Nonlocal Continuum Modeling of Granular Flow

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Main Goal

Q: What’s the constitutive law for well-developed granular flow?
Inertial Flow Rheology: Simple Shear

• Inertial number: \[ I = \dot{\gamma} d \sqrt{\frac{\rho_s}{P}} \]
  \( d \) = mean particle diameter
  \( \rho_s \) = the density of a grain

• Inertial rheology: \( \mu = \mu(I) \)
  [Da Cruz et al 2005, Jop et al 2006, related to “Bagnold Scaling” (1954)]

Fit to a straight line:
\[ \mu(I) = \mu_s + bI \]
Dense granular flows are cooperative. Not captured by local rheology.

Grain-size-dependent flow fields
Split-bottom cell

Thin layers - $h_{stop}$ effect
Midi (2004)

Secondary Rheology
van Hecke (2010)

Drainage flows
Grain size affects steady flow profiles

Annular Shear:

Koval et al., PRE (2009)

Fill with grains of diameter $d$

$d$ and $P_{out}$ held fixed

- $\mu$ vs. $I$ is not one-to-one
- Flow occurs below $\mu_s$
Physical picture of cooperativity

Basic idea: *Flow induces flow.* Plastic dynamics are spatially cooperative.

*Kinetic Elasto-Plastic* (KEP) model
Bocquet et al., *PRL* (2009)

- **Elastic deformation**
- **Plastic event**
- **Stress redistribution**

Proposed for emulsions
Nonlocal rheology background

The form of the nonlocal rheology is derived from a *Landau-like PDE* for the fluidity (Related idea: “partial fluidization” Aranson and Tsimring, *PRE* (2001)):

**Order parameter:**

\[ g = \frac{\dot{\gamma}}{\mu} = \text{Relative susceptibility to flow} \]

\[ \text{Rearrangement rate} = \text{Indicator of the bulk stress state} \]

**Dynamical PDE:**

\[ \frac{\partial g}{\partial \tilde{t}} = A^2 d^2 \nabla^2 g - \left[ (\mu_s - \mu)g + b \sqrt{\frac{\rho_s d^2}{P}} \mu g^2 \right] \]

(Akin to a “coarse-grain energy derivative”)

(Henann and Kamrin, *IJP* (2014))

Reduce to *steady-state-only* model:

Expanding around steady, stable, homogeneous flow solutions

\[ g = g_{1oc}(\mu, P) + \xi(\mu)^2 \nabla^2 g \]

Local rheology (standard inertial flow relation):

\[ g_{1oc}(\mu, P) = \frac{\dot{\gamma}_{1oc}}{\mu} = \begin{cases} \sqrt{P/\rho_s d^2}(\mu - \mu_s)/b\mu & \text{if } \mu > \mu_s \\ 0 & \text{if } \mu \leq \mu_s \end{cases} \]

Cooperativity length:

\[ \xi(\mu) = \frac{A}{\sqrt{|\mu - \mu_s|}} d \]
Cooperativity Length

Theoretical form for cooperativity length:

$$\xi(\mu) = \frac{A}{\sqrt{|\mu - \mu_s|}} d$$

[Similar to Bocquet PRL 2009]

Extract $\xi$ directly from DEM tests using our proposed fluidity PDE:

$$g = g_{loc}(\mu, P) + \xi(\mu)^2 \nabla^2 g$$

For our 2D DEM disks, we find: $A=0.70$

Only new material constant is $A$, the nonlocal amplitude. Local law constants all carry over.
The Diverging Length-Scale

M. van Hecke, Cond. Mat. (2009)


Pre-avalanche zone sizes for inclined plane flow:

\[ \theta = 5^\circ \]

\[ \theta = 15^\circ \]
Model for steady dry flow

\[ \mathbf{\sigma} = -P \mathbf{1} + 2 \frac{P}{g} \mathbf{D} \] stress constitutive relation

\[ \text{tr}(\mathbf{D}) = 0 \] constant-volume constraint

\[ \xi^2 \nabla^2 g = g - g_{\text{loc}} \] nonlocal rheology

Local rheology (standard inertial flow relation):

\[ g_{\text{loc}} = \begin{cases} \sqrt{P/\rho_s d^2} (\mu - \mu_s)/b \mu & \text{if } \mu > \mu_s \\ 0 & \text{if } \mu \leq \mu_s \end{cases} \]

Cooperativity length:

\[ \xi = \frac{A}{\sqrt{|\mu - \mu_s|}} d \]

2 local material parameters: \( \{\mu_s, b\} \)

2 grain parameters: \( \{d, \rho_s\} \)

1 new nonlocal amplitude: \( \{A\} \)

Already known for many materials

Our calibration: Glass beads \( A=0.48 \)
DEM disks \( A=0.70 \)
2D Results: Validating the Nonlocal Model

Check predictions against DEM data from Koval et al. (PRE 2009) in the annular couette cell.

\[
P(r) \approx P_{out}
\]

\[
\mu(r) = \mu_{wall}(R/r)^2
\]

DEM (symbols)
Nonlocal Rheology (lines)
Local Rheology (---)

Kamrin & Koval PRL (2012)
2D Results: Same model in different geometries. Velocity profiles (theory compared to DEM disks)

Kamrin and Koval, PRL 2012

{Lines: Model predictions  Symbols: DEM results}
Comparison to 3D Experiments: Flow of beads in the split-bottom Couette cell

No previous continuum model has correctly predicted the flows in this geometry

Schematic:

Filled with grains:

Flow in the split-bottom Couette cell – surface flow

The normalized revolution-rate: \( \omega = \left( \frac{v_\theta}{r} \right) / \Omega \)

Viewed from the top down:

Flow in the split-bottom Couette cell – surface flow

Flow fields on the top surface for five values of $H$:

Universal flow profile:

$(d = 0.35 \text{ mm}, R_s = 85 \text{ mm})$

Nonlocal model in the split-bottom Couette cell

- Custom-wrote an FEM subroutine to simulate the nonlocal model via Abaqus User-Element (UEL).
- Performed many sims varying $H$ and $d$.

Simulated flow field in the $r-z$ plane:

$H = 10 \text{ mm}$

$H = 30 \text{ mm}$


Flow field on top surface

Henann and Kamrin, *PNAS* 2013
Nonlocal model in the split-bottom Couette cell

Define:
- $R_c$ = Location of shear-band center
- $W$ = Width of shear-band
- $\lambda = \frac{(r-R_c)}{W}$ = Normalized position

Surface flow results for all 22 combinations of $H$ and $d$ tried:

Collapse to error-function flow field

Non-diffusive scaling of $W$ with $H$
Remove the inner wall and let $H$ increase

The nonlocal model correctly captures the transition from shallow to deep behavior.

Henann and Kamrin, *PNAS* 2013
Existing 3D wall-shear data using glass beads

Annular shear flow:

Linear shear flow with gravity:

Same exact model and parameters used in split-bottom cases also captures these flows.

Henann and Kamrin, PNAS 2013
More on the cooperativity length

Suppose we try different power laws in the cooperativity length formula:

\[ \xi(\mu) = \frac{A}{(|\mu - \mu_s|)^\alpha d} \]
Secondary Rheology

Experiment:

Reddy et. al PRL, 2009

- When inner wall is stationary, rod has a yield force:
  \[ v_{\text{creep}} = 0 \quad \text{unless} \quad F > F_c \]

- When wall is moving, rod creeps for any \( F \). Given \( r/d \),
  \[ v_{\text{creep}} \approx C_1 v_{\text{wall}} \sinh(C_2 F) \]

Nonlocal model (Simplified 2D geometry):

Stainless inner wall

\[
\frac{V_{\text{creep}}}{\Omega R_i} = 0.0069
\]

\[
\frac{F}{F_c} = 0.75
\]
**$H_{\text{stop}}$ Phenomenon:**
A different kind of particle-size dependence

- Another famous granular size-effect is the “extra strengthening” granular layers gain when they are thin. Effect is easy to see in the inclined chute geometry.

- Produces so-called “$H_{\text{stop}}$ curve”.

