Shear modulus caused by stress avalanches for jammed granular materials under oscillatory shear

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Contents

• Introduction for jamming transition and shear modulus
• Simulation
  o Movie, crossover, scaling plots
• How can we understand exponents?
• Discussion and conclusion
Introduction

- Granular materials behaves as unusual solids and liquids.

Flow of mustard seeds @Chicago group  Kamigamo shrine (Kyoto!)
Jamming transition

- Granular materials cannot flow above a critical density.
- Above the critical density, the granules have rigidity and behave as a solid.
- This transition is known as the jamming transition.

Ikeda-Berthier-Sollich (2012)
Liu & Nagel (1998)
Characterization of jamming

- Rigidity of jammed solid is characterized by the shear modulus, \( G = S/\gamma \), where \( S \) is the shear stress.
- (Storage) modulus becomes nonzero above the jamming transition.
G above the jamming

- The shear modulus is believed to behave as

\[ G \sim (\phi - \phi_J)^{1/2} \]

where \( \phi \) and \( \phi_J \) are the volume fraction and the jamming fraction (O’Hern et al. 2002).

- Some people suggested a different scaling

\[ G \sim \gamma_0^{-c} (\phi - \phi_J) \quad c=1/2 \]

We would like to clarify the relationship between two different scalings. For this purpose, we perform simulation of frictionless granular particles under oscillatory shear. See, M. Otsuki & HH, PRE 90, 042202 (2014).
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Simulation setup

- Number of grains 16,000.
- Linear or Hertzian spring model
- Shear strain

$$\gamma(t) = \gamma_0 \{1 - \cos(\omega t)\}$$

$$\frac{dr_i}{dt} = \frac{p_i}{m} + \dot{\gamma}(t)y_i e_x,$$

$$\frac{dp_i}{dt} = \sum_{j \neq i} \{f^{(el)}_{ij} + f^{(dis)}_{ij} \} - \dot{\gamma}(t)p_i y e_x,$$
Simulation movie
Avalanches in simulation

- (a) $\gamma = 0$
- (b) $\gamma = 1.2 \times 10^{-4}$
- (c) $\gamma = 4.8 \times 10^{-4}$
- (d) $\gamma = 7.5 \times 10^{-4}$
Storage modulus

- Storage modulus strongly depends on the amplitude of oscillatory shear.
Scaling ansatz

\[ G(\phi, \gamma_0) = G_0(\phi - \phi_J)^a \mathcal{G} \left( \gamma_0 / (\phi - \phi_J)^b \right) \]

\[ \lim_{x \to 0} G(x) = \text{const.}, \quad \lim_{x \to \infty} G(x) = x^{-c}. \]
Scaling for linear spring

\[ a = 0.50 \pm 0.02, \quad b = 0.98 \pm 0.02. \]
Scaling for Hertzian model

\[ C(\phi - \phi_j) = \frac{C_0}{\gamma_0 (\phi - \phi_j)^b} \]

\[ a = 0.99 \pm 0.02, \quad b = 0.98 \pm 0.01 \]

\[ c = 1/2 \]
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Exponent \( a \)

\[
f \sim k_{\text{eff}} \delta \sim \delta^\Delta \\
\delta \sim \phi - \phi_J \\
G/k_{\text{eff}} \sim \delta z \sim \sqrt{\phi - \phi_J} \\
G \sim (\phi - \phi_J)^{\Delta - 1/2} \quad a = \Delta - 1/2
\]
Stress avalanche and exponent $c$

- Simulation suggests $c = 1/2$

- We may understand this from stress avalanches

$$G = \int_{0}^{\infty} ds \tilde{G}(\gamma_0, s) \rho(s)$$

$
\tilde{G}(\gamma_0, s)$ Shear modulus of an individual element of stress drop $s$

$\rho(s)$ Probability density of stress drop $s$
Stress of individual element

\[ \tilde{S}(s, t) = \begin{cases} 
  G_0 \gamma(t) & (0 \leq \theta(t) < \theta_c) \\
  0 & (\theta_c \leq \theta(t) < \pi) \\
  -G_0 (\gamma(t) - 2\gamma_0) & (\pi \leq \theta(t) < \pi + \theta_c) \\
  0 & (\pi + \theta_c \leq \theta(t) < 2\pi), 
\end{cases} \]

\[ \theta(t) = \omega t. \]

\[ \theta_c \left( \frac{s}{G_0 \gamma_0} \right) = \cos^{-1} \left( 1 - \frac{s}{G_0 \gamma_0} \right). \]
Stress drop distribution

- The stress drop distribution may obey (Dahmen et al, PRE\textbf{58}, 1494 (1998)):

\[
\rho(s) = A(\phi)s^{-3/2}e^{-s/s_c(\phi)}
\]
Shear modulus

• From the combination of two contributions we obtain

\[ G \approx A(\phi)G_0^{1/2} \gamma_0^{-1/2} \int_0^\infty dx \ x^{-3/2} F(x) \]

• Then we reach

\[ c = 1/2 \]

• If size distribution is \( \rho(s) \sim s^{-\tau} \) then \( c = 1 - \tau \).
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Yamaguchi et al., J. Phys. Condens. 21, 205105 (2009)
Perspective

• Some experiments such as previous one suggest that the exponent of stress avalanches $3/2$ is not universal.

• There are various possibilities to obtain the exponent not equal to $3/2$.
  o Renomalization method for elastic interfaces
  o Levy process or trapped diffusion or Bessel process etc

• Currently, we do not know what the physical mechanism for non-trivial exponent is.

• Contribution from frictional force => loss modulus and discontinuous change of stress
Conclusion

- We perform simulations for frictionless grains under oscillator shear.
- We found a crossover from the known exponent for the jamming to the non-trivial behavior.
- Non-trivial exponent can be understood by the mean field theory for stress avalanches.
- See M. Otsuki and HH, *PRE*90, 042202 (2014) for the details.
Derivation of stress drop function

We assume that the mean field theory can be used.

\[ \sigma_i = KVt + Ju - (K + J)u_i, \]

\[ \bar{u} = \frac{1}{N'} \sum_{j=1}^{N'} u_j / N', \]

\[ \sigma = \frac{1}{N'} \sum_{i=1}^{N'} \sigma_i. \]
Derivation (2)

\[ \delta u_i = -\frac{\sigma_y - \sigma_a}{K + J}, \]

\[ s_{\text{self}} = -(\sigma_y - \sigma_a), \]

local yield stress \( \sigma_y \)

‘arrest stress’ \( \sigma_a \)

the stress drop

\[ s = (1 - C')(\sigma_y - \sigma_a)n/N'. \]

\[ C = \frac{J}{J + K}. \]

\[ s_{\text{oth}} = C(\sigma_y - \sigma_a)/N'. \]
Bernoulli's trial

\[ X_n = \sigma_i(n+1) \]

\[ \delta X_n = X_n - X_{n-1}, \]

Poisson distribution

\[ \rho_X(\delta X_n) = \frac{N'}{\sigma_y - \sigma_a} e^{-\frac{N'}{\sigma_y - \sigma_a} \delta X_n} \]

\( i(n) \) is the index of the site that has the \( n \)th largest
Derivation

\[ Z_n = X_n - (\sigma_y - ns_{oth}). \quad \delta Z_n = -\delta X_n + s_{oth} \]

\[ \delta Z_n = Z_n - Z_{n-1} \]

\[ \mu_Z = (2p - 1)\Delta x = -(1 - C) \frac{\sigma_y - \sigma_a}{N'} \]

\[ V_Z = 4\Delta x^2 p(1 - p) = \frac{(\sigma_y - \sigma_a)^2}{N'^2} \]
Solution of Bernoulli trial

• The solution of Bernoulli’s trial is thus given by

\[ \lambda_{2n-1} = 0, \]

\[ \lambda_{2n} = \frac{1}{2p} \binom{1/2}{n} (-1)^{(n+1)} \{4p(1 - p)\}^n. \]

• In the continuum limit, it is reduced to

\[ \lambda_{2n-1} = \frac{1}{4\pi^{1/2} p} \frac{1}{n^{3/2}} e^{-n/n_c} \]

\[ n_c = 1/\log(1 + (C - 1)^2) \]

\[ = -1/\log(4p - 4p^2) \]
Time sequence of stress
Stress-strain relation

\[ \frac{\mathcal{S}(t)}{(k_0 \sigma_0^{-1})} \]

\[ \gamma(t) \]
\[ \tilde{G}(\gamma_0, s) = -\frac{\omega}{\pi} \int_0^{2\pi/\omega} dt \frac{\tilde{S}(s, t) \cos(\omega t)}{\gamma_0}. \] (23)

Substituting Eq. (18) into Eq. (23), we obtain

\[ \tilde{G}(\gamma_0, s) = G_0 F\left( \frac{s}{G_0 \gamma_0} \right), \] (24)

where

\[ F(x) = \begin{cases} 
1 & (x \geq 1), \\
T(x)/\pi & (x < 1),
\end{cases} \] (25)

with

\[ T(x) = \theta_c(x) - 2 \sin \theta_c(x) + \frac{\sin 2\theta_c(x)}{2}. \] (26)

Substituting Eqs. (21) and (24) into Eq. (22), we obtain

\[ G = A(\phi)G_0 \int_{s_0}^{\infty} ds \ s^{-3/2} e^{-s/s_c(\phi)} F\left( \frac{s}{G_0 \gamma_0} \right). \] (27)