Minimal integer automaton behind crystal plasticity

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Joint work with Umut Salman,
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Plastic yielding is associated with states that are only marginally stable.

Heat production at infinitely slow loading.

The area under the stress-strain curve has little to do with the stored energy.
Fluctuations

- Fluctuation statistics carry important information about the ‘health’ of the sample
- Fluctuations prevent precise control of small samples during forming

(a and b) Cleared stress–time series for the less ductile Zr$_{65}$Cu$_{15}$Ni$_{10}$Al$_{10}$ BMG and the more ductile Cu$_{47.5}$Zr$_{47.5}$Al$_{5}$ BMG samples, respectively, for the region indicated in the original stress–strain plot.

R. Sarmah et al. / Acta Materialia 59 (2011)
Plasticity experiment

Dimiduk et al., Phil. Mag., 2010

Plasticity experiment

Zaiser and Hahner, 1999
Discrete dislocation dynamics

M. Zaiser et al., 2011
0D model

\[ f(\varepsilon, \alpha) = \frac{E}{2}(\varepsilon - \alpha)^2 + \sigma_M \alpha - k \delta \cos \left( \frac{\alpha}{\delta} \right) \]

\[ \nu \dot{\alpha} = E(\varepsilon - \alpha) - \sigma_M - k \sin \left( \frac{\alpha}{\delta} \right) \]

\[ v = \dot{\varepsilon} \]
0D model

\[ \mu \dot{\alpha} = -\frac{\partial f}{\partial \alpha}, \quad \dot{\varepsilon} = 1 \]

\[ f(\varepsilon, \alpha) = \frac{E}{2}(\varepsilon - \alpha)^2 + \sigma_M \alpha - k \delta \cos \left( \frac{\alpha}{\delta} \right) \]

\[ \mu = \frac{\nu}{\gamma} \to 0 \]

\[ D = \nu \lim_{\Delta \varepsilon \to 0} \left[ \frac{\hat{f}}{d\varepsilon} \right]_{\Delta \varepsilon} = \nu \lim_{\Delta \varepsilon \to 0} \left[ \frac{\hat{f}}{\Delta \varepsilon} \right] \]

\[ \dot{\gamma} = -\lim_{\Delta \varepsilon \to 0} \left[ \frac{\hat{f}}{\Delta \varepsilon} \right] \]

\[ D = \hat{g} |\hat{\alpha}| \]

\[ \alpha = \hat{\alpha}(\varepsilon), \quad f = \hat{f}(\varepsilon) \]
1D model
1D model

Marginally stable states
1D model (RFIM)
2D model

Relabeling

Without Relabeling
Two representations of a dislocation

One dislocation

Two dislocations

no relabeling

relabeling
A simple discrete model

\[ \nu \dot{u} = -\partial \Phi(u) / \partial u \]

\[ \Phi(u) = \sum_{i,j} \phi(\theta, \xi) \]

\[ \theta(i, k) = u(i + 1, k) - u(i, k) \]

\[ \xi(i, k) = u(i, k + 1) - u(i, k) \]

\[ \phi(\theta, \xi) = g(\xi) + \frac{K}{2} \theta^2 - h_1 \xi - h_2 \theta \]

Loading

\[ \sum_{k=0}^{N-1} \xi(i, k) = t \]

\[ \sum_{i=0}^{N-1} \theta(i, k) = 0 \]
Numerical solution

\[
\hat{u}_{i,j} = K (D_x^- D_x^+ u_{i,j}) + D_y^- \mu (g(D_y^+ u_{i,j})).
\]

\[
c \hat{u}(q) = \left\{ \frac{D_x^- D_x^+ u_{i,j}}{\Delta x^2} \right\}_q = 2 (\cos(q_i) - 1) \hat{u}(q),
\]

\[
s_y^+ \hat{u}(q) = \left\{ \frac{D_y^+ u_{i,j}}{\Delta x} \right\}_q = (\cos(q_j) + I \sin(q_j) - 1) \hat{u}(q)\]

\[
s_y^- \hat{u}(q) = \left\{ \frac{D_y^- u_{i,j}}{\Delta x} \right\}_q = (1 - \cos(q_j) + I \sin(q_j)) \hat{u}(q)\]

\[
\dot{u}(q) = K c \hat{u}(q) + \mu s^- \left\{ g(\{s^+ \hat{u}(q)\}_q)^{-1} \right\}_q
\]

\[
\hat{u}^{t+1}(q) = \frac{\hat{u}^t(q) + \Delta t \mu s^- \left\{ g(\{s^+ \hat{u}^t(q)\}_q)^{-1} \right\}_q}{1 + \Delta t K c(q)}_q,
\]
Dislocation nucleation
Typical configuration of a dislocated body
Stress-strain curves, shakedown

Fast time movie

Slow time movie
Temporal correlations: power law

\[ P(E) \sim E^{-\epsilon} \]

\[ \epsilon = 1.6 \pm 0.05 \]
Temporal correlations

Dissipated energy

Slip area

Avalanche durations

Salman, LT, PRL, 2011
Stress field

Slow time movie
Power spectrum: $1/f$ noise
Spatial correlations

![Graph showing spatial correlations with a log-log plot and a map on the right side.](image-url)
Spatial energy distribution

\[ P(E_S) \sim E_S^{-\zeta} \]
\[ \zeta = 1.45 \pm 0.1 \]
Origin of correlations?
Equivalent automaton model

\[ \nu \dot{u} = -\frac{\partial \Phi(u)}{\partial u} \]

\[ \nu \rightarrow 0 \]

\[ \frac{\partial \Phi}{\partial u} = 0 \]

piece-wise quadratic potential

\[ g(\xi) = \frac{1}{2} (\xi - d)^2 \]

integer valued field: plastic strain

\[ d(i, j) \]
Elimination of elastic fields

\[ \hat{\xi}(q) = (s_y^+(q)s_y^-(q)\hat{d}(q) + \hat{H}(q))/\hat{\lambda}(q) \]

quenched disorder

\[ \hat{H}(q) = s_x^-(q)s_x^+(q)\hat{h}_1(q) + s_y^-(q)s_y^+(q)\hat{h}_2(q) \]

elasticity

\[ s_{a\pm}^\mp(q) = \pm(1 - \cos(q_a) \pm i\sin(q_a)) \]

\[ \hat{\lambda}(q) = 2K(\cos(q_x) - 1) + s_y^-(q)s_y^+(q) \]
Thresholds

\[ \Delta \xi = \xi - (\xi_0 + \xi_h) \]

\[ \hat{\xi}_0(q) = t \delta(q) \]

\[ \hat{\xi}_h(q) = \frac{\hat{H}(q)}{\hat{\lambda}(q)} \]

\[ -\xi^0 - \xi(h, t) < \Delta \xi(i, j) < \xi^0 - \xi(h, t) \]

\[ \xi(h, t) = [\hat{\xi}_h]^{-1}_q + t \]
Automaton

\[ d \rightarrow d + M(\Delta \xi) \]

\[ M(\Delta \xi) = \begin{cases} 
+ 1, & \text{if } \Delta \xi > \xi^0 - \xi(h, t), \\
- 1, & \text{if } \Delta \xi < -\xi^0 - \xi(h, t) \\
0, & \text{otherwise.} 
\end{cases} \]

\[ \hat{\Delta} \xi \rightarrow \hat{\Delta} \xi - \hat{L}(q) \hat{M}(\Delta \xi) \]

\[ \hat{L}(q) = \frac{\sin(q_y/2)^2}{\sin(q_y/2)^2 + K \sin(q_x/2)^2} \]
Macroscopic response
Scaling relations in automaton model

\[ P(E) = E^{-\epsilon} \varphi\left(\frac{E}{E_c}\right) \]

\[ E_c \sim N^\delta \]
Scaling collapse of avalanche shapes

\[< E(t)_T > = T^{\gamma-1} e(t/T) \]

\[< E(t)_{E_0} > = E_0^{1-\alpha} \tilde{c}(t/E_0^\alpha) \]
Conclusions

• Model of crystal plasticity allowing one to study cooperative effects in dislocation dynamics without additional assumptions such as nucleation rules and kinetic relations. The model shows an excellent agreement with experimental observations of fluctuations during plastic yielding in HCP crystals.

• In the limit of slow driving continuous dynamics can be reduced to an integer automaton. Despite partial linearization implied by the use of piece-wise quadratic potential and the replacement of fast stages of dynamics by a sequence of jumps, the automaton model exhibits the same critical behavior as the original ODE based model.

• Our numerical study suggests that the automaton model has some form of Abelian property which makes it amenable in principle to rigorous mathematical treatment.

References: