Local statistics of
the abelian sandpile model
David B. Wilson

The looping rate and sandpile density
of planar graphs
Joint work with Adrien Kassel

Spanning trees of graphs on surfaces and
the intensity of loop-erased random walk
on planar graphs
Joint work with Richard Kenyon
[sandpile demo]
Equivalence between the Abelian sandpile model and the q → 0 limit of the Potts model
Abstract: We establish an equivalence between the undirected Abelian sandpile model and the q → 0 limit of the q-state Potts model. The equivalence is valid for arbitrary finite graphs. Two-dimensional Abelian sandpile models thus correspond to a conformal field theory ...
Cited by 236  Related articles  All 9 versions  Cite  Save

Abelian sandpile model on the Bethe lattice
Abstract: We study Bak, Tang and Wiesenfeld's Abelian sandpile model of selforganised criticality on the Bethe lattice. Exact expressions for various distribution functions including the height distribution at a site and the joint distribution of heights at two sites separated by ...
Cited by 158  Related articles  All 8 versions  Cite  Save

Height correlations in the Abelian sandpile model
Abstract: We study the distribution of heights in the self-organized critical state of the Abelian sandpile model on a d-dimensional hypercubic lattice. We calculate analytically the concentration of sites having minimum allowed value in the critical state. We al-
Cited by 136  Related articles  All 6 versions  Cite  Save

Abelian sandpile model
HF Chau - Physical Review E, 1993 - aipsabs.harvard.edu
Abstract: A systematic and simple method to find the correlation function of the Abelian sandpile model up to any finite order is developed. In addition, an algorithm for evaluating the distribution function of the avalanche size P (a) exactly is also discovered along the ...
Cited by 1  Related articles  All 6 versions  Cite  Save

Rare events and breakdown of simple scaling in the Abelian sandpile model
M De Menech, AL Stella, C Tebaldi - Physical Review E, 1998 - APS
Abstract: Due to intermittency and conservation, the Abelian sandpile in two dimensions obeys multifractal, rather than finite size scaling. In the thermodynamic limit, a vanishingly small fraction of large avalanches dominates the statistics and a constant gap scaling is ...
Cited by 113  Related articles  All 8 versions  Cite  Save

Formation of avalanches and critical exponents in an Abelian sandpile model
VB Priezzhev, DV Kitarev, EV Ivashechkin - Physical review letters, 1996 - APS
Abstract: The structure of avalanches in the Abelian sandpile model on a square lattice is analyzed. It is shown that an avalanche can be considered as a sequence of waves of decreasing sizes. Being more simple objects, waves admit a representation in terms of spanning trees ...
Cited by 8  Related articles  All 6 versions  Cite  Save
Underlying graph

Uniform spanning tree
Uniform spanning tree on infinite grid

Pemantle: limit of UST on large boxes converges as boxes tend to $\mathbb{Z}^d$

Pemantle: limiting process has one tree if $d \leq 4$, infinitely many trees if $d > 4$
UST and LERW on $\mathbb{Z}^2$

Benjamini-Lyons-Peres-Schramm: UST on $\mathbb{Z}^d$ has one end if $d>1$, i.e., one path to infinity
Local statistics of UST can be computed via determinants of transfer impedance matrices (Burton—Pemantle).

Why doesn’t this give local statistics of sandpiles?
Infinite volume limit

• Infinite volume limit exists (Athreya—Jarai ’04)

• \( \Pr[h=0] = \frac{2}{\pi^2} - \frac{4}{\pi^3} \) (Majumdar—Dhar ’91)

• Other one-site probabilities computed by Priezzhev (’93)

\[
P(2) = \frac{1}{2} - \frac{3}{2\pi} - \frac{2}{\pi^2} + \frac{12}{\pi^3} + \frac{I_0}{4}
\]

with

\[
I_0 = \frac{1}{(2\pi)^4} \iiint_0^{2\pi} \frac{i \sin (\beta_1) \det (M)}{D(\alpha_1, \beta_1)D(\alpha_2, \beta_2)D(\alpha_1 + \alpha_2, \beta_1 + \beta_2)} \times d\alpha_1 \, d\alpha_2 \, d\beta_1 \, d\beta_2
\]

\[
D(\alpha, \beta) = 2 - \cos (\alpha) - \cos (\beta)
\]

and \( M \) is a 4 × 4 matrix

\[
M = \begin{pmatrix}
1 & 1 & e^{i\alpha_2} & 1 \\
3 & e^{i(\beta_1 + \beta_2)} & e^{i(\alpha_2 - \beta_2)} & e^{i\beta_1} \\
(4/\pi - 1) & e^{i(\alpha_1 + \alpha_2)} & 1 & e^{-i\alpha_1} \\
(4/\pi - 1) & e^{-i(\alpha_1 + \alpha_2)} & e^{2i\alpha_2} & e^{i\alpha_1}
\end{pmatrix}
\]

The numerical evaluation of the integral (2) leads to

\[
P(2) = 0.1739 \ldots
\]

The solution is based on an analogy
\[ P(2) = \frac{1}{2} - \frac{3}{2\pi} - \frac{2}{\pi^2} + \frac{12}{\pi^3} + \frac{I_1}{4} \]  
\[ P(3) = \frac{1}{4} + \frac{3}{2\pi} + \frac{1}{\pi^2} - \frac{12}{\pi^3} - \frac{I_1}{2} - \frac{3I_2}{32} \]  
\[ P(4) = \frac{1}{4} - \frac{1}{\pi^2} + \frac{4}{\pi^3} + \frac{I_1}{4} + \frac{3I_2}{32} \]

Here \( I_v, \ v = 1, 2, \) are integrals:
\[ I_v = \frac{1}{(2\pi)^4} \iint_0^{2\pi} \frac{i \sin(\beta_1) \det(M_v)}{D(\alpha_1, \beta_1) D(\alpha_2, \beta_2) D(\alpha_1 + \alpha_2, \beta_1 + \beta_2)} d\alpha_1 \ d\alpha_2 \ d\beta_1 \ d\beta_2 \]

where
\[ D(\alpha, \beta) = 2 - \cos(\alpha) - \cos(\beta) \]

and \( M_1, M_2 \) are matrices,
\[
M_1 = \begin{pmatrix}
1 & 1 & e^{i\alpha_2} & 1 \\
3 & e^{i(\beta_1 + \beta_2)} & e^{i(\alpha_2 - \beta_2)} & e^{-i\beta_1} \\
4/\pi - 1 & e^{i(\alpha_1 + \alpha_2)} & 1 & e^{-i\alpha_1} \\
4/\pi - 1 & e^{-i(\alpha_1 + \alpha_2)} & e^{2i\alpha_2} & e^{i\alpha_1}
\end{pmatrix}
\]

and
\[
M_2 = \begin{pmatrix}
e^{i\beta_2} & e^{-i(\alpha_1 + \alpha_2) - i(\beta_1 + \beta_2)} & e^{i\beta_1} \\
e^{-i\alpha_2} & 1 & e^{-i\alpha_1} \\
e^{i\alpha_2} & e^{-2i(\alpha_1 + \alpha_2)} & e^{i\alpha_1}
\end{pmatrix}
\]