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**Experiment**
(glass beads, rough floor):


\[ \mu = \tan \theta \]

- Pouliquen, Phys Fluids, 1999

\[ H_{\text{stop}}(\theta) \]

\[ \frac{H}{d} \]

\[ \theta \]
\( H_{stop} \) phenomenon.

Stability of the no-flow solution

Let the fluidity be a small perturbation from the global \( g=0 \) solution and check stability of global no-flow solution.

Theory

Experiment, glass beads (Pouliquen 1999)

Note: Same continuum parameters are used here that were used for split-bottom flow field predictions.

Concluding remarks

- The nonlocal fluidity model is capable of describing multiple incarnations of nonlocal phenomenology in granular media.

- Model has captured complex phenomena untenable to local rheological models, such as:
  1. Grain-size dependent shear features in steady flows
  2. Secondary rheology
  3. “Smaller-is-stronger” $H_{\text{stop}}$ size-effect

- Work in progress: Predicting silo flow rates? More conclusive theoretical understanding?

Relevant papers:

**Flow and flow-stoppage in drainage geometries**

Big open question:
Can you predict how fast an hourglass will flow?

“Beverloo Correlation” (in quasi-2D):

\[ -kd^{3/2} \]

- vs hole size in Silo geometry
- Continuum simulation (Material Point Method) of local law:
- [c/o Sachith Dunatunga]
New tests with DEM disks:

Dependence of continuum model variables on the surface friction of the grains, $\mu_{\text{surf}}$
Local law dominates when particle surface friction vanishes?

\[ \mu_{surf} = 0.8 \]

\[ \mu_{surf} = 0.1 \]

\[ \mu_{surf} = 0 \]
The Diverging Length-Scale

M. van Hecke, Cond. Mat. (2009)


Pre-avalanche zone sizes for inclined plane flow:

\[ \theta = 5^\circ \]

\[ \theta = 15^\circ \]
Cooperativity Length

Theoretical form:

\[ \xi(\mu) = \frac{A}{\sqrt{|\mu - \mu_s|}}d \]

Direct tests: Steady-flow DEM data in 3 geometries (annular shear, vertical chute, shear w/ gravity):

For our 2D DEM disks, we find:  \( A=0.70 \)

Only new material constant is \( A \), the nonlocal amplitude. Local law constants all carry over.
Developed packing fraction on 5d scale

\[ \phi \cong \phi(I) \]

Rycroft, Kamrin, and Bazant (JMPS 2009)
Stage 2: Fixing the problem. Accounting for size-effects

Past nonlocal granular flow approaches:

- Self-activated process (Pouliquen & Forterre 2009)
- Cosserat continuum (Mohan et.al 2002)
- Partial fluidization (Aronson & Tsimring 2001)
Flow not spreading enough

Local model predictions are:
- Qualitatively ok:
  - Shear band at inner wall
  - Flow roughly invariant in the vertical direction
- Quantitatively flawed:
  - No sharp flow/no flow interface.
  - Wall speed slows to zero, local law says shear band width goes to zero.

Velocity field:

Stage 1: Local modeling.

Static zones and flowing zones.

Samadani & Kudrolli (2002) (Experiment)

Kamrin, Rycroft, & Bazant (2007) (Discrete Element Method [DEM])
Cauchy stress field

Eigendirections of stress:

Theory

DEM simulation
Rycroft, Kamrin, and Bazant (JMPS 2009)
Constitutive Process

Conservation of linear and angular momentum require

\[ \nabla \cdot \mathbf{T} + \rho \mathbf{g} = \rho \frac{D\mathbf{v}}{Dt} \]

\[ \mathbf{T} = \mathbf{T}^T \]

To close the system:

(1D picture)

(3D version, Kroner-Lee decomposition)

Spring: Jiang-Liu granular elasticity law [Jiang and Liu, PRL, 2003]
Results
FEM Simulations

- Simulations performed using ABAQUS/Explicit finite element package.
- Constitutive model applied as a user material subroutine (VUMAT).
- All flow parameters in the model are from the papers of Jop et. al. (2006) and Jiang and Liu (2003).
Continuing on

• **More 3D tests:** Need to check the relation in more geometries with less symmetry; e.g. silo/hopper flow. This will help determine if there are any non-negligible anisotropic size-effects.

• **Transient strengthening/dilation:** With well-developed flow response quantified, should include a model for the transient behavior. Has a long history in the soils literature (i.e. Critical State Theory).

• **Improved understanding of fluidity BC’s:** Sims have used transparent fluidity BC’s, but better theory is needed to inform BC assignment.

• **Dynamic nonlocality:** So far have studied well-developed flows. There are known nonlocal dynamic effects in thin layers, such as thickness-dependent startup angle for inclined chutes.

• **More concrete theoretical understanding:** Statistical arguments have been proposed, but a rigorous continuum mechanical derivation is lacking. Possibly an order-parameter in the free-energy function?
Granular Continuum?

Some evidence for continuum treatment:
- Space-averaged fields vary smoothly when coarse-grained at width 5d
- Certain deterministic relationships between fields appear in elements 5d wide
Rough Inclined Chute

Since this model does have elasticity, we always know all stresses, even in regions of 0 plastic flow.
Mathematical Form

\[ T = \frac{R^e M R^e T}{(\det F)} \]

\[ M = \frac{\partial \psi(E^e)}{\partial E^e} \]

\[ D^p = d^p(M) \frac{M_0}{\tau} \]

**Enforce:**
- Frame indifference
- 2nd law of thermodynamics
- Coaxiality of flow and stress

**Definitions:**
\[ R^e \equiv F^e \left(F^e T F^e\right)^{-1/2} \]
\[ M_0 \equiv M - (1/3)(\text{tr}M)1 \]
\[ \tau \equiv \frac{\sqrt{(M_0 : M_0)}/2}{\tau} \]

To close, must choose isotropic scalar functions \( \psi \) and \( d^p \) with \( d^p \geq 0 \).
**Nonlocal Fluidity Model**

**Existing theory for emulsions**

**Define:** Fluidity $\equiv f = \dot{\gamma}/\tau$ (Inverse viscosity)

Local law for flow in large bulk:
Herschel-Bulkley law:

$$\dot{\gamma}_{bulk} = \begin{cases} (\tau - \tau_y)^b A & \text{for } \tau > \tau_y \\ 0 & \text{otherwise} \end{cases}$$

$\rightarrow f_{bulk}(\tau) = \dot{\gamma}_{bulk}/\tau$

Nonlocal add-on:

$$f = f_{bulk}(\tau)$$

0 when $\tau < \tau_y$

Micro-level length-scale proportional to $d$

**Extend to granular media**

**Define:** Fluidity $\equiv f = \dot{\gamma}/\mu$

Local law for flow in large bulk:

Granular flow law:

$$\dot{\gamma}_{bulk} = \begin{cases} \sqrt{P(\mu - \mu_s)} A & \text{for } \mu > \mu_s \\ 0 & \text{otherwise} \end{cases}$$

$\rightarrow f_{bulk}(\mu, P) = \dot{\gamma}_{bulk}/\mu$

Nonlocal law:

$$f = f_{bulk}(\mu, P) + \xi(\mu)^2 \nabla^2 f$$
Quantifying Steady Slow Flow (2D)

- Inertial number: \[ I = \frac{\gamma d \sqrt{\frac{\rho_s}{P}}}{\text{cm}} \]
- Inertial flow rheology says: \[ \mu = f(I) \]

Koval et al. (PRE 2009)

Fill with disks of mean diameter \( d \).

\( d \) and \( P_{\text{out}} \) held fixed

Fix \( V_{\text{wall}} \), Vary \( R/d \)

Fix \( R/d \), Vary \( V_{\text{wall}} \)
Flow and Stresses

Stress ratio ($\mu$):

Dem simulation
Rycroft, Kamrin, and Bazant (JMPS 2009)

Theory

DEM simulation
Rycroft, Kamrin, and Bazant (JMPS 2009)
Flow not spreading enough

Elasto-plastic model predicts:

1. Shear band at inner wall
2. Flow roughly invariant in the vertical direction

Known experimental data verify 1) and 2) but contradict 3).

Velocity field: