Improved Method to extract Nucleon Helicity Distributions using Event Weighting

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Preproposal for a KITP program

Frontiers in Nuclear Physics

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We also plan to have a workshop related to topic 3.

Outline

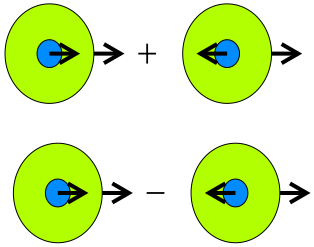
- Deep ineleastic scattering & helicity distributions
- Different methods to determine a polarisation P from event rates:
  (P : polarisation, parton helicity distribution,
  precession frequency)
- Application to quark helicity distributions

Deep Inelastic Scattering & Helicity Distributions

Helicity Distributions

$q = q^{\uparrow} + q^{\downarrow}$:

$\Delta q = q^{\uparrow} - q^{\downarrow}$:



accessible in deep inelastic scattering:

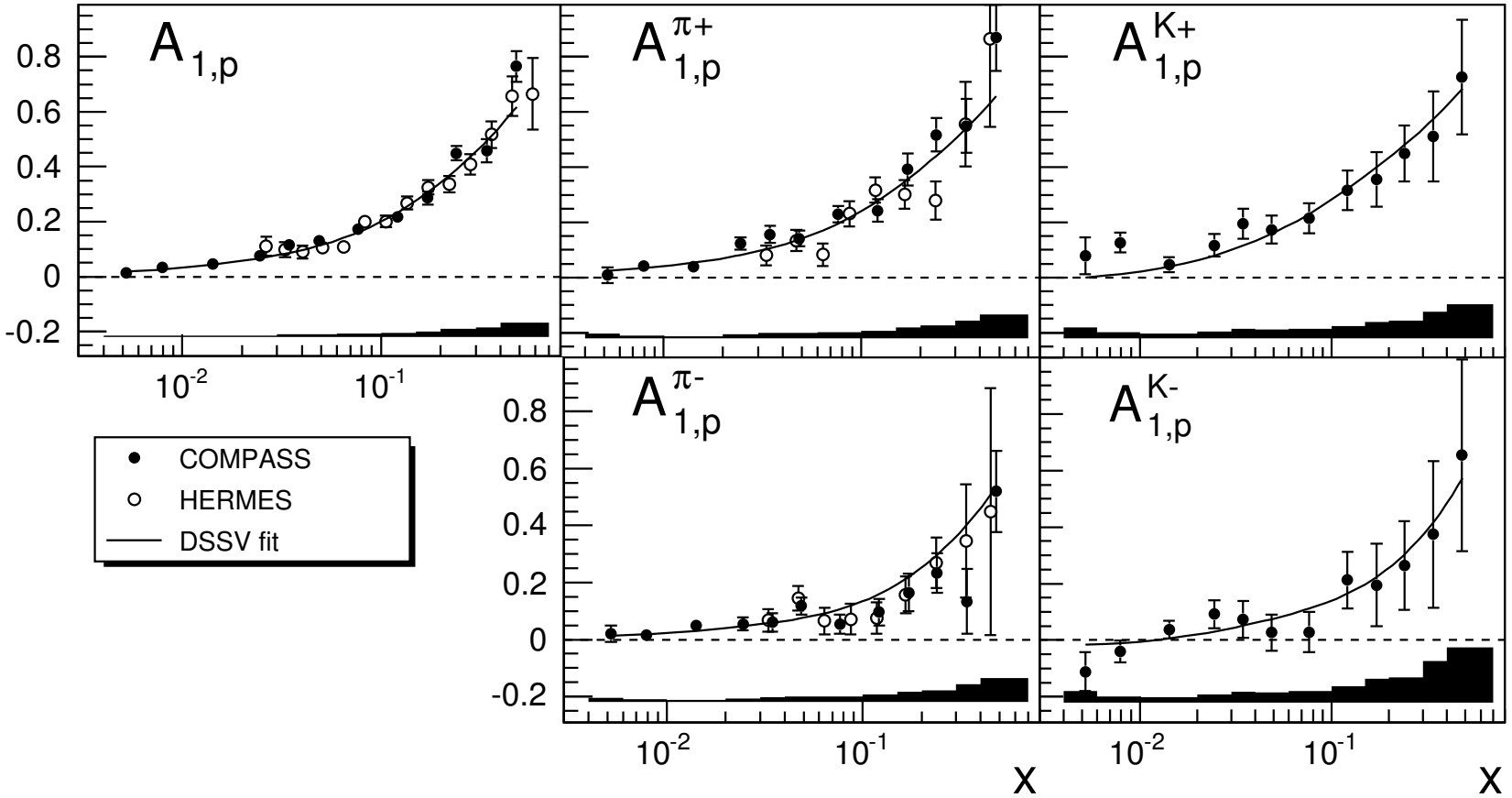
inclusive	
$\ell + N \to \ell' + X$	$F_1 = \frac{1}{2}\sum_q e_q^2(q(x) + \bar{q}(x))$
polarized inclusive	
$\vec{\ell} + \vec{N} \to \ell' + X$	$g_1 = \frac{1}{2}\sum_q 1 e_q^2(\Delta q(x) + \Delta\bar{q}(x))$
polarized semi-inclusive	
$\vec{\ell} + \vec{N} \to \ell' + h + X$	$\sum_q e_q^2(\Delta q(x) D_q^h(z) + \Delta\bar{q}(x) D_{\bar{q}}^h(z))$

Asymmetries

$$A^h = \frac{n_h^{\uparrow\downarrow} - n_h^{\uparrow\uparrow}}{n_h^{\uparrow\downarrow} + n_h^{\uparrow\uparrow}} \propto \frac{\sum_q e_q^2 \left(\Delta q(x) D_q^h(z) + \Delta \bar{q}(x) D_{\bar{q}}^h(z) \right)}{\sum_q e_q^2 \left(q(x) D_q^h(z) + \bar{q}(x) D_{\bar{q}}^h(z) \right)}$$

- $n_h^{(\uparrow\downarrow)\uparrow\uparrow}$ number of hadrons with target and beam polarisation (anti-)parallel
- D_q^h: fragmentation function,
  $D_q^h(z)\mathrm{d}z =$ number of hadrons of type h produced from a quark q with energy fraction in $[z, z + \mathrm{d}z]$
- use $D_u^{\pi^+} > D_{\bar{u}}^{\pi^+}$ to distinguish between u and $\bar{u}$

Input: Asymmetries



+ asymmetries on deuteron target

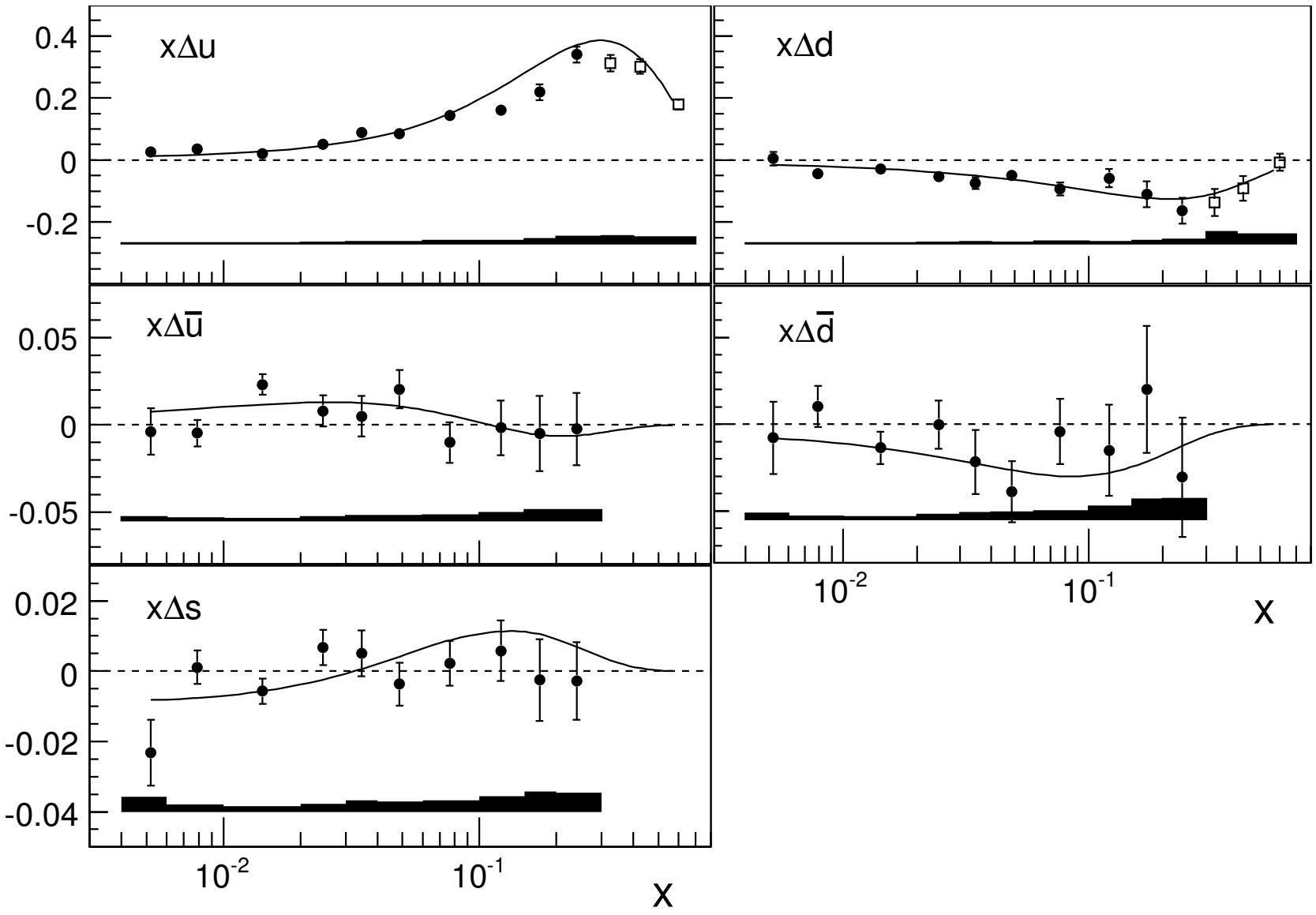
Asymmetries $\rightarrow$ Δq's

Solve:

$$\vec{A} = B\Delta\vec{q}$$

- $\vec{A} = (A_p, A_p^{\pi^+}, A_p^{K^+}, \ldots, A_d, \ldots, A_d^{K^-})$
- further input: $B(q(x), \int D_q^h(z)\mathrm{d}z)$
- $\rightarrow \Delta\vec{q} = (\Delta u, \Delta d, \Delta s, \Delta\bar{u}, \Delta\bar{d}, \Delta\bar{s})$

Output: $\Delta u(x), \Delta d(x), \Delta s(x) = \Delta\bar{s}(x), \Delta\bar{u}(x), \Delta\bar{d}(x)$



COMPASS data

Polarisation Measurements

What is the problem?

Consider events distributed according to

$$n^{\pm}(\vartheta) = \alpha(\vartheta)(1 \pm \beta(\vartheta)P)$$

Goal: Determine P, given the analyzing power $\beta(\vartheta)$ and the event rates $n^{\pm}(\vartheta)$ in two different polarisation states

$\alpha(\vartheta)$ contains flux, acceptance factor, generally not well known.

Examples

- $\vec{p}C$ scattering

  $n(\vartheta, \Phi)^{\pm} = n(\vartheta)\,(1 \pm A(\vartheta)\sin(\Phi)P_{\perp})$

  ϑ: polar angle, $n^{\pm}$ number of events with proton polarisation up/down
- muon decay $(g-2)$

  $$f(t, y) = \frac{n(y)}{\tau}\,\mathrm{e}^{-t/\tau}(1 + A(y)\sin(\omega t))$$

  t: time, $y = E_{e^-}/E_{\mu^+}$, ω: $g-2$ precession frequency
- helicity distributions Δq, ΔG

  $$n_{\pi^+}^{\uparrow\downarrow(\uparrow\uparrow)}(x, z) =$$

  $$a\Phi n\sigma\left(1 \pm P_T P_B f D\left(\frac{e_u^2 D_u^{\pi^+}(z)\Delta u(x) + e_d^2 D_d^{\pi^+}\Delta d(x) + \ldots}{e_u^2 D_u^{\pi^+}(z)u(x) + e_d^2 D_d^{\pi^+}(z)d(x) + \ldots}\right)\right)$$

  $a\Phi n\sigma$: acceptance, flux, target density, unpolarized cross section

  $P_T P_B f D$: target, beam polarisation, dilution factor, depolarisation factor

Several Methods to determine P

Compare:

1. Counting rate asymmetry
2. Binning
3. **Weighting**
4. Maximum Likelihood Method

with respect to their

Figure of Merit (FOM) = (statistical uncertainty)$^{-2}$

1.) Counting rate asymmetry

Count all events in certain ϑ-range:

$$\langle N^{\pm} \rangle = \int n^{\pm}(\vartheta)\mathrm{d}\vartheta = \int \alpha(\vartheta)\mathrm{d}\vartheta \pm P \int \alpha(\vartheta)\beta(\vartheta)\mathrm{d}\vartheta$$

Observed events are $N^{\pm}$
Consider

$$\left\langle \frac{N^{+} - N^{-}}{N^{+} + N^{-}} \right\rangle = \langle \beta \rangle \; P$$

$$\langle \beta \rangle = \frac{\int \alpha\beta\mathrm{d}\vartheta}{\int \alpha\mathrm{d}\vartheta} \approx \frac{\sum_{+} \beta(\vartheta_i) + \sum_{-} \beta(\vartheta_i)}{N^{+} + N^{-}}$$

Estimator

$$\hat{P} = \frac{N^{+} - N^{-}}{\sum_{+} \beta(\vartheta_i) + \sum_{-} \beta(\vartheta_i)}$$

1.) Counting rate asymmetry: Uncertainty

Statistical uncertainty ($\langle\beta\rangle\, P \ll 1$):

$$\text{FOM} = \sigma_P^{-2} = \langle\beta\rangle^2\, N$$

$N = N^+ + N^-$: total number of events

More familiar:

$$\sigma_P = \frac{1}{\langle\beta\rangle\,\sqrt{N}}$$

Can one do better? Yes!

2.) Bins in ϑ

Consider a bin j in ϑ:

$$\mathrm{FOM}_j = \langle \beta_j \rangle^2 \, n_j$$

Note: P does not depend on ϑ.

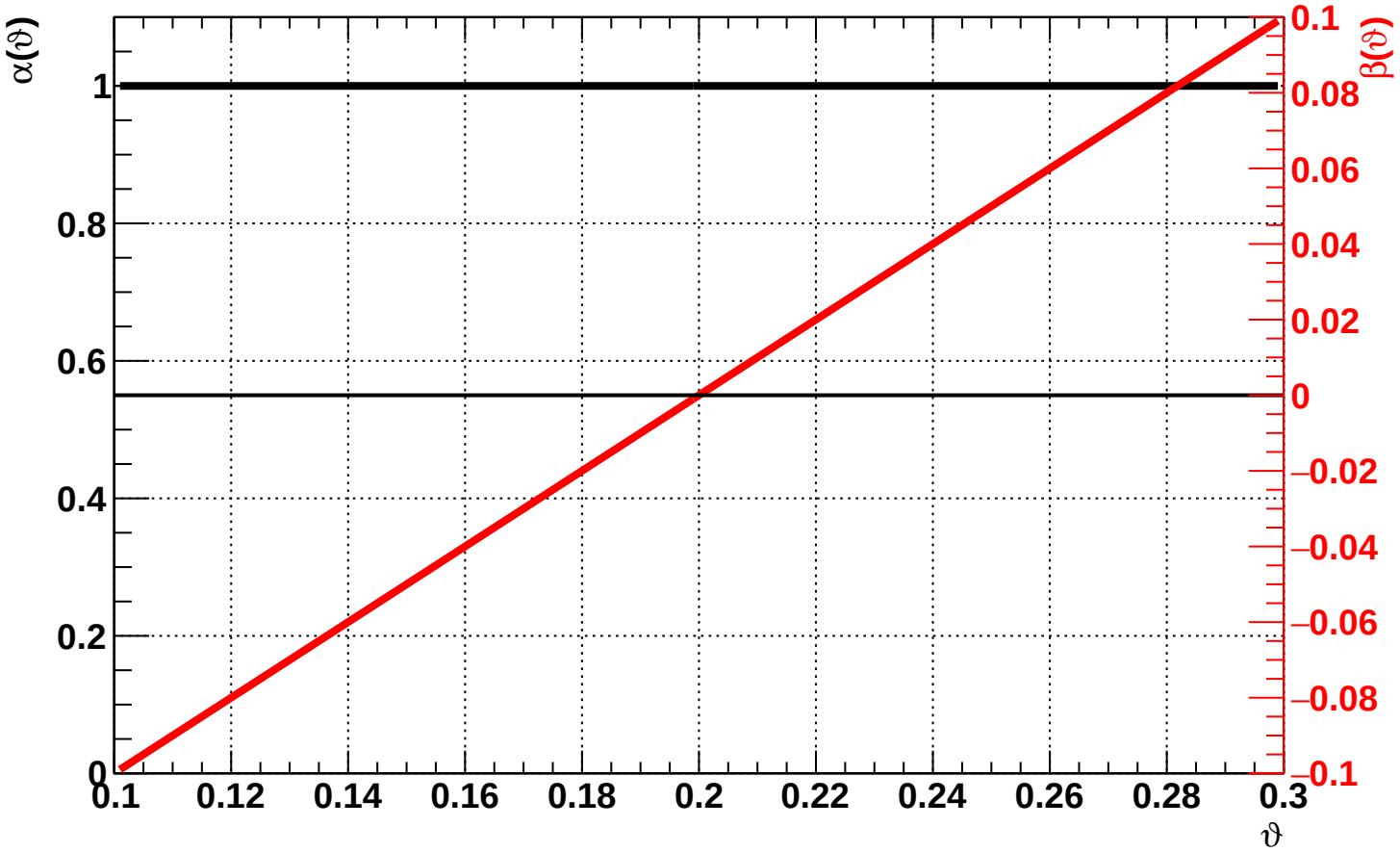
Now combine all bins (uncorrelated events)

$$\mathrm{FOM} = \sum_j \mathrm{FOM}_j = \sum_{j=1}^{N_{bin}} \langle \beta_j \rangle^2 \, n_j \overset{N_{bin}\to\infty}{=} \sum_j n_j \beta_j^2 = N \left\langle \beta^2 \right\rangle$$

Gain in FOM compared to counting rate method: $\dfrac{\langle \beta^2 \rangle}{\langle \beta \rangle^2} \geq 1$.

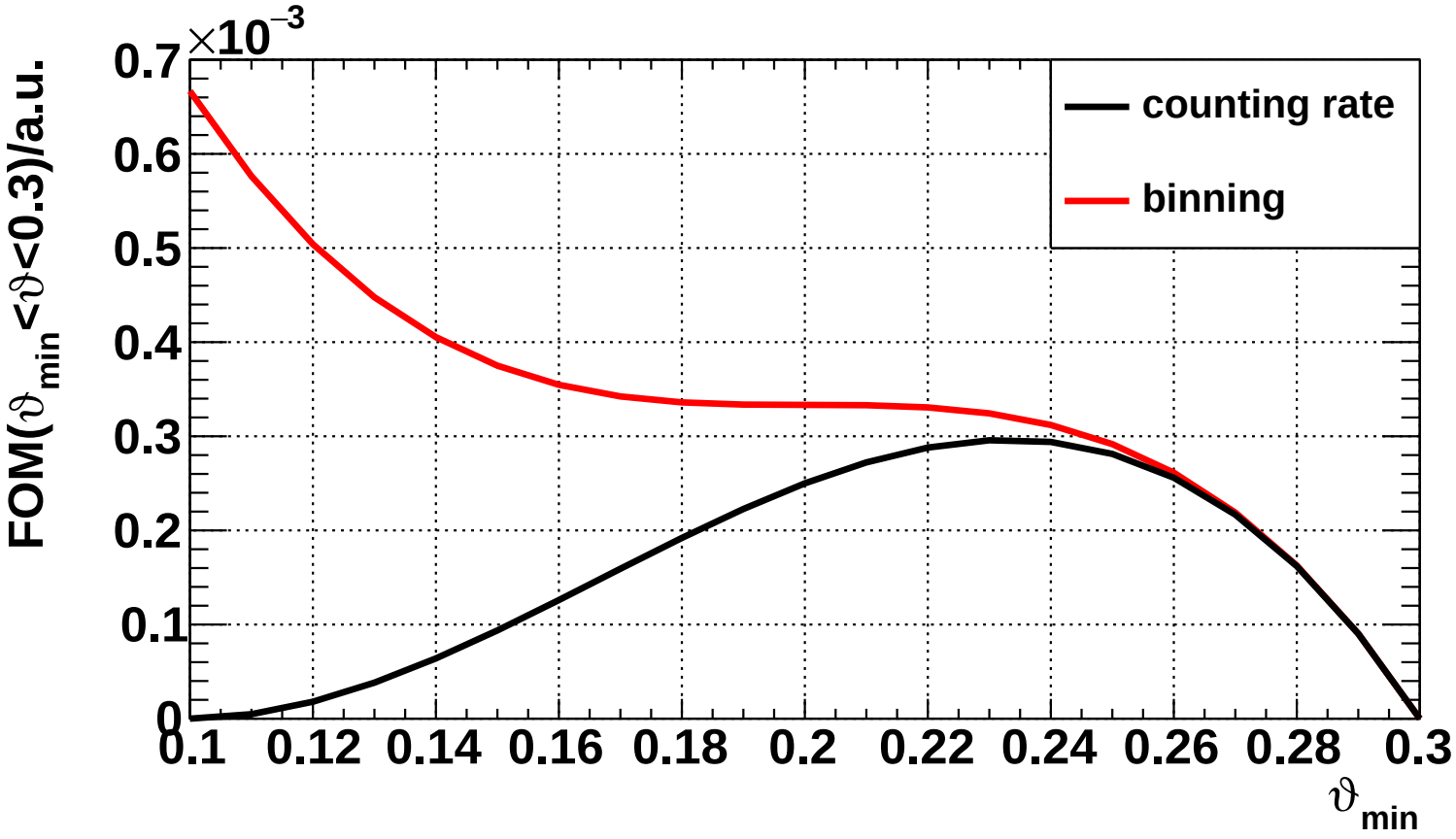
(Academic) Example

$\alpha(\vartheta) \propto \sigma =$ const.
$\beta(\vartheta) = \vartheta - 0.2$, in region $\vartheta_{min} = 0.1$ to $\vartheta_{max} = 0.3$.



(Academic) Example

FOM($\vartheta_{min} < \vartheta < 0.3$)



2.) Bins in ϑ

With binning FOM can be improved.
but: binning is sometimes inconvenient:

- Too few bins $\Rightarrow$ FOM not maximal
- Too many bins $\Rightarrow$ Empty bins

Is there an alternative? Yes!

3.) Event Weighting

Instead of just counting events, assign to every event an (for the moment) arbitrary) weight factor $w(\vartheta_i)$.
Consider the following **estimator** for P:

$$\hat{P} = \frac{\sum_+ w_i - \sum_- w_i}{\sum_+ w_i \beta_i + \sum_- w_i \beta_i}$$

where $w_i = w(\vartheta_i)$
Easy to show: $\left\langle \hat{P} \right\rangle = P$ independent of w
In words: Whatever you choose for w, you always get the correct result, **but** with different uncertainties.

3.) Event Weighting: Uncertainty

Simple error propagation leads to

$$\mathrm{FOM}_w = N \frac{\langle w\beta \rangle^2}{\langle w^2 \rangle}$$

Reminder: $\langle w\beta \rangle = \frac{\int \alpha w \beta \mathrm{d}\vartheta}{\int \alpha \mathrm{d}\vartheta} \approx \frac{\sum w_i \beta_i}{N}$

Two different weights

$w = 1$, for $\vartheta > \vartheta_{min}$, $w = 0$ else

$$\mathrm{FOM}_{w=1} = N \langle \beta \rangle^2$$

$\rightarrow$ like counting rate asymmetry in one bin

$w = \beta$

$$\mathrm{FOM}_{w=\beta} = N \left\langle \beta^2 \right\rangle$$

$\rightarrow$ same as infinite number of bins

Counting rates vs. Weighted events

	counting, $w = 1$	Binning, $w = \beta$, MLH
FOM	$N \langle \beta \rangle^2$	$N \langle \beta^2 \rangle$

Gain in FOM: $\dfrac{\langle \beta^2 \rangle}{\langle \beta \rangle^2}$

An event with a large analyzing power β tells you more about P than an event with lower β. It should thus enter the analysis with more weight.

Can one do better? No!

Connection to Maximum Likelihood Method

Maximum likelihood (MLH) method is known to give the best FOM (Cramér-Rao bound)
In general

$$\mathcal{L} = \prod_i p(\vartheta_i; P)\,,\ P \text{ follows from } \frac{\partial \mathcal{L}}{\partial P} \stackrel{!}{=} 0, \quad \text{FOM} = -\left\langle \frac{\partial^2 \ln \mathcal{L}}{\partial P^2} \right\rangle$$

In our case event rates n instead of probabilities p:
Extended likelihood method (Fermi) has to be applied

Log-likelihood function:

$$\begin{aligned} \ell = \log(\mathcal{L}) \quad &= \quad \sum_{+} \ln\left(\alpha_i(1+\beta_i P)\right) - \langle n^+ \rangle(P) \\ &+ \quad \sum_{-} \ln\left(\alpha_i(1-\beta_i P)\right) - \langle n^- \rangle(P)\,. \end{aligned}$$

Connection to Maximum Likelihood Method

$$\text{FOM} = -\left\langle \frac{\partial^2 \ln \mathcal{L}}{\partial P^2} \right\rangle = \left\langle \sum_{+} \frac{\beta_i}{(1+\beta_i P)^2} + \sum_{-} \frac{\beta_i}{(1-\beta_i P)^2} \right\rangle$$

$$\overset{\beta P \ll 1}{=} \left\langle \sum_{+} \beta_i^2 + \sum_{-} \beta_i^2 \right\rangle = N \left\langle \beta^2 \right\rangle$$

FOM of likelihood method is the same as in binning or weighting method

First Summary

- weighting the events with their analyzing power β gives the largest FOM
- Gain with respect to just counting events is $\frac{\mathrm{FOM}_{w=\beta}}{\mathrm{FOM}_{cnt}} = \frac{\langle \beta^2 \rangle}{\langle \beta \rangle^2}$

Application to Helicity Distributions

Back to helicity distributions

$$A^h = \frac{n_h^{\uparrow\downarrow} - n_h^{\uparrow\uparrow}}{n_h^{\uparrow\downarrow} + n_h^{\uparrow\uparrow}} \propto \frac{\sum_q e_q^2 \left(\Delta q(x) D_q^h(z) + \Delta \bar{q}(x) D_{\bar{q}}^h(z) \right)}{\sum_q e_q^2 \left(q(x) D_q^h(z) + \bar{q}(x) D_{\bar{q}}^h(z) \right)}$$

To simplify consider a proton at a given Bjorken-x with $u = 2$ and $d = 1$.
Consider only event rates of π^+ and π^-
(notation: $+$ for π^+ and $-$ for π^-)

$$\begin{aligned}
n_+^{\uparrow\downarrow}(z) &\propto \alpha^+(1 + \beta_u^+(z)\Delta u + \beta_d^+(z)\Delta d) \\
n_+^{\uparrow\uparrow}(z) &\propto \alpha^+(1 - \beta_u^+(z)\Delta u - \beta_d^+(z)\Delta d) \\
n_-^{\uparrow\downarrow}(z) &\propto \alpha^-(1 + \beta_u^-(z)\Delta u + \beta_d^-(z)\Delta d) \\
n_-^{\uparrow\uparrow}(z) &\propto \alpha^-(1 - \beta_u^-(z)\Delta u - \beta_d^-(z)\Delta d)
\end{aligned}$$

α's and β's

with

$$\alpha^+ = \frac{4}{9} u D_{fav} + \frac{1}{9} d D_{unf}\,, \qquad \alpha^- = \frac{4}{9} u D_{unf} + \frac{1}{9} d D_{fav}\,,$$

$$\beta_u^+ = \frac{4 D_{fav}}{4 u D_{fav} + d D_{unf}}\,, \qquad \beta_d^+ = \frac{D_{unf}}{4 u D_{fav} + d D_{unf}}\,,$$

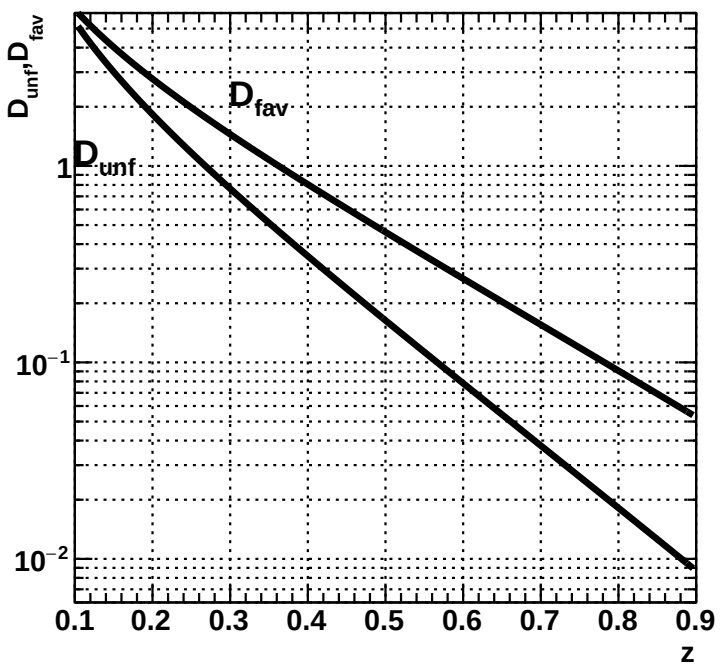
$$\beta_u^- = \frac{4 D_{unf}}{4 u D_{unf} + d D_{fav}}\,, \qquad \beta_d^- = \frac{D_{fav}}{4 u D_{unf} + d D_{fav}}\,.$$

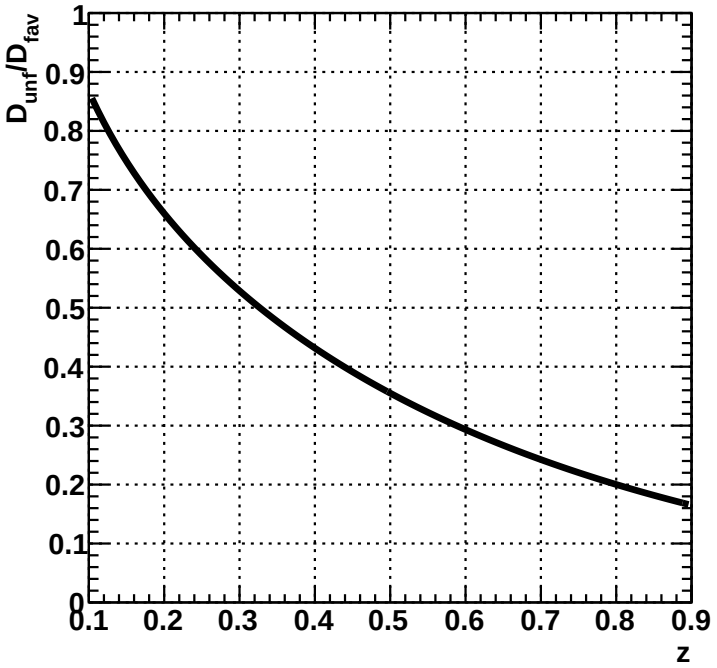
only two different fragmentation function are present:

favored $\quad D_{fav} = D_u^{\pi^+} = D_d^{\pi^-}$

unfavored $\quad D_{unf} = D_u^{\pi^-} = D_d^{\pi^+}$

Fragmentation Functions

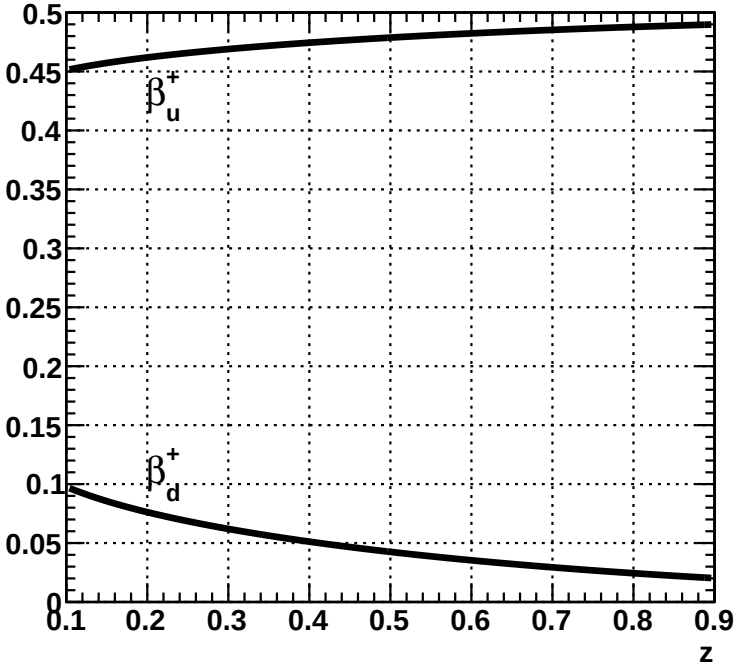




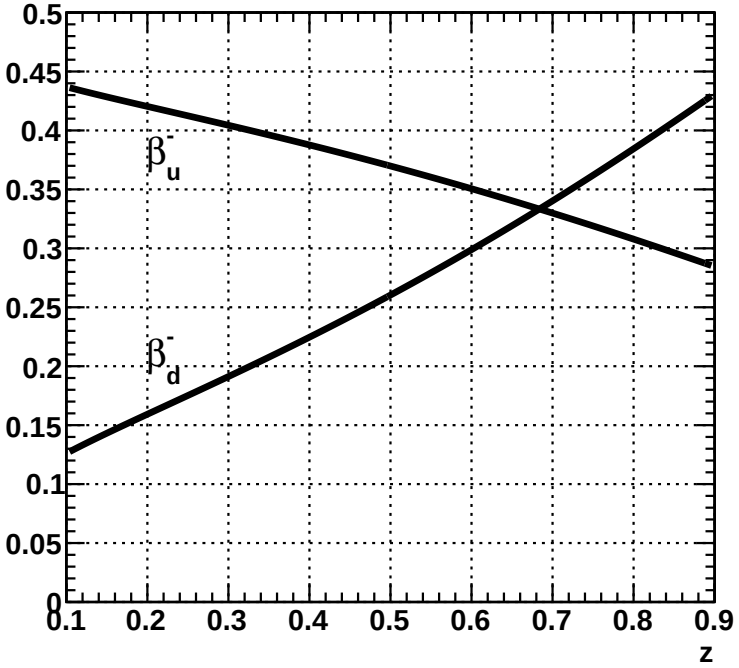
Event at large z carries more information on struck quark.

Weight factors

$\beta_u^+(z), \beta_d^+(z)$



$\beta_u^-(z), \beta_d^-(z)$



Connection to discussion before

Mathematically:

before:	1 unknown, 2 event rates
now:	N unknowns, $2 \times M$ event rates
	to simplify notation:
	$N = 2$ (Δu and Δd) and
	$M = 2 \times 2$ event rates ($n_+^{\uparrow\downarrow}, n_+^{\uparrow\uparrow}, n_-^{\uparrow\downarrow}, n_-^{\uparrow\uparrow}$)

Weighted Asymmetries

Consider:

$$a_{\beta_u^+} := \frac{\sum_{\uparrow\downarrow} \beta_u^+(z_i) - \sum_{\uparrow\uparrow} \beta_u^+(z_i)}{\sum_{\uparrow\downarrow}(\beta_u^+(z_i))^2 + \sum_{\uparrow\uparrow}(\beta_u^+(z_i))^2},$$

similar expression for $a_{\beta_u^-}$ $a_{\beta_d^+}$ $a_{\beta_d^-}$, i.e. there is one asymmetry per quark flavor and observed hadron.
Solve

$$\vec{a} = B(\beta)\Delta\vec{q}\,, \quad \vec{a} = (a_{\beta_u^+}, a_{\beta_u^-}, a_{\beta_d^+}, a_{\beta_d^-})$$

FOM

$$\mathrm{FOM}_{\Delta u} = \left(N_+ \left\langle (\beta_u^+)^2 \right\rangle + N_- \left\langle (\beta_u^-)^2 \right\rangle \right) (1 - \rho^2)$$

with

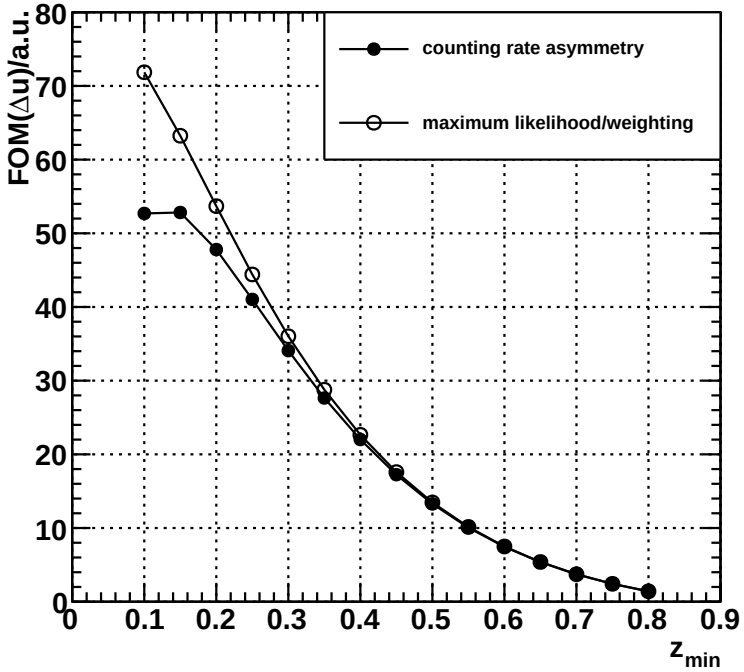
$$\rho = -\frac{N_+ \left\langle \beta_u^+ \beta_d^+ \right\rangle + N_- \left\langle \beta_u^- \beta_d^- \right\rangle}{\sqrt{\left(N_+ \left\langle (\beta_u^+)^2 \right\rangle + N_- \left\langle (\beta_u^-)^2 \right\rangle \right) \left(N_+ \left\langle (\beta_d^+)^2 \right\rangle + N_- \left\langle (\beta_d^-)^2 \right) \right\rangle}}$$

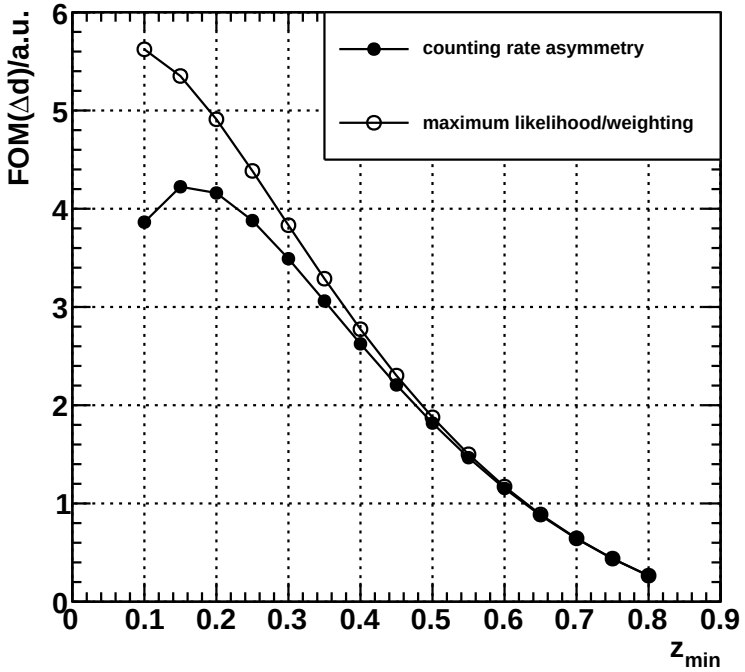
ρ turns out to be the correlation coefficient $\mathrm{cov}(\Delta u, \Delta d)$

$N_+ = N_+^{\uparrow\downarrow} + N_+^{\uparrow\uparrow}$, $N_- = N_-^{\uparrow\downarrow} + N_-^{\uparrow\uparrow}$

Figure of Merit for Δu and Δd

$z_{min} < z < 0.9$





- $\text{FOM}_{w=\beta} \geq \text{FOM}_{w=1}$
- adding data at low z decreases the FOM for the counting rate asymmetry
- Gain up to 30%

Summary

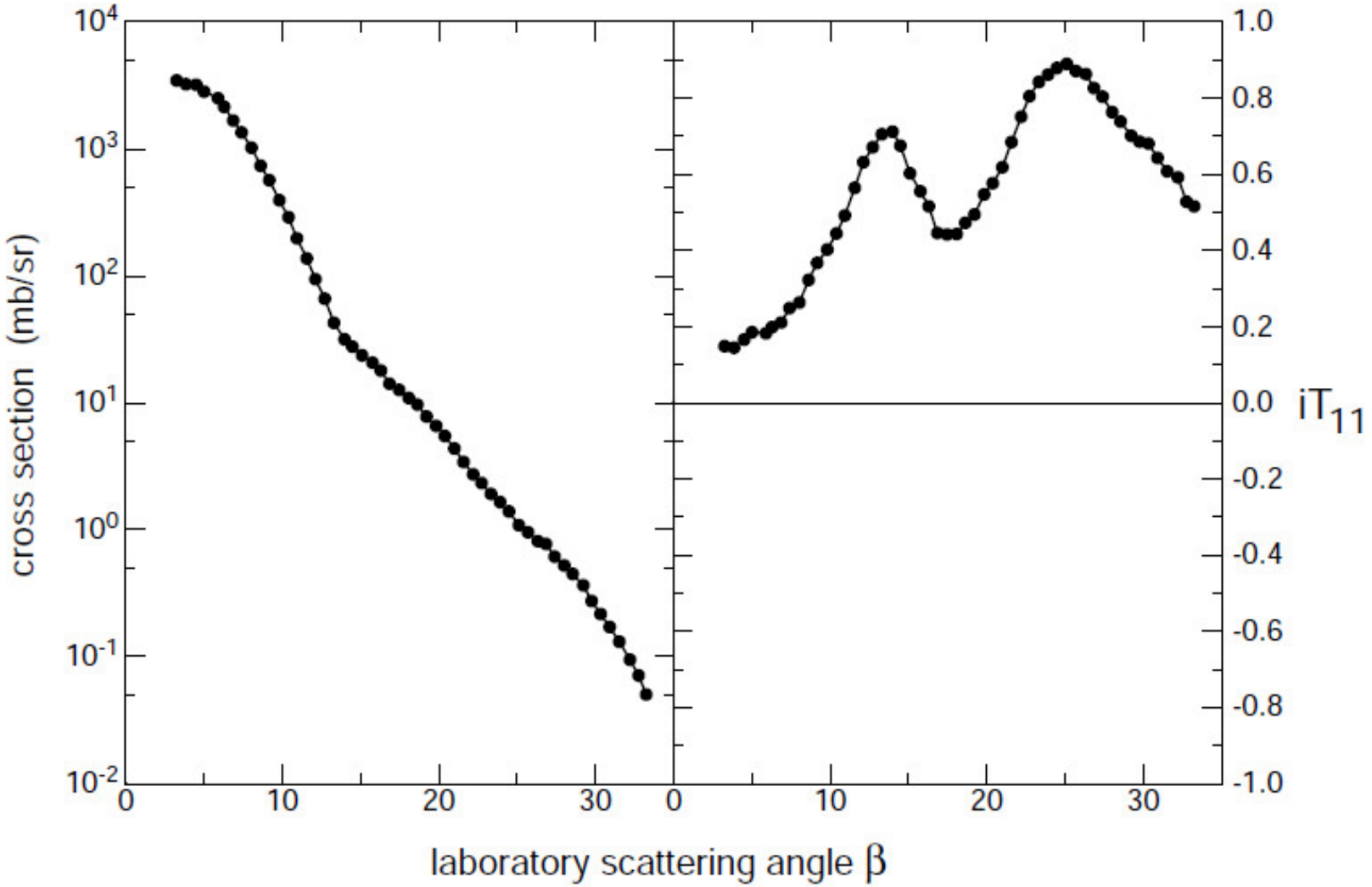
Method	Counting Rates	MLH	weighting	binning in z
FOM	non-optimal	optimal	optimal	optimal for $N_{bin} \to \infty$
drawbacks	see above $\uparrow$	CPU intensive, correlation (π^+, π^-) hard to implement		empty bin problem

References

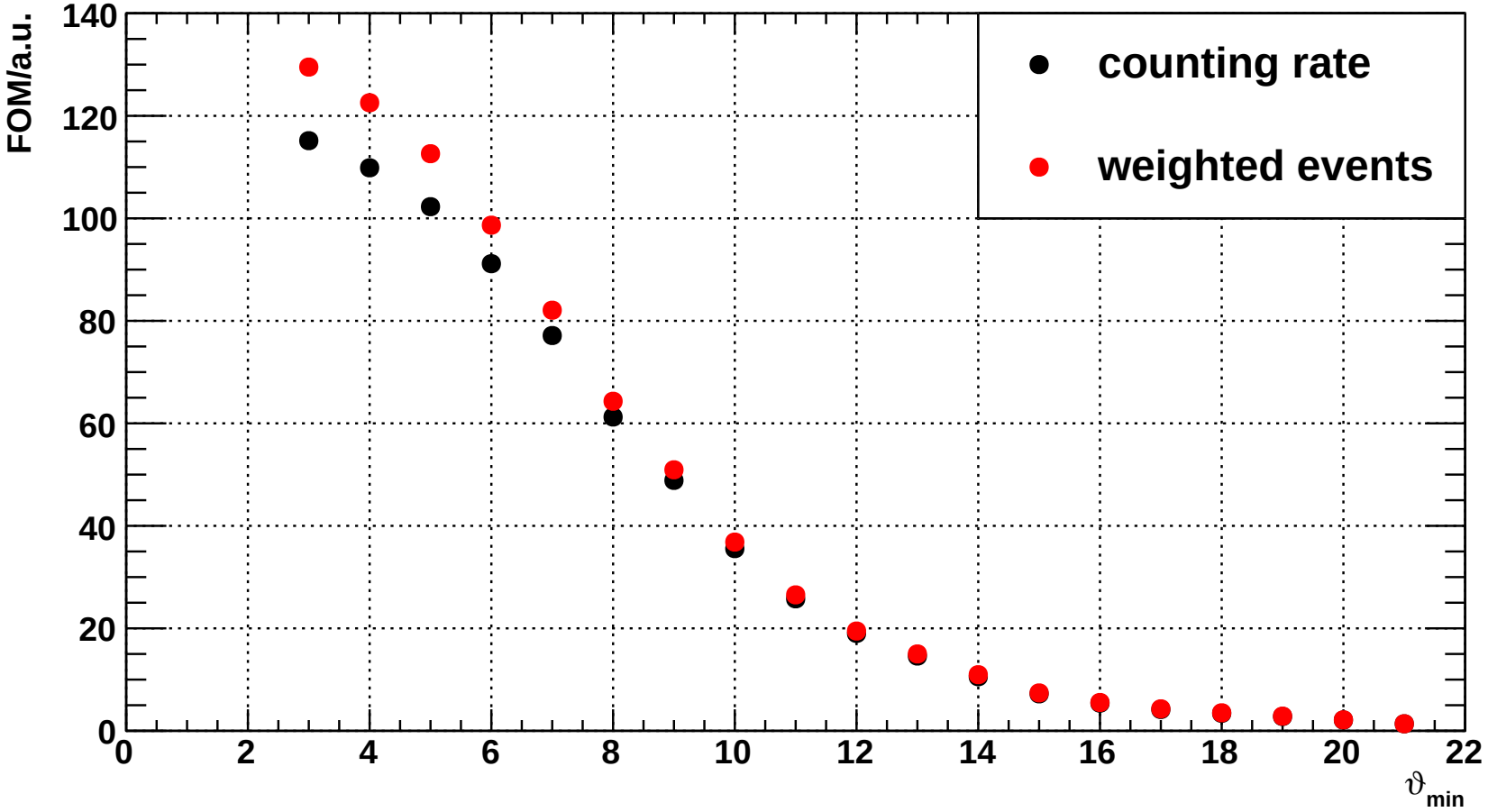
- "Comparison of methods to extract an asymmetry parameter from data," JP,
  Nucl. Instrum. Meth. A **659** (2011) 456
  arXiv:1104.1038
- "Simultaneous Determination of Signal and Background Asymmetries,"
  JP and J. M. Le Goff,
  Nucl. Instrum. Meth. A **602** (2009) 594
  arXiv:0811.1426

Spare

Example: Elastic deuteron carbon scattering at $T = 270\text{MeV}$



Example


FOM/a.u.
140
120
100
80
60
40
20
0
0
2
4
6
8
10
12
14
16
18
20
22
ϑ_{min}
counting rate
weighted events

FOM for arbitrary $\langle\beta\rangle$ P

$$\begin{aligned}
\mathrm{FOM}_{\hat{P}_{LH}} &= N\left\langle \frac{\beta^2}{1-\beta^2 P^2} \right\rangle \\
\mathrm{FOM}_{\hat{P}_{w=\beta}} &= N\frac{\left\langle \beta^2 \right\rangle}{1 - P^2 \frac{\left\langle \beta^4 \right\rangle}{\left\langle \beta^2 \right\rangle}} \\
\mathrm{FOM}_{\hat{P}_{iw}} &= N\frac{\left\langle \frac{\beta^2}{1-\beta^2 P_0^2} \right\rangle^2}{\left\langle \beta^2 \frac{1-\beta^2 P^2}{(1-\beta^2 P_0^2)^2} \right\rangle}
\end{aligned}$$

FOM for arbitrary $\langle\beta\rangle$ P