= \quad \langle \beta^2 \rangle \quad \cong \mathbb{Z}/2, \\ \mathbb{K} \quad &\stackrel{\text{def}}{=} \quad \mathbb{D}/Z(\mathbb{D}) \quad \cong \mathbb{Z}/2 \times \mathbb{Z}/2, \end{aligned}$$

and that β^2 acts on T as a translation by a 2-torsion point.

Since

$$M = T/\mathbb{D} = (T/Z(\mathbb{D}))/\mathbb{K},$$

it follows that

$$M = \mathbb{T}/\mathbb{K}$$

where $\mathbb{T}$ is the 3 dimensional torus $\mathbb{T} := T/Z(\mathbb{D})$.

Thus

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_1(\mathbb{T}) & \longrightarrow & \pi_1(M) & \longrightarrow & \mathbb{K} & \longrightarrow & 0. \\ & & \| & & & & \| & & \\ & & \mathbb{Z}^3 & & & & \mathbb{Z}/2 \times \mathbb{Z}/2 & & \end{array}$$

Explicitly $\pi_1(M)$ is generated by elements $\mathfrak{a}$, $\mathfrak{b}$ subject to the relations

$$\begin{array}{rclcrcl} \mathfrak{a}a\mathfrak{a}^{-1} & = & a^{-1} & \qquad & \mathfrak{b}a\mathfrak{b}^{-1} & = & a^{-1} \\ \mathfrak{a}b\mathfrak{a}^{-1} & = & b^{-1} & & \mathfrak{b}b\mathfrak{b}^{-1} & = & b \\ \mathfrak{a}c\mathfrak{a}^{-1} & = & c & & \mathfrak{b}c\mathfrak{b}^{-1} & = & c^{-1} \end{array}$$

where $a := (\mathfrak{a}\mathfrak{b})^2, b = \mathfrak{b}^2, c = \mathfrak{a}^2$, and a, b, c commute. Note that in this setup $\pi_1(\mathbb{T})$ is freely generated by a, b and c.

Using this presentation of $\pi_1(M)$ one can easily compute $\mathrm{Loc}_n(M)$ and $\mathrm{Loc}^o(M)$. For instance:

$\mathrm{Loc}_1(M) =$ 16 points;

$\mathrm{Loc}_2^o(M) =$ 4 connected components: 3 isolated points and a trident comprising three copies of $\mathbb{C}$ touching at the origin.

Hilbert scheme of points on Y:

The difficulty here lies in identifying the correct mirror for X. One possibility is to note that as a complex manifold

$$X = (\mathbb{C}^\times \times \mathbb{C}^\times \times \mathbb{C}^\times)/\mathbb{D},$$

and try to construct the mirror as the quotient of the mirror of $(\mathbb{C}^\times)^3$ by a suitably defined dual action of $\mathbb{D}$.

This can be carried out but gives an incorrect answer for the mirror. To analyze what is going on we can look at this procedure from the Strominger-Yau-Zaslow point of view.

Recall that

$$\begin{aligned} X &= \mathrm{tot}(T^\vee T)/\mathbb{D} \\ &= \mathrm{tot}(T^\vee \mathbb{T})/\mathbb{K} \\ &= \mathrm{tot}(T^\vee M). \end{aligned}$$

The natural projection

$$\mathrm{tot}(T^\vee M) \to T_0^\vee T = T_0^\vee \mathbb{T},$$

descends to a slag torus fibration

$$\pi : X \to B := (T_0^\vee T)/\mathbb{D},$$

where $T_0^\vee T = \mathbb{R}(dx_1)_0 \oplus \mathbb{R}(dx_2)_0 \oplus \mathbb{R}(dx_2)_0$ and α, β act as

$$\alpha = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

In particular B is singular with a singularity locus shaped again like a trident: it is the image of the three coordinate axes in $\mathbb{R}^3 = T_0^\vee T$ corresponding to the basis $(dx_1)_0, (dx_2)_0, (dx_3)_0$.

The action of $\mathbb{D}$ on $T_0^\vee T = T_0^\vee \mathbb{T}$ factors through $\mathbb{K}$ and the three rays in the trident are labeled by the stabilizers of the coordinate axes in $\mathbb{K}$. These are precisely the three subgroups of index two in $\mathbb{K}$.

Again the main difficulty in undersanding the mirror of X lies in understanding what T-duality does looks like at the singular fibers of π.

Note: The fibers of π naturally fall into classes:

$\Diamond$ If $b \in B$ is a smooth point, then the fiber X_b of π is isomorphic to the torus $\mathbb{T}$;

$\Diamond$ If $b \in B$ is a point on the ray of the trident labeled by a cyclic subgroup $A \subset \mathbb{K}$, then the fiber X_b is isomorphic to the quotient $\mathbb{T}/A$. This fiber occurs in the fibration with multiplicity 2;

$\Diamond$ If $b \in B$ is the vertex of the trident, then the fiber X_b is isomorphic to $\mathbb{T}/\mathbb{K} = M$ and occurs in the fibration with multiplicity 4.

Naively we can try to construct the mirror of X in two steps:

- Dualize the smooth slag torus fibration

$$\mathrm{tot}(T^{\vee}T) \to T_0^{\vee}T$$

  on the covering Calabi-Yau $T^{\vee}T$.

- Divide the resulting Calabi-Yau by the dual action of the group $\mathbb{D}$.

We have $\mathrm{tot}(T^{\vee}T) = T \times T_0^{\vee}T$ and so the T-dual of $\pi : \mathrm{tot}(T^{\vee}T) \to T_0^{\vee}T$ is

$$\widehat{T} \times T_0^{\vee}T,$$

where

$$\widehat{T} = \text{ moduli of } U(1) \text{ local systems on } T.$$

T-dual action of $\mathbb{D}$: old action on $T_0^{\vee}T$ and the pullback action on $U(1)$-local systems on T.

Therefore the action of $\mathbb{D}$ on $\widehat{T} \times T_0^\vee T$ factors trough $\mathbb{K}$, i.e. $\mathbb{D}$ acts non-effectively.

This suggests that as a mirror of X we should take the stack quotient

$$[(\widehat{T} \times T_0^\vee T)/\mathbb{D}].$$

But as an algebraic manifold

$$\widehat{T} \times T_0^\vee T \cong (\mathbb{C}^\times)^3$$

with $\mathbb{D}$ acting by

$$\begin{aligned}\alpha(\lambda_1, \lambda_2, \lambda_3) &= (\lambda_1^{-1}, \lambda_2^{-1}, \lambda_3)\\ \beta(\lambda_1, \lambda_2, \lambda_3) &= (\lambda_1^{-1}, \lambda_2, \lambda_3^{-1}).\end{aligned}$$

Hence $[(\mathbb{C}^\times)^3/\mathbb{D}]$ is a $\mathbb{Z}/2$-gerbe over the smooth orbifold $[(\mathbb{C}^\times)^3/\mathbb{K}]$.

If this was the correct mirror one expects to have an isomorphism

$$\mathsf{Loc}_n(M) \cong \mathsf{Hilb}_n([(\mathbb{C}^\times)^3/\mathbb{D}]).$$

We have verified the correctness of this satement for $n = 1$ and we are working on verifying it for higher n's.

Remark: Taking $[(\mathbb{C}^\times)^3/\mathbb{D}]$ as the answer for the mirror of X requires interpreting the string orbifolds as string compactifications on stacks. A detailed analysis of such an interpretation has been carried out recently in a series of papers by Eric Sharpe.

One can try to stay within the conventional framework of string orbifolds and only work with gerbes on spaces. For this one needs to use the McKay correspondence and replace the CY stack $[(\mathbb{C}^\times)^3/\mathbb{D}]$ with a gerbe over a crepant resolution of the singular space $(\mathbb{C}^\times)^3/\mathbb{K}$.

Caution: Since we are interested in moduli spaces (of physical D-branes) rather than categories (of topological D-branes), it matters which resolution we take*.

Natural choice: Following Bridgeland-King-Reid we may take the $\mathbb{K}$-Hilbert scheme

$$\mathbb{K} - \mathsf{Hilb}((\mathbb{C}^\times)^3)$$

as our crepant resolution.

However, it turns out that one does not get the right moduli spaces of B-branes from this crepant resolution - there is a mismatch already with $\mathsf{Loc}_1(M)$.

*Alternatively one may work with any resolution but choose an appropriate stability notion in the sense of Douglas-Bridgeland.

Two ways out:

- Find the correct crepant resolution of $(\mathbb{C}^\times)^3$, or

- Work with coherent perverse sheaves on $\mathbb{K}-\mathsf{Hilb}((\mathbb{C}^\times)^3)$.

We have a proposal for a different crepant resolution, which is built as a moduli space of objects on X itself.

Recall: away from the orbifold points the mirror CY stack $[(\mathbb{C}^{\times})^3/\mathbb{D}]$ parameterizes A-branes on X, namely flat $U(1)$ bundles $\mathcal{L}$ on the smooth fibers X_b of the slag fibration π.

The idea is to view these as deformations of the flat $U(1)$ connections on the most singular fiber of π - the multiplicity 4 fiber sitting at the vertex of the trident.

To get a rigorous description of such connections recall that $X = \text{tot}(T^{\vee}M)$. If $b \in B$ is a smooth point, then X_b is a copy of the torus $\mathbb{T}$ embeded in X, so that the natural projection map to M:

$$\begin{tikzcd} X_b \arrow[r, hook] \arrow[dr, "p_b"'] & X \arrow[d, "p"] \\ & M \end{tikzcd}$$

becomes an unramified cover with Galois group $\mathbb{K}$. In other words any pair $(X_b, \mathcal{L})$ can be viewed as a flat counterpart of the spectral data that we described in the holomorphic situation.

Since we can push-forward local systems by smooth maps, we can mimick the spectral correspondence from the holomorphic setup and construct its flat version here.

This allows us to descrie each pair $(X_b, \mathcal{L})$ in terms of the equivalent data $(\mathcal{V}, \phi)$ where

- $\mathcal{V} = (\mathbb{V}, \nabla)$ is a complex unitary local system of rank 4 on M;

- $\phi \in \Gamma(M, \mathrm{End}(\mathbb{V} \otimes A^1_M))$ is a horizontal section satisfying $\phi \wedge \phi = 0$, $\phi^* = -\phi$ and $\mathrm{tr}(\phi^2) = 0$.

Here the horizontality of ϕ is taken with respect to the natural flat connection on $\mathrm{End}(\mathbb{V} \otimes A^1_M)$, which is induced from the unitary connection on $\mathbb{V}$ and the flat metric connection on M.

Note: This is related to our description of A-branes wrapping M as complex flat connection:

• One checks that the horizontality of ϕ is equivalent to the harmonicity of ϕ w.r.t. ∇. Thus a pair $(\mathcal{V}, \phi)$ as above but without the $\mathrm{tr}(\phi^2) = 0$ condition corresponds precisely to a complex rank 4 flat connection on M.

• The condition $\mathrm{tr}(\phi^2) = 0$ turns out to be equivalent to the fact that the complex flat connection preserves a non-degenerate quadratic form on $\mathbb{V}$.

Conclusion: The flat spectral construction on $T^\vee M$ identifies the moduli space Y of $U(1)$ flat connections on smooth fibers of $\pi : X \to B$ with a Zariski open set of a component of the moduli space of $SO(4)$-configurations of A-branes wrapping $M \subset X$.

As we saw this moduli space is most naturally interpreted as the moduli of $SO(4, \mathbb{C})$ flat Higgs bundles on M. It comes naturally equipped with a Hitchin map

$$h : Y \to \mathbb{R}^4$$

given by the invariant polynomials on ϕ.

It is straightforward to check that the image of h is an algebraic hypersurface isomorphic to B and that the smooth fibers of h are canonically isomorphic to the dual torus $\widehat{T}$. Thus Y can be thought of as a resolution of $(\mathbb{C}^{\times})^3/\mathbb{K}$ and is a good prospect for the mirror of X.

In fact, there is a natural $\mathbb{Z}/2$ gerbe on Y and we have checked that its length one and two Hilbert schemes agree with $\mathrm{Loc}_1(M)$ and $\mathrm{Loc}_2(M)$ as predicted by mirror symmetry.