The numerical evaluation of integrals in Eq. (6) leads to
\[ P(2) = 0.1739..., \quad P(3) = 0.3063..., \quad P(4) = 0.4461..., \]

in good agreement with the high-statistics data.
\[ P_2 = \frac{1}{2} - \frac{1}{\pi} - \frac{3}{\pi^2} + \frac{12}{\pi^3} - \frac{\pi - 2}{2\pi} J_2 \approx 0.1739, \quad (4.10) \]

\[ P_3 = \frac{1}{4} + \frac{2}{\pi} - \frac{12}{\pi^3} - \frac{8 - \pi}{4\pi} J_2 \approx 0.3063. \quad (4.11) \]

\[ J_2 = \frac{4}{\pi^2} - \frac{14}{\pi} - 8 - \frac{4\sqrt{2}}{\pi^2} \int_0^\pi \frac{d\beta_1}{\sqrt{3 - \cos \beta_1}} \int_{-\pi}^\pi \frac{d\beta_2}{1 - t_1 t_2 t_3} \sin \frac{\beta_1 - \beta_2}{2} \left[ \cos \frac{\beta_1 - \beta_2}{2} - 2 \cos \frac{\beta_1 + \beta_2}{2} \right] \times \left[ (3 - \cos \beta_1 + \cos \beta_2) \cos \frac{\beta_1}{2} - 2 \sin \beta_2 \sin \frac{\beta_1}{2} \right], \quad (4.16) \]

where \( t_i = y_i - \sqrt{y_i^2 - 1}, \ y_i = 2 - \cos \beta_i \) and \( \beta_3 = -(\beta_1 + \beta_2) \). This integral expression has been used for the numerical evaluation of \( J_2 \), yielding \( J_2 = 0.5 + o(10^{-12}) \).

Remarkably these values imply an even simpler formula for the mean height in the bulk,

\[ \langle h \rangle = P_1 + 2P_2 + 3P_3 + 4P_4 = \frac{25}{8}, \quad (4.15) \]

a value conjectured by Grassberger \cite{3}. The striking simplicity of this result clearly calls for a
Sandpile density and LERW

Conjecture: path to infinity visits neighbor to right with probability 5/16 (Levine—Peres, Poghosyan—Priezzhev)
Sandpile density and LERW

Theorem: path to infinity visits neighbor to right with probability $5/16$
(Poghosyan-Priezzhev-Ruelle, Kenyon—W)

JPR integral evaluates to $\frac{1}{2}$
(Caracciolo—Sportiello)

Proof involving only rationals
(Kassel—W)
<table>
<thead>
<tr>
<th>lattice</th>
<th>discrete-time LERW looping rate $\rho = \tau + \frac{1}{2} \Pr[e \in T]$</th>
<th>sandpile density $\bar{\sigma} = (\delta \rho + \delta - 1)/2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>square</td>
<td>$\frac{5}{16}$, 0.3125</td>
<td>$\frac{17}{8}$, 2.125</td>
</tr>
<tr>
<td>triangular</td>
<td>$\frac{5}{18}$, 0.277778…</td>
<td>$\frac{10}{3}$, 3.333333…</td>
</tr>
<tr>
<td>honeycomb</td>
<td>$\frac{13}{36}$, 0.361111…</td>
<td>$\frac{37}{24}$, 1.541667…</td>
</tr>
<tr>
<td>kagomé / trihexagonal</td>
<td>$\frac{1}{3}$, 0.333333…</td>
<td>$\frac{13}{6}$, 2.166667…</td>
</tr>
<tr>
<td>dice / rhombille</td>
<td>$\frac{5}{16}$, 0.3125</td>
<td>$\frac{17}{8}$, 2.125</td>
</tr>
<tr>
<td>Fisher / truncated hexagonal</td>
<td>$\frac{359}{900}$, 0.398889…</td>
<td>$\frac{959}{600}$, 1.598333…</td>
</tr>
<tr>
<td>triakis triangular</td>
<td>$\frac{7}{25}$, 0.28</td>
<td>$\frac{167}{50}$, 3.34</td>
</tr>
<tr>
<td>square-octagon / truncated square</td>
<td>$\frac{3 \arccsc(3)}{8} - \frac{\arccsc(3)^2}{12 \sqrt{2 \pi}} + \frac{\arccsc(3)^2}{8 \pi^2}$</td>
<td>$\frac{25 \arccsc(3)}{16} - \frac{3 \arccsc(3)^2}{8 \sqrt{2 \pi}} + \frac{3 \arccsc(3)^2}{16 \pi^2}$</td>
</tr>
<tr>
<td>tetrakis square</td>
<td>$\frac{7 \arccsc(3)}{24} - \frac{\arccsc(3)^2}{12 \sqrt{2 \pi}} + \frac{\arccsc(3)^2}{16 \pi^2}$</td>
<td>$\frac{27 \arccsc(3)}{8} - \frac{3 \arccsc(3)^2}{4 \sqrt{2 \pi}} + \frac{3 \arccsc(3)^2}{16 \pi^2}$</td>
</tr>
</tbody>
</table>
\[ \text{det} \begin{bmatrix} w_{1,2} + w_{1,3} & -w_{1,2} & -w_{1,3} \\ -w_{2,1} & w_{2,1} + w_{2,3} & -w_{2,3} \\ -w_{3,1} & -w_{3,2} & w_{3,1} + w_{3,2} \end{bmatrix} = w_{1,2}w_{2,3} + w_{2,1}w_{1,3} + w_{1,3}w_{2,3} \]
\[ F_2(G) = \sum_{v \neq s} \det \Delta_{v,s}^{u,s} - \sum_{u \sim v \atop u,v \neq s} w_{u,v} \det \Delta_{u,v,s}^{u,v,s} \]

\[ \frac{F_2(G)}{F_1(G)} = \sum_{v \neq s} \det G_v^v - \sum_{u \sim v \atop u,v \neq s} w_{u,v} \det G_{u,v}^{u,v} \]

\[ = \sum_{v \neq s} G_{v,v} - \sum_{u \sim v \atop u,v \neq s} w_{u,v} [G_{u,u} G_{v,v} - G_{u,v}^2] \]

\[ G^{(s)}_{u,v} = \begin{cases} \left[ (\Delta_{s,s}^{s})^{-1} \right]_{u,v} & u, v \neq s, \\ 0 & u = s \text{ or } v = s \end{cases} \]
\[
\frac{F_2(G)}{F_1(G)} = \sum_{u \sim v} w_{u,v} \left[ (A^{(s)}_{u,v} - A^{(s)}_{v,u})^2 + A^{(s)}_{u,v}A^{(s)}_{v,u} \right]
\]

\[
A^{(s)}_{u,v} = G^{(s)}_{u,u} - G^{(s)}_{u,v}
\]
\[ \rho = \text{discrete-time LERW looping rate} = \frac{\text{weighted sum of oriented CRST's}}{\text{weighted sum of marked oriented CRST's}} \]

\[ \tau = \frac{\text{weighted sum of oriented CRST's with cycle length } \geq 3}{\text{weighted sum of marked oriented CRST's}} \]

\[ \rho - \tau = \frac{1}{2} \Pr[\text{random edge } e \in \text{random tree } T] \]

\[ \tau = \sum_{u^* \sim v^*} \frac{w_{u^*,v^*}}{1/w_{u^*,v^*}} \left( A_{u^*,v^*}^{(s^*)} A_{v^*,u^*}^{(s^*)} + \left( A_{u^*,v^*}^{(s^*)} - A_{v^*,u^*}^{(s^*)} \right)^2 \right) \]
\[ T_G(x, y) = \sum_{E' \subseteq E} (x - 1)^{k(E') - 1} (y - 1)^{k(E') + |E'| - |V|} \]

\[ \sum_{\text{recurrent sandpiles } \sigma} y^{\text{level}(\sigma)} = T_G(1, y) \]

\[ \sum_{\text{recurrent sandpiles } \sigma} \binom{\text{level}(\sigma)}{j} = \frac{1}{j!} \frac{d^j}{dy^j} T_G(1, y) \bigg|_{y = 1} = \text{# connected subgraphs of } G \text{ with } |V| + j - 1 \text{ edges.} \]

\[ \mathbb{E}[\text{level}(\sigma)] = \frac{\text{# unicycles of } G}{\text{# spanning trees of } G} = \tau \times |E| \]
Joint distribution of heights at two neighboring vertices

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>9/32 - 9/2π + 47/π² - 48/π³ + 32/π⁴</th>
<th>-33/64 + 191/32π - 383/16π² + 81/2π³ - 47/2π⁴</th>
<th>15/64 - 47/32π + 39/16π² + 7/2π³ - 17/2π⁴</th>
</tr>
</thead>
<tbody>
<tr>
<td>-33/64 + 191/32π - 383/16π² + 81/2π³ - 47/2π⁴</td>
<td>83/32 - 243/8π + 537/4π² - 256/π³ + 619/4π⁴ + 59/π⁵</td>
<td>-107/32 + 617/16π - 1259/8π² + 291/2π³ - 375/π⁴ - 108/π⁵</td>
<td>105/64 - 421/32π + 753/16π² - 175/2π³ + 225/4π⁴ + 49/π⁵</td>
<td></td>
</tr>
<tr>
<td>15/64 - 47/32π + 39/16π² + 7/2π³ - 17/2π⁴</td>
<td>-15/32 + 97/16π - 227/8π² + 111/2π³ - 79/4π⁴ - 27/π⁵</td>
<td>105/64 - 421/32π + 753/16π² - 175/2π³ + 225/4π⁴ + 49/π⁵</td>
<td>-33/32 + 129/16π - 161/8π² + 65/2π³ - 28/π⁴ + 22/π⁵</td>
<td></td>
</tr>
</tbody>
</table>

0.     0.0103411  0.0238479  0.0394473
0.0103411  0.0260442  0.0525221  0.0849925
0.0238479  0.0525221  0.0930601  0.136861
0.0394473  0.0849925  0.136861  0.184871
Higher dimensional marginals of sandpile heights

\[
\Pr[3,2,1,0 \text{ in } 4\times1 \text{ rectangle}] =
\]
\[
\frac{61815}{128} + \frac{1856395}{128\pi} - \frac{99783277}{576\pi^2} + \frac{964096235}{864\pi^3} - \frac{5588534021}{1296\pi^4} + \frac{5014047485}{486\pi^5}
- \frac{10884136816}{729\pi^6} + \frac{25765891840}{2187\pi^7} - \frac{23058546688}{6561\pi^8} - \frac{319225856}{729\pi^9} \div 0.00169649.
\]
Sandpiles on hexagonal lattice

\[ \Pr[h = 0] = \frac{1}{12} \]
\[ \Pr[h = 1] = \frac{7}{24} \]
\[ \Pr[h = 2] = \frac{5}{8} \]

(One-site probabilities also computed by Ruelle)
Sandpiles on triangular lattice

Pr[h = 0] = \(-\frac{25}{648} - \frac{55}{72\sqrt{3}\pi} + \frac{7}{3\pi^2} + \frac{11\sqrt{3}}{\pi^3} - \frac{90}{\pi^4} + \frac{54\sqrt{3}}{\pi^5}\) ≈ 0.053623

Pr[h = 1] = \(\frac{47}{1296} + \frac{301}{24\sqrt{3}\pi} - \frac{193}{6\pi^2} - \frac{29\sqrt{3}}{\pi^3} + \frac{405}{\pi^4} - \frac{270\sqrt{3}}{\pi^5}\) ≈ 0.091525

Pr[h = 2] = \(\frac{3}{8} - \frac{5929}{144\sqrt{3}\pi} + \frac{1441}{12\pi^2} - \frac{9\sqrt{3}}{\pi^3} - \frac{720}{\pi^4} + \frac{540\sqrt{3}}{\pi^5}\) ≈ 0.137356

Pr[h = 3] = \(\frac{3427}{2592} + \frac{6515}{144\sqrt{3}\pi} - \frac{2125}{12\pi^2} + \frac{91\sqrt{3}}{\pi^3} + \frac{630}{\pi^4} - \frac{540\sqrt{3}}{\pi^5}\) ≈ 0.189037

Pr[h = 4] = \(-\frac{2663}{1296} - \frac{71\sqrt{3}}{16\pi} + \frac{1331}{12\pi^2} - \frac{94\sqrt{3}}{\pi^3} - \frac{270}{\pi^4} + \frac{270\sqrt{3}}{\pi^5}\) ≈ 0.242307

Pr[h = 5] = \(\frac{1175}{864} - \frac{365}{144\sqrt{3}\pi} - \frac{289}{12\pi^2} + \frac{30\sqrt{3}}{\pi^3} + \frac{45}{\pi^4} - \frac{54\sqrt{3}}{\pi^5}\) ≈ 0.286152
Spanning tree in which path from start to boundary uses given directed edge

\[
\Pr[\text{LERW uses directed edge}] = \frac{Z(1,3|2,4)}{Z}
\]
grove of type $1/2$

Kirchhoff: $R_{1,2} = \frac{Z(1/2)}{Z}$
grove of type \( 1 \big/ 2, 3 \)

\[
\frac{Z(1/2,3)}{Z} = \frac{1}{2} R_{1,2} + \frac{1}{2} R_{1,3} - \frac{1}{2} R_{2,3}
\]

\[
\frac{Z(1/2/3)}{Z} = \frac{R_{1,2} R_{1,3} + R_{1,2} R_{2,3} + R_{1,3} R_{2,3}}{2} - \frac{R_{1,2}^2 + R_{1,3}^2 + R_{2,3}^2}{4}
\]
4 or more nodes

No formula of the above type that holds for general graphs.

Pairwise resistances do not determine \( \frac{Z(1,2/3,4)}{Z} \) for general graphs.

If graph is planar and all nodes on the same face, then \( \frac{Z(0)}{Z} \) is a polynomial in the \( R_{ij} \)'s.

(KW)
Line Bundle Laplacian

\[ \Delta = \begin{bmatrix}
1 & 2 & 3 \\
1 & -5/2 & -2 \\
2 & -5Z & 8 \\
3 & -2 & -3 \\
3 & -2 & 5
\end{bmatrix} \]

Forman:

\[ \text{det} \Delta = 30 (2 - Z - \frac{1}{2}Z) \]

\[ = \text{weighted sum of cycle-rooted spanning forests} \]
\[ \text{det} \Delta = 480 - 30z - 30z^2 \\
-72 - 150 - 40 \\
= 30(2-z-\frac{1}{2}) + 158 \]
Cartis - Ingerman - Morrow

det \( L_{i,j,k}^{4,5,6} \) = \( Z(14/25/36) - Z(14/26/35) \) 

\[ \begin{align*} 
&+ Z(15/26/34) - Z(15/24/36) \\
&+ Z(16/24/35) - Z(16/25/34) \\
&- Z(1/2/3/4/5/6) 
\end{align*} \]

general graph with \( 2n \) vertices
Response matrix

\[ L_{ij} : \text{apply voltage at } i, \text{ measure current at } j. \]

Fact: \[ \sum_i L_{ij} = 0 \]

(\#) \[ L_{ij} \text{ variables} \]

(\#) \[ R_{ij} \text{ variables} \]
\[ \det L_{4,5,6} = Z(4 \| 5 \| 6) - Z(4 \| 6 \| 5) \]
\[ + Z(5 \| 6 \| 4) - Z(5 \| 4 \| 6) \]
\[ + Z(6 \| 4 \| 5) - Z(6 \| 5 \| 4) \]
\[ \frac{Z(1 \| 2 \| 3 \| 4 \| 5 \| 6)}{Z(1 \| 2 \| 3 \| 4 \| 5 \| 6)} \]

Includes parallel transports

Cycle rooted graphs

(proof similar to C-I-M proof, but starts with Forman's MTT